THE THEORY OF THE COST-OF-LIVING INDEX AND THE MEASUREMENT OF WELFARE CHANGE

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SUMMARY

The Consumer Price Index is often regarded as an approximation to a Cost-of-Living Index. This paper reviews the theoretical foundations of the Cost-of-Living Index and the closely related problems involved in measuring changes in economic welfare.

The Cost-of-Living Index for a single person is defined as the minimum cost of achieving a certain standard of living during a given period divided by the minimum cost of achieving the same standard of living during a base period. In order to numerically construct an individual's Cost-of-Living Index, it is necessary to know his or her preferences over economic goods. Since these preferences are essentially unobservable, it is necessary to construct approximations to the Cost-of-Living Index. This topic is discussed in Section 2 of the paper.

The remaining sections of the paper discuss a number of related topics, including: the closely related problems involved in measuring a group Cost-of-living index and changes in the welfare of a group, the fixed based versus the chain principle, the choice of a functional form for the Cost-of-Living Index, the treatment of durable goods, such as housing and the treatment of taxes and labour supply in a Cost-of-Living Index.

RÉSUMÉ

L'indice des prix à la consommation est souvent perçu comme une approximation de l'indice du coût de la vie. Cette communication passe en revue les fondements théoriques
de l'indice du coût de la vie et les problèmes étroitement connexes liés à la mesure des variations du bien-être économique.

L'indice du coût de la vie pour une personne seule se définit comme le coût minimum associé à un certain niveau de vie au cours d'une période donnée divisé par le coût minimum du même niveau de vie au cours d'une période de base. Afin de construire numériquement l'indice du coût de la vie d'un particulier, il faut connaître ses préférences quant aux biens économiques. Puisque ces préférences sont essentiellement inobservables, il y a lieu de construire des approximations de l'indice du coût de la vie. C'est cette question qui fait l'objet de la section 2 de la présente communication.

À la section 3, l'auteur étudie le problème très connexe d'obtenir des indicateurs de la variation d'utilité ou de bien-être pour un ménage individuel est étudié.

La section 4 traite du principe de l'enchaînement utilisé dans la construction des nombres-indices comme l'indice des prix à la consommation et le met en contraste avec le principe de la base fixe.

Les sections 5 et 6 traitent d'autres concepts possibles pour un indice collectif du coût de la vie. L'indice de prix démocratique est une moyenne simple des indices du coût de la vie des ménages individuels. L'indice du coût social de la vie est défini par Prais et Pollak comme la dépense minimale requise pour atteindre un certain niveau de bien-être pour chaque ménage de l'économie lorsque l'économie fait face à des prix de période 1 par rapport à la dépense minimale requise pour atteindre le même niveau de bien-être pour chaque ménage lorsque l'économie fait face à des prix de période 0. Afin d'établir des approximations de l'indice de prix démocratique, il faut disposer de données sur les consommateurs individuels. Cependant, on peut obtenir des approximations du coût social de la vie en utilisant des données agrégatives.

La section 7 traite des problèmes épineux que comporte la formation d'indices collectifs de bien-être.

La section 8 aborde les questions suivantes: (i) comment peut-on définir un sous-indice
de l'indice du coût de la vie, et (ii) comment faut-il regrouper les sous-indices pour obtenir une approximation de l'indice global du coût de la vie?

Comment faut-il traiter les économies et les questions de finances des consommateurs dans l'indice du coût de la vie? Ces questions font surface à la section 9, qui traite de la théorie des indices intertemporels du coût de la vie. Cependant, même si l'on peut étudier théoriquement les indices intertemporels, ils sont difficiles à réaliser en pratique, puisqu'ils dépendent des attentes des ménages à l'égard des prix à venir, qui sont essentiellement inobservables.

Les indices spatiaux du coût de la vie, qui comparent le niveau des prix d'un endroit géographique avec celui d'un autre endroit au même moment, font une brève apparition à la section 10.

La section 11 présente un examen plus approfondi du problème de la modélisation de l'offre de loisirs et de main-d'œuvre dans un indice du coût de la vie.

La section 12 traite du problème de la modélisation des biens durables (comme le logement et l'automobile) dans l'indice du coût de la vie. Elle développe l'approche du coût d'utilisation à l'égard de ce problème. Les attentes du ménage pour ce qui est des prix à venir, la situation financière du ménage (est-il emprunteur ou prêteur, et à quels taux), et la situation du ménage face à l'impôt sur le revenu sont autant d'éléments qui jouent un rôle crucial dans cette approche du coût d'utilisation. Ainsi, les données qu'exige cette approche sont malheureusement très volumineuses.

La section 13 présente quelques brèves observations sur le problème des nouveaux biens.

La section 14 conclut avec quelques recommandations.

La section 15 est une annexe qui présente les preuves des nouveaux résultats théoriques.

1. Introduction
As the title of the paper indicates, we will investigate the theoretical foundations of the
cost-of-living index. This seems appropriate in a conference about the Consumer Price Index (CPI), since the CPI is now being used as a proxy for the cost-of-living index in indexing contracts and as an inflation measure.\footnote{We shall also discuss the closely related issues involved in measuring welfare changes, both for individual consumers and for groups of consumers.} The economic theory of the cost-of-living index for a single household is reviewed in Section 2.

In Section 3, we study the closely related problem of obtaining single household indicators of utility or real income change.

In Section 4, we discuss the costs and benefits of using the chain principle in the construction of index number formulae versus the fixed base principle.

In Sections 5 to 7, we discuss various concepts that have been proposed for group cost-of-living and welfare indexes.

In Sections 8 and 9, we outline the theory of subindexes of the cost-of-living index and the related idea of an intertemporal cost-of-living index.

In Section 10, we consider the problem of constructing price indexes that compare the level of prices in different locations.

Labour and durable goods in the cost-of-living index make their appearance in Sections 11 and 12 respectively.

Section 13 discusses the new goods problem and Section 14 concludes with some recommendations.

In a companion paper, Dievert [1983], we discuss price and output indexes from the viewpoint of producer theory. The reader may be aware of the old Hicks [1940; 1958; 1981] - Samuelson [1950; 1961] measurement of real income controversy; i.e., is there such an
animal as real income, and if so, should it be measured from the consumer or producer point of view? From the consumer point of view, our conclusion is that real income is a very subjective animal and hence it probably does not exist, unless we are willing to give explicit numerical weights to the welfares of different household classes. On the producer side, the situation is more encouraging: although real output is not a useful concept, the closely related concept of total factor productivity does turn out to be useful. For the details of this “new” approach to measurement total factor productivity (which in fact was suggested many years ago by Hicks [1961] [1981; 192-3]), see Caves, Christensen and Diewert [1982b] and Diewert [1983].

Section 15 is an Appendix that collects proofs of new theorems.²

2. The Single Household Cost-of-Living Index

We assume that the household or individual has recurring preferences over combinations of N goods that may be represented by a utility function $F(x)$ where $u = F(x)$ is the utility level or standard of living that can be attained if the individual consumes the consumption vector $x \equiv (x_1, x_2, \ldots, x_N)^T \geq 0_N^3$.

We assume that the utility function $F$ satisfies Conditions I which are technical enough to relegate to a footnote.⁴

We shall assume that the consumer maximizes his utility function $F(x)$ subject to a budget constraint of the form $p \cdot x = \sum_{n=1}^{N} p_n x_n \leq y$ where $p > 0_N$ is a positive vector of commodity (rental) prices and $y > 0$ is expenditure on the N commodities.

The consumer’s utility maximization problem can be decomposed into two stages. In the first stage, the consumer attempts to minimize the cost of achieving a given utility level, and in the second stage, he chooses the maximal utility level that is just consistent with his budget constraint.

The solution to the first-stage problem defines the consumer’s cost function $C$: for $u \geq 0, p > 0_N$
Given that $F$ satisfies Conditions I, $C$ will satisfy Conditions II (which we relegate to another footnote). Moreover, if we are given a cost function $C$ satisfying Conditions II, $C$ may be used in order to construct the underlying preference function $F$ which will satisfy Conditions I.\footnote{See footnote.}

Our interest in $C$ stems from the fact that it may be used to define the Knüsen [1924] cost-of-living index $P_K$: for $p^0 >> 0_{ N}$, $p^1 >> 0_{ N}$ and $u > 0$ define

$$P_K(p^0, p^1, u) = C(u, p^1)/C(u, p^0).$$

The $P_K$ depends on three variables: (i) $p^0$, a vector of period 0 or base period prices, (ii) $p^1$, a vector of period 1 or current period prices, and (iii) $u$, a number that indexes the reference indifference surface. Thus $P_K(p^0, p^1, u)$ is the minimum cost of achieving the standard of living indexed by $u$ when the consumer faces period 1 prices $p^1$ relative to the minimum cost of achieving the same standard of living when the consumer faces period 0 prices $p^0$. If there is only one good, then it can be seen that $P_K(p^0, p^1, u) = p^1/p^0$ for all $u > 0$. In this case, there is obviously no index number problem.

In the general case when there is more than one good, the functional form for the cost-of-living index $P_K$ obviously depends on the functional form for the consumer’s cost function $C$, which in turn is determined by the form of the consumer’s preference function $F$. Our fundamental problem is that we do not know what the functional forms for $F$ or $C$ and hence $P_K$ are. Our primary task in this section will be to see if we can find adequate bounds or approximations to the true cost-of-living index $P_K$ that depend only on observable market price and quantity data. However, before we turn to this primary task, we state a theorem which provides necessary and sufficient conditions for a given function $P(p^0, p^1, u)$ of $2N + 1$ variables to be interpretable as a cost-of-living index.
Theorem 1

Let P be a function of \(2N+1\) positive variables that satisfies the following properties: for all \(u > 0, p^0_i > 0_N, p^1_i > 0_N,\) and \(p^2_i > 0_N,\) we have (i) \(P(p^0, p^1, u) > 0\) (positivity), (ii) \(P(p^0, p^1, u) = 1/P(p^1, p^0, u)\) (time reversal property), (iii) \(P(p^0, p^2, u) = P(p^0, p^1, u)P(p^1, p^2, u)\) (circularity or transitivity) and (iv) for some \(p^* > 0_N, C(u, p) \equiv uP(p^*, p, u)\) regarded as a function of \(u\) and \(p\) satisfies Conditions II for a cost function. Then \(P\) is the cost-of-living index that corresponds to the preferences that are dual to \(C;\) i.e., \(P = P_K\) satisfies (2). Moreover, \(C\) satisfies the following money metric⁷ scaling of utility property:

\[C(u, p^*) = u \text{ for all } u > 0.\]  \hspace{1cm} (3)

Conversely, given a cost function \(C\) satisfying Conditions II and the money metric property (3), then \(P = P_K\) defined by (2) satisfies properties (i) to (iv) listed above in the theorem.

The above theorem is very closely related to some results in Pollak [1983], who stressed that the mathematical properties of \(P_K\) are completely characterized by the mathematical properties of \(C.\)

The following theorem⁸ provides observable bounds on \(P_K(p^0, p^1, u).\)

Theorem 2

(Lerner [1935-36], Joseph [1935-36], Samuelson [1947; p.159]). \(P_K\) lies between the lowest price ratio and the highest price ratio for any reference indifference surface indexed by \(u > 0;\) i.e.,

\[
\min_{i=1,\ldots,N} \left\{ \frac{p^1_i}{p^0_i} \right\} \leq P_K(p^0, p^1, u) \leq \max_{i=1,\ldots,N} \left\{ \frac{p^1_i}{p^0_i} \right\}.
\]  \hspace{1cm} (4)

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The above limits are wide, but they are not useless. For example, if prices vary in strict proportion so that $p^1 = \lambda p^0$ for some $\lambda > 0$, then the upper and lower limit in (4) is $\lambda$ and hence $P_K(p^0, \lambda p^0, u) = \lambda$ also.

In order to make further progress, we shall assume cost minimizing behaviour on the part of the consumer during periods 0 and 1. We shall also assume that we can observe the consumer's quantity choices $x^0 > 0_N$ and $x^1 > 0_N$ made during periods 0 and 1 in addition to the corresponding price vectors $p^0 > > 0_N$ and $p^1 >> 0_N$. Thus we assume:

$$p^0 \cdot x^0 = C[F(x^0), p^0]; p^1 \cdot x^1 = C[F(x^1), p^1].$$

(5)

We have introduced the concept of the Konüs cost-of-living index $P_K(p^0, p^1, u)$ without saying much about the choice of the reference indifference surface indexed by $u$. It would appear that there are two natural choices for $u$: namely $F(x^0)$ or $F(x^1)$. Thus the Laspeyres-Konüs cost-of-living index is defined as:

$$P_K(p^0, p^1, F(x^0)) = C[F(x^0), p^1]/C[F(x^0), p^0]$$

(6)

while the Paasche-Konüs cost-of-living index is defined as:

$$P_K(p^0, p^1, F(x^1)) = C[F(x^1), p^1]/C[F(x^1), p^0].$$

(7)

In order to understand why the indexes (6) and (7) are named after Laspeyres and Paasche it is first necessary to introduce the concept of a mechanical price index formula. This is simply a function $P$ of the observable price and quantity vectors for the two periods, $p^0, p^1, x^0, x^1$, of known functional form. Two examples of such formulae are the Laspeyres price index $P_L$ defined by

$$P_L(p^0, p^1, x^0, x^1) = p^1 \cdot x^0/p^0 \cdot x^0$$

(8)

and the Paasche price index $P_P$ defined by
\[ P_p(p^0, p^1, x^0, x^1) = p^1 \cdot x^1 / p^0 \cdot x^0. \]  

(9)

Irving Fisher [1922] gives hundreds of examples of mechanical price index formulae. The axiomatic characterization of these indexes may be found in Eichhorn [1976; 1978] and Eichhorn and Voeller [1976; 1983]. Note that these indexes are functions of \( 4N \) arguments whereas the price index that appeared in Theorem 1 had only \( 2N + 1 \) arguments.

The following theorem relates the (unobservable) Laspeyres-Konüs cost-of-living index defined by (6) to the (observable) Laspeyres price index defined by (8), and the Paasche-Konüs cost-of-living index defined by (7) to the Paasche price index defined by (9).

**Theorem 3**

(Konüs [1924; pp.17-19]): Assuming (5), cost minimizing behaviour during periods 0 and 1, we have:

\[ P_K(p^0, p^1, F(x^0)) \leq p^1 \cdot x^0 / p^0 \cdot x^0 = P_L; \]  

(10)

\[ P_K(p^0, p^1, F(x^1)) \geq p^1 \cdot x^1 / p^0 \cdot x^1 = P_P. \]  

(11)

**Corollary** (Pollak [1983]):

\[ \min\{p^1 / p^0\} \leq P_K(p^0, p^1, F(x^0)) \leq P_L \equiv p^1 \cdot x^1 / p^0 \cdot x^0; \]  

(12)

\[ P_P \equiv p^1 \cdot x^1 / p^0 \cdot x^1 \leq P_K(p^0, p^1, F(x^1)) \leq \max\{p^1 / p^0\}. \]  

(13)

The above corollary follows combining Theorems 2 and 3. The Laspeyres-Konüs index \( P_K(p^0, p^1, F(x^0)) \equiv C(F(x^0), p^1) / C(F(x^0), p^0) = p^1 \cdot x^0^* / p^0 \cdot x^0 \) is illustrated in Figure 1 in the two good case along with the bounds in (12). Note that \( x^0^* \) is the solution to the problem of minimizing the cost of achieving the utility level \( u^0 \equiv F(x^0) \) when the consumer is faced with period 1 prices \( p^1 \). Although \( p^1 \cdot x^0^* \) is not observable, the upper bound \( p^1 \cdot x^0 \)
(see the dashed straight line through $x^0$) is observable. The lower bound to $p^1 \cdot x^{0*}$ is $p^1 \cdot x^0$, where $x^0$ is the point where the budget line $\{ x: p^0 \cdot x = p^0 \cdot x^0 \}$ intersects the $x_1$ axis, and it too is observable (see the dashed line through $x^0$). These upper and lower bounds correspond to the upper and lower bounds in (12). Analytically, the upper bound rests on the fact that the point $x^0$ is an inner approximation to the consumer's true indifference set $L(u^0) = \{ x: x \geq F(x^0) = u^0 \}$ while the set that is on or above the period 0 budget line $x: p^0 \cdot x \geq p^0 \cdot x^0$ intersected with the non-negative orthant $x: \{ x: x \geq 0_N \}$ is an outer approximation to the true indifference set $L(u^0)$.

Figure 1.

A similar analysis can be carried out for the Paasche-Konüs price index $P_{K}(p^0, p^1, F(x^1))$ $\equiv C[F(x^1), p^1]/C[F(x^1), p^0] = p^1 \cdot x^1/p^0 \cdot x^{1*}$. From Figure 1, it can be seen that $p^0 \cdot x^1$ is an upper bound for $p^0 \cdot x^{1*}$ while $p^0 \cdot x^1$ is a lower bound for $p^0 \cdot x^{1*}$. These upper and lower bounds for $p^0 \cdot x^{1*}$ yield the lower and upper bounds in (13).
It can be seen from Figure 1, that the Laspeyres index $P_L$ will be rather close to $P_K(p^0, p^1, F(x^0))$ provided that the indifference surface $x: F(x) = F(x^0)$ is not too linear around $x^0$. (The perfectly linear case corresponds to the perfect substitutes case). Similarly, the Paasche index $P_P$ will be close to $P_K(p^0, p^1, F(x^1))$ provided that the indifference surface $x: F(x) = F(x^1)$ is not too linear around $x^1$. This is an encouraging observation, but it still does not tell us how close $P_L$ is to $P_K(p^0, p^1, F(x^0))$ or how close $P_P$ is to $P_K(p^0, p^1, F(x^1))$.

In order to make further progress, we may proceed in three directions: (i) introduce additional observations $(p^2, x^2), ..., (p^T, x^T)$ and use the revealed preference techniques associated with Samuelson [1947] and Afriat [1967; 1972; 1979] in order to form non-parametric approximations to the consumer's preferences, (ii) make specific functional form assumptions about $F$ or $C$, or (iii) choose the reference utility level $u$ that occurs in $P_K(p^0, p^1, u)$ in an empirically convenient manner. We will pursue only possibilities (ii) and (iii) in this paper.

It is an empirical fact that the Laspeyres and Paasche indexes, $P_L(p^0, p^1, x^0, x^1)$ and $P_P(p^0, p^1, x^0, x^1)$, are often rather close to each other numerically (we will return to this point in Section 4). Thus the following theorem is extremely important from a practical point of view.

**Theorem 4**

(Konüs [1924; pp.20-21]): Let the consumer's utility function $F$ satisfy Conditions I and suppose that the observed data for periods 0 and 1, $(p^0, x^0)$ and $(p^1, x^1)$ respectively, satisfy the cost minimization assumptions (5). Then there exists a reference utility level $u^*$ that lies between the base utility level $u^0 = F(x^0)$ and the period 1 utility level $u^1 = F(x^1)$ such that the consumer's true cost-of-living index for this reference utility level, $P_K(p^0, p^1, u^*)$, lies between $P_L = p^1 \cdot x^0 / p^0 \cdot x^0$ and $P_P = p^1 \cdot x^1 / p^0 \cdot x^1$; i.e., we have

$$P_P \leq P_K(p^0, p^1, u^*) \leq P_L \quad \text{if } P_P \leq P_L \quad \text{or} \quad \text{(14)}$$
\[ P_L \leq P_K(p^0, p^1, u^*) \leq P_P \quad \text{if} \quad P_L \leq P_P. \quad (15) \]

In most applications of index number theory, we would be quite happy if we knew \( P_K(p^0, p^1, u^0) \) or \( P_K(p^0, p^1, u^1) \) or \( P_K(p^0, p^1, u^*) \) for some \( u^* \) between \( u^0 \) and \( u^1 \). Hence if \( P_L \) is numerically close to \( P_P \), \( P_P(p^0, p^1, u^*) \) will be squeezed in by these two numbers and we will have the consumer’s true cost of living between periods 0 and 1 for all practical purposes.

It would be pleasant if we could extend Theorem 4 to conclude that there exists a \( u^* \) between \( u^0 \) and \( u^1 \) such that \( P_K(p^0, p^1, u^*) \) equals some specific average of \( P_L \) and \( P_P \), such as Irving Fisher’s [1922] ideal index number formula \( P_2 \) which is defined as the geometric average of \( P_L \) and \( P_P \); i.e.,

\[ P_2(p^0, p^1, x^0, x^1) = [p^1 \cdot x^0 / p^0 \cdot x^0]^{1/2} [p^1 \cdot x^1 / p^0 \cdot x^1]^{1/2}. \]

Unfortunately, we cannot draw such a conclusion in general. The problem is that \( P_K(p^0, p^1, u) \) could be rather close to either \( P_L \) or \( P_P \) for all \( u \) between \( u^0 \) and \( u^1 \) and hence if \( P_L \) and \( P_P \) are rather different, then their geometric mean \( P_2 \) could lie above or below \( P_K(p^0, p^1, u) \) for all \( u \) between \( u^0 \) and \( u^1 \). However, if \( P_L \) and \( P_P \) are “close” to each other, then there exists a \( u^* \) between \( u^0 \) and \( u^1 \) such that \( P_K(p^0, p^1, u^*) \) is “close” to the Fisher index \( P_2 \). This provides a somewhat informal justification for the use of \( P_2 \) as an approximation to \( P_K \).

A more formal justification for the use of \( P_2 \) that rests on a specific functional form for \( C(u, p) \) is also possible. Let us suppose that \( C(u, p) = uc^r(p) \) where \( c^r \) is defined by:

\[ c^r(p) = \left( \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} p_i^{r/2} p_j^{r/2} \right)^{1/r} \quad \text{for} \quad r \neq 0, \quad (16) \]

where the parameters \( b_{ij} \) satisfy the restrictions \( b_{ij} = b_{ji} \) for \( 1 \leq i < j \leq N \). The unit cost function \( c^r \) defined by (16) is Denny’s [1974] quadratic mean of order \( r \) unit cost function.
which can provide a second order approximation to an arbitrary twice continuously differentiable unit cost function.11

Define the base period expenditure shares $s_i^0 = \frac{p_i^0}{p^{0.1}}$ of the period 1 expenditure shares $s_i^1 = \frac{p_i^1}{p^{1.1}}$ for $i = 1, \ldots, N$. For each $r \neq 0$, define the mechanical price index formula $P_r$ by:

$$
P_r(p^0, p^1, x^0, x^1) = \left[ \frac{1}{N} \sum_{i=1}^{N} s_i^0 \frac{p_i^0}{p_i^{0.1}} \right]^{1/r} \left[ \frac{1}{N} \sum_{j=1}^{N} s_j^1 \frac{p_j^1}{p_j^{0.1}} \right]^{-1/r}. \tag{17}
$$

It can be verified that when $r = 2$, $P_r$ defined by (17) coincides with the Fisher index $P_2$ defined earlier.

The following theorem relates $c^r$ defined by (16) to the price index formula $P_r$ defined by (17).

**Theorem 5**

(Diewert [1976; p.133]): Suppose $C(u, p) = uc^r(p)$ and the observed data $(p^0, x^0)$ and $(p^1, x^1)$ for periods 0 and 1 satisfy the cost minimization assumption (5). Then for any reference utility level $u > 0$, we have

$$P_K(p^0, p^1, u) = c^r(p^1)/c^r(p^0) = P_r(p^0, p^1, x^0, x^1). \tag{18}$$

Thus $P_K(p^0, p^1, u)$ may be precisely determined by using the mechanical index number formula $P_r$ defined by (17). Diewert [1976] calls a price index formula $P(p^0, p^1, x^0, x^1)$ **superlative** if it correctly evaluates the Konüs price index $P_K(p^0, p^1, u) = C(u, p^0)/C(u, p^1)$ for some cost function $C$ of the form $C(u, p) = uc(p)$ where the unit cost function $c$ can provide a second order approximation to an arbitrary unit cost function. Equations (16) to (18) show that the price indexes $P_r$ are superlative for each $r \neq 0$. 

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Theorem 5 provides a strong economic justification for the use of the indexes $P_\tau$ in empirical applications. In particular, we have obtained an economic justification for the use of the Fisher index $P_2$.

However, Theorem 5 is subject to a serious defect: namely, if $C(u,p) = uc^T(p)$, then the preference function $F^T$ that is dual to this cost function is homothetic; in particular $F^T(x)$ is linearly homogeneous in $x$. Theorem 6 below is not subject to this defect.

First, let us define the translog cost function $C^0$ by:

$$\ln C^0(u,p) = a_0 + \sum_{i=1}^{N} a_i \ln p_i + 1/2 \sum_{i=1, j=1}^{N} a_{ij} \ln p_i \ln p_j + a_{00} \ln u$$

$$+ \sum_{i=1}^{N} a_{0i} \ln p_i \ln u + 1/2 a_{000} (\ln u)^2$$

(19)

where the parameters $a_{ij}$ satisfy the following restrictions.

$$\sum_{i=1}^{N} a_i = 1; a_{ij} = a_{ji}; \sum_{j=1}^{N} a_{ij} = 0 \text{ for } i = 1, \ldots, N; \sum_{i=1}^{N} a_{0i} = 0.$$  \hspace{1cm} (20)

Define the translog price index $P_{012}$ by

$$P_{0} (p^0, p^1, x^0, x^1) = \prod_{i=1}^{N} \left( \frac{p^1_i}{p^0_i} \right)^{a_{0i}/2}$$

(21)

**Theorem 6**

(Diewert [1976, p.122]): Suppose the consumer's cost function $C$ equals the translog cost function $C_0$ defined by (19) and suppose the observed data $(p^0, x^0)$ and $(p^1, x^1)$ satisfy the cost minimization assumptions (5) where $F$ is the utility function dual to $C$. Define $u^0 = F(x^0)$, $u^1 = F(x^1)$ and $u^* = (u^0 u^1)^{1/2}$. Then the true cost of living index $P_k(p^0, p^1, u^*)$
evaluated at the intermediate utility level \( u^* \) may be calculated by evaluating the translog price index \( P_0 \); i.e., we have

\[
P_K(p^0, p^1, u^*) = C^0(u^*, p^1)/C^0(u^*, p^0) = P_0(p^0, p^1, x^0, x^1).
\]

(22)

We note that the translog cost function \( C^0 \) defined by (19) can provide a second order approximation to an arbitrary twice continuously differentiable cost function, so that Theorem 6 is not restricted to the homothetic case. Thus Theorem 6 provides a very strong economic justification for the use of the translog price index \( P_0 \) as an approximation to the true cost of living \( P_K(p^0, p^1, u^*) \).

Theorems 4 and 5 together provided a strong justification for the use of the Fisher index \( P_2 \) while Theorem 6 justified the use of the translog index \( P_0 \). Which one should we use? We will return to this question in Section 4, but first, it is useful to study the problem of measuring changes in the consumer's welfare (as opposed to the problem of measuring changes in the levels of prices that he faces).

3. Single Household Welfare Indicators

A first approach to measuring changes in the consumer's welfare would be to use the Konüs cost-of-living index \( P_K(p^0, p^1, u) \) as a deflator for the consumer's expenditure ratio between the two periods, \( p^1\cdot x^1/p^0\cdot x^0 \). Hence we define the Pollak [1971; p.64] Implicit Quantity Index \( \bar{Q}_K \) as

\[
\bar{Q}_K(p^0, p^1, x^0, x^1, u) = p^1\cdot x^1/p^0\cdot x^0 P_K(p^0, p^1, u).
\]

(23)

If we make our usual cost minimization assumption (5), and if we use definition (2), we find that we can rewrite (23) as

\[
\bar{Q}_K(p^0, p^1, x^0, x^1, u) = \frac{C(u^1, p^1)}{C(u, p^1)} \cdot \frac{C(u, p^0)}{C(u^0, p^0)}.
\]

(24)
where as usual \( u^0 = F(x^0) \) and \( u^1 = F(x^1) \).

There are two natural choices for the reference utility level \( u \); namely, \( u^0 \) and \( u^1 \). Inserting these choices into (24) leads to the following formulae:

\[
\tilde{Q}_K(p^0, p^1, x^0, x^1, u^0) = C(u^1, p^1)/C(u^0, p^1);
\]

\[
\tilde{Q}_K(p^0, p^1, x^0, x^1, u^1) = C(u^1, p^0)/C(u^0, p^0). \tag{26}
\]

Diewert [1981: p.170] shows that \( \tilde{Q}_K \) defined by (24) has the correct ordinal properties if the reference utility level \( u \) is chosen to be any level between \( u^0 \) and \( u^1 \) (including \( u^0 \) and \( u^1 \) as well); i.e., if \( u^0 \leq u \leq u^1 \) with at least one strict inequality, then \( \tilde{Q}_K(p^0, p^1, x^0, x^1, u) > 1 \) while if \( u^0 \geq u \geq u^1 \) with at least one strict inequality, then

\[
\tilde{Q}_K(p^0, p^1, x^0, x^1, u) < 1.
\]

The special cases of (23) defined by (25) and (26) may be illustrated with reference to Figure 1. The index defined by (25) is equal to the ratio OD/OB while the index defined by (26) is equal to the ratio OC/OA.

The special cases (25) and (26) are also special cases of another class of quantity indexes. For \( x^0 > 0_N, x^1 > 0_N \) and \( p > 0_N \), define the Allen [1949; p.199] Quantity Index as

\[
Q_A(x^0, x^1, p) = C[F(x^1), p]/C[F(x^0), p]. \tag{27}
\]

When \( p = p^1 \), (27) reduces to (25), and when \( p = p^0 \), (27) reduces to (26). The reader will also be able to verify that the implicit quantity index \( \tilde{Q}_K \) defined by (24) is a product of two Allen quantity indexes.

The Pollak and Allen quantity indexes, \( \tilde{Q}_K \) and \( Q_A \), are studied in greater detail (and
bounds are derived) in Pollak [1971] and Diewert [1981]. We shall not dwell on their properties here since neither index is the most natural concept for a quantity index or a welfare indicator. If \( x^1 = \lambda x^0 \) for some \( \lambda > 0 \), it would be desirable if our quantity index took on the value \( \lambda \). Neither \( \bar{Q}_K^* \) nor \( Q_A \) has this desirable homogeneity property. However, the Malmquist [1953; 232] Quantity Index \( Q_M \) does have this homogeneity property. In order to define \( Q_M \), we must first define the deflation or distance function \( D(u,x) \) that corresponds to the consumer's utility function \( F \). For \( u > 0 \) and \( x > > 0_N \), define

\[
D(u,x) = \max_k \{ k : F(x/k) \geq u, k > 0 \}.
\]  

(28)

Thus \( D(u,x^1) \) is the deflation factor \( k_1 \) say that will just reduce the vector \( x^1 \) proportionately so that \( F(x^1/k_1) = u \). If \( F \) satisfies Conditions I, then \( D \) will satisfy certain regularity conditions (Conditions III say) and a \( D \) satisfying these conditions will uniquely characterize \( F \).\(^{14}\)

For \( u > 0, x^0 > > 0_N, x^1 > > 0_N \), define the Malmquist Quantity Index \( Q_M \) as

\[
Q_M(x^0,x^1,u) = D(u,x^1)/D(u,x^0).
\]  

(29)

In general, the Malmquist quantity index \( Q_M(x^0,x^1,u) \) will depend on the reference indifference surface indexed by \( u \). As usual, two natural choices for \( u \) are \( u^0 = F(x^0) \) and \( u^1 = F(x^1) \). Thus the Laspeyres-Malmquist quantity index is defined as

\[
Q_M(x^0,x^1,u^0) = D(u^0,x^1)/D(u^0,x^0)
\]

(30)

\[
= D(u^0,x^1)
\]

where (30) follows since \( D(u^0,x^0) = 1 \). The Paasche-Malmquist quantity index is defined as

\[
Q_M(x^0,x^1,u^1) = D(u^1,x^1)/D(u^1,x^0)
\]

(31)

\[
= 1/D(u^1,x^0)
\]
since $D(u^1,x^1) = 1$.

Geometric interpretations for the general Malmquist index (29) and the two special indexes (30) and (31) may be obtained from Figure 2 for the case of two goods. An observed quantity vector $x^0$ is the point $A$ on the $u^0$ indifference curve while the other observed quantity vector $x^1$ is the point $F$ on the $u^1$ indifference curve. The reference indifference curve indexed by $u$ is indicated by the dashed indifference curve.

Figure 2.

The reader can confirm that $Q_M(x^0,x^1,u)$ defined by (29) is equal to $[OF/0E]/[OA/OB]$, that $Q_M(x^0,x^1,u^0) = OF/OD$ and that $Q_M(x^0,x^1,u^1) = OC/OA$. 

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Note that the assumption of cost minimizing behaviour is not required in order to define
the Malmquist quantity index.

From Figure 2, we see that as long as the reference indifference surface indexed by \(u\) remains between \(u^0\) and \(u^1\), the Malmquist index \(Q_M(x^0, x^1, u)\) will correctly indicate whether welfare has increased or decreased going from \(x^0\) to \(x^1\). This property holds in
general as the following result indicates.\(^{15}\)

**Theorem 7**

(Diewert [1981; p.174]): If \(F\) satisfies Conditions I, \(x^0 > > 0_N\), \(x^1 > > 0_N\) and \(u\) is between \(F(x^0)\) and \(F(x^1)\), then (i) \(Q_M(x^0, x^1, u) = 1\) if \(F(x^0) = F(x^1)\), (ii) \(Q_M(x^0, x^1, u) < 1\) if \(F(x^0) > F(x^1)\), and (iii) \(Q_M(x^0, x^1, u) < 1\) if \(F(x^0) < F(x^1)\).

Comparing definitions (2), which defined \(P_K\) as a ratio of cost functions, with definition (29), which defined \(Q_M\) as a ratio of distance functions, it could be conjectured that
\(Q_M\) will satisfy more or less the same mathematical properties as \(P_K\), except that the role of prices and quantities is interchanged. This conjecture is correct. Thus we could write
down a \(Q_M\) version of Theorem 1, where \(Q_M\) replaces \(P_K\), \(x^\prime\)'s replace \(p^\prime\)'s, \(D\) replaces \(C\), and Conditions III for distance functions \(D\) replace Conditions II for cost functions \(C\). Similarly, Theorem 8 below is an exact analogue to Theorem 2.

**Theorem 8**

(Diewert [1981; p.175]): If \(F\) satisfies Conditions I and \(x^0 > > 0_N\), \(x^1 > > 0_N\), \(u > 0\), then

\[
\min_i \{x^1_i/x^0_i \} \leq Q_M(x^0, x^1, u) \leq \max_i \{x^1_i/x^0_i \}.
\]

The above theorem provides a start to the problem of providing observable points for
the essentially unobservable index \(Q_M\). The above theorem did not require the assumption of cost minimizing behaviour on the part of the consumer. The following theorem does.
Theorem 9

(Malmquist [1953; p.231]): Suppose F satisfies Conditions I and (5) holds. Define \( u^0 = F(x^0) \) and \( u^1 = F(x^1) \). Then

\[
Q_M(x^0,x^1,u^0) \leq p^0 \cdot x^1 / p^0 \cdot x^0 \equiv Q_L(p^0,p^1,x^0,x^1), \text{ and} \tag{33}
\]

\[
Q_M(x^0,x^1,u^1) \geq p^1 \cdot x^1 / p^1 \cdot x^0 \equiv Q_P(p^0,p^1,x^0,x^1). \tag{34}
\]

Leontief [1936; pp.58-59] illustrated the above bounds in the two good case. Note that the right-hand side of (33) defines the Laspeyres quantity index \( Q_L \) and the left-hand side of (33) is the Laspeyres-Malmquist quantity index defined earlier by (30). Thus \( Q_L \) is an observable upper bound to the essentially unobservable Malmquist index \( Q_M(x^0,x^1,u^0) \). In Figure 2, \( Q_M(x^0,x^1,u^0) = 0F/0D \leq 0F/0G = p^0 \cdot x^1 / p^0 \cdot x^0 = Q_L \).

The right-hand side of (34) defines the Paasche quantity index \( Q_P \) and the left-hand side of (34) is the Paasche-Malmquist quantity index defined earlier by (31). Thus \( Q_P \) is an observable lower bound to \( Q_M(x^0,x^1,u^1) \).

Corollary

\[
\min_i \{ x_i^1 / x_i^0 \} \leq Q_M(x^0,x^1,F(x^0)) \leq Q_L \equiv p^0 \cdot x^1 / p^0 \cdot x^0; \tag{35}
\]

\[
p^1 \cdot x^1 / p^1 \cdot x^0 \equiv Q_P \leq Q_M(x^0,x^1,F(x^1)) \leq \max_i \{ x_i^1 / x_i^0 \}. \tag{36}
\]

Note that \( Q_M(p^0,p^1,x^0,x^1) = P_P(p^0,p^1,x^0,x^1) = p^1 \cdot x^1 / p^0 \cdot x^0 = Q_P(p^0,p^1,x^0,x^1) \) \( P_L(p^0,p^1,x^0,x^1) \). Thus if the Paasche and Laspeyres price indexes \( P_P \) and \( P_L \) are numerically close, then the Paasche and Laspeyres price indexes \( P_P \) and \( P_L \) are numerically close, then the Paasche and Laspeyres quantity indexes \( Q_P \) and \( Q_L \) will also be close. Thus the upper bound in (35) will often be close to the lower bound in (36). Hence the following theorem is extremely useful.
Theorem 10

(Diewert [1981; p.176]): Suppose $F$ satisfies Conditions I and the cost minimization assumption (5) holds. Then there exists a reference utility level $u^*$ between $u^0 = F(x^0)$ and $u^1 = F(x^1)$ such that the Malmquist quantity index $Q_M(x^0, x^1, u^*)$ for this reference utility level $u^*$ lies between $Q_L$ and $Q_P$.

Theorem 10 is a counterpart to Theorem 4 (and in fact is proved in the same manner). Thus if $Q_L$ and $Q_P$ are "close" to each other, we may take an average of them (such as Irving Fisher's ideal quantity index $Q_2 = (Q_L Q_P)^{1/2}$) and obtain a "close" approximation to the unobservable Malmquist quantity index $Q_M(x^0, x^1, u^*)$ where $u^*$ is some utility level between $u^0$ and $u^1$.

Note that Theorem 10 made no assumptions about the shape of the consumer's indifference surfaces (other than our usual general regularity conditions on the consumer's utility function $F$). However our choice of the reference utility level $u^*$ was somewhat limited.

In order to make further progress, it is necessary to make specific functional form assumptions. Thus let $f^r(x)$ be the quadratic mean of order $r$ utility function for $r \neq 0$, defined in a manner analogous to the definition of $c^r(p)$ (recall (16)) and define the quadratic mean of order $r$ mechanical quantity index formula $Q_r$ by

$$Q_r(p^0, p^1, x^0, x^1) = P_r(x^0, x^1, p^0, p^1)$$  \hspace{1cm} (37)$$

where $P_r$ was defined by (17). Note that we have interchanged the role of prices and quantities in the right-hand side of (37). It may be verified that when $r = 2$, $Q_2$ defined by (37) reduces to the Fisher quantity index, $(Q P Q_L)^{1/2}$.

The following theorem is the quantity counterpart to Theorem 5.
Theorem 11

(Diewert [1976; p. 132]): Suppose \( F = f^r \) for some \( r \neq 0 \) and the data \((p^0, x^0), (p^1, x^1)\) satisfy the cost minimization assumption (5). Then for any reference utility level \( u > 0 \), we have

\[
Q_M(x^0, x^1, u) = f^r(x^1)/f^r(x^0) = Q_r(p^0, p^1, x^0, x^1). \tag{38}
\]

Thus \( Q_M(x^0, x^1, u) \) may be precisely determined by using the mechanical index number formula \( Q_r \) defined by (37), provided that \( F = f^r \). Since \( f^r \) can provide a second order approximation to an arbitrary twice continuously differentiable homogeneous utility function, Diewert [1976] calls \( Q_r \) a superlative quantity index number formula.

Although Theorem 11 provides a strong economic justification for the use of the superlative indexes \( Q_r \), the result is subject to the same defect that occurred in Theorem 5; namely, the preferences that correspond to \( f^r \) are homothetic. Theorem 12 below is not subject to this limitation.

First, we define the translog distance function \( D^0(u, x) \) by setting \( \text{ln}D^0(u, x) \) equal to the right-hand side of (19) where \( x_i \) replaces \( p_i \). Define the translog quantity index \( Q_0 \) by

\[
\text{ln}Q_0(p^0, p^1, x^0, x^1) = \frac{1}{2} \sum_{i=1}^{N} \left( \frac{p_i^{0,0}}{p^0_0 x^0_i} + \frac{p_i^{1,1}}{p^1_1 x^1_i} \right) \ln\left(\frac{x^1_i}{x^0_i}\right). \tag{39}
\]

Theorem 12

(Diewert [1976; p. 123]): Suppose the consumer's distance function \( D \) equals the translog distance function \( D^0 \) defined above. Let \( F \) be the corresponding utility function and \( C \) the corresponding cost function. Let the observed data \((p^0, x^0), (p^1, x^1)\) satisfy the cost minimization assumptions (5). Define \( u^0 \equiv F(x^0), u^1 \equiv F(x^1) \) and \( u^* \equiv (u^0 u^1)^{1/2} \). Then the Malmquist quantity index \( Q_M(x^0, x^1, u^*) \) evaluated at the intermediate utility level \( u^* \) is precisely equal to the translog quantity index \( Q_0 \); i.e.,
\[ Q_M(x_0^0, x_1^0, u^*) = Q_0(p_0^0, p_1^0, x_0^0, x_1^0). \]

Since the translog distance function \( D^0 \) can provide a second order approximation to an arbitrary twice continuously differentiable distance function, Theorem 12 provides a very strong economic justification for the use of the translog quantity index \( Q_0 \) as an approximation to the Malmquist quantity index \( Q_M(x_0^0, x_1^0, u^*) \).

4. Fixed Base versus Chain Indexes

In Section 2, we found a family of mechanical index number formulae, \( P_r(p_0^0, p_1^0, x_0^0, x_1^0) \) for each number \( r \), which had \textit{a priori} good properties from an economic point of view. To each such \( P_r \), there corresponds an implicit quantity index \( Q_r \) that may be defined as follows:

\[ \tilde{Q}_r(p_0^0, p_1^0, x_0^0, x_1^0) = p_1^1/x_1^1/p_0^0/x_0^0 P_r(p_0^0, p_1^0, x_0^0, x_1^0). \]  

(40)

In Section 3, we found a family of mechanical quantity indexes, \( Q_r(p_0^0, p_1^1, x_0^0, x_1^1) \) for each number \( r \), which had \textit{a priori} good properties from an economic point of view. To each such \( Q_r \), there corresponds an implicit price index \( \tilde{P}_r \) defined by:

\[ \tilde{P}_r(p_0^0, p_1^1, x_0^0, x_1^1) = p_1^1/x_1^1/p_0^0/x_0^0 Q_r(p_0^0, p_1^1, x_0^0, x_1^1). \]  

(41)

Our problem now is that we have too many index number formulae that have desirable properties. Which price index formula should we choose from the double infinity of candidates of the form \( P_r \) or \( \tilde{P}_s \) for some \( r \) or \( s \)? The following theorem leads to an answer to this question.

Theorem 13

(Diewert [1978]): The functions \( P_r(p_0^0, p_1^1, x_0^0, x_1^1) \) and \( \tilde{P}_s(p_0^0, p_1^1, x_0^0, x_1^1) \) differentially approximate each other to the second order around any point where \( p_0^0 = p_1^1 > 0 \) and \( x_0^0 = x_1^1 > 0 \). A similar statement holds for the quantity indexes \( Q_s \) and \( Q_r \).
Thus we have for all $r$ and $s$;

\[ P_r(p^0, p^0, x^0, x^0) = P_s(p^0, p^0, x^0, x^0) \quad (42) \]

\[ \nabla P_r(p^0, p^0, x^0, x^0) = \nabla P_s(p^0, p^0, x^0, x^0) \quad \text{and} \quad (43) \]

\[ \nabla^2 P_r(p^0, p^0, x^0, x^0) = \nabla^2 P_s(p^0, p^0, x^0, x^0) \quad (44) \]

where $\nabla P_r$ stands for the $4N$ dimensional vector of first order partial derivatives of the function $P_r$ and $\nabla^2 P_r$ is the $4N$ by $4N$ matrix of second order partials of $P_r$.

Hence for "normal" time series data, the indexes $P_r(p^0, p^1, x^0, x^1)$ and $P_s(p^0, p^1, x^0, x^1)$ will all give the same answer to a very high degree of approximation. Some empirical evidence on this issue is presented in Dievert [1978; 1983b] and in Généreux [1983]. Thus the answer to the question posed before Theorem 13 is: it does not matter very much which formula we use.

Dievert [1978] also shows that the Paasche and Laspeyres indexes that figured prominently in sections 2 and 3 approximate the superlative indexes to the first order, i.e., for $p^0 > > 0_N$ and $x^0 > > 0_N$, we have

\[ P_r(p^0, p^0, x^0, x^0) = P_L(p^0, p^0, x^0, x^0) = P_P(p^0, p^0, x^0, x^0) \quad \text{and} \quad (45) \]

\[ \nabla P_r(p^0, p^0, x^0, x^0) = \nabla P_L(p^0, p^0, x^0, x^0) = \nabla P_P(p^0, p^0, x^0, x^0) \quad (46) \]

but we do not obtain the equality of the second order partial derivatives as we did in (44).

It is worth emphasizing that the above results hold without the assumption of optimizing behaviour on the part of consumers; i.e., they are theorems in numerical analysis rather than economics.\(^{16}\)

Theorems 4 and 10 above showed that it is very useful to have the Paasche and Laspeyres indexes close to each other since this will lead to a very close approximation to the Konüs
cost-of-living index $P_K(p^0, p^1, u^*)$ and to the Malmquist quantity index $Q_M(x^0, x^1, u^*)$. This implies that we should use the chain principle for constructing indexes rather than the fixed base principle. The difference between the two principles may be illustrated for the case of three observations. In the fixed base principle, given a mechanical index number formula $P$, the aggregate level of prices for the three periods would be

$$1, P(p^0, p^1, x^0, x^1) \text{ and } P(p^0, p^2, x^0, x^2).$$  \hspace{1cm} (47)

Using the chain principle, the aggregate price level for the three periods would be

$$1, P(p^0, p^1, x^0, x^1) \text{ and } P(p^0, p^1, x^0, x^1) P(p^1, p^2, x^1, x^2).$$

Using the chain principle, variations between prices and quantities tend to be smaller and hence the Paasche, Laspeyres and all of the superlative indexes will approximate each other much more closely than is the case when we use a fixed base. This is borne out in empirical computations.\(^\text{17}\)

If we had to choose a particular index number formula for a price index from the families $P_t$ and $P_s$, we could perhaps narrow the choice to three indexes: (i) $P_0$, because of Theorem 6, (ii) $P_0$, because of Theorem 12, or (iii) $P_2$, because it lies between the Paasche and Laspeyres bounds.\(^\text{18}\)

Of course, there is a high empirical cost associated with the use of a superlative index number formula as opposed to using a simple fixed based Laspeyres index of the form $p^1 x^0 / p^0 x^0$. The advantage of this latter index is that it requires quantity information for only the base period. Its disadvantage is that it will probably\(^\text{19}\) not approximate the consumer’s true cost-of-living index $P_K(p^0, p^1, u^0)$ very closely as $p^t$ moves further away from $p^0$.

Having discussed price and welfare indexes for an individual consumer or household at great length, it is time now to turn to the construction of group indexes.
5. The Democratic Consumer Price Index

Assume that there are H distinct households, where household h has utility function $F^h(x)$ satisfying Conditions I noted in Section 2 with a dual cost function $C^h(u_h, p)$.

There are many possible ways for constructing a group cost-of-living index. Our first method will be to construct a simple average of the individual household Konüs cost-of-living indexes $C^h(u_h, p^1)/C^h(u_h, p^0)$. This is what Prais [1959] and Muellbauer [1974] call a democratic price index. Letting $u \equiv (u_1, \ldots, u_H)^T$ denote a vector of reference utility levels, we define the democratic cost-of-living index $P_D$ as

$$P_D(p^0, p^1, u) = \frac{1}{H} \sum_{h=1}^{H} \frac{C^h(u_h, p^1)}{C^h(u_h, p^0)} \quad (49)$$

Let $p^0 > 0 \forall i$ and $p^1 > 0 \forall i$ be the period 0 and 1 price vectors as usual, let $x^t_h = (x^t_{1h}, x^t_{2h}, \ldots, x^t_{Nh})^T$ be consumer h’s observed quantity vector for period t, $t = 0, 1$ and for $h = 1, \ldots, H$, let $u^0_h = F^h(x^0_h)$ and $u^1_h = F^h(x^1_h)$ be the period 0 and 1 utility levels attained by household h, and define the utility vectors $u^1 = (u^1_1, \ldots, u^1_H)^T$ and $u^0 = (u^0_1, \ldots, u^0_H)^T$. Assume cost minimizing behaviour for each household during both periods; i.e.,

$$p^0 \cdot x^0_h = C^h(u^0_h, p^0) = C^h[F^h(x^0_h), p^0] \quad \text{for } h = 1, \ldots, H \quad (50)$$

and

$$p^1 \cdot x^1_h = C^h(u^1_h, p^1) = C^h[F^h(x^1_h), p^1] \quad \text{for } h = 1, \ldots, H.$$  

Then we may immediately apply Theorems 2 and 3 in order to obtain the following bounds on the Laspeyres-Democratic index $P_D(p^0, p^1, u^0)$ and on the Paasche-Democratic index $P_D(p^0, p^1, u^1)$:
\begin{align*}
\min_h \{\frac{p^1_i}{p^0_i}\} \leq P_D(p^0, p^1, u^0) & \leq \frac{H}{\sum_{h=1}^{H} \frac{1}{p^0_{i-h}}}, \\
\sum_{h=1}^{H} \frac{1}{p^0_{i-h}} & \leq P_D(p^0, p^1, u^1) \leq \max_i \{\frac{p^1_{i}}{p^0_{i}}\}. 
\end{align*}
(51)

Note that the right-hand side of (51) is an arithmetic average of the individual household Laspeyres price indexes, \(\sum_h P_L(p^0, p^1, x^0_{i-h}, x^1_{i-h})/H\), which we denote by \(\bar{P}_L\). The left-hand side of (52) is an arithmetic average of the individual Paasche indexes, \(\sum_h P_P(p^0, p^1, x^0_{i-h}, x^1_{i-h})/H\), which we denote by \(\bar{P}_P\).

The following theorem provides a group counterpart to Theorem 4 (and in fact is proven in the same manner).

**Theorem 14**

Let each consumer's utility function \(u^h\) satisfy Conditions 1. Suppose that the observed data for periods 0 and 1, \((p^0, x^0_{i-h})\) and \((p^1, x^1_{i-h})\) for \(h = 1, \ldots, H\), satisfy the cost minimization assumption (50). Then there exists a reference utility vector \(u^* = (u^*_1, \ldots, u^*_H)\) such that each component \(u^*_h\) lies between \(u^0_{i-h}\) and \(u^1_{i-h}\) and the Prais-Muellbauer group cost-of-living index evaluated at this reference utility vector \(P_D(p^0, p^1, u^*)\), lies between \(\bar{P}_L\) and \(\bar{P}_P\).

For typical data, we would expect \(\bar{P}_L\) and \(\bar{P}_L\) to be very close to each other (closer than the individual indexes \(P^h\) and \(P^h\) since \(\bar{P}_L\) and \(\bar{P}_P\) are averages of the individual \(P^h\), and \(P^h\)), so \(P_D(p^0, p^1, u^*)\) may be very closely approximated by either \(\bar{P}_L\) or \(\bar{P}_P\).

The practical difficulty with using the democratic price index \(P_D(p^0, p^1, u^*)\) as a general measure of inflation between periods 0 and 1 is that \(\bar{P}_L\) and \(\bar{P}_P\) can only be constructed if we have individual household data. The group index defined in the following section has bounds that can be constructed from aggregate data.
6. The Plutocratic Cost-of-Living Index

Pollak [1981; p.328] defines what he calls a Scitovsky-Laspeyres cost-of-living index for a group of consumers as the ratio of the total expenditure required to enable each household to attain its reference indifference curve at period 1 prices to that required at period 0 prices. This same concept of a group cost-of-living index was suggested by Prais [1959] in less precise language. Prais referred to his concept as a *plutocratic price index*. Making the same assumptions and using the same notation as in the previous section, this Prais-Pollak *plutocratic cost-of-living index* may be defined as:

\[
P_{PP}(p^0, p^1, u) = \frac{\sum_{h=1}^{H} c^h(u_h, p^1)}{\sum_{h=1}^{H} c^h(u_h, p^0)} \quad (53)
\]

\[
= \sum_{h=1}^{H} \frac{s^h(u, p^0)}{c^h(u_h, p^1) / c^h(u_h, p^0)} \quad (54)
\]

where the consumer \( h \) share of total expenditure at reference utility levels \( u \) prices \( p \) is defined as

\[
s^h(u, p) = \frac{c^h(u_h, p)}{\sum_{h=1}^{H} c^h(u_h, p)} \quad \text{for } h = 1, \ldots, H. \quad (55)
\]

The *Laspeyres-Pollak cost-of-living index* is defined as \( P_{PP}(p^0, p^1, u^0) \) while the *Paasche-Pollak cost-of-living index* is defined as \( P_{PP}(p^0, p^1, u^1) \). Note that each household is given the same weight \( 1/H \) in the democratic price index defined by (49), while household \( h \) is given the expenditure share weight \( s^h(u, p^0) \) in the plutocratic price index defined by (54).

It is convenient to define aggregate consumption vectors \( \bar{x}^i \) as follows:

\[
\bar{x}^0 = \sum_{h=1}^{H} x^0_h; \quad \bar{x}^1 = \sum_{h=1}^{H} x^1_h. \quad (56)
\]
Armed with the above definitions, we may now prove the usual Paasche and Laspeyres bounding theorem.

**Theorem 15**

Suppose each consumer's utility function \( f^h \) satisfies Conditions 1 and suppose the cost minimization assumptions (5) hold. Then

\[
\min_i \left\{ \frac{p^1_i}{p^0_i} \right\} \leq P_{PP}(p^0, p^1, u^0) \leq p^1 \cdot \bar{x}^0 / p^0 \cdot \bar{x}^0 = P_L \tag{57}
\]

\[
P_P = p^1 \cdot \bar{x}^1 / p^0 \cdot \bar{x}^1 \leq P_{PP}(p^0, p^1, u^1) \leq \max_i \left\{ \frac{p^1_i}{p^0_i} \right\}. \tag{58}
\]

Note that \( P_L \) in (57) is the usual Laspeyres price index involving the aggregate base period consumption vector \( \bar{x}^0 \), while \( P_P \) in (58) is the usual Paasche price index involving the aggregate period 1 consumption vector \( \bar{x}^1 \). The consumer price index constructed by statistical agencies is usually an approximation to \( P_L \). All that is required to construct \( P_L \) is: the current vector of prices \( p^1 \), the base price vector \( p^0 \), and the base period aggregate consumption vector \( \bar{x}^0 = \Sigma_h x^0_h \).

Usually, the upper bound \( P_L \) in (57) will be close to \( P_{PP}(p^0, p^1, u^0) \) while the lower bound \( P_P \) in (58) will be close to \( P_{PP}(p^0, p^1, u^1) \) and to \( P_L \).

**Theorem 16**

Under the conditions of Theorem 15, there exists a reference utility vector \( u^* = (u^*_1, ..., u^*_H) \) such that each component \( u^*_h \) lies between \( u^0_h \) and \( u^1_h \) and the Prais-Pollak group cost-of-living index evaluated at this reference utility vector, \( P_{PP}(p^0, p^1, u^*) \), lies between \( P_L = p^1 \cdot \bar{x}^0 / p^0 \cdot \bar{x}^0 \) and \( P_P = p^1 \cdot \bar{x}^1 / p^0 \cdot \bar{x}^1 \).

As was the case for the Prais-Muehlbauer index \( P_D \), we would expect \( P_L \) and \( P_P \) to be very close to each other provided that \( p^0 \) and \( p^1 \) are not "too" different, and hence \( P_{PP}(p^0, p^1, u^*) \) may be closely approximated by either \( P_L \) or \( P_P \) (or say by the Fisher ideal \( P_2 = P_L P_P^{1/2} \)) under these circumstances.
We have noted that the bounds for the Pollak index, \( P_L \) and \( P_P \), may be computed provided only that we have aggregate data on prices and quantities, while the bounds for the Muellbauer index, \( \overline{P}_L \) and \( \overline{P}_P \), require detailed household data for their computation. Are there other important differences between the two concepts of the group price index?

An important conceptual difference emerges if we compare the formula (49) for \( P_D \) and the formula (54) for \( P_{PP} \). In (49), each household’s individual Konüs cost-of-living index, \( C^h(u_{h0},p^1)/C^h(u_{h0},p^0) \), is given the same weight, \( 1/H \). In (54), when \( u = u^0 \), household \( h \)'s Laspeyres-Konüs cost-of-living index, \( C^h(u_{h0}^0,p^1)/C^h(u_{h0}^0,p^0) \), is given the weight \( s^h(u_{h0}^0,p^0) \), which is household \( h \)'s share of total expenditure during period 0. Thus high expenditure households will tend to get weighted more heavily than low expenditure households in the construction of the plutocratic index. This is why Prais and Muellbauer call the index (49) a democratic index, since it weights each household’s cost-of-living index equally.

Which concept of a group price index should be used as a measure of average inflation between periods 0 and 1? Unless we have quantity data by household class, we cannot evaluate the bounds for the democratic index, so in this case we are stuck with the plutocratic index. If we do have detailed data by household class, then instead of calculating bounds for the democratic index, we should calculate the usual Paasche and Laspeyres bounds for the individual households’ true cost-of-living indexes, \( C^h(u_{h0},p^1)/C^h(u_{h0},p^0) \).

Deaton and Muellbauer [1980; p.178] note that the simplification of working with a single price index can be very dangerous. They cite the Great Bengal Famine of 1943 when between three and five million people died of starvation. An average price index was not very relevant to the problems of low-income households under those circumstances.

Statistics Canada [1982; p.89] has taken the first step in the direction of providing consumer price indexes that are household specific in that they now construct an experimental CPI for low-income families. They are to be commended for this effort and they should be given additional resources to construct additional price indexes by household class.

We turn now to a brief discussion of the mirror image to the problem of constructing group price indexes - the problem of constructing aggregate welfare indexes.
7. Social Welfare Indexes

In Section 3, we found that the Malmquist quantity index, \( Q_M(x^0, x^1, u) = D(u, x^1)/D(u, x^0) \), was a very satisfactory quantity index or welfare indicator for a single household. Let us suppose that household \( h \)'s preferences may be represented by the deflation function \( D^h(u_h, x^h) \) that is dual to the utility function \( F^h(x^h) \), where \( F^h \) satisfies Conditions I as usual.

Define the \( N \) by \( H \) matrix of period 0 (1) consumer choices by \( X^0 (X^1) \): i.e.,

\[
X^0 = [x^0_1, x^0_2, \ldots, x^0_H]; \quad X^1 = [x^1_1, x^1_2, \ldots, x^1_H]
\]

where \( x^t_h = [x^t_{1h}, x^t_{2h}, \ldots, x^t_{Nh}]^T \) is consumer \( h \)'s observed consumption vector during period \( t \) for \( t = 0, 1 \) and \( h = 1, 2, \ldots, H \).

It is natural to try and aggregate individual welfare changes into a single scalar measure of overall welfare change. A simple social index of welfare change that respects individual preferences is

\[
w(x^0, x^1, u) = \sum_{h=1}^{H} \beta^h(u) \frac{D^h(u_h, x^1_h)}{D(u_h, x^0_h)}
\]

where \( u = (u_1, u_2, \ldots, u_H)^T \) is a vector of individual reference utility levels and the weight function \( \beta^h \) satisfies the following restriction:

\[
\sum_{h=1}^{H} \beta^h(u) = 1 \quad \text{for any vector of reference utilities } u.
\]

The reader will note that our indicator of social welfare change defined by (60) is a quantity counterpart to the Pollak cost-of-living index defined by (54) (which is an indicator of price change).
Our reason for imposing the restriction (61) is the following: if \( x_h^1 = x_h^0 \) for \( h = 1, \ldots, H \) so that there is no change in consumption between periods 0 and 1, then we want our indicator of social welfare change (or our aggregate quantity index) to indicate that there has been no change; i.e., we want \( W(X^0, x^0, u) = 1 \). Hence we must have (61).

In general, the weights \( \beta_h(u) \) do not have to be positive or even non-negative. However, if the weights are non-negative, then we obtain the following bounds for \( W \) applying Theorem 8:

\[
\min_{i,h} \{ x_{ih}^1 / x_{ih}^0 \} \leq W(X^0, x^0, u) \leq \max_{i,h} \{ x_{ih}^1 / x_{ih}^0 \}. \quad (62)
\]

As usual, we define the Laspeyres Social Welfare Indicator \( W(X^0, x^1, u^0) \) by selecting our reference utility vector to be \( u^0 \equiv [F^0(x_1^0), \ldots, F^0(x_H^0)] \equiv [u_1^0, \ldots, u_H^0] \) and the Paasche Social Welfare indicator \( W(X^0, x^1, u^1) \) by selecting the reference utility vector to be \( u^1 \equiv [u_1^1, u_2^1, \ldots, u_H^1] \).

The following theorem provides a social welfare counterpart to Theorem 9.

**Theorem 17**

Let each household utility function \( f^h \) satisfy Conditions I and suppose the cost minimization assumptions (50) hold. Define the utility weights \( \beta_h^0 \equiv \beta(u^0) \) and the utility weights \( \beta_h^1 \equiv \beta_h(u^1) \) for \( h = 1, 2, \ldots, H \). Then if the weights \( \beta_h^0 \) and \( \beta_h^1 \) are non-negative,

\[
W(X^0, x^1, u^0) \leq \sum_{h=1}^{H} \beta_h^0 \frac{p_h^0 \cdot x_h^0}{p^0} x_h^0 = \sum_{h=1}^{H} \beta_h^0 Q_h^0 \quad (63)
\]

and

\[
\sum_{h=1}^{H} \beta_h^1 Q_h^1 = \sum_{h=1}^{H} \beta_h^1 \frac{p_h^1 \cdot x_h^1}{p^1} x_h^0 \leq W(X^0, x^1, u^1). \quad (64)
\]

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Thus the Laspeyres social welfare indicator \( W(X^0, X^1, u^0) \) is bounded from above by a weighted average of the individual household Laspeyres quantity indexes \( Q^h_L \), and the Paasche social welfare indicator \( W(X^0, X^1, u^1) \) is bounded from below by a weighted average of the individual household Paasche quantity indexes \( Q^h_P \).

The following theorem provides a social welfare analogue to Theorem 10.

**Theorem 18**

Suppose that the hypotheses of Theorem 17 are satisfied. Suppose also that the household weighting functions \( \beta_h(u) \) are continuous as \( u \) varies linearly between \( u^0 \) and \( u^1 \). Then there exists a reference utility vector \( u^* = (1-\lambda^*)u^0 + \lambda^*u^1 \) for some \( \lambda^* \) between 0 and 1 such that the social welfare change indicator \( W(X^0, X^1, u^*) \) lies between \( \Sigma_h = \sum_h \beta^0_h Q^h_L \) (the upper bound in (63)) and \( \Sigma_h = \sum_h \beta^1_h Q^h_P \) (the lower bound in (64)).

**Corollary**

Suppose \( \beta^0_h = p^0 \cdot x^0_h / \sum_k p^0 \cdot x^0_k \) (the share of household \( h \) in base period expenditure) and \( \beta^0_h = p^1 \cdot x^1_h / \sum_k p^1 \cdot x^1_k \) (the share of household \( h \) in expenditure using period 1 prices and period 0 quantities) for \( h = 1, \ldots, H \). Then the social welfare change indicator \( W(X^0, X^1, u^*) \) defined in the theorem lies between the aggregate Laspeyres and Paasche quantity indexes, \( p^0 \cdot (\Sigma x^1_h) / p^0 \cdot (\Sigma x^0_h) \) and \( p^1 \cdot (\Sigma x^1_h) / p^1 \cdot (\Sigma x^0_h) \), respectively.

Thus if the aggregate Paasche and Laspeyres quantity indexes are close to each other, an average of them such as \( Q_2 = (Q_L Q_P)^{1/2} \) (Fisher's ideal quantity index), will yield a close approximation to the welfare change indicator \( W(X^0, X^1, u^*) \) described in the above corollary. Thus we have provided a justification of sorts for Pigou's [1920; p.84] cautious recommendation of the Fisher quantity index as an indicator of aggregate welfare change.

However, I do not think that the corollary to Theorem 18 should be taken too seriously. We need very special welfare weights \( \beta_h(u) \) in order to obtain the corollary. These special weights need not correspond to anybody's idea of a just society.
The paragraph above illustrates a problem with the social welfare approach: it is difficult to come to a consensus on what the weights $\beta H(\mu)$ should be. Hence perhaps we should concentrate on the calculation of welfare change by household class rather than attempting to construct somewhat arbitrary measures of aggregate welfare change. For alternative attempts to construct aggregate cost-of-living indexes and aggregate measures of welfare change, see Blackorby and Donaldson [1983] and Jorgenson and Slesnick [1983].

We leave the world of group indexes in the following sections in order to focus on some special problems that are associated with single household price and quantity indexes.

8. The Theory of Subindexes

Pollak [1975a] noted that in many instances, economists are interested in subindexes of the cost-of-living index; i.e., in indexes that do not cover the whole spectrum of consumer goods, but only selected subsets. He notes several interesting classes of subindexes: (1) a food index in the context of a complete cost-of-living index defined over all consumer goods, (2) a one-period index in the context of a multi-period world, (3) a consumption goods subindex in the context of a consumer choice model that included not only the consumption decision, but also the labour supply decision, (4) a consumption goods subindex in the context of a model where the consumer has preferences defined over not only consumer goods but also environmental variables (such as pollution) and also public goods (such as roads and parks).

Pollak's discussion summarized above indicated that the concept of a subindex in the cost-of-living index is not without applications. We must now face up to two problems: (i) how do we define a subindex rigorously, and (ii) how may we combine subindexes in order to form an approximation to the true overall cost-of-living index.\footnote{20}

Some new notation and a new concept are required. As usual, think of $x$ as being the consumer's overall consumption vector. We now partition the vector $x$ into $M$ subvectors (of varying dimension) which we denote by $(x_1, x_2, ..., x_M) \equiv x$ (so $x_m$ is the $m$th subvector). Partition the overall price vector $p$ in an analogous manner. We may now define
Pollak's [1969]\textsuperscript{21} conditional expenditure or cost function for the mth subgroup of goods as:

$$C^m(u, p_m, x_1, x_2, \ldots, x_{m-1}, x_m + 1, x_{m+2}, \ldots, x_M)$$

$$= \min_{x_m} \left\{ p_m \cdot x_m : F(x) \geq u \right\} \quad m = 1, 2, \ldots, M,$$

where $u$ is the consumer's reference utility level, $F$ is his overall utility function satisfying the usual conditions, and the overall consumption vector is $x = (x_1, x_2, \ldots, x_{m-1}, x_m, x_{m+1}, \ldots, x_M)$. Thus $C^m$ defined by (65) is the minimum group $m$ cost of achieving utility level $u$, given that the consumer has available $x_1$ units of group 1 goods, $x_2$ units of group 2 goods, $\ldots$, $x_{m-1}$ units of group $m-1$ goods, $x_{m+1}$ units of group $m+1$ goods, $\ldots$, and $x_M$ units of group $M$ goods. In order to save space, we shall write $C^m$ defined by (65) as $C^m(u, p_m, x)$ where $x$ is the entire consumption vector, but it should be understood that $C^m(u, p_m, x)$ is constant with respect to variations in the $x_m$ components of $x$; i.e., $C^m(u, p_m, x)$ does not depend on the $x_m$ components of $X$.

The regularity properties of $C^m$ with respect to the vector of price variables $p_m$ are exactly the same properties that were given in Conditions II for the vector of price variables $p$.\textsuperscript{22} Thus we may define Pollak's [1975; 147] Subindex of the Cost of Living Index for Group $m$ goods as:

$$p^m(p^0_m, p^1_m, u, x) = C^m(u, p^1_m, x) / C^m(u, p^0_m, x); \quad m = 1, \ldots, M,$$

where $p^0 = (p^0_1, p^0_2, \ldots, p^0_M)$ is the base period price vector (remember each $p^0_m$ is a vector pertaining to group $m$ goods), $p^1 = (p^1_1, p^1_2, \ldots, p^1_M)$ is the period 1 price vector, $u$ is a reference utility level, and $x = (x_1, x_2, \ldots, x_M)$ is a reference quantity vector.

Since Theorem 2 depended only on the regularity properties of $C(u, p)$ with respect to the price vector $p$ and since $C^m(u, p_m, x)$ satisfies these same regularity properties, it can
be seen that $P^m$ satisfies the usual Lerner-Joseph-Samuelson bounds for any reference vector $(u,x)$:

$$\min_i \left\{ \frac{p_{m}^{0}}{p_{m}^{0}} \right\} \leq P^m(p^0_m, p^1_m, u, x) \leq \max_i \left\{ \frac{p_{m}^{1}}{p_{m}^{0}} \right\}$$

where the index $i$ runs through the components of the group $m$ price vectors.

As usual, we can pick out particular reference vectors of interest. Define the Laspeyres-Pollak Group $m$ Subindex as $P^m(p^0_m, p^1_m, u^0, x^0)$ where $x^0$ is the consumer's period 0 choice vector and $u^0 = F(x^0)$ is his period 0 utility level. Define the Paasche-Pollak Group $m$ Subindex as $P^m(p^0_m, p^1_m, u^1, x^1)$ where $x^1$ is the consumer's period 1 choice vector and $u^1 = F(x^1)$ is his period 1 utility level. Assuming optimizing behaviour during the two periods, we may derive the following subindex counterpart to Theorem 3.

**Theorem 19**

Assuming (5), the $m$th subindex $P^m$ defined by (66) satisfies the following inequalities when $(u,x) = (u^0, x^0)$ and $(u^1, x^1)$ respectively:

$$P^m(p^0_m, p^1_m, u^0, x^0) \leq P^m = p^1_m / p^0_m, \quad m = 1, \ldots, M \text{ and } (67)$$

$$P^m(p^0_m, p^1_m, u^1, x^1) \geq P^m = p^1_m / p^0_m, \quad m = 1, \ldots, M. \quad (68)$$

Thus we obtain the usual Laspeyres and Paasche bounds for the subindexes, and hence we may obtain the following adaptation of Theorem 4.

**Theorem 20**

Assume that $F$ satisfies Conditions I, and assume that (5) holds; i.e., that there is overall utility maximizing behaviour during the two periods. Then there exists a reference utility-consumption vector $(u^m, x^m) = (1 - \lambda^*_m)(u^0, x^0) + \lambda^*_m(u^1, x^1)$ where $0 < \lambda^*_m < 1$ and $u^0 = F(x^0), u^1 = F(x^1)$ such that the $m$th Pollak subindex evaluated at this reference vector, $P^m(p^0_m, p^1_m, u^m, x^m)$, lies between the group $m$ Laspeyres index $P^m_L$ defined in (67) and
the group m Paasche index $P^m_p$ defined in (68).

Thus under normal circumstances when $P^m_p$ is close to $P^m_L$, we may obtain a rather good estimate of the subindex $P^m(p^0_m, p^1_m, u^m, x^m, x^m)$. 

How may we combine the subindexes in order to obtain an estimate of the overall cost-of-living $P_K(p^0, p^1, u)$?

Consider the Laspeyres-Pollak subindexes $P^m(p^0_m, p^1_m, u^0, x^0)$. A natural way of weighting these subindexes to form an overall cost-of-living index would be to use the base period expenditure share $p^0_m x^0_m / p^0_m x^0$ to weight the mth subindex $P^m(p^0_m, p^1_m, u^0, x^0)$. Call the resulting overall cost-of-living index $P(0)$. Thus

$$P(0) = \sum_{m=1}^{M} (p^0_m x^0_m / p^0_m x^0)p^m(p^0_m, p^1_m, u^0, x^0)$$

$$= \sum_{m=1}^{M} (p^0_m x^0_m / p^0_m x^0) c^m(u^0_m, p^1_m, x^0) / c^m(u^0_m, p^0_m, x^0)$$

$$= \sum_{m=1}^{M} (p^0_m x^0_m / p^0_m x^0) c^m(u^0_m, p^1_m, x^0) / p^0_m x^0$$

using (A6) in the appendix

$$\leq p^1 x^0 / p^0 x^0 = P_L$$ using (A8) in the appendix. (69)

Thus the aggregate two-stage index $P(0)$ is bounded from above by the aggregate Laspeyres price index $P_L$. Note that if $P^m$ were replaced by the Laspeyres group m subindex $p^1_m x^0_m / p^0_m x^0$, then (69) would collapse to $P_L$, the aggregate Laspeyres price index.

Consider now the Paasche-Pollak subindexes $P^m(p^0_m, p^1_m, u^1, x^1)$. A natural way of weighting these indexes to form an overall cost-of-living index would be to use period 1 expenditure shares $p^1_m x^1_m / p^1_m x^1$ as weights for the subindexes $P^m(p^0_m, p^1_m, u^1, x^1)$. However, instead of forming a simple weighted average, we shall form a harmonic mean:
\[ P(1) = \left[ \sum_{m=1}^{M} p_m^{1,1} \left( \frac{1}{s_m^1} \right) \left[ p_m^{0,0} \right] \right]^{-1} \]
\[ = \left[ \sum_{m=1}^{M} C_m^0 u^1_{m,1} s_m^1 \right]^{-1} \]
\[ = \left[ \sum_{m=1}^{M} C_m^0 u^1_{m,1} s_m^1 \right]^{-1} \hspace{1cm} \text{using (A7)} \]
\[ \geq \frac{p}{p_0} \hspace{1cm} \text{using (A9).} \]

Thus the two stage-index \( P(1) \) is bounded from below by the aggregate Paasche index \( P_P \). Note that if \( p_m^0 \) were replaced by the Paasche group \( M \) subindex \( p_m^1 x_m^1 / p_m^0 x_m^1 \), then (70) would collapse to \( P_P \), the aggregate Paasche price index. Thus the overall Paasche price index \( P_P \) may be calculated as a weighted harmonic average of Paasche subindexes.

For each \( \lambda \) between 0 and 1, define the intermediate aggregate two-stage index \( P(\lambda) \) by

\[ P(\lambda) = \left[ \sum_{m=1}^{M} (1-\lambda) s_m^0 + \lambda s_m^1 \right] \left[ p_m^{0,0} p_m^{1,1} (1-\lambda) u^0 + \lambda u^1 (1-\lambda) x^0 + \lambda x^1 \right]^{1/r} \]  

where \( s_m^0 = p_m^0 x_m^0 / p_m^0 x_m^0, s_m^1 = p_m^1 x_m^1 / p_m^1 x_m^1 \) and the exponent \( r \) that appears in (71) is defined by \( r = 1 - 2\lambda \).

It can be verified that \( P(0) \) and \( P(1) \) defined by (71) coincide with the \( P(0) \) and \( P(1) \) defined in (69) and (70) and moreover, \( P(\lambda) \) is continuous for \( 0 \leq \lambda \leq 1 \). Thus we may apply the usual Konüs [1924] proof (see the proof of Theorem 14) and conclude that there exists a \( \lambda^* \) between 0 and 1 such that

\[ P_P \leq P(\lambda^*) \leq P_L \text{ or } P_L \leq P(\lambda^*) \leq P_P. \]  

Thus \( P(\lambda^*) \), a weighted average of the subindexes \( p_m^{1,1} (1-\lambda^*) u^0 + \lambda^* u^1, (1-\lambda^*) x^0 + \lambda^* x^1 \) for \( m = 1, \ldots, M \) has the usual overall Paasche and Laspeyres bounds and hence will be "close" to the one-stage Konüs cost-of-living index, \( P_K(p^0, p^1, u^0) \), whose existence was given in Theorem 4 above.
Diewert [1978] showed that certain index number formulae, such as the superlative mechanical price index formulae $P_t(p^0, p^1, x^0, x^1)$ and $P_s(p^0, p^1, x^0, x^1)$ defined by (17) and (41), had a second order consistency in aggregation property. Diewert's proof was highly technical and lacked economic motivation. Perhaps the results in this section cast some light on why "good" index number formulae may be expected to aggregate up to the "right" aggregate value when we use a "good" weighting scheme: the Paasche and Laspeyres bounds occur in an appropriate two-stage procedure as well as in the usual single-stage procedure for constructing a true cost-of-living index.

9. An Intertemporal Cost-of-Living Index

Pollak [1975a] developed the theory of subindexes and in Pollak [1975b], he attempted to apply his general theory of subindexes to the intertemporal context. However, in this section, we shall attempt to show that the general theory of subindexes cannot be very readily applied to the intertemporal context.

We begin with a single consumer who has a horizon that extends over $T + 1$ Hicksian [1946] periods when we first observe his behaviour in period 0. Let his utility function be given by $F(x_0, x_1, \ldots, x_T)$ where $x_t$ is the period $t$ vector of purchases that the consumer plans to make in period $t$. Assume for simplicity that the consumer can borrow or lend a dollar from period 0 to period 1 at the interest rate $r^0_1$ and he expects the interest rate from period $t$ to $t + 1$ to be $r^0_{t+1}$ for $t = 1, 2, \ldots, T-1$. Define the sequence of period 0 expected discount factors to be $\delta^0_1 = 1/(1 + r^0_1)$, $\delta^0_2 = \delta^0_1/(1 + r^0_2)$, $\ldots$, $\delta^0_T = (\delta^0_{T-1}/1 + r^0_T)$. Suppose that from the vantage point of period 0, the consumer expects the spot price vector $p^0_t > 0_N$ to prevail during period $t$ for $t = 1, 2, \ldots, T$. The price vector $p^0_0 > 0_N$ is the observable vector of market prices prevailing during period 0. Let $p^0$ be the vector of discounted expected prices

$$p^0 = (p^0_0, \delta^0_1 p^0_1, \delta^0_2 p^0_2, \ldots, \delta^0_T p^0_T).$$
We assume that \( x^0 = (x^0_0, x^0_1, \ldots, x^0_T) \) solves the period 0 expected utility maximization problem:

\[
\max_{x} \left\{ F(x) : p^0 \cdot x \leq p^0 \cdot x^0 = W^0 \right\}
\]

where \( W^0 > 0 \) is the consumer's initial wealth. Letting \( C \) be the cost function, we have the following equation (where \( \delta^0_0 \equiv 1 \)):

\[
W^0 = C[F(x^0), p^0]
\]

\[
= \min_{x} \left\{ p^0 \cdot x : F(x) \geq F(x^0) \right\}
\]

\[
= \min_{x_0, \ldots, x_T} \left\{ \sum_{t=0}^{T} \delta^0_t p^0_t x_t : F(x_0, \ldots, x_T) \geq F(x^0_0, \ldots, x^0_T) \right\}
\]

\[
= \sum_{t=0}^{T} \delta^0_t p^0_t x^0_t
\]

\[
= \sum_{t=0}^{T} C^t[F(x^0), \delta^0_t p^0_t x^0_t]
\]

\[
= C^0[F(x^0), p^0_0 x^0_0] + \sum_{t=1}^{T} \delta^0_t C^t[F(x^0), p^0_t x^0_t]
\]

(74)

where (74) follows from the homogeneity properties of the conditional cost functions \( C^t \) which are defined in a manner analogous to \( C^m \) in (65). In period 0, we may observe the consumer's initial wealth \( W^0 \), the vector of prevailing market prices \( p^0_0 \), the consumer's vector of actual period 0 purchases \( x^0_0 \), his savings \( W^0 - p^0_0 x^0_0 \), and perhaps the ex ante period 0 interest rate \( r^0_1 \) (and the corresponding discount rate \( \delta^0_1 = 1/(1 + r^0_1) \)). All of the other variables are unobservable.
Now suppose that we can observe what happens during period 1. There will be new expectations about nominal interest rates \( r_t^1 \) for \( t = 2, \ldots, T \) and a new sequence of discount factors \( \delta_t^1 = \delta_{t-1}^1/(1 + r_t^1) > 0 \), the consumer will have a new initial wealth \( W^1 \), there will be a new spot price vector for goods \( p_1^1 \) (which may or may not be equal to the consumer’s period 0 expectation about this vector of spot prices \( p_1^0 \)), and the consumer will have new expectations about future spot prices, \( p_2^1, p_3^1, \ldots, p_T^1 \). We now assume that \( x_1^1, x_2^1, \ldots, x_T^1 \) solves the following period 1 expected utility maximization problem:

\[
\max_{x_1^1, \ldots, x_T^1} \left\{ F(x_0^0, x_1^1, \ldots, x_T^1); p_1^1 \cdot x_1^1 + \delta_2^1 p_2^1 \cdot x_2^1 + \cdots + \delta_T^1 p_T^1 \cdot x_T^1 \right\} \leq p_1^1 \cdot x_1^1 + \delta_2^1 p_2^1 \cdot x_2^1 + \cdots + \delta_T^1 p_T^1 \cdot x_T^1 = W^1. \tag{75}
\]

Note that the period 0 decision vector \( x_0^0 \) appears in (75), since we are stuck with it in period 1. Define \( x^1 = (x_0^0, x_1^1, \ldots, x_T^1) \). Proceeding in a manner analogous to our derivation of (74), we find that

\[
W^1 = C^1[F(x^1), p_1^1, x^1] + \sum_{t=2}^{T} \delta_t^1 C^t[F(x^1), p_t^1, x^1]. \tag{76}
\]

In period 1, we may observe the consumer’s period 1 wealth \( W^1 \), the period 1 vector of actual market prices \( p_1^1 \), the consumer’s purchases of goods during the period \( x_1^1 \), his saving \( W^1 - p_1^1 \cdot x_1^1 \), and the ex post rate of return that was earned going from period 0 to period 1, \( r_1^1 \) (remember the corresponding ex ante expected rate of return was \( r_1^0 \)).

Using the techniques outlined in the previous section, we may establish the following observable bounds for two intertemporal subindexes:

\[
\min_i \left\{ \frac{p_1^1}{p_0^0} \right\} \leq \frac{C^0[F(x^0), p_1^1, x^0]}{C^0[F(x^0), p_0^0, x^0]} \leq p_1^1 \cdot x_0^0 / p_0^0 \cdot x_0^0 = P_L;
\]

\[
P_p = p_1^1 \cdot x_1^1 / p_0^0 \cdot x_1^0 \leq \frac{C^1[F(x^1), p_1^1, x^1]}{C^1[F(x^1), p_0^0, x^1]} \leq \max_i \left\{ \frac{p_1^1}{p_0^0} \right\}.
\]
However, the bounds established above do not allow us to answer any really interesting questions.

Although we cannot compute a complete intertemporal cost-of-living index in general, it is possible to make a comparison of the expected welfare of the consumer during periods 0 and 1. Recall that the \textit{ex post} period 0 rate of return was defined to be $r^1_1$. Define the \textit{ex post} discount rate $\delta^1_1 = 1/(1 + r^1_1)$. Then we may rewrite the period 1 expected utility maximization problem (75) in the following manner:

$$
\max_{x_1, \ldots, x_T} \{ F(x_0^0, x_1^0, \ldots, x_T^0): p_0^0 x_0^0 + \delta^1_1 p_1^1 x_1 + \delta^1_1 \delta^1_2 p_2^1 x_2 + \ldots \}
$$

$$
\quad + \delta^1_1 \delta^1_2 p_2^1 x_T \leq p_0^0 x_0^0 + \delta^1_1 \mathcal{W}^1 \}.
$$

The objective functions in (75) and (77) are the same and the constraint in (77) is obtained by multiplying both sides of the constraint in (75) by $\delta^1_1$ and then adding the constant $p_0^0 x_0^0$ to both sides.

The consumer's period 0 expected utility maximization problem (73) may be rewritten as (78) when we fix $x^0 = x_0^0$:

$$
\max_{x_1, \ldots, x_T} \{ F(x_0^0, x_1^0, \ldots, x_T^0): p_0^0 x_0^0 + \delta^0_1 p_1^0 x_1 + \delta^0_2 p_2^0 x_2 + \ldots + \delta^0_T p_T^0 x_T \leq \mathcal{W}^0 \}.
$$

If the consumer's period 0 expectations equal his period 1 expectations about future prices and interest rates and if the expected \textit{ex ante} first period rate of return $r^0_1$ equals the \textit{ex post} rate of return $r^1_1$, then it can be verified that the prices appearing in the constraint of (77) are identical to the prices appearing in the constraint of (78). Under these conditions, we could say that the choice set of the consumer (and hence his welfare) has increased going from period 0 to period 1 if

$$
p_0^0 x_0^0 + \delta^1_1 \mathcal{W}^1 > \mathcal{W}^0
$$

(79)
where $\delta_1^1 = 1/(1 + r_1^1)$ and $r_1^1$ is the ex post one-period rate of return on assets going from period 0 to period 1.

The criterion for an expected welfare increase (or for an increase in real wealth) may be rewritten as

$$w^1 > (w^0 - p_0^0 \cdot x_0^0)(1 + r_1^1)$$

$$= (\text{first period savings})(1 + \text{ex post rate of return}).$$

This criterion for an increase in real wealth is due to Hicks [1946; p.175]. However, its validity does require the assumption of constant expectations, an assumption that is unlikely to be fulfilled under present economic conditions.

The problem of measuring welfare changes in a general Hicksian intertemporal choice model seems to be inherently difficult. Virtually all of the index number techniques that we have surveyed and developed in this paper rely on the twin assumptions of optimizing behaviour on the part of the consumer and observability of market prices and the consumer's quantity choices (or we require assumptions about the constancy of expectations that are unlikely to be met in practice). Hence in order to apply the traditional theory of index numbers in the intertemporal context, it appears to be necessary to place a priori restrictive assumptions on the form of the intertemporal utility function $F(x_0, x_1, \ldots, x_T)$ such as intertemporal additivity; i.e., $F(x_0, x_1, \ldots, x_T) = \sum_{t=0}^{T} f(x_t)$. If we do this, then we may apply traditional index number theory in order to measure changes in the one-period utility function $f(x_t)$; i.e., we could approximate the change in $f(x_1^1)/f(x_0^0)$ using the period 0 and 1 price and quantity data in the usual manner.

10. Spatial Cost-of-Living Indexes

The basic problem to be considered in this section is the problem of comparing the level of prices in different cities or localities. The problem is isomorphic to the usual problem of making international comparisons.26
If we are willing to assume that a certain class of consumers in one location (location 0 say) has the same one-period preferences as another class of consumers in another location (location 1 say) and if there are no significant differences in environmental (non-market) variables in the two locations, then we may simply apply the theory outlined in Sections 2-4 above, where the superscripts 0 and 1 will now refer to locations. However, there were two rather big "ifs" in the previous sentence.

We may relax the restrictiveness of the second "if" by using a Pollak subindex (recall Section 8) of the form

\[
C^I[F(x^0_0, p^0_1, x^0_2)] / C^I[F(x^0_0, p^0_1, x^0_2)] = p^1(p^0_1, p^1_1, F(x^0_0, x^0_2)) \quad \text{and} \quad (80)
\]

\[
C^I[F(x^1_0, p^1_1, x^1_2)] / C^I[F(x^1_0, p^1_1, x^1_2)] = p^1(p^0_1, p^1_1, F(x^1_0, x^1_2)) \quad \text{(81)}
\]

where \( P^1 \) is a Pollak subindex of the cost of living over market goods (recall (66)), \( p^0_1 \) and \( p^1_1 \) are vectors of observed market prices in locations 0 and 1 respectively, \( F(x) = F(x_1, x_1) \) is the consumer’s utility function defined over combinations of market goods \( x_1 \) and non-market locational amenities \( x_2 \), \( x^0_1 \) and \( x^1_1 \) are the observed market choice vectors for consumers 0 and 1 respectively, \( x^0_2 \) is the amenity vector in location 0, \( x^1_2 \) is the amenity vector in location 1, \( x^0 = (x^0_1, x^0_2) \) and \( x^1 = (x^1_1, x^1_2) \).

The bounds derived in Section 8 (see (67) and (68)) are applicable to the subindexes defined in (80) and (81) under the usual optimizing behaviour assumptions:

\[
\min_i \{ p^1_1 / p^0_1 \} \leq p^1(p^0_1, p^1_1, F(x^0_0, x^0_2)) \leq p^1_1 x^0_1 / p^0_1 x^0_1 \equiv P_L \quad \text{and} \quad (82)
\]

\[
P_D = p^1_1 x^1_1 / p^0_1 x^1_1 \leq p^1(p^0_1, p^1_1, F(x^1_0, x^1_2)) \leq \max_i \{ p^1_1 / p^0_1 \}. \quad (83)
\]

Theorem 19 may be applied in the present context as well.

The assumption that the consumers in the two locations have the same preferences may also be relaxed; see Caves, Christensen and Diewert [1982b; p.1410] and Denny and Fuss [1983] for various "translog approaches".
11. Leisure and Labour Supply in the Cost-of-Living Index

Consider a household that can supply various kinds of labour service. It is natural to assume that the household has one-period preferences defined over various combinations of market goods \( x = (x_1, \ldots, x_N) \geq 0_N \) and labour supplies \( y = (y_1, \ldots, y_M) \leq 0_M \) where \( y_M \leq 0 \) is the negative of the number of hours of the mth type of work supplied by the household. The preferences are summarized by the utility function \( F(x,y) \). Suppose now that the household faces the positive commodity price vector \( p > 0_N \) and the positive (after tax marginal) wage vector \( w > 0_M \). Then we may define the household’s cost function \( C \) in the usual manner:

\[
C(u,p,w) = \min_{x,y} \{ p \cdot x + w \cdot y : F(x,y) \geq u \}, \tag{84}
\]

Given period 0 and 1 price vectors, \((p^0, w^0)\) and \((p^1, w^1)\), we may be tempted to define the Konüs cost-of-living index in the usual manner:

\[
P_K(p^0, w^0, p^1, w^1, u) = C(u, p^1, w^1)/C(u, p^0, w^0). \tag{85}
\]

If we could be assured that \( C(u, p, w) > 0 \) for \( u > 0 \), and \( p > 0_N \), and \( w > 0_M \), then there would be no problem with definition (85), and in fact we could derive the usual bounds that we derived in Section 2. However, \( C[F(x^0, y^0), p^0, w^0] > 0 \) corresponds to a situation where the value of household consumption in period 0, \( p^0 \cdot x^0 \) exceeds the value of household labour supply, \( -w^0 \cdot y^0 \). There is no reason for this to be the case. Worse yet, the two values could coincide (i.e., we could have \( p^0 \cdot x^0 + w^0 \cdot y^0 = 0 \)) in which case \( P_K(p^0, w^0, p^1, w^1, F(x^1, y^1)) \) becomes undefined (since we are dividing by zero in definition (85)).

We could attempt to avoid these problems by assuming that the household has preferences defined over different combinations of market goods \( x \) and leisure, where the leisure vector is defined by \( b + y \geq 0_M \) and \( y \leq 0_M \) represents hours of work and \( b > 0_M \) is a positive vector that could perhaps represent the maximum hours that could be supplied of the various types of household labour services. This approach is pursued in Riddell [1983]. However, it will be difficult to come to an agreement on just what value we should take
for the vector $b$ (particularly if members of the household are holding multiple jobs). Hence the resulting Konüs Cost of Living index would be somewhat arbitrary, and should therefore not be used as an inflation measure.\textsuperscript{28}.

However, it is still possible to use index number techniques in order to obtain approximations to the change in the household's welfare which occurred going from period 0 to period 1. Essentially, what we shall do is measure welfare changes in terms of proportional changes in consumption goods.

First we define the household's conditional consumption deflation function $D$ by

$$D(u,x,y) = \max_k \{ k : F(x/k,y) \geq u, k > 0 \} \quad (86)$$

where $u > 0$ is a reference utility level, $x > 0_N$ is a consumption vector, $y \leq 0_M$ is a vector of labour supplies indexed negatively and $F$ is the household utility function. If $D(u,x,y,) > 1$ ($<1$), then the household joint consumption labour supply vector yields a utility level greater (less) than $u$, while if $D(u,x,y) = 1$, then $(x,y)$ yields precisely the utility level $u$. In general $D(u,x,y)$ tells us the proportion $k^* > 0$ that we have to deflate the consumption vector $x$ so that $(x/k^*,y)$ will yield utility level $u$ for the household.

A Malmquist [1953] consumption quantity index may now be defined in the usual manner:

$$Q(x^0,y^0,x^1,y^1,u) = D(u,x^1,y^1)/D(u,x^0,y^0). \quad (87)$$

A geometric interpretation of the quantity index defined by (87) may be found on Figure 3 for the case of one consumption good ($N = 1$) and one type of labour supply ($M = 1$).
Figure 3  The Malmquist Quantity Index in the Labour Supply Context.

For the reference utility level u (which is between \( u^0 = F(x^0, y^0) \) and \( u^1 = F(x^1, y^1) \)), it can be verified that \( Q(x^0, y^0, x^1, y^1, u) = [DA/CA]/[FE/GE] \).

As usual, it is useful to let the references utility level \( u \) in (87) be either the base utility level \( u^0 = F(x^0, y^0) \) or the period 1 utility level \( u^1 = F(x^1, y^1) \). Thus define the Laspeyres-Malmquist consumption quantity index by \( Q(x^0, y^0, x^1, y^1, F(x^0, y^0)) \) and the Paasche-Malmquist consumption quantity index by \( Q(x^0, y^0, x^1, y^1, F(x^1, y^1)) \). In Figure 2, the first of these indexes is DA/BA while the second is HE/FE.

Many of the theorems that were stated in Section 3 go through in the present context. In particular, a counterpart to Theorem 7 is true, so if the reference utility level in (87) is between \( u^0 \) and \( u^1 \), \( Q(x^0, y^0, x^1, y^1, u) > 1 \) \((< 1, = 1)\) indicates that household utility has increased (decreased, remained constant) going from period 0 to period 1.

We also obtain the following counterpart to Theorem 9.
Theorem 21

Suppose \( F \) satisfies the modified Conditions I and the household's observed period 0 and period 1 choices \((x^0, y^0)\) and \((x^1, y^1)\) are consistent with utility maximizing or cost minimizing behaviour; i.e., \( x^0, y^0 \) satisfies

\[
p^0 \cdot x^0 + w^0 \cdot y^0 = C[F(x^0, y^0), p^0, w^0]
\]  
(88)

where \((p^0, w^0)\) are the observed positive period 0 prices and \( C \) is the cost function defined by (84), and \((x^1, y^1)\) satisfies

\[
p^1 \cdot x^1 + w^1 \cdot y^1 = C[F(x^1, y^1), p^1, w^1]
\]  
(89)

where \((p^1, w^1)\) are the observed positive period 1 prices. Then provided that \( p^0 \cdot x^0 + w^0 \cdot (y^0 - y^1) > 0 \) and \( p^1 \cdot x^1 + w^1 \cdot (y^1 - y^0) > 0 \),

\[
Q(x^0, y^0, x^1, y^1, F(x^0, y^0)) \leq p^0 \cdot x^1 / [p^0 \cdot x^0 + w^0 \cdot (y^0 - y^1)] \equiv \alpha \quad \text{and} \quad (90)
\]

\[
Q(x^0, y^0, x^1, y^1, F(x^1, y^1)) \geq [p^1 \cdot x^1 + w^1 \cdot (y^1 - y^0)] / p^1 \cdot x^0 \equiv \beta.
\]  
(91)

The bounds in (90) and (91) may be illustrated by referring to Figure 3. (90) becomes \( DA/BA \leq DA/JA \) and (91) becomes \( HE/FE \geq KE/FE \). As the reader can observe, the bounds are rather close to the appropriate theoretical index. It is also true that the bounds \( \alpha \) and \( \beta \) will often be close to each other. Hence the following counterpart to Theorem 10 is of some practical interest.

Theorem 22

Assume the regularity conditions of the previous theorem and define the base utility \( u^0 \equiv F(x^0, y^0) \) and the period 1 utility level \( u^1 \equiv F(x^1, y^1) \). Then there exists a reference utility level \( u^* \) between \( u^0 \) and \( u^1 \) such that the Malmquist consumption quantity index \( Q(x^0, y^0, x^1, y^1, u^*) \) lies between \( \alpha \) and \( \beta \), the bounds defined in (90) and (91).
Thus if $\alpha$ and $\beta$ are close, we will be able to obtain a good estimate of $Q(x^0, y^0, x^1, y^1, u^*)$ by averaging $\alpha$ and $\beta$.

For an empirical implementation of the above material, see Riddell [1983].

12. Durables in the Cost-of-Living Index

The treatment of consumer durables in the CPI and in the true cost-of-living index is an interesting and controversial issue.\(^{29}\)

The basic issues can readily be explained. Consider a durable good that can be purchased at the beginning of the current period at the spot price $q^0$. A consumer can purchase this good at the beginning of the period, use the good during period 0, but because of the good's durable nature, some of it will be left over at the beginning of the following period. The consumer could sell his used durable good at a (possibly hypothetical) second-hand market at an expected price of $q^{01}$. Assuming that the consumer can lend or borrow at the rate of return $r$, we may follow Hicks [1946] and compute the present value of the cost of buying one unit of the good, using it for one period, and selling it next period. The resulting user cost $p$ is

$$p = q^0 - q^{01}/(1+r) = (q^0 r + (q^0 - q^{01}))/[1+r]. \quad (92)$$

The first term on the right-hand side of (82) is an interest cost while the second term combines the effects of anticipated capital gains and depreciation. We can separate out these two effects if we let $q^1$ be the spot price of a new unit of the durable that the consumer expects to prevail during period 1. Hence we may write $(q^0 - q^{01}) = (q^1 - q^{01}) - (q^1 - q^0)$ = depreciation - capital gains. Thus the user cost (92) becomes

$$p = (1+r)^{-1}[q^0 r + (q^1 - q^{01}) - (q^1 - q^0)]. \quad (93)$$

The above derivation of the user cost of a durable in discrete time was essentially obtained by Diewert [1974; p.504] and Pollak [1975b]. The term $(1+r)^{-1}$ may strike the reader
as being a bit odd, but we need it so that when \( q^{01} = 0 \) (so that the good is actually a non-durable), then the user cost \( p \) collapses to the period 0 purchase price \( q^0 \).

Unfortunately, there are many problems associated with the user cost formulae (92) or (93).

The first problem is that the prices \( q^{01} \) and \( q^1 \) are not market prices - they are the consumer’s period 0 expectations of what the market prices will actually be in period 1. It is not realistic to assume that our non-renting consumer will be able to accurately forecast these future prices. If consumers were able to accurately forecast future prices, then the existing rental market price for the services of the durable would equal the user cost \( p \) defined by (92) where the expected price \( q^{01} \) in (92) is replaced by the observed ex-post market price. Hendershott [1980; 406] demonstrates that rental prices for houses do not track ex post user costs for housing very closely. However, all is not lost if there are some rental markets for the class of durable goods under consideration.

If we do have some rental market information on the durable then we could assume that the rental price equals the user cost \( p \) that appears in (92) and we could use (92) to solve for the expected price \( q^{01} \) in terms of the observed market prices \( p, q^0 \) and \( r \). Thus expectational information gleamed in this way for classes of durables that have rental markets could be applied to forecast expectations of future prices for classes of “similar” durables that do not have rental markets.

Most goods can only be purchased in integral numbers, and for most goods, this does not cause major problems. However, some durable goods such as cars and houses may be purchased only in integer units, and such purchases would form a large share of the consumer’s total expenditure. Hence we cannot neglect the lumpiness problem for such classes of durables. How may we apply traditional “continued” utility and index number theory to this situation?

A possible solution is illustrated in Figure 4 below. Assume \( x_1 \) represents units of a “continuous” good, ice cream say, while \( x_2 \) represents the number of television sets that a household holds during a period.
In Figure 4, we have graphed two indifference "curves" for the household. Only the "kinky" points on the curves plus the line segments parallel to the \( x_2 \) axis are actual feasible choices for the household, but for all practical purposes, we can replace the original preferences defined only over integer combinations of TV sets by continuous preferences with "kinks".\(^1\) The resulting preference function \( F(x) \) may be treated in the normal manner as far as index number theory is concerned. Note that the economic effect of the "kinks" will be to make the consumer change his durable holdings only after relatively large changes in the rental prices of the durables relative to non-durable goods; i.e., responses will be "sticky". This point should be taken into account in econometric work, but it need not concern us from the viewpoint of index number theory.

Another difficulty (which does create problems for us from the viewpoint of index number theory) is that the expected buying and selling price of the durable may not be the same; i.e., there may be significant transactions costs associated with buying and selling units of the durable. The effect of this difference in buying and selling prices for the durable will be to put a "link" in the consumer's budget constraint around his initial holdings of the durable. See Figure 5 below.

---

**Figure 4.**

```
\[ x_2 \]
```

**Figure 5.**

```
\[ x_1 \]
```
If the consumer is observed at $x^0$ during period 0, the correct price of the durable to use in an index number comparison lies somewhere between the buying and selling price. For our purposes, we shall probably have to settle for taking an average of the two prices. Note that the effect of this difference in buying and selling prices will again have the effect of making the consumer's demand to hold durables "sticky" around his initial holdings.\[^{32}\]

Differences in borrowing and lending rates for a household may also have the effect of introducing a "kink" into the budget set when the purchase of a large consumer durable is contemplated. Deaton and Muellbauer [1980; Ch.13] discuss the effect of a borrowing constraint, and in fact, they have an excellent discussion of the problems involved in modelling the demand for consumer durables.

The final problem that we wish to discuss is the problem of calculating the user cost of a durable when there are tax considerations involved.

At the marginal investment point, we assume that the durable holder invests in a market asset that earns a before tax rate of return $r$. Suppose that consumer's marginal tax rate is $\tau$. The present value of the cost of using the durable for one period is

$$
p = q^0 - \frac{q_1^0}{1 + (1 - \tau)r} + \frac{\tau \alpha (q_1^0 - q^0)}{1 + (1 - \tau)r} - \frac{\tau \beta r_M q^0}{1 + (1 - \tau)r} + \frac{tq^0}{1 + (1 - \tau)r} - \frac{\tau \delta q^0}{1 + (1 - \tau)r} \tag{94}
$$

where $q^0$ and $q_1^0$ are the same as before, $\alpha$ is the proportion of capital gains on the durable that is taxable, $\beta$ is the proportion of (mortgage) interest that is deductible from taxable income (and $r_M$ is the appropriate mortgage interest rate), $t$ is a user tax rate on holdings of the durable (e.g., a property tax), $\delta$ is the depreciation rate that is allowed for taxation purposes, and all taxes are assumed to be payable in the following period (and hence they are discounted). If we assume $\alpha = \beta = \delta = 0$, then (94) reduces to

$$
p = \frac{[(1 - \tau)rq^0 + tq^0 - (q_1^0 - q^0)]}{1 + (1 - \tau)r}. \tag{95}
$$
Thus the higher is the marginal tax rate, the lower will be the user cost $p$. If the durable holder is a borrower (at the mortgage rate $r_M$ say), then the discount factor in (94) and (95) must be replaced by $1 + r_M$. Under these conditions, (95) becomes

$$p = r_M q^0 + t q^0 - (q^{01} - q^0)/[1 + r_M].$$

(96)

which will be much higher than the user cost defined by (95).

Thus the tax and financial situation of the consumer plays an essential role in the calculation of user costs for durables. This is a very unfortunate result, since the informational requirements for implementing a user cost formula such as (94) are very high. In particular, the expected price $q^{01}$ that appears in (94) is not directly observable. Also it may be very difficult to determine precisely what is the appropriate opportunity cost of the marginal investment ($r$) for the consumer under consideration. These problems are particularly acute in the case of housing, since housing expenditures are generally a large proportion of a typical household budget. In order to avoid the difficult measurement problems associated with the user cost approach, Gillingham [1982] suggests essentially that the price quantity data that pertains to the rental segment of the housing market be extrapolated to the entire housing market. The problem with this rather sensible suggestion is that the rental segment of the housing market is generally not representative of the entire housing market. A possible reason for this non-representativeness emerges if we compare the user cost for a house for a rich individual with a high marginal tax rate $\delta$ (see (95)) with the user cost formula for a less well-off individual who has no non-labour income and holds a mortgage on his house (recall (96)): the same house will cost the rich individual far less in terms of user cost than the poor individual. Furthermore, the tax laws in most Western countries will generally make it more profitable for a rich individual to own and live in his house rather than rent out his house and live in a rental house of comparable quality. Thus the rental price information for the rental portion of the housing market will not be representative for the market as a whole (due to these tax considerations). Moreover, as nominal interest rates and marginal tax rates changed over time, we could expect the price of rental housing to systematically deviate from the appropriate user cost index of non-rental housing. Thus I do not believe that the informational difficulties that are inherent in the user
cost formula (94) can be avoided. For further discussion on the role of tax considerations in the construction of user costs, see Darrough [1983].

We conclude this section with a reminder that all of our bounds on the consumer’s true cost-of-living index rested on the assumption of utility maximizing behaviour subject to a budget constraint. The prices that appear in the budget constraint are observable market prices in the case of non-durable goods, but for a durable good which is owned by the consumer, the appropriate price must be an ex ante user cost of the form (94), which depends on the (unobservable) anticipated price of the depreciated durable which is expected to prevail in the following period. It is not in general appropriate to use an ex post user cost formula of the form (94) where the anticipated price $q^{01}$ is replaced by an (observable) market price, since the resulting ex post user cost may well be negative if there is an unanticipated inflation in the price of the durable. Thus the existence of unanticipated inflation (or deflation) and the non-neutrality of the income tax with respect to the treatment of durables make it very difficult to construct a Konüs cost-of-living index (or bounds to it) for a household. Thus there are costs due to unanticipated inflation and the non-neutrality of the present tax system.

13. The New Goods Problem

The standard approach to the new goods problem in the context of consumer theory dates back to Hicks [1940]: the period after a new good appears (period 1 say), we attempt to impute a price for the new good in period 0 which would just make the consumer’s demand for the good equal to 0 in period 0. The details of an econometrically implementable approach are outlined in Diewert [1980; pp.501-503].

However, a simpler method that is more practical is readily available: as soon as a new good appears, start collecting price and quantity information on it. Since initially the quantity purchased will be low and the price will often be rather high, a quick introduction of the good into the CPI universe of prices would solve most of the practical problems associated with the current neglect of the quality change problem in official CPIs. It is true that many new goods quickly disappear, but this causes no particular theoretical problems. However, we must concede that linking the prices of similar new goods that vary
in quality poses some practical problems.

14. Conclusion

The main conclusions which emerge from this paper are listed below.

(i) In addition to the usual Laspeyres based CPI, a Paasche based CPI should also be published as frequently as possible, since an appropriate true cost-of-living index lies between a Paasche and Laspeyres index. This means that household surveys where quantity information is collected should be undertaken more frequently, say every second year.

(ii) CPIs should be constructed on a more disaggregated basis (by household demographic characteristics, by income and by region).

(iii) Labour supply and leisure should be introduced into the CPI framework on an experimental basis. The consumer-worker’s income tax position will play an important role here.

(iv) Intertemporal CPIs are too problematical to be introduced at this time.

(v) The treatment of consumer durables, particularly housing, is not very satisfactory at present. Various user cost and rental equivalent alternatives should be tried on an experimental basis. The importance of the household’s tax and financial position should not be overlooked.

(vi) The treatment of seasonal commodities is also unsatisfactory. A more satisfactory treatment of seasonal commodities from the viewpoint of economic theory is outlined in Diewert [1983b].

(vii) New goods should be introduced into the CPI universe of prices as soon as possible. The neglect of new goods provides an upward bias to the existing CPI of a possibly major magnitude.
If the above recommendations are implemented, then not only will we have much more accurate information on how inflation affects different consumer groups, we will also be able to simulate how changes in tax policy affect the welfare of the different household groups.

Appendix: Proofs of Theorems

Proof of Theorem 1: Define \( C(u, p) = uP(p^*, p, u) \) for \( u > 0 \) and \( p > 0 \). Let \( u > 0 \), \( p^0 > 0 \), and \( p^1 > 0 \). Then

\[
\frac{C(u, p^1)}{C(u, p^0)} = \frac{uP(p^*, p^1, u)}{uP(p^*, p^0, u)}
\]

\[
= \frac{P(p^*, p^1, u)}{P(p^*, p^0, u)}
\]

\[
= \frac{P(p^0, p^*, u)P(p^*, p^1, u)}{P(p^*, p^0, u)}
\]

using the time reversal property (ii)

\[
= P(p^0, p^1, u)
\]

using the circularity property (iii).

Note that properties (ii) and (iii) imply \( P(p^*, p^*, u) = 1 \). Hence \( C(u, p^*) = uP(p^*, p^*, u) = u \) for all \( u > 0 \) which is (3).

The proof of the converse part of the theorem is straightforward. I owe this method of proof to David Donaldson.

Proof of Theorem 14: Define \( h(\lambda) = P_D(p^0, p^1, (1-\lambda)u^0 + \lambda u^1) \) for \( 0 \leq \lambda \leq 1 \). Note that \( h(0) = P_D(p^0, p^1, u^0) \) and \( h(1) = P_D(p^0, p^1, u^1) \). There are 24 possible a priori inequality relations that are possible between the four numbers \( h(0) \), \( h(1) \), \( \bar{P}_L \) and \( \bar{P}_P \). However, (51) and (52) imply that \( h(0) \leq \bar{P}_L \) and \( \bar{P}_P \leq h(1) \). This means that there are only six
possible inequalities between the four numbers:

1. \( h(0) \leq \bar{P}_L \leq \bar{P}_P \leq h(1) \),
2. \( h(0) \leq \bar{P}_L \leq \bar{P}_P \leq h(1) \),
3. \( h(0) \leq \bar{P}_L \leq h(1) \leq \bar{P}_L \),
4. \( \bar{P}_L \leq h(0) \leq \bar{P}_L \leq h(1) \),
5. \( \bar{P}_L \leq h(1) \leq h(0) \leq \bar{P}_L \), and
6. \( \bar{P}_P \leq h(0) \leq h(1) \leq \bar{P}_L \).

Since the individual cost functions \( C^h(u_h, p^i) \) are continuous in \( u_h \), it can be seen that \( P_D(p^0, p^1, u) \) defined by (49) is continuous in the vector of utility variables \( u \). Hence \( h(\lambda) \) is a continuous function for \( 0 \leq \lambda \leq 1 \) and assumes all intermediate values between \( h(0) \) and \( h(1) \). By inspecting cases (1) to (6) above, it can be seen that we can choose \( \lambda \) between 0 and 1 (call this number \( \lambda^* \)) so that \( \bar{P}_L \leq h(\lambda^*) \leq \bar{P}_P \) for case (1) or so that \( \bar{P}_L \leq h(\lambda^*) \leq \bar{P}_L \) for cases (2) to (6). Now define \( u^* = (1 - \lambda^*) u^0 + \lambda^* u^1 \) and the proof is complete.

**Proof of Theorem 15:** Using expression (54) for \( P_{P^2}(p^0, p^1, u^0) \) and \( P_{P^2}(p^0, p^1, u^1) \), noting that the shares \( s^h(u^0, p^0) \) and \( s^h(u^1, p^0) \) defined by (55) are non-negative and sum to 1, and using Theorem 2, we may readily establish the left inequality in (57) and the right inequality in (58).

Consider now the right inequality in (57). This follows readily from definition (53), the equalities \( C^h(u^0, p^0) = p^0 \cdot x^0_h \) for \( h = 1, \ldots, H \) and the inequalities

\[
C^h(u^0, p^0) = \min_x \left\{ p^0 \cdot x^0_h : F^h(x) \geq F^h(x^0_h) = u^0_h \right\}
\]

\[\leq p^0 \cdot x^0_h\]  \hspace{1cm} (A1)

which follow since \( x^0_h \) is feasible for the minimization problem.

The left inequality in (58) follows from definition (53) when \( u = u^1 \), the equalities \( C^h(u^1, p^1) = p^1 \cdot x^1_h \), and the inequalities

\[
C^h(u^1, p^0) = \min_x \left\{ p^0 \cdot x^1_h : F^h(x) \geq F^h(x^1_h) = u^1_h \right\}
\]

\[\leq p^0 \cdot x^1_h\]  \hspace{1cm} (A2)
Proof of Theorem 16: The proof of this theorem is identical to the proof of Theorem 14 if we replace $P_D$ by $P_{PP}$, $F_L$ by $P_L$ and $P_P$ by $P_P$.

Proof of Theorem 17: First note that for each household $h$,

$$D^h(u^0_h, x^1_h) = \max_k \{ k : F^h(x^1_h/k) \geq u^0_h \}$$

$$= k^1_h \text{ where } F^h(x^1_h/k^1_h) = u^0_h.$$ 

Thus

$$p^0 \cdot x^0_h = C^h(u^0_h, p^0)$$

$$= \min_x \{ p^0 \cdot x : F^h(x) \geq u^0_h \}$$

$$\leq p^0 \cdot x^1_h/k^1_h$$

since $x^1_h/k^1_h$ is feasible for the cost minimization problem. Thus (A3) yields the following inequality:

$$D(u^0_h, x^1_h) \leq p^0 \cdot x^1_h/p^0 \cdot x^0_h \quad \text{for } h = 1, \ldots, H.$$ 

Repeating the above argument interchanging the superscripts 0 and 1 yields the following inequality:

$$D(u^1_h, x^0_h) \leq p^1 \cdot x^0_h/p^1 \cdot x^1_h \quad \text{for } h = 1, \ldots, H. \quad \text{(A5)}$$

From definition (60), we have
\[ W(X^0, X^1, u^0) = \sum_{h=1}^{H} \beta_h^0 D^h(u_h^0, x_h^1)/D^h(u_h^0, x_h^0) \]
\[ = \sum_{h=1}^{H} \beta_h^0 D^h(u_h^0, x_h^1) \quad \text{since } D^h(u_h^0, x_h^0) = 1 \]
\[ \geq \sum_{h=1}^{H} \beta_h^0 p^1 \cdot x_h^1/p^0 \cdot x_h^0 \]

using (A4) and \( \beta_h^0 \geq 0 \), which establishes (63). Similarly

\[ W(X^0, X^1, u^0) = \sum_{h=1}^{H} \beta_h^1 D^h(u_h^1, x_h^1)/D^h(u_h^1, x_h^0) \]
\[ = \sum_{h=1}^{H} \beta_h^1 1/D^h(u_h^1, x_h^0) \quad \text{since } D^h(u_h^1, x_h^0) = 1 \]
\[ \geq \sum_{h=1}^{H} \beta_h^1 p^1 \cdot x_h^1/p^1 \cdot x_h^0 \quad \text{using (A5) and } \beta_h^1 \geq 0. \]

**Proof of Theorem 18:** Define \( h(\lambda) \equiv W(X^0, X^1, (1-\lambda)u^0 + \lambda u^1) \) and repeat the proof of Theorem 14, where \( \sum_{h=1}^{H} \beta_h^0 Q_h^L \) replaces \( P^L \) and \( \sum_{h=1}^{H} \beta_h^1 Q_h^P \) replaces \( P^P \).

**Proof of Theorem 19:** First note that

\[ C(F(x^0), p^0) \equiv \min_{x} \{ p^0 \cdot x: F(x) \geq F(x^0) \} \]
\[ = p^0 \cdot x^0 \quad \text{by (5)} \]
\[ = \sum_{m=1}^{M} p_m^0 \cdot x_m^0 \]
\[ = \min_{x_1} \left\{ p_1^0 \cdot x_1 + \sum_{m=2}^{M} p_m^0 \cdot x_m^0 : F(x_1, x_2, \ldots, x_M) \right\} \]
\[ \geq F(x_1^0, x_2^0, \ldots, x_M^0) \]
\[ = C^L(F(x^0), p_1^0 \cdot x_1^0) + \sum_{m=2}^{M} p_m^0 \cdot x_m^0. \]
Hence \( C^1(F(x^0), F^0_1, x^0_1) = p^0_1 \cdot x^0_1 \). In a similar manner, we obtain the following equalities:

\[
C^m(F(x^0), p^0_m, x^0_m) = p^0_m \cdot x^0_m, \quad m = 1, 2, \ldots, M \quad \text{and} \quad \tag{A6}
\]

\[
C^m(F(x^1), p^1_m, x^1_m) = p^1_m \cdot x^1_m, \quad m = 1, 2, \ldots, M. \quad \tag{A7}
\]

Since \( x^0_m \) is feasible for the conditional cost minimization problem defined by \( C^m(F(x^0), p^0_m, x^0) \), we have

\[
C^m(F(x^0), p^1_m, x^0_m) \leq p^1_m \cdot x^0_m, \quad m = 1, \ldots, M. \quad \tag{A8}
\]

Since \( x^1_m \) is feasible for \( C^m(F(x^1), p^0_m, x^1) \),

\[
C^m(F(x^1), p^0_m, x^1_m) \leq p^0_m \cdot x^1_m, \quad m = 1, \ldots, M. \quad \tag{A9}
\]

The definition of \( P^m(p^0_m, p^1_m, u^0, x^0) = C^m(F(x^0), p^1_m, x^0 \div C^m(F(x^0), p^0_m, x^0), (A6) \) and \( (A8) \) yield (67) while \( P^m(p^0_m, p^1_m, u^1, x^1) = C^m(F(x^1), p^1_m, x^1 \div C^m(F(x^1), p^0_m, x^1), (A7) \) and \( (A9) \) yield (68).
Footnotes

1 The conceptual framework for the Canadian CPI is nicely explained in Statistics Canada [1982]. The CPI and the Implicit Consumption Price Index for Canada are compared and contrasted in Loyns [1972], who was mainly interested in their inflation measuring capabilities. My focus will be more welfare-oriented.

2 Unfortunately, much of the material presented in this paper is a bit technical. Two useful references that lead the reader into the technical aspects of index number and growth measurement theory in a gentle fashion are Allen [1975], and Usher [1980]. More technical discussions may be found in Konüs [1924], Samuelson [1947; pp.146-162], Malmquist [1953], Pollak [1971], Afriat [1977] and Diewert [1981].

3 Notation: $x^T$ denotes the transpose of the column vector $x$, $p^T x = p \cdot x \equiv \sum_{n=1}^{N} p_n x_n$ denotes the inner product of the vectors $p$ and $x$, $x \geq 0_N$ means each component of the vector $x$ is non-negative, $x > 0_N$ means each component if positive, and $x > 0_N$ means $x \geq 0_N$ but $x \neq 0_N$.

4 $F$ is a function defined over the non-negative orthant \( \{ x : x \geq 0_N \} \) that has the following properties: (i) continuity, (ii) increasingness; i.e., if $x'' > x' > 0_N$, then $F(x'') > F(x')$, (iii) quasiconcavity; i.e., for each utility level $u$, the upper level set $L(u) = \{ x : F(x) \geq u \}$ is convex, (iv) $F(0_N) = 0$ and (v) $F(x)$ tends to $+\infty$ as the components of $x$ all tend to $+\infty$.

5 $C(u,p)$ is defined for $u \geq 0$, $p > 0_N$ and has the following properties: (i) it is continuous, (ii) $C(0,p) = 0$ for every $p > 0_N$, (iii) for every $p > 0_N$, $C(u,p)$ is increasing in $u$ and $C(u,p)$ tends to $+\infty$ as $u$ tends to $+\infty$, (iv) $C(u,p)$ is positively linearly homogeneous in $p$ for fixed $u$, i.e., for $u \geq 0$, $p > 0_N$, $\lambda > 0$, $C(u,\lambda p) = \lambda C(u,p)$, (v) $C(u,p)$ is concave in $p$ for fixed $u$, (vi) $C(u,p)$ is increasing in $p$ for fixed $u > 0$, i.e., if $p'' > p' > 0_N$, $u > 0$, then $C(u,p'') > C(u,p')$ and (vii) $C$ is such that the function $F^*(x) \equiv \max_u \{ u : p \cdot x \geq C(u,p) \}$ for every $p > 0_N$ is continuous for $x \geq 0_N$.

6 This is a version of the Shephard [1953] Duality Theorem; see Diewert, [1982].

7 The term is due to Samuelson [1974].

8 Throughout this section, we assume that $F$ satisfies Conditions I. Many of the theorems in this section can be proven under much weaker regularity conditions; e.g., see Diewert [1981].

9 See also Diewert [1973] and Varian [1982].

10 If $u^0 = u^1$, then $u^* = u^0 = u^1$.

11 Thus $c^F$ is a flexible functional form to use Diewert's [1974] terminology.

12 This corresponds to the terminology used in Christensen, Cummings and Jorgenson [1980]. Diewert [1981; p.187] called $P_0$ the Törnqvist price index, but the term translog price index seems to be more descriptive.

13 The Allen quantity index is closely related to: (i) Samuelson's [1974] money metric scaling for a consumer's utility function, and (ii) Hicks' [1941-42] consumer surplus measures, which are defined in terms of differences of cost functions rather than ratios of cost functions.
The appropriate regularity conditions are listed in Diewert [1982; p.560], and references to the literature on duality theorems between F and D may be found there also. Essentially, D(u,x) has the same regularity properties as the cost function C(u,p) where x replaces p, except that D(u,x) decreases in u while C(u,p) increases in u.

Theorem 7 is not necessarily true if $u > \max \{u^0, u^1\}$ or if $u < \min \{u^0, u^1\}$.

Related approximation theorems have been obtained by Samuelson and Swamy [1974] and Vartia [1978]. It should be noted that Diewert's [1978] results were derived using some results due to Vartia [1976].

For example, see Diewert [1978; p.894], Génèreux [1983] and Szulc [1983].

In fact Allen and Diewert [1981; p.435] provide an even stronger case for the use of the Fisher ideal formula, since it is the only superlative index number formula that is consistent with both the Hicks [1946; pp. 312-313] and Leontief [1936; pp. 54-57] composite commodity theorems.

We cannot even evaluate how good or bad the approximation is unless we are also given current period quantity information $x^f$.

Pollak [1975a] demonstrates that it is difficult if not impossible to combine the subindexes into the true cost-of-living index under general conditions on the underlying preferences. This is not exactly the relevant issue, since we cannot calculate the true cost-of-living index anyway in general. However, if we can use the subindexes to form a close approximation to the overall true cost-of-living index, then this is all that we require. Of course, under restrictive assumptions on preferences, subindexes can be combined to give precisely the correct overall cost-of-living index. The first result of this type was obtained by Shephard [1953], who assumed a special structure of preferences that is now called homothetic separability. For generalizations of the Shephard result and reference to the literature, see Blackorby, Primont and Russell [1978; ch. 9].

In Pollak [1975a; p.145], $C^M$ is called a generalized conditional expenditure function for category m.

$C^m(u,p_m,x)$ is non-decreasing in u and non-increasing in the components of x. See McFadden [1978] for a detailed analysis of the properties of $C^m$.

Remember that $C^m(u,p_m,x)$ does not actually depend on the mth subvector in $x, x^*_m$, and hence $p^0_m(p^1_m, x^*, x^*)$ does not actually depend on $x^*_m$, the mth subvector in $x^* = (x^*_1, x^*_2, ..., x^*_M)$.

Vartia [1976] defines an index number formula to be consistent in aggregation if the value of the index calculated in two stages coincides with the value of the index calculated in a single stage. A careful analysis of this concept may be found in Blackorby and Primont [1980]. Empirical evidence on the closeness of two-stage aggregates with the corresponding single-stage aggregates may be found in Diewert [1978, 1983b].

If we are willing to use econometric techniques, then it is not necessary to assume intertemporal additivity in order to estimate the consumer's intertemporal preference function; i.e., see Diewert [1974] and Darrough [1977]. This suggests that it may be possible to adapt the usual index number techniques to the intertemporal context as well.
26 In the consumer context, see Ruggles [1967], and in the producer context, see Denny and Fuss [1981] and Caves, Christensen and Diewert [1982a].

27 We are assuming that the household derives disutility from supplying additional hours of work. This may not be true for \((x,y)\) vectors where \(y\) is close to \(0_{M}\). Formally, we assume that \(F\) satisfies Conditions I except that now the domain of definition of \(F\) is \((x,y): x \geq 0_{N}, y \leq y \leq 0_{M}\) where \(y \leq 0_{M}\).

28 However, it could still be used as a deflator for the household's nominal "full" income ratio, \((p^{1}x^{1} + w^{1}b)/(p^{0}x^{0} + w^{0}b)\), to form Pollak implicit quantity indexes as was done in the beginning of Section 3 above. On the concept of "full" income, see Becker [1965].

29 Recent papers on this issue include McFadyen and Hobart [1978], Rymes [1979], Blinder [1980], Hendershott [1980], Hughes [1980], Gordon [1981], Dougherty and Van Order [1982] and Gillingham [1982].

30 In fact when one works with user cost formulae of the type defined by (93) and evaluates the expected prices by using \textit{ex post} market prices, one will often find negative user costs for housing.

31 In technical terms, we replace the original preferences by the convex free disposal hull of the original preferences.

32 Other types of transactions costs will also have this effect.

33 Comprehensive accounts of the quality change problem may be found in Triplett [1982] and Hodgins [1982].

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