Preventing Self-Fulfilling Debt Crises: The Role of Expectations

Additional Results

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1 Introduction

This document contains additional derivations that have been omitted in the main appendix, but which a reader might be interested in. In Section 1 I derive the expression for the change in the default threshold A^* when the government implements the policy change with probability $p \in (0,1)$. In Section 2 I solve the complete information model and characterize the fragility region. I also derive the sufficient bound for B_1 that ensures that the government always wants to borrow a positive amount in the fragility region, and the sufficient bound on Z that ensures that the government's desired unconstrained borrowing is strictly increasing in A.

2 Multiplier under uncertainty: Derivations

In this section I derive the change in the default threshold when households and lenders are uncertain as to whether the policy change will be implemented. The households and lenders assign probability p to the government implementing the new policy. As in the case of no uncertainty (Section B of the main Appendix) I am interested in understanding the effect of an announcement of a small policy change on the default threshold. To do so, I start by considering a situation where with probability $(1-p)$ the policy parameter takes value ψ (which I associate with the case when the policy change is not implemented) and with probability p the policy parameter takes value ψ' (which I associate with the new level of the policy parameter if the policy is implemented). I then compute the effect of a further change in ψ', and I impose the condition that initially $\psi' = \psi$. By following these steps, I obtain the effect of an announcement of a change in the policy parameter when such a change will take place with probability p.

Let A^* be the threshold if the policy parameter takes value ψ (i.e., the policy change is not implemented) and $A^{*\prime}$ be the policy threshold when the policy parameter takes value ψ' (i.e., the policy is implemented).[1] Then the equilibrium conditions can be written as

$$
\begin{aligned}
(1-p)\, I\left(A^{*}+\kappa\varepsilon, A^{*}, k_{2}^{*}(\kappa), \psi\right)+p I\left(A^{*}+\kappa\varepsilon, A^{*\prime}, k_{2}^{*}(\kappa), \psi^{\prime}\right) &= 0 \quad (1)\\
(1-p)\, I\left(A^{*\prime}+\kappa\varepsilon, A^{*}, k_{2}^{*\prime}(\kappa), \psi\right)+p I\left(A^{*\prime}+\kappa\varepsilon, A^{*\prime}, k_{2}^{*\prime}(\kappa), \psi^{\prime}\right) &= 0 \quad (2)\\
(1-p)\, L\left(A^{*}, x^{*}, \psi\right)+p L\left(A^{*\prime}, x^{*}, \psi^{\prime}\right) &= 0 \quad (3)\\
\Delta V\left(A^{*};\left\{k_{2}^{*}(\kappa)\right\}_{\kappa\in[0,1]}, x^{*}, A^{*}, \psi\right) &= 0 \quad (4)\\
\Delta V\left(A^{*\prime};\left\{k_{2}^{*}(\kappa)\right\}_{\kappa\in[0,1]}, x^{*}, A^{*\prime}, \psi^{\prime}\right) &= 0, \quad (5)
\end{aligned}
$$

where $k_2^*(\kappa)$ denotes an individual household's equilibrium investment when that household's productivity is equal to $A^*+\kappa\varepsilon$, while $k_2^{*\prime}$ denotes the individual household's equilibrium investment when that household's productivity is equal $A^{*\prime}+\kappa\varepsilon$.

When households and lenders are uncertain whether an announced policy will be implemented, there are additional equilibrium equations compared to the case considered in Section B in the Appendix to the paper. This is because we need now to determine the default threshold both when the policy in implemented and when it is not (the possibility of a policy change also affects the threshold even if in the end the policy is not implemented). In particular, to compute the equilibrium default threshold when the policy parameter takes value ψ, we need both the government default condition and the household investment decisions evaluated at ψ (Equations 1 and 4). Similarly, to compute the equilibrium default threshold when the policy parameter takes value ψ', we need both the government default condition and the household investment decisions evaluated at ψ' (Equations 2 and 5).

To compute the effect of a policy announcement when the policy is expected to be implemented with probability p, I follow an approach similar to the one in Section B of th Appendix. That is, I begin by considering the total derivatives of all equilibrium equations with respect to ψ'. Let I_i^{**} denote the partial derivative of $I(\cdot)$ with respect to its ith argument evaluated at the average productivity level A^* and when the households' belief is $A^{**}=A^*$, $I_i^{**\prime}$ denote the partial derivative of $I(\cdot)$ with respect to its ith argument evaluated at the average productivity level A^* and when households' belief is $A^{**}=A^{*\prime}$, $I_i^{*\prime *}$ when productivity is $A^{*\prime}$ and households' belief is $A^{**}=A^*$ and, finally, $I_i^{*\prime*\prime}$ when productivity is $A^{*\prime}$ and households' belief is $A^{**}=A^{*\prime}$. Similarly, let ΔV_j be the partial derivative of $\Delta V\left(A^*;\left\{k_2^*(\kappa)\right\}_{\kappa\in[0,1]}, x^*, \psi\right)$ with respect to the jth argument, and $\Delta V_j'$ be the partial derivative of $\Delta V\left(A^*;\left\{k_2^*(\kappa)\right\}_{\kappa\in[0,1]}, x^*, \psi'\right)$ with respect to the jth argument. Then,

[1] For example if the relevant policy parameter is a tax rate τ and the government contemplates increasing tax rate to $\tau'>\tau$ then $\psi=\tau$ while $\psi'=\tau'$.

$$(1-p)\left[I_1^{**}\frac{dA^*}{d\psi'}+I_2^{**}\frac{dA^{**}}{d\psi'}+I_3^{**}\frac{dk_2^*}{d\psi'}\right]+pI\left(I_1^{**\prime}\frac{dA^*}{d\psi'}+I_2^{**\prime}\frac{dA^{**}}{d\psi'}+I_3^{**\prime}\frac{dk_2^*}{d\psi'}+I_4^*\right) = 0 \quad (6)$$

$$(1-p)\left[I_1^{*\prime *}\frac{dA^*}{d\psi'}+I_2^{*\prime *}\frac{dA^{**}}{d\psi'}+I_3^{*\prime *}\frac{dk_2^{*\prime}}{d\psi'}\right]+p\left(I_1^{*\prime *\prime}\frac{dA^*}{d\psi'}+I_2^{*\prime *\prime}\frac{dA^{**}}{d\psi'}+I_3^{*\prime *\prime}\frac{dk_2^{*\prime}}{d\psi'}+I_4^{*\prime *\prime}\right) = 0 \quad (7)$$

$$(1-p)\left[L_1\frac{dA^{**}}{d\psi'}+L_2\frac{dx^*}{d\psi'}\right]+p\left[L_1\frac{dA^{**\prime}}{d\psi'}+L_2\frac{dx^*}{d\psi'}+L_3\right] = 0 \quad (8)$$

$$\Delta V_1\frac{dA^*}{d\psi'}+\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{dk_2^*(\kappa)}{d\psi'}d\kappa+\Delta V_3\frac{dx^*}{d\psi'} = 0 \quad (9)$$

$$\Delta V_1'\frac{dA^{*\prime}}{d\psi'}+\int_{-1}^{1}\Delta V_2'(\kappa)\frac{1}{2}\frac{dk_2^{*\prime}(\kappa)}{d\psi'}d\kappa+\Delta V_3'\frac{dx^*}{d\psi'}+\Delta V_4' = 0 \quad (10)$$

The above equations constitutes the system of linear equations in: $dk_2^*(\kappa)/d\psi'$, $dk_2^{*\prime}(\kappa)/d\psi'$, $dx^*/d\psi'$, $dA^*/d\psi'$, $dA^{*\prime}/d\psi'$. Below I solve the above system for $dA^*/d\psi'$ and $dA^{*\prime}/d\psi'$.

2.1 Detailed Derivations

Start with $dx^*/d\psi'$. Rearranging Equation (8) yields

$$\frac{dx^*}{d\psi'}=-\frac{(1-p)L_1^{**}}{(1-p)L_2^{**}+pL_2^{**\prime}}\frac{dA^{**}}{d\psi'}-\frac{pL_1^{**\prime}}{(1-p)L_2^{**}+pL_2^{**\prime}}\frac{dA^{**\prime}}{d\psi'}-\frac{pL_3^{**\prime}}{(1-p)L_2^{**}+pL_2^{**\prime}}$$

Note that we are interested in computing the effect of a change in ψ' from the initial situation when $\psi'=\psi$ and $A^{*\prime}=A^*$ (i.e., a situation where a policy parameter is fixed at its initial level). Hence, it follows that $L_2^{**}=L_2^{**\prime}$ and $L_1^{**}=L_1^{**\prime}$ and, thus, the above condition simplifies to

$$\frac{dx^*}{d\psi'}=(1-p)\frac{L_1^{**}}{L_2^{**}}\frac{dA^{**}}{d\psi'}+p\frac{L_1^{**\prime}}{L_2^{**}}\frac{dA^{**\prime}}{d\psi'}+\frac{pL_3^{**\prime}}{L_2^{**}}.$$

Then,

$$\frac{dx^*}{d\psi'}=(1-p)\left[\frac{\partial x^*}{\partial A^{**}}\right]^c\frac{dA^{**}}{d\psi'}+p\left[\frac{\partial x^*}{\partial A^{**\prime}}\right]^c\frac{dA^{**\prime}}{d\psi'}+p\left[\frac{\partial x^*}{\partial\psi}\right]^c,$$

where $[\partial x^*/\partial A^{**}]^c$ indicates that a given partial derivative is defined as the corresponding partial derivative under no uncertainty (or under "certainty"). In what follows I define all partial derivatives in the same way they were defined under no policy uncertainty and, to save on notation, I omit the superscript c. Thus, I write

$$\frac{dx^*}{d\psi'}=(1-p)\frac{\partial x^*}{\partial A^{**}}\frac{dA^{**}}{d\psi'}+p\frac{\partial x^*}{\partial A^{**\prime}}\frac{dA^{**\prime}}{d\psi'}+p\frac{\partial x^*}{\partial\psi},$$

where it should be understood that $\partial x^*/\partial A^{**}$ corresponds to the partial effect of a change in belief regarding A^{**} under no policy uncertainty and so on. I use the same convention below.

Defining all partial effects in this way makes it easy to compare the case with uncertainty about the policy implementation with the case of no uncertainty.[2]

Next, consider $dk_2^*/d\psi$ and note that when $\psi' = \psi$ then $A^{*\prime} = A^*$, and hence it follows that $I_1^{**} = I_1^{**\prime}$, $I_2^{**} = I_2^{**\prime}$, $I_3^{**} = I_3^{**\prime}$ and $I_4^{**} = I_4^{**\prime}$. Therefore, one can write $dk_2^*/d\psi'$ as

$$\frac{dk_2^*}{d\psi'} = -(1-p)\frac{\partial k_2^*}{\partial A^*}\frac{dA^*}{d\psi'} - (1-p)\frac{\partial k_2}{\partial A^{**}}\frac{dA^{**}}{d\psi'} - p\frac{\partial k_2^*}{\partial A^*}\frac{dA^*}{d\psi'} - p\frac{\partial k_2^*}{\partial A^{**\prime}}\frac{dA^{**\prime}}{d\psi'} - p\frac{\partial k_2^*}{\partial \psi'},$$

where the partial effects are defined in the same way as under no uncertainty (see the discussion above and Footnote 2).

Now, substituting the above expressions $dx^*/d\psi'$ and $dk_2^*/d\psi'$ into Equation 10 and rearranging we obtain

$$\frac{dA^*}{d\psi'} = \frac{-(1-p)\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(\kappa)}{\partial A^{**}}\frac{dA^{**}}{d\psi'}d\kappa - p\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\left[\frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}}\frac{dA^{**\prime}}{d\psi'} + \frac{\partial k_2^*}{\partial \psi'}\right]d\kappa}{\left[\Delta V_1 + \int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(\kappa)}{\partial A^*}d\kappa\right]} + \frac{-(1-p)\Delta V_3\frac{\partial x^*}{\partial A^{**}}\frac{dA^{**}}{d\psi'} - p\Delta V_3\frac{\partial x^*}{\partial A^{**\prime}}\frac{dA^{**\prime}}{d\psi'} - p\Delta V_3^*\frac{\partial x^*}{\partial \psi}}{\left[\Delta V_1 + \int_{-1}^{1}\Delta V_2(\kappa)\frac{1}{2}\frac{\partial k_2^*(\kappa)}{\partial A^*}d\kappa\right]}$$

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[2] To be precise, let superscript u denote the partial derivative under uncertainty and c denote the partial effect under no uncertainty (or certainty). Then, the partial effect of an increase in ψ' on x^* under uncertainty, when initially $\psi' = \psi$, is

$$\begin{aligned}\left[\frac{\partial x^*}{\partial \psi'}\right]^u &= -p\frac{L_3^{**\prime}}{(1-p)L_2^{**} + pL_2^{**\prime}} = -p\frac{\frac{\partial}{\partial \psi'}L(A^{**\prime}, x^*, \psi')\Big|_{\{A^{**\prime}=A^*, \psi'=\psi\}}}{(1-p)\frac{\partial}{\partial x^*}L(A^{**}, x^*, \psi)\Big|_{A^{**}=A^*} + \frac{\partial}{\partial x^*}pL(A^{**\prime}, x^*, \psi')\Big|_{\{A^{**\prime}=A^*, \psi'=\psi\}}} \\ &= -p\frac{L_1(A^*, x^*, \psi)}{L_2(A^*, x^*, \psi)}\end{aligned}$$

since all the partial derivatives are computed when $\psi = \psi'$ and $A^{**} = A^{**\prime} = A^* = A^{*\prime}$. On the other hand

$$\left[\frac{\partial x^*}{\partial \psi'}\right]^c = -\frac{\frac{\partial}{\partial \psi}L(A^{**}, x^*, \psi)\Big|_{\{A^{**\prime}=A^*\}}}{\frac{\partial}{\partial x^*}L(A^{**}, x^*, \psi)\Big|_{\{A^{**\prime}=A^*\}}} = -\frac{L_1(A^*, x^*, \psi)}{L_2(A^*, x^*, \psi)}.$$

Comparing the above expressions we note that

$$\frac{\partial x^*}{\partial \psi'}^u = p\frac{\partial x^*}{\partial \psi'}^c.$$

In a similar fashion, one can show that

$$\left[\frac{\partial x^*}{\partial A^{**}}\right]^u = (1-p)\left[\frac{\partial x^*}{\partial A^{**}}\right]^c$$

and so on. The same argument applies to the households' investment decisions.

As in the case of no uncertainty, I define

$$\int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa \equiv -\frac{\int_{-1}^{1} \frac{1}{2} \Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa}{\Delta V_1 + \int_{-1}^{1} \Delta V_2(\kappa) \frac{1}{2} \frac{\partial k_2^*(\kappa)}{\partial A^*} d\kappa}$$

$$\int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}} d\kappa \equiv -\frac{\int_{-1}^{1} \frac{1}{2} \Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}} d\kappa}{\Delta V_1 + \int_{-1}^{1} \frac{1}{2} \Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^*} d\kappa},$$

and

$$\int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi'} d\kappa \equiv -\frac{\int_{-1}^{1} \frac{1}{2} \Delta V_2(\kappa) \frac{\partial k_2^*}{\partial \psi'} d\kappa}{\Delta V_1 + \int_{-1}^{1} \frac{1}{2} \Delta V_2(\kappa) \frac{\partial k_2^*}{\partial A^*} d\kappa}$$

With the above definitions, and recognizing that in equilibrium $A^{**} = A^*$ (and thus $dA^*/d\psi' = dA^{**}/d\psi$), we obtain

$$\frac{dA^*}{d\psi'} = \frac{p \frac{\partial A^*}{\partial x^*} \frac{\partial x^*}{\partial \psi'} + p \int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi'} d\kappa}{1 - (1-p) \frac{\partial A^*}{\partial x^*} \frac{\partial x^*}{\partial A^{**}} - (1-p) \int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa}$$
$$+ \frac{p \frac{\partial A^*}{\partial x^*} \frac{\partial x^*}{\partial A^{**\prime}} + p \int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}} d\kappa}{1 - (1-p) \frac{\partial A^*}{\partial x^*} \frac{\partial x^*}{\partial A^{**}} - (1-p) \int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa} \frac{dA^{*\prime}}{d\psi'}$$

Following the same steps as above, we obtain the expression for $dA^{*\prime}/d\psi'$ which is given by

$$\frac{dA^{*\prime}}{d\psi'} = \frac{\frac{\partial A^{*\prime}}{\partial \psi} + p \frac{\partial A^{*\prime}}{\partial x^*} \frac{\partial x^*}{\partial \psi'} + p \int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi'} d\kappa}{1 - p \frac{\partial A^{*\prime}}{\partial x^*} \frac{\partial x^*}{\partial A^{**\prime}} - p \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}} d\kappa}$$
$$+ \frac{(1-p) \frac{\partial A^{*\prime}}{\partial x^*} \frac{\partial x^*}{\partial A^{**}} + (1-p) \int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}} d\kappa}{1 - p \frac{\partial A^{*\prime}}{\partial x^*} \frac{\partial x^*}{\partial A^{**\prime}} - p \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}} d\kappa} \frac{dA^*}{d\psi'}$$

Substituting the expression for $dA^*/d\psi'$ into the expression for $dA^{*\prime}/d\psi'$ and rearranging yields

$$\frac{dA^{*\prime}}{d\psi'} = \frac{\frac{\partial A^{*\prime}}{\partial \psi} \left[1 - (1-p) \frac{\partial A^{*\prime}}{\partial x^*} \frac{\partial x^*}{\partial A^{**}} - (1-p) \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa \right]}{1 - p \frac{\partial A^{*\prime}}{\partial x^*} \frac{\partial x^*}{\partial A^{**\prime}} - p \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}} d\kappa - (1-p) \frac{\partial A^{*\prime}}{\partial x^*} \frac{\partial x^*}{\partial A^{**}} - (1-p) \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa} \tag{11}$$
$$+ \frac{p \frac{\partial A^{*\prime}}{\partial x^*} \frac{\partial x^*}{\partial \psi'} + p \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi'} d\kappa}{1 - p \frac{\partial A^{*\prime}}{\partial x^*} \frac{\partial x^*}{\partial A^{**\prime}} - p \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}} d\kappa - (1-p) \frac{\partial A^{*\prime}}{\partial x^*} \frac{\partial x^*}{\partial A^{**}} - (1-p) \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa}$$

Equation (11) then provides the expression for the effect of a policy announcement on the default threshold when households and lenders believe that the policy will be implemented with probability $p \in [0,1]$ and the policy ends up being implemented.

The above expression for $dA^{*\prime}/d\psi'$ can be used to derive Equation (5) in the main text. Let $\frac{dA^{*\prime}}{d\psi'}(p)$ denote the change in the default threshold when the policy is implemented with probability p so that $\frac{dA^{*\prime}}{d\psi'}(1)$ corresponds to the change in the default threshold when the policy is always implemented and $\frac{dA^{*\prime}}{d\psi'}(0)$ to the case when it is never implemented. Recall that all the partial derivatives are calculated at $A^{*\prime} = A^*$ and $\psi' = \psi$. Therefore,

$$\frac{\partial x^*}{\partial A^{**\prime}} = \frac{\partial x^*}{\partial A^{**}} \text{ and } \frac{\partial A^{*\prime}}{\partial x^*} = \frac{\partial A^*}{\partial x^*}$$

and

$$\int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}} d\kappa = \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa$$

Thus, the denominator in Equation (11) can be simplified as follows

$$\begin{aligned} & 1 - p\frac{\partial A^{*\prime}}{\partial x^*}\frac{\partial x^*}{\partial A^{**\prime}} - p\int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**\prime}} d\kappa - (1-p)\frac{\partial A^{*\prime}}{\partial x^*}\frac{\partial x^*}{\partial A^{**}} - (1-p)\int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa \\ = \; & 1 - \frac{\partial A^{*\prime}}{\partial x^*}\frac{\partial x^*}{\partial A^{**\prime}} - \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa \end{aligned}$$

while the numerator can be written as

$$\begin{aligned} & \frac{\partial A^{*\prime}}{\partial \psi}\left[1 - (1-p)\frac{\partial A^{*\prime}}{\partial x^*}\frac{\partial x^*}{\partial A^{**}} - (1-p)\int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa\right] + p\frac{\partial A^{*\prime}}{\partial x^*}\frac{\partial x^*}{\partial \psi'} + p\int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi'} d\kappa \\ = \; & (1-p)\frac{\partial A^{*\prime}}{\partial \psi}\left[1 - \frac{\partial A^{*\prime}}{\partial x^*}\frac{\partial x^*}{\partial A^{**}} - \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa\right] + p\left[\frac{\partial A^{*\prime}}{\partial \psi} + \frac{\partial A^{*\prime}}{\partial x^*}\frac{\partial x^*}{\partial \psi'} + p\int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi'} d\kappa\right] \end{aligned}$$

Thus,

$$\frac{dA^{*\prime}}{d\psi'} = p\frac{\frac{\partial A^{*\prime}}{\partial \psi} + \frac{\partial A^{*\prime}}{\partial x^*}\frac{\partial x^*}{\partial \psi'} + p\int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi'} d\kappa}{1 - \frac{\partial A^{*\prime}}{\partial x^*}\frac{\partial x^*}{\partial A^{**\prime}} - \int_{-1}^{1} \frac{1}{2} \frac{\partial A^{*\prime}}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa} + (1-p)\frac{\partial A^{*\prime}}{\partial \psi}$$

or

$$\frac{dA^{*\prime}}{d\psi'} = p\frac{dA^{*\prime}}{d\psi'}(1) + (1-p)\frac{dA^{*\prime}}{d\psi'}(0)$$

3 Complete Information Model

In this section I solve the complete information version of the model. In the complete information game the productivity parameter **A is common knowledge** among all agents and hence there is no role for signals. Moreover, I assume that there is **no dispersion in productivity across households** which implies that all households are identical. Otherwise, the model is the same as in the paper. In what follows, I use superscript R to denote the case where the government repays the debt and superscript D to denote the case where the government defaults on its debt.

3.1 Borrowing Decision

I start by investigating the government's borrowing decision if the government decides to repay the debt. The government can borrow at interest rate r or save at the risk free interest rate $r_F = 0$. In Section 2.5 I derive condition under which the government will always want to borrow a non-negative amount.

The government's optimal unconstrained borrowing solves:

$$\begin{aligned} \max_{B_2 \in \mathbb{R}} & \sum_{t=1,2} \log\left(c_t^R\right) + \log\left(g_t^R\right) \\ \text{s.t. } & g_1^R = \tau Y_1^R - B_1 + B_2 \\ & g_2^R = \tau Y_2^R - (1+r) B_2 \\ & c_1^R = (1-\tau) e^A f(k_1) - k_2 \\ & c_2^R = (1-\tau) e^A f(k_2) \end{aligned}$$

where Y_1^R and Y_2^R denote the aggregate output in periods 1 and 2, respectively, if the government repays the debt and are given by

$$Y_1^R = e^A f(k_1) \quad \text{and} \quad Y_2^R = e^A f(k_2).$$

Let $B_2^{R,u}$ be the government's optimal unconstrained borrowing choice. If the government's borrows a positive amount, then it does so at an interest rate r, while if it lends, it charges lenders a risk-free interest rate. It follows that the government unconstrained borrowing policy in repayment is given by

$$B_2^{R,u} = \begin{cases} B_2^{R,bor} & \text{if } B_2^{R,bor} \geq 0 \\ 0 & \text{if } B_2^{R,bor} < 0 \text{ and } B_2^{R,len} \geq 0 \\ B_2^{R,len} & \text{if } B_2^{R,len} < 0 \end{cases},$$

where

$$B_2^{R,bor} = \frac{(1+r) B_1 + \tau Y_2 - (1+r) \tau Y_1}{2(1+r)} \quad \text{and} \quad B_2^{R,len} = \frac{B_1 + \tau Y_2 - \tau Y_1}{2}.$$

Finally, consider the government's optimal unconstrained borrowing if it decides to default (the assumed timing allows such a possibility), denoted by $B_2^{D,u}$. In this case, the government will always borrow all the funds available in the market and use them to increase the government spending at time 1.[3] That is, $B_2^{D,u} = S$ where S is the total supply of funds in the market.

[3] An implicit assumption is that the government has no access to a saving technology.

3.2 Fragility Region

In this section I compute the "fragility region". To do this I have to see what the households' and lenders' optimal choices are when: (1) they expect that the government repays the debt and (2) when they expect a default. In what follows, I assume that, in the case the government repays its debt, it always wants to borrow a non-negative amount, that is $B_2^{R,u} \geq 0$. Below, I provide sufficient conditions that ensure that this is the case.

3.2.1 Households

It is straightforward to see that if a household expects a default, then its optimal choice of capital is given by

$$k_2^D(A) = \frac{Z(1-\tau)e^A f(k_1)}{1+\frac{1}{\alpha}},$$

where superscript D denotes default, while if a household expects repayment, then it invests

$$k_2^R(A) = \frac{(1-\tau)e^A f(k_1)}{1+\frac{1}{\alpha}} = \frac{1}{Z}k_2^D(A),$$

where superscript R denotes repayment. Therefore, if the households expect the government to default, they decrease their investment for any given productivity level A.

3.2.2 Lenders

The lenders choose between supplying their funds to the bond market and investing in a risk-free asset. The risk-free asset delivers a sure net return of 0. On the other hand, the net return on the government debt is r if the government repays its debt, and 0 otherwise. It follows that the lenders will never supply their funds to the bond market if they expect a default. If lenders' expect to be repaid and $r \geq 0$ then they will be willing to lend to the government.

It follows that if lenders expect default then the supply of funds to the market is given by $S^D = 0$ while if they expect repayment and $r \geq 0$ then $S^R = b$.

3.3 The Fragility Region

Let $V_1^R(A, k_2, S)$ and $V_1^D(A, \mathbf{k}_2, S)$ be the value to the government from repaying its debt and defaulting on its debt, respectively, when the productivity is A, all households chose to invest the identical amount, k_2, the supply of funds is equal to S, and the government chose earlier an interest $r \geq 0$. The government repays its debt if and only if

$$\Delta V(A, k_2, S; r) = V_1^R(A, k_2, S; r) - V_1^D(A, k_2, S; r) \geq 0.$$

It is straightforward to see that $\Delta V(A, k_2, S)$ is strictly increasing in $\mathbf{k}_2$ for all $A \in \mathbb{R}_{++}$. Thus, a higher investment by the households decreases the government's default incentive. On the other hand, the impact of a higher supply of funds to the bond market S is ambiguous. Clearly, a higher S increases the value of repaying the debt as it allows the government to smooth the repayment of its debt across the two periods. However, a higher S also increases the value of defaulting, by increasing government's spending at $t = 1$ in default (recall, the government will choose to borrow $B_2^D = S$ and then default also on its new borrowing). It follows that for S close to $B_2^{R,u}$ the effect of a higher S is to necessarily decrease the government's incentive to repay the debt. This complicates the solution without adding much to the model, and hence I consider the case as $\xi \to 1$, where this effect is absent. In this case, a higher S unambiguously leads to an increase in the government's incentive to repay.[4]

When $\xi \to 1$ the government has the strongest incentives to default when all households choose $k_2 = k_2^D$ and the supply of funds is $S = 0$. It follows that, for a given interest rate r, the government will repay the debt irrespectively of agents' action if the productivity A is greater or equal to $\overline{A}$, where $\overline{A}$ is the unique solution to

$$\Delta V\left(\overline{A}, \mathbf{k}_2^D, 0; r\right) = 0 \tag{12}$$

Note that $\overline{A}$ does not depend on r, since with $S = 0$ the government cannot borrow a positive amount in period 1.

Similarly, the government has the lowest incentive to default when all households choose $k_2 = k_2^R$ and when it can borrow an unconstrained optimal amount $B_2^{R,u}$. It follows that the government will always default on its debt if $A < \underline{A}(r)$, where $\underline{A}(r)$ is the unique solution to

$$\Delta V\left(\underline{A}(r), \mathbf{k}_2^R, B_2^u(A); r\right) = 0 \tag{13}$$

Note that, in contrast to the upper bound, the lower bound of the fragility region depends on the interest rate. This is because a higher interest rate decreases the government's incentive to repay by increasing the cost of repaying the debt in period 2.[5]

Finally, note that even if r is high enough so that $B_2^{R,u} = 0$ the government is still exposed to self-fulfilling equilibria, solely caused by changes in the households' beliefs. Let $\overline{r}$ be such that

$$B_2^{R,u}\left(\overline{r}; \underline{A}(\overline{r}), k_2^R\right) = 0. \tag{14}$$

Then for all $r \geq \overline{r}$ the government default decision is only driven by the households' expectations: That is if $r \geq \overline{r}$, then lenders play no role in determining whether the government defaults or not. The next lemma provides a characterization of the fragility region. Its proof is straightforward and is left to the reader.

[4] One can analyze the model with $0 < \xi < 1$ under the condition that ξ is large enough. In that case the fragility region will be a bit smaller, as now, there will be an additional region of A with unique equilibrium where lenders and the government follow mixed strategies. This, however, substantially complicates the analysis, particularly in the case of global game version of the model.

[5] By monotonicity of $\Delta V(A, k_2, S; r)$ in A, k_2 and S it follows that for all $A \in \left[\underline{A}(r), \overline{A}\right)$ the outcome of the model depends on the households' and lenders' choices. Thus, for all $A \in \left[\underline{A}(r), \overline{A}\right)$ there are two pure strategy equilibria: one in which the government indeed defaults and one in which the government repays the debt.

Lemma 1 (Fragility Region) *Let $\xi \to 1$. For any given interest rate $r \in \mathbb{R}_+$, the "fragility region" is given by $\left[\underline{A}(r), \overline{A}\right)$, where $\underline{A}(r)$ and $\overline{A}$ are uniquely determined by equations* (13) *and* (12)*, respectively. Moreover:*

1. *$\underline{A}(r)$ is strictly increasing for all $r < \overline{r}$ and constant for all $r \geq \overline{r}$.*
2. *$\underline{A}(r) < \overline{A}$ for all $r \in \mathbb{R}_{++}$.*
3. *$\overline{r}$ is the unique solution to equation* (14).

3.4 Equilibrium

Lemma 2 (Equilibrium) *Let $\xi \to 1$ and fix the interest rate $r \geq 0$.*

1. *If $A > \overline{A}$ then the government never defaults. Households choose to invest k_2^R and all lenders supply all of their funds to the bond market.*
2. *If $A \in \left[\underline{A}(r), \overline{A}\right)$ then there are two pure strategy equilibria:*
   (a) *a default equilibrium, where all households choose to invest k_2^D, lenders invest all the funds in risk-free asset and the government defaults on its debt.*
   (b) *a repayment equilibrium, where all households choose to invest k_2^R, lenders supply all their funds to the bond market and the government repays its debt.*
3. *If $A < \underline{A}(r)$ then the government always defaults. Households choose to invest k_2^D and all lenders invest all the funds in risk-free asset.*

Proof. Follows from the above discussion. ■

3.5 Assumptions 1 and 3

In this section I derive a sufficient condition for the government's desired borrowing in repayment, $B_2^{R,u}$, to be non-negative and increasing in A for all $A > \underline{A}(0)$.

3.5.1 Non-negative Borrowing

Lemma 3 *A sufficient condition for the government borrowing to be non-negative is that its initial debt B_1 is greater than $\underline{B}_1$ where*

$$\underline{B}_1 \equiv \frac{1+\alpha}{\alpha} \frac{\tau}{(1-\tau)} k_1$$

Proof. Note that the government budget constraint in period 1, in the case the government repays its debt, is given by

$$g_1 = \tau Y_1^R - B_1 + B_2.$$

Since $g_1 > 0$, B_2 can be negative if and only if $\tau Y_1^R > B_1$ or $e^A > B_1/\tau f(k_1)$. Moreover, recall that k_2 is increasing in A. Thus, if we can show that at $A = \log(B_1/(\tau f(k_1)))$ the government wants to borrow a positive amount it would follow that the government would never find it optimal to borrow negative amount whenever it repays B_1.[6]

From the expression for B_2 we see that the government will find it optimal to borrow positive amount if

$$f(k_2) > f(k_1)$$

Using the definition of k_2 we get

$$\frac{\alpha}{1+\alpha}(1-\tau)e^A f(k_1) > k_1$$

At $e^A = B_1/(\tau f(k_1))$ the above inequality becomes

$$B_1 > \frac{1+\alpha}{\alpha}\frac{\tau}{(1-\tau)}k_1$$

Setting

$$\underline{B}_1 \equiv k_1\frac{1+\alpha}{\alpha}\frac{\tau}{(1-\tau)}k_1$$

completes the proof. ■

I assume in the paper below that B_1 satisfies the above condition.The above condition is only sufficient and one can find a lower bound for B_1 under which the borrowing is non-negative.

In order for a self-fulfilling crisis that originates from the lenders' expectations to exist we also need the government's optimal unconstrained borrowing to be strictly positive for some $A < \overline{A}^R$. The next, lemma defines interest rate $\overline{r}$ such that for all $r \in [0, \overline{r})$ the government would like to borrow a strictly positive amount at $A = \overline{A}^R$, and hence, by the continuity of its desired borrowing in A, for some $A < \overline{A}^R$.

Lemma 4 *If $r \in [0, \overline{r})$ then the government borrows a strictly positive amount at $A = \overline{A}^R$. If $r > \overline{r}$ then the government will borrow nothing at $A = \overline{A}^R$ The interest $\overline{r}$ is defined as*

$$\overline{r} \equiv \frac{1}{Z^3(Z(1+\alpha)-\alpha)}\frac{f\left(k_2\left(\overline{A}^R\right)\right)}{f(k_1)} - 1$$

where $k_2\left(\overline{A}^R\right)$ denotes capital at $t = 2$ when the productivity is $A = \overline{A}^R$.

[6] I focus here on $r = 0$ because that is the interest rate at which the government can lend. Thus, if at $r = 0$ its desired borrowing is positive it means that the government does not want to lend. However, whether it borrows a strictly positive amount or chooses to borrow nothing depends on the interest rate at which it can borrow.

Proof. Let $r = \overline{r}$ where

$$\overline{r} \equiv \frac{1}{Z^3 (Z(1+\alpha) - \alpha)} \frac{f\left(k_2\left(\overline{A}^R\right)\right)}{f(k_1)} - 1.$$

When $r = \overline{r}$, the government's desired borrowing at $A = \overline{A}^R$ is given by

$$B_2 = \frac{(1+\overline{r}) B_1 + \tau e^{\overline{A}^R} f\left(k_2\left(\overline{A}^R\right)\right) - (1+\overline{r}) \tau e^{\overline{A}^R} f(k_1)}{2(1+\overline{r})}.$$

Using the definition of $\overline{r}$ and of $\overline{A}^R$ we get

$$B_2 = \frac{\frac{1-Z^3(Z(1+\alpha)-\alpha)}{Z^3(Z(1+\alpha)-\alpha)} \tau e^{\overline{A}^R} f\left(k_2\left(\overline{A}^R\right)\right) + \tau e^{\overline{A}^R} f\left(k_2\left(\overline{A}^R\right)\right) - \frac{1}{Z^3(Z(1+\alpha)-\alpha)} \tau e^{\overline{A}^R} f\left(k_2\left(\overline{A}^R\right)\right)}{2(1+\overline{r})} = 0$$

Finally, note that

$$\frac{\partial B_2}{\partial r} = -\frac{\tau e^A f(k_2)}{4(1+r)^2} < 0$$

Since B_2 is continuous in A and decreasing in r it follows that for all $r \in [0, \overline{r})$ there exist values of productivity smaller than $\overline{A}^R$ such that B_2 is strictly positive. ■

3.5.2 Sufficient condition for the desired borrowing to be increasing in A

Lemma 5 *Suppose that* $(1+\alpha) - \frac{1}{Z^3(Z(1+\alpha)-\alpha)} \geq 0$. *Then* $\partial B_2^{R,u} / \partial A > 0$ *for all* $A > \underline{A}^R(0)$.

Proof. Fix $r \in [0, \overline{r})$. Substituting $\underline{A}^R(r)$ into the expression for the desired borrowing yields

$$B_2^{R,u}\left(\underline{A}^R(r); r\right) = \frac{2\tau e^{\underline{A}^R(r)} f(k_2) - 2\tau e^{\underline{A}^R(r)} \sqrt{(1+r) f(k_1) f(k_2) (Z^3(1+\alpha) - \alpha)}}{2(1+r)}$$

By the definition of $\underline{A}^R(r)$, and by the fact that $r < \overline{r}$, we know that $B_2^{R,u}\left(\underline{A}^R(r); r\right) \geq 0$. Therefore by rearranging above expression for $B_2^{unc}\left(\underline{A}^R(r); r\right)$ we get

$$(1+r) \leq \frac{1}{Z^3(Z(1+\alpha) - \alpha)} \frac{f\left(k_2\left(\underline{A}^R(r)\right)\right)}{f(k_1)}$$

Fix r and consider $\frac{\partial B_2^{R,u}(A,r)}{\partial A}|_{\underline{A}^R(r)}$:

$$\begin{aligned} \frac{\partial B_2^{R,u}(A,r)}{\partial A}|_{\underline{A}^R(r)} &= \frac{(1+\alpha)\tau e^{\underline{A}^R(r)} f\left(k_2\left(\underline{A}^R(r)\right)\right) - (1+r)\tau e^{\underline{A}^R(r)} f(k_1)}{2(1+r)} \\ &\geq \frac{(1+\alpha)\tau e^{\underline{A}^R(r)} f\left(k_2\left(\underline{A}^R(r)\right)\right) - \frac{1}{Z^3(Z(1+\alpha)-\alpha)}\tau e^{\underline{A}^R(r)} f\left(k_2\left(\underline{A}^R(r)\right)\right)}{2(1+r)} \end{aligned}$$

and hence

$$\frac{\partial B_2^{R,u}(A,r)}{\partial A}|_{A=\underline{A}^R(r)} \geq 0 \text{ if } (1+\alpha) \geq \frac{1}{Z^3(Z(1+\alpha)-\alpha)}$$

Finally, note that if $\frac{\partial B_2^{R,u}(A,r)}{\partial A} \geq 0$ at $A = \underline{A}^R(r)$, then $\frac{\partial B_2^{R,u}(A,r)}{\partial A} > 0$ for all $A > \underline{A}^R(r)$ (since the second derivative is strictly positive). This implies that the condition

$$(1+\alpha) \geq \frac{1}{Z^3 (Z(1+\alpha) - \alpha)}$$

is sufficient for the desired borrowing to be strictly increasing in A for all $A > \underline{A}^R(0)$. ■