SEPARABILITY AND A GENERALIZATION OF THE
COBB-DOUGLAS COST, PRODUCTION AND
INDIRECT UTILITY FUNCTIONS

by

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In modelling the supply and demand for labour, it is possible to use consumer theory to model the supply of labour (see the studies [2] and [9] in this series) and production theory to model the demand for labour (see the studies [1] and [11] in this series).

It is useful to have functional forms for utility and production functions which do not impose a priori values for the elasticities of substitution between pairs of commodities. For example, the Leontief functional form for a production function imposes an elasticity of substitution equal to zero between every pair of inputs. The use of this functional form in the context of modelling labour demand would lead us to assume that labour demand was totally unresponsive to changes in other input prices when this may not be the case. The Cobb-Douglas functional form for a production function imposes an elasticity of substitution equal to unity between every pair of inputs. The use of this functional form would lead us to conclude that each type of labour input always accounted for a constant share of the total cost of production when this may not be the case.

The functional form developed in this study allows for an arbitrary pattern of elasticities of substitution and thus should be useful in modelling the demand for labour.

The views expressed are solely those of the author, and do not necessarily correspond to any policy or position of the Department of Manpower and Immigration.

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The following generalization of the N factor Cobb-Douglas production function is defined: \( y = k \prod_{i=1}^{n} \prod_{j=1}^{n} \left( \frac{1}{2} x_i + \frac{1}{2} x_j \right)^{\alpha_{ij}} \) where \( y \) is output, \( x_1, \ldots, x_n \) are quantities of the \( n \) inputs and the parameters \( \alpha_{ij} \) satisfy the following restrictions in the constant returns to scale case: \( \alpha_{ij} = \alpha_{ji} \) \( 1 \leq i, j \leq n \) and \( \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} = 1 \). If \( \alpha_{ij} = 0 \) for all \( i \neq j \), the above functional form reduces to the \( n \) factor Cobb-Douglas production function. Note that \( \ln y = \ln k + \sum \alpha_{ij} \ln \left( \frac{1}{2} x_i + \frac{1}{2} x_j \right) \) and thus the parameters \( \alpha_{ij} \) may be estimated using linear regression techniques.

The same functional form may be used in order to define a generalization of the Cobb-Douglas unit cost function. Let the input prices be \( p_1, \ldots, p_n \) and let \( c(p_1, \ldots, p_n) \) be defined as the minimum cost of producing one unit of output. Then the Generalized Cobb-Douglas unit cost function is defined by: \( c(p_1, \ldots, p_n) = k^* \prod_{i=1}^{n} \prod_{j=1}^{n} \left( \frac{1}{2} p_i + \frac{1}{2} p_j \right)^{\beta_{ij}} \) where \( \beta_{ij} = \beta_{ji} \) and \( \sum_{i} \sum_{j} \beta_{ij} = 1 \). Application of Shephard's Lemma yields the following system of equations in the factor expenditure shares \( s_i = p_i x_i / \left( \sum_{j=1}^{n} p_j x_j \right) \): \( s_i = \sum_{j=1}^{n} \beta_{ij} p_j / \left( \frac{1}{2} p_i + \frac{1}{2} p_j \right) \); \( i = 1, 2, \ldots, n \), which is a system of equations which are linear in the unknown parameters.

We shown that the above functional forms can provide a second order approximation to an arbitrary production function in the constant returns to scale case.
PAPERS IN THIS SERIES


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SEPARABILITY AND A GENERALIZATION OF THE COBB-DOUGLAS COST,
PRODUCTION AND INDIRECT UTILITY FUNCTIONS*

by

W. E. Dievert**

1. Introduction

Suppose we are given a production function of the form

\[ y = f^*(x_1, x_2, \ldots, x^M) \]

where \( f^* \) is the production function, \( y \) is output and \( x^i \) is an \( N_i \) dimensional vector of inputs for \( i = 1, 2, \ldots, M \).

If \( f^*(x_1, x_2, \ldots, x^M) = g[f^1(x_1), f^2(x_2), \ldots, f^M(x^M)] \), then \( f^* \) is said to be weakly separable \(^1\) with respect to the partition of inputs \( x_1, x_2, \ldots, x^M \).

Let us refer to the function \( g \) as the macro (production) function and the functions \( f^i \) as the micro (production) functions.

The assumption of weak separability occurs quite frequently in production theory. For example, consider Sato's two level C.E.S. production function [36]. Another example of a weak separability assumption is given by the Cordon [10], Sims [39], Arrow [3] approach to estimating real value added production functions where the gross output production function \( f^*(K, L, M) \) is assumed to be of the form \( f^*(K, L, M) = g[f^1(K, L), M] \), where \( K \) is capital services input, \( L \) is labour input and \( M \) is intermediate input. The micro function \( f^1(K, L) \) is interpreted as a real value added production function.

---

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**University of British Columbia and the Department of Manpower and Immigration, Ottawa. I would like to thank E.R. Berndt, A.D. Woodland and T.J. Wales for their helpful comments. Thanks are also due to L.J. Lau for stimulating this research.
The assumption of weak separability occurs even more frequently in consumer theory. (We now interpret $y$ as utility, $f^u$ as a (macro) utility function and the vectors $x^i$, $i = 1, 2, \ldots, M$, as commodity vectors. For example, Pollak and Wales [30] assume that there is a micro utility function $f^1$, which has as its arguments consumption of nondurables in a given period, which is separable from durables consumption in the same period. In the theory of intertemporal consumer demand, many authors (e.g., Strotz [41], Gorman [18], Christensen [7], Heien [21]) assume that the intertemporal utility function is separable. In Becker's [4] theory of the allocation of time, the one period utility function is assumed to be of the form $g[f^1(x^1), f^2(x^2), \ldots, f^M(x^M)]$ where the "basic commodities" $z^i = f^i(x^i)$ are produced by inputs of time and rentals of consumer goods, and thus the one period utility function is separable. As a final example of separability in consumer theory, consider Lloyd's [27] demand for money model, where real cash balances appear in the consumer's one period utility function, but in a weakly separable manner.

Given that examples of weak separability abound in both the theory of production and in the theory of consumer demand, we now ask how may such separable functions be estimated econometrically.

In the production case, if one has data on output $y$ and on input vectors $x^1, x^2, \ldots, x^M$, then one can simply assume functional forms for the macro function $g$ and for the micro functions $f^1, \ldots, f^M$ and use nonlinear regression techniques on the equation $y = g[f^1(x^1), \ldots, f^M(x^M)]$. For an example which uses this procedure, see Sato [36]. Another possible
procedure is to assume a general functional form for the production function \( f^* \) where \( y = f^*(x^1, x^2, \ldots, x^N) \) and then either test the validity of or impose various (nonlinear, in general) restrictions on the parameters of \( f^* \) which convert \( f^* \) into a separable production function of the form \( g[f^1(x^1), \ldots, f^M(x^M)] \). Berndt and Christensen [6] have recently indicated how this last procedure could be applied using the translog production function.\(^2\)

However, in the context of consumer theory, neither of the above two procedures can be applied since "utility" \( y \) is not an observable variable. Moreover, even in the context of production theory, we often can observe only a subset of the vectors of variables \( x^1, x^2, \ldots, x^M \) and thus we can only hope to estimate a subset of the micro functions \( f^1, f^2, \ldots, f^M \). In the remainder of this paper, we shall be concerned only with estimating the first micro function, \( f^1(x^1) \), which we will henceforth denote as \( f(x) = f(x_1, x_2, \ldots, x_N) \).

For the sake of definiteness, we will use primarily the language of production theory in the remainder of this paper, but we note that the results of section 2 and section 4 of this paper are equally applicable in the consumer theory context.

If the macro function \( g \) is increasing in its first argument, we see that the producer will want to make the micro output \( z \equiv f^1(x^1) \equiv f(x_1, x_2, \ldots, x_N) \) as large as possible for a given factor expenditure

\[
\sum_{i=1}^{N} p_i x_i = k \text{ where } p_i \text{ is the rental price of input } i, \ x_i \text{ is the quantity}
\]
of input \( i \). Conversely, the producer will want to minimize the cost of producing a given amount of the micro output \( z \). Thus for a positive vector of input prices \( p \gg 0_N \) and a nonnegative level of micro output \( z \), we define the producer's total minimum cost function for the production of \( z \) as:

\[
1-1 \quad C(z; p) \equiv \min_x \{p^T x : f(x) \geq z; x \geq 0_N\}.
\]

If the micro function \( f \) satisfies the following Conditions I:

- \( f \) is a positive, nondecreasing, \( \lambda \) (positively) linear homogeneous, \( \delta \) concave function over the positive orthant in \( N \) space, then the cost function defined by 1-1 factors into the following form:

\[
1-2 \quad C(z; p) = c(p)z \quad \text{where} \quad c(p) \equiv \min_x \{p^T x : f(x) \geq 1; x \geq 0_N\}.
\]

Moreover, the unit cost function \( c \) satisfies the same regularity conditions as \( f \) and it may be used to define the production function by means of the following definition, where \( \bar{x} \gg 0_N \):

\[
1-3 \quad f(\bar{x}) = \max_z \{z : c(p)z \leq p^T \bar{x} \quad \text{for every} \quad p \geq 0_N\} \equiv \min_p \left\{ \frac{1}{c(p)} : p \geq 0_N, p^T \bar{x} \leq 1 \right\}.
\]

The above assertions are a version of the Shephard\(^9\) Duality Theorem between cost and production functions. We need one more theoretical result known as Shephard's Lemma [37]: if the cost function \( C(z; p) \) satisfies
the appropriate regularity conditions\textsuperscript{10} (so that it can be interpreted as the solution to the cost minimization problem 1-1) and is differentiable with respect to factor prices, then

\begin{equation}
1-4 \quad x_i(z; p) \equiv \frac{\partial c(z; p)}{\partial p_i} ; \ i = 1, 2, \ldots, N
\end{equation}

where \( x_i(z; p) \) is the cost minimizing demand for input \( i \) needed to produce output \( z \geq 0 \), given factor prices \( p >> 0 \).

With the above preliminaries out of the way, we may return to the problem of estimating the parameters of the micro production function, \( z = f(x_1, \ldots, x_N) \), given that we have data on inputs \( x = (x_1, x_2, \ldots, x_N)^T \) and on input prices \( p = (p_1, p_2, \ldots, p_N)^T \) but not on the micro output \( z \). We will distinguish two approaches to this problem: Method I and Method II.

Method I is due to Arrow [3] although it is implicit in the work of Shephard [38; 145-6] and others. We assume that the micro production function \( f \) is subject to constant returns to scale. In particular, we assume that \( f \) satisfies Conditions I and thus \( f \) has a total cost function of the form given by 1-2: \( C(z; p) = c(p)z \) where \( c(p) \) is the unit cost function which also satisfies Conditions I. We note that the expenditure share on the \( i \)th factor \( s_i \), defined by \( s_i = p_i x_i / p^T x \) is an observable variable given that \( p \) and \( x \) are observable. Now if \( c(p) \) is differentiable with respect to input prices \( p \), we may apply Shephard's Lemma 1-4 and we find \( x_i = z \partial c(p) / \partial p_i \). Since \( c(p) \) is
linear homogeneous in $p$, by Euler's Theorem, $c(p) = \sum_{i=1}^{N} p_i \frac{\partial c(p)}{\partial p_i}$.

Putting all of this together yields the following system of equations in the factor shares $s_i$:

$$1-5 \quad s_i = \frac{p_i x_i}{p^T x} = \frac{p_i \frac{\partial c(p)}{\partial p_i}}{c(p)} \quad ; \quad i = 1, 2, \ldots, N .$$

Thus, given a functional form for the unit cost function $c(p)$ which is consistent with Conditions I and in addition is differentiable with respect to input prices $p$, we may form the system of equations 1-5, estimate the unknown parameters of $c(p)$, and then the micro function $f$ may be calculated by means of 1-3.11 Econometric estimation will be facilitated if the system of equations given by 1-5 turns out to be linear in the unknown parameters. In section 2 of this paper, we pursue this method of estimating the micro function $f$.

If the micro function $f$ is not subject to constant returns to scale, then we can apply Method II in order to estimate $f$. Method II is the following modification of standard techniques used in consumer demand analysis. Let us suppose that the micro function satisfies the following Conditions II: $f(x)$ is a nonnegative, nondecreasing, continuous, quasiconcave12 function for $x \geq 0_N$, and $f(x) > 0$ if $x >> 0_N$. Then the inverse, indirect micro function $h(v)$ is defined for $v >> 0_N$ by:

$$1-6 \quad h(v) \equiv \min_{x} \left\{ \frac{1}{f(x)} : v^T x \leq 1, x \geq 0_N \right\} .$$
We see that $h(v) = 1/[\max_x \{f(x): v^T x \leq 1, x \geq \underline{0}_N\}] = 1/g(v)$ where $g(v)$ is the indirect utility or production function which corresponds to the direct utility or production function $f$; i.e., given that the consumer has income $Y > 0$ to spend on the commodities $x$ which may be rented at prices $p$, the function $g(v)$ gives the maximum (micro) utility which the consumer can attain given the normalized prices $v = p/y$. The domain of definition of the function $h$ can be extended to the nonnegative orthant and it turns out that if $f$ satisfies Conditions II above, then $h$ also satisfies Conditions II and moreover, $f$ may be calculated from $h$ using the following formula, where $\bar{x} >> \underline{0}_N$:

$$f(\bar{x}) = \min_v \left\{ \frac{1}{h(v)}: \bar{x}^T v \leq 1, v \geq \underline{0}_N \right\}.$$  

Thus there is an equivalence or duality between the (direct) micro function $f$ and the inverse indirect micro function $h$. We need an additional theoretical result which is a variant of Roy's Identity [32; 219]. As usual, let $x > \underline{0}_N$ be a vector of observed factor inputs, let $p >> \underline{0}_N$ be a positive vector of input prices and define the normalized price vector by $v = p/p^T x$. We assume that the inverse, indirect micro function $h$ satisfies Conditions II and in addition is once differentiable with respect to the normalized prices with $\nabla h(v) \neq \underline{0}_N$ where $\nabla h(v)$ is the vector of first order partial derivatives of $h$ evaluated at $v$. Let $x(v)$ denote the solution to the
micro maximization problem: \( \max_x \{ f(x) : v^T x \leq 1, x \geq 0_N \} \) where \( f \) is the micro function which corresponds to \( h \) via 1-7. Then we have:

\[
1-8 \quad x(v) = \frac{Vh(v)}{v^T h(v)}.
\]

Thus in order to obtain a system of derived demand equations \( x(v) \) which is a solution to the micro maximization problem \( \max_x \{ f(x) : v^T x \leq 1, x \geq 0_N \} \), we need only postulate a functional form for \( h \), the inverse indirect micro function, which satisfies the appropriate regularity conditions, differentiate \( h \) and then estimate the unknown parameters of \( h \) using the system of equations 1-8. Once \( h \) is determined, the micro function \( f \) may be found using 1-7. In section 4 of this paper, we pursue this method of estimating the micro function \( f \). This relatively "painless" method of generating systems of derived demand equations consistent with output or utility maximization subject to an expenditure constraint is due to Houthakker [23].

We can now summarize the contents of the remainder of this paper.

Recall that the \( N \) factor constant returns to scale Cobb-Douglas [9] production function is given by \( f(x) = k^{a_1 a_2 \ldots a_N} x_{1}^{a_1} x_{2}^{a_2} \ldots x_{N}^{a_N} \) where \( k > 0 \), \( a_1 > 0 \) and \( \sum_{i=1}^{N} a_i = 1 \) and \( c(p) = \min_x \{ p^T x : f(x) \geq 1, x \geq 0_N \} \), the corresponding unit cost function, is given by \( c(p) = k^* p_1^{a_1} p_2^{a_2} \ldots p_N^{a_N} \) where \( k^* = \frac{a_1}{a_2} \ldots \frac{a_N}{k} \). The corresponding inverse indirect production function is given by \( h(v) = k^* v_1^{a_1} v_2^{a_2} \ldots v_N^{a_N} \).
In section 2 below, we generalize the Cobb-Douglas unit cost function; in section 3, we generalize the Cobb-Douglas production function and in sections 4 and 5, we provide a generalization of the Cobb-Douglas inverse indirect production function. The analysis of section 4 is mostly concerned with the case where we cannot observe the micro output (or utility), \( z = f(x) \), but at the end of section 4, we assume that we can observe \( z \). We note that the functional forms given in sections 2 and 4 generalize the Cobb-Douglas production function in a certain direction, while the functional form defined in section 3 provides a generalization in a different direction; i.e., the direct micro function \( f \) defined in section 3 does not in general correspond to the inverse indirect function \( h \) defined in section 4.

2. **A Generalization of the Cobb-Douglas Unit Cost Function**

Let us define the following generalization of the Cobb-Douglas unit cost function:

\[
2.1 \quad c(p_1, p_2, \ldots, p_N) = k^* \prod_{i=1}^{N} \prod_{j=1}^{N} \left( \frac{1}{2} p_i + \frac{1}{2} p_j \right)^{a_{ij}}
\]

where \( p_i \) is the price of the \( i \)th input, \( k^* > 0 \), \( a_{ij} = a_{ji} > 0 \) and \( \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} = 1 \). Note that \( c(p) \) defined by 2-1 is a geometric mean of the arithmetic means \( (1/2)p_i + (1/2)p_j \). It is easy to see that \( c(p) \) is positive if \( p > q_N \), nondecreasing, and (positively) homogeneous of degree one in \( p \). Thus \( c(p) \) defined by 2-1 will satisfy Conditions I
if we can show that it is a concave function for \( p >> \frac{1}{N} \). Note that

the function \( \prod_{i=1}^{N} \prod_{j=1}^{N} x_{ij} \) where \( \alpha_{ij} > 0 \), \( \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} = 1 \) is nondecreasing in its arguments \( x_{ij} \) and since it is a mean of order zero (see Hardy, Littlewood and Pólya [20; 13]), it is a concave function in its arguments \( x_{ij} \) (using Minkowski's Inequality [20; 30]). The functions \( x_{ij}(p) \equiv (1/2)p_i + (1/2)p_j \) are linear in \( p \) and hence concave functions in \( p \). The concavity of \( c(p) \) follows readily since a nondecreasing, concave function in concave functions of \( p \) is also concave.\(^{17}\) Thus \( c(p) \) satisfies Conditions I and we may apply Shephard's Lemma 1-4 and we obtain the following system of equations in the factor shares \( s_i \equiv p_i x_i / p^T x \) where \( x = (x_1, x_2, \ldots, x_N)^T \) is the vector of observed factor inputs:

\[
2.2 \quad s_i = \sum_{j=1}^{N} \frac{\alpha_{ij}p_i}{\left( \frac{1}{2}p_i + \frac{1}{2}p_j \right)} \quad ; \quad i = 1, 2, \ldots, N
\]

Notice that the system of equations 2-2 is **linear** in the unknown \( \alpha_{ij} \) parameters and if \( \alpha_{ij} = 0 \) for \( i \neq j \), then 2-2 reduces to the Cobb-Douglas case, \( s_i = \alpha_{ii} \) for \( i = 1, 2, \ldots, N \).\(^{18}\)

The nonnegativity restrictions \( \alpha_{ij} > 0 \) may be relaxed but then the functional form given by 2-1 will not satisfy Conditions I for all \( p >> \frac{1}{N} \). Thus we look for a region of \( p \)'s where Conditions I are satisfied. If \( S \) is a closed, convex subset of the positive orthant in \( N \) space and \( c(p) \) is defined by 2-1 where \( k^* > 0 \) and \( \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} = 1 \),
then $c(p)$ will satisfy Conditions I for all $p \in S$ if:

$$2-3 \quad \nabla c(p) > 0 \quad \text{and} \quad \nabla^2 c(p)$$

is a negative semidefinite matrix for all $p \in S$

where $\nabla c(p)$ is the vector of first order partial derivatives of $c$ evaluated at $p$ and $\nabla^2 c(p)$ is the matrix of second order partial derivatives of $c$ evaluated at $p$. We note that if the following condition is satisfied for some $p^* \gg 0_N$:

$$2-4 \quad \nabla c(p^*) \gg 0_N \quad \text{and} \quad \nabla^2 c(p^*)$$

is a negative semidefinite matrix of rank $N-1$.

then there will exist a closed convex set $S^*$ which contains $p^*$ in the interior of $S$ and conditions 2-3 will be satisfied for this $S^*$.

(The matrix $\nabla^2 c(p^*)$ cannot be of rank $N$ since $c(p)$ is a linear homogeneous function).

In any case, if 2-3 holds for a closed convex set $S$ of prices for a Generalized Cobb-Douglas unit cost function $c(p)$ of the form given by 2-1 (where we do not impose the nonnegativity restrictions $a_{ij} \geq 0$), then $c(p)$ may be used to define a production function by means of the following definition:

$$2-4 \quad f(\bar{x}) \equiv \max_y \{ y : c(p)y \leq p^T \bar{x} \text{ for every } p \in S \} \text{ where } \bar{x} \gg 0_N .$$
It can be verified that $f$ defined by 2-4 satisfies Conditions I and the unit cost function $c^*(p) = \min_x \{ p^T x : f(x) \geq 1, x \geq 0_N \}$ which $f$ generates has the property that $c^*(p) = c(p)$ for $p \in S$. Thus if $c(p)$ satisfies 2-3 for a neighbourhood of prices, then the domain of definition of $c$ can be extended from that neighbourhood to the entire positive orthant and this extension of $c$ will satisfy Conditions I.

We now show that the Generalized Cobb-Douglas unit cost function defined by 2-1 where $k^* > 0$, $\alpha_{ij} = \alpha_{ji}$ and $\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} = 1$ can provide a good local approximation to an arbitrary twice continuously differentiable unit cost function.

2-5 Theorem: Given an arbitrary unit cost function $c^*$ (consistent with Conditions I) which is twice continuously differentiable at the point $p^* \gg 0_N$ where $c^*_i = \partial c^*(p^*)/\partial p_i$ for $i = 1, 2, \ldots, N$ and $c^*_{ij} = \partial^2 c(p^*)/\partial p_i \partial p_j$ for $1 \leq i < j \leq N$, then there exists a Generalized Cobb-Douglas unit cost function which provides a second order approximation to $c^*$ at the point $p^* = (p_1^*, p_2^*, \ldots, p_N^*)^T$.

Proof: Since $c^*$ satisfies Conditions I, it is linear homogeneous in $p$ and thus by Euler's Theorem on homogeneous functions, we have $c^*(p^*) = \sum_{i=1}^{N} p_i c^*_i$. Also if we define $c^*_{ij} = \partial^2 c^*(p^*)/\partial p_i \partial p_j$, then by Young's Theorem, $c^*_{ij} = c^*_{ji}$ and by Euler's Theorem, $\sum_{j=1}^{N} p_j c^*_j = 0$ for $i = 1, 2, \ldots, N$. Thus $c^*(p^*)$ and $c^*_{ii}$ can be determined from a knowledge of the first order partial derivatives $c^*_i$ for $i = 1, 2, \ldots, N$ and the
second order partial derivatives \( c_{i,j}^* \) for \( 1 \leq i < j \leq N \). Now take \( c(p) \) to be given by 2-1, partially differentiate \( c(p) \) with respect to \( p_i \) at the point \( p^* \) and upon setting the resulting partial derivative equal to \( c_i^* \), we obtain the following system of equations:

\[
\sum_{j=1}^{N} a_{i,j} \frac{c(p^*)}{(\frac{1}{2} p_i^* + \frac{1}{2} p_j^*)} = c_i^* \quad ; \quad i = 1, 2, \ldots, N .
\]

If we differentiate \( \frac{\partial c(p^*)}{\partial p_i} \) with respect to \( p_j \) for \( i \neq j \) and set the resulting partial derivative equal to \( c_{i,j}^* \), we obtain the following system of equations, assuming that equations 2-6 hold:

\[
-\frac{1}{2} a_{i,j} c(p^*) (\frac{1}{2} p_i^* + \frac{1}{2} p_j^*)^{-2} + \frac{c_{i,j}^* c_i^*}{c(p^*)} = c_{i,j}^* \quad ; \quad 1 \leq i < j \leq N .
\]

Let us for the moment assume that \( c(p^*) = \sum_{i=1}^{N} p_i^* c_i^* \) and replace \( c(p^*) \) by \( \sum_{i=1}^{N} p_i^* c_i^* \) in equations 2-6 and 2-7. Thus we obtain a system of linear equations in the unknown \( a_{i,j} \). We may use equations 2-7 to determine \( a_{i,j} \) for \( 1 \leq i < j \leq N \), set \( a_{j,i} = a_{i,j} \) and then the \( a_{i,i} \) may be determined from equations 2-6. The parameter \( k^* \) which occurs in 2-1 is then determined by solving the equation \( c(p^*) = \sum_{i=1}^{N} p_i^* c_i^* \)

\[
k^* \prod_{i=1}^{N} \prod_{j=1}^{N} ((\frac{1}{2}) p_i^* + (\frac{1}{2}) p_j^*) a_{i,j} \text{ for } k^*. \]

It remains to show that the \( a_{i,j} \) solution to 2-6 and 2-7 satisfies the homogeneity constraint,

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} a_{i,j} = 1 . \text{ If we replace } c(p^*) \text{ in 2-6 by } \sum_{k=1}^{N} p_k^* c_k^* \text{ and multiply}
\]

\[
\sum_{k=1}^{N} \frac{c_k^*}{c(p^*)} = 1 .
\]
both sides of the \( i^{th} \) equation of 2-6 by \( p_i^* / \sum_{k=1}^{N} p_k^* c_k^* \), we obtain the equivalent system:

\[
2-8 \quad \sum_{j=1}^{N} \frac{\alpha_{ij} p_i^*}{(\frac{1}{2} p_i^* + \frac{1}{2} p_j^*)} = \frac{p_i^* c_i^*}{(\sum_{k=1}^{N} p_k^* c_k^*)} ; \quad i = 1, 2, \ldots, N .
\]

If we sum equations 2-8 for \( i = 1, 2, \ldots, N \), we obtain:

\[
2-9 \quad 1 = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\alpha_{ij} p_i^*}{(\frac{1}{2} p_i^* + \frac{1}{2} p_j^*)} = \sum_{i=1}^{N} \alpha_{ii} + \sum_{1<i<j<N} \frac{[\alpha_{ij} p_i^* + \alpha_{ij} p_j^*]}{(\frac{1}{2} p_i^* + \frac{1}{2} p_j^*)}.
\]

\[
= \sum_{i=1}^{N} \alpha_{ii} + \sum_{1<i<j<N} \frac{[2\alpha_{ij}]}{(\frac{1}{2} p_i^* + \frac{1}{2} p_j^*)} \quad (\text{using } \alpha_{ij} = \alpha_{ji})
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} .
\]

Thus we have shown that \( c(p^*) = c^*(p^*) \) and the first and second order partial derivatives of \( c \) and \( c^* \) coincide at \( p^* \), where \( c(p) \) is defined by 2-1. Q.E.D.

It turns out that we can relate the parameter \( \alpha_{ij} \) for \( i \neq j \) which occurs in 2-1 to the partial elasticity of substitution \( \sigma_{ij} \) between inputs \( i \) and \( j \). Uzawa [44] has shown that \( \sigma_{ij} \) may be defined
in terms of the partial derivatives of the cost function as follows, where \( c_i(p) \equiv \partial c(p)/\partial p_i \) and \( c_{ij}(p) \equiv \partial^2 c(p)/\partial p_i \partial p_j \):

\[
\sigma_{ij}(p) = \frac{c(p)c_{ij}(p)}{c_i(p)c_j(p)}.
\]

If we make use of Shephard's Lemma 1-4, we see that \( x_i(z; p) \), the cost minimizing demand for input \( i \) needed to produce output \( z \) given factor prices \( p >> 0_N \), is equal to \( c_i(p)z \) and thus

\[
\sigma_{ij}(p) = \frac{c(p)z \partial x_i(z; p)/\partial p_j}{x_i(z; p)x_j(z; p)}
\]

which is simply a normalization of \( \partial x_i(z; p)/\partial p_j \), the normalization being chosen so that \( \sigma_{ij}(p) = \sigma_{ji}(p) \) and so that \( \sigma_{ij} \) is invariant under scale changes in units.

If \( c(p) \) is defined by 2-1, then a straightforward calculation shows that:

\[
\sigma_{ij}(p) = 1 - \frac{\frac{1}{2} \sigma_{ij}[c(p)]^2}{(\frac{1}{2}p_i + \frac{1}{2}p_j)^2 c_i(p)c_j(p)}
\]

Thus if \( \sigma_{ij} = 0 \) for \( i \neq j \), then \( \sigma_{ij}(p) = 1 \); i.e., if we set the parameter \( \sigma_{ij} = 0 \) a priori, then we force \( \sigma_{ij} = 1 \).

If we wish to force certain elasticities of substitution to be identical at a given price vector \( p^* >> 0_N \) where the observed vector of factor demands is \( x^* >> 0_N \), then this may be accomplished by
substituting \( x_i^\star /z^\star \) for \( c_i(p^\star) \), where \( z^\star \) is the (unobserved)
output which corresponds to the input vector \( x^\star \equiv (x_1^\star, x_2^\star, \ldots, x_N^\star)^T \),
\( x_j^\star /z^\star \) for \( c_j(p^\star) \) and \( p^\star x^\star /z^\star \) for \( c(p^\star) \) in 2.11. We obtain
the following equation which is linear in \( a_{ij} \):

\[
2.12 \quad \sigma_{ij}(p^\star) = 1 - \frac{1}{2} \cdot \frac{a_{ij}(p^\star x^\star)^2}{(\frac{1}{2} p_i + \frac{1}{2} p_j)^2 x_i^\star x_j^\star} ; \quad 1 \leq i < j \leq N .
\]

Thus a priori restrictions on elasticities of substitution at a
given point \( (p^\star, x^\star) \) may be rather easily imposed on the Generalized Cobb-
Douglas unit cost function 2-1 using 2-12.

We conclude this section by briefly considering some of the econo-
metric problems involved in estimating the parameters of the system of
factor shares equations, 2-2. Let us assume that we have \( K \) observa-
tions on input shares and on input prices. Let \( s^k \equiv (s_1^k, s_2^k, \ldots, s_N^k)^T \)
be the observed vector of factor shares \( \epsilon^k \equiv (\epsilon_1^k, \epsilon_2^k, \ldots, \epsilon_N^k) \) be a vector
of unobserved errors in period \( k \) and let \( \alpha = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1N}; \alpha_{23}, \ldots, \alpha_{2N}; \alpha_{33}, \ldots, \alpha_{NN})^T \) be the \((N+1)N/2\) vector of \( \alpha_{ij} \)'s, after the symmetry conditions \( \alpha_{ij} = \alpha_{ji} \) are imposed. Then if we put time superscripts on the \( s_i \) and \( p_i \) which appear in 2-2 and add the error term \( \epsilon_i^k \) to the \( i \)th equation of 2-2 for \( i = 1, 2, \ldots, N \), we obtain the following system of equations:

\[
2-13 \quad s^k = X^k \alpha + \epsilon^k \quad ; \quad k = 1, 2, \ldots, K
\]

where the \( N \) by \((N+1)N/2\) matrix \( X^k \) is formed in an obvious manner using the observed input price vector for period \( k \), \( p^k = (p_1^k, p_2^k, \ldots, p_N^k) \). We may regard the vector of prices \( p^k \) as being exogenous, the vector of shares \( s^k \) as being endogenous and \( \epsilon^k \) as a vector of random variables. However, because the observed shares \( s^k \) sum to unity in each period, (i.e., \( 1^T s^k = 1 \) where \( 1 \) is a vector of ones), we must have \( 1^T \epsilon^k = 1 \) and \( 1^T X^k \alpha \), a constant. Thus the covariance matrix of \( \epsilon^k \) must be singular for \( k = 1, 2, \ldots, K \), and we cannot apply the usual least squares sampling theory to 2-13.

Theil [43; 276-287] has outlined a relatively simple procedure for dealing with this singularity problem. Let the \( N \) by \( N-1 \) matrix \( F \) be defined as \( F = [f_1, f_2, \ldots, f_{N-1}] \) where the \( N \) dimensional vectors \( f_i \) satisfy \( f_i^T f_j = 0 \) if \( i \neq j \), \( f_i^T f_i = 1 \) and \( f_i^T 1 = 0 \); that is, the matrix \( F \) consists of \( N-1 \) mutually orthonormal vectors which are all orthogonal to the \( N \) dimensional vector of ones, \( 1 \). We now assume that the \( N \) dimensional disturbance vector \( \epsilon^k \) in 2-13 is such that:
2-14 \[ \varepsilon^k = F\eta^k \; ; \; E\eta^k = 0_{N-1} \; \text{for} \; k = 1,2,\ldots,K \; ; \]
\[ E\eta_i^i(\eta_j^j)^T = \delta_{ij} \sigma^2 I_{N-1} \]

where \( E \) denotes the expectation operator, \( \eta^k \) is an \( N-1 \) dimensional vector of disturbances which are distributed independently over time, and \( \delta_{ij} \) is the Kronecker delta where \( i,j = 1,2,\ldots,K \). Note that \( \frac{1}{k} \varepsilon^k \equiv 0 \) for \( k = 1,2,\ldots,K \) since the columns of \( F \) are all orthogonal to the vector \( 1 \). Thus we are assuming that the \( N \) dimensional vector of disturbances \( \varepsilon \) is generated by an \( N-1 \) dimensional vector of random variables \( \eta \) and we initially assume that the expectation of \( \eta \) is \( 0_{N-1} \) and the variance covariance matrix of \( \eta \) is \( \sigma^2 I_{N-1} \). Now premultiply both sides of 2-13 by \( F^T \) and we obtain the following system of equations (since \( F^T \varepsilon^k = \eta^k \)):

2-15 \[ F^T \varepsilon^k = F^T \alpha + \eta^k \; ; \; k = 1,2,\ldots,K \; . \]

Let us combine observations and rewrite 2-15 as:

2-16 \[ y = X\alpha + \eta \; ; \; E\eta = 0_{(N-1)K} \; ; \; E\eta^T = \sigma^2 I_{(N-1)K} \]

where \( y^T \equiv [(F^T \varepsilon_1)^T, (F^T \varepsilon_2)^T, \ldots, (F^T \varepsilon_K)^T] \), \( X \) is an \((N-1)K\) by \((N+1)N/2\) matrix and \( \eta^T \equiv [\eta^T_1, \eta^T_2, \ldots, \eta^T_K] \). We have already imposed the symmetry constraints \( \alpha_{ij} = \alpha_{ji} \) in 2-16, but we have not imposed the linear homogeneity constraint \( \sum_{i=1}^N \sum_{j=1}^N \alpha_{ij} = 1 \). This last constraint may
be written as:

\[2-17 \quad \alpha_1 + 2\alpha_{12} + \ldots + 2\alpha_{1N} + \alpha_2 + 2\alpha_{23} + \ldots + 2\alpha_{2N} + \ldots + \alpha_{NN} = 1\]

or

\[r^T \alpha = 1\]

where

\[r^T \equiv [1, 2, \ldots, 2, 1, 2, \ldots, 2; \ldots, 1]\]

\[a^T \equiv [\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1N}; \alpha_{22}, \alpha_{23}, \ldots, \alpha_{2N}; \ldots, \alpha_{NN}]\]

We now estimate the vector of unknown parameters \(\alpha\) which occurs in 2-16 by least squares, imposing the constraint 2-17. If \((X^TX)\) is nonsingular, the relevant estimator \(\hat{\alpha}\) is given by (see Theil [43; 285]):

\[2-18 \quad \hat{\alpha} \equiv (X^TX)^{-1}X^TY + (X^TX)^{-1}r^T(X^TX)^{-1}r^{-1}[1 - r^T(X^TX)^{-1}X^TY]\]

and the variance covariance matrix of \(\hat{\alpha}\) is given by:

\[2-19 \quad E(\alpha - \bar{\alpha})(\hat{\alpha} - \alpha)^T = \sigma^2[(X^TX)^{-1} - (X^TX)^{-1}r^T(X^TX)^{-1}r^{-1}r^T(X^TX)^{-1}]\]

The constrained least squares estimator of \(\alpha\) given by 2-18 possesses the following optimality property: \(\hat{\alpha}\) is the best linear unbiased estimator of \(\alpha\), where linear means linear in the original observations \(s \equiv [s^T, 2^T, \ldots, K^T]^T\) and a constant, where \(s^k\) is defined by 2-13 for \(k = 1, 2, \ldots, K\). To prove this last assertion, we need only verify that Theil's [43; 285] regularity conditions are satisfied by our model. The only tricky regularity condition which needs to be verified (see Theil [43; 278]) arises from the fact that \(\frac{\epsilon T}{\epsilon} \equiv 0\) and thus if we premultiply 2-13 by \(\frac{\epsilon T}{\epsilon}\), we obtain:
\[
1 = \frac{1}{2} s^T k = \frac{1}{2} x^T k \quad ; \quad k = 1, 2, \ldots, K
\]

In general, the system of equations 2.20 will impose \( K \) additional constraints on the vector of parameters \( \alpha \); however, when the symmetry conditions \( \alpha_{ij} = \alpha_{ji} \) and the linear homogeneity condition \( r^T \alpha = 1 \) are imposed on \( \alpha \), it can be verified (see equation 2.9) that equations 2.20 will automatically be satisfied.

Thus the estimator \( \hat{\alpha} \) defined by 2-18 has the following properties: \( \hat{\alpha} \) satisfies the symmetry and linear homogeneity conditions, it is best linear unbiased, it treats all \( N \) of the original factor share equations 2-13 in a symmetric manner (i.e., we do not asymmetrically drop any one of the \( N \) equations), and finally, if we assume that \( \eta \) is multivariate normal with mean \( 0_{N-1} \) and covariance matrix \( \sigma^2 I_{N-1} \), then \( \hat{\alpha} \) is also distributed normally with mean vector \( \alpha \) and covariance matrix given by 2-19 and thus we do not have to appeal to asymptotic theory in order to obtain confidence intervals for \( \alpha \) or to test hypotheses about \( \alpha \).

If we wish to replace the assumption that \( E \eta^T \eta = \delta_{ij} \sigma^2 I_{N-1} \) by the less restrictive assumption \( E \eta^T \eta = \delta_{ij} \Sigma \) where \( \Sigma \) is an \( N-1 \) by \( N-1 \) positive definite symmetric matrix, then we must first obtain an estimator, \( \hat{\Sigma} \), for \( \Sigma \). Define the \((N-1)K\) matrix \( \hat{\Omega} = \hat{\Sigma} \otimes I_K \). Then conditional upon our estimate for \( \Sigma \), the appropriate generalized least squares estimator for \( \alpha \) is given by (see Theil [43; 285]):

\[
\hat{\alpha}^* = (X^T \hat{\Omega}^{-1} X)^{-1} X^T \hat{\Omega}^{-1} y + (X^T \hat{\Omega}^{-1} X)^{-1} r (r^T (X^T \hat{\Omega}^{-1} X)^{-1} r)^{-1} (r^T (X^T \hat{\Omega}^{-1} X)^{-1} X^T \hat{\Omega}^{-1} y - \frac{1}{2} s^T k)
\]
One method of obtaining an estimator for $\Sigma$ proceeds as follows: define the $(N-1)K$ vector of observed residuals $e = y - \hat{x}$ where $\hat{x}$ is defined by 2-18. Write $e$ as $e^T = [e_1^T, e_2^T, \ldots, e^K]$ where the vectors $e^k$ are $N-1$ dimensional for $k = 1, 2, \ldots, K$. Define $\hat{\Sigma} = K^{-1} \sum_{k=1}^K e^k e^k^T$ and now the second stage estimator $\hat{\alpha}^k$ is defined by 2-21. The asymptotic properties of such two stage estimators have been considered by Zellner [47] and Theil [43; 399-403].

3. **A Generalization of the Cobb-Douglas Production Function**

Obviously, the functional form given by 2-l can be used to define a production function, which also generalizes the Cobb-Douglas production function:

$$f(x_1, x_2, \ldots, x_N) = k \prod_{i=1}^{N} \prod_{j=1}^{N} \left( \frac{1}{2} x_i + \frac{1}{2} x_j \right)^{\alpha_{ij}}$$

where $x_i$ is quantity of the $i^{th}$ input, $k > 0$, $\alpha_{ij} = \alpha_{ji}$ and $\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} = 1$

If all $\alpha_{ij} \geq 0$, then $f(x)$ defined by 3-1 will satisfy Conditions I for all $x > 0$. If not all $\alpha_{ij} > 0$, then we look for a region where Conditions I are satisfied.

Consider the following generalization of 3-1:

$$f(x_1, x_2, \ldots, x_N) = k \prod_{i=0}^{N} \prod_{j=0}^{N} \left( \frac{1}{2} x_i + \frac{1}{2} x_j \right)^{\alpha_{ij}}$$

where $x_0 = 1$, $x_i$ is quantity of the $i^{th}$ input for $i = 1, 2, \ldots, N$, $k > 0$, $\alpha_{ij} = \alpha_{ji}$ for $0 \leq i, j \leq N$ and $\alpha_{00} = 0$.22
Recall regularity Conditions II: \( f(x) \) is a nonnegative, non-decreasing, continuous, quasiconcave function for \( x > \frac{0}{N} \) and \( f(x) > 0 \) if \( x \gg \frac{0}{N} \). If all \( a_{ij} \geq 0 \), then it can be verified that the Generalized Cobb-Douglas production function defined by 3-2 satisfies Conditions II. The only difficult condition to verify is quasiconcavity. Define
\[
\beta = \sum_{i=0}^{N} \sum_{j=0}^{N} a_{ij}, \text{ where all } a_{ij} \geq 0. \text{ If } \beta = 0, \text{ then } f(x) \equiv k \text{ and } f \text{ satisfies Conditions II. Assume } \beta > 0 \text{ and define } g(x_0, x_1, \ldots, x_N) = k^{1/\beta} \sum_{i=0}^{N} \sum_{j=0}^{N} ((1/2)x_i + (1/2)x_j)^{a_{ij}/\beta}.
\]
Using the results of section 2, it can be verified that \( g \) is a concave function and thus \( f(x_1, x_2, \ldots, x_N) = [g(1, x_1, x_2, \ldots, x_N)]^{\beta} \) is a quasi concave function.

If not all \( a_{ij} > 0 \), then we look for a region of \( x \)'s where Conditions II are satisfied. If \( S \) is a closed, convex subset of the positive orthant in \( N \)-space and \( f(x) \) is defined by 3-2, then \( f(x) \) will satisfy Conditions II for all \( x \in S \) if:

3-3 (i) \( f(x) > 0 \) for all \( x \in S \),

(ii) \( \forall f(x) > 0 \) for all \( x \in S \) and

(iii) \( f(x) \) is a quasiconcave function over the set \( S \).

The problem of obtaining necessary and sufficient conditions for a twice continuously differentiable function \( f \) to be quasiconcave over a convex set is somewhat delicate and is considered in the appendix to this paper, section 6.
Suppose that we have found a closed convex subset of the positive orthant \( S \) such that \( f \) defined by 3-2 satisfies 3-3. We indicate how the domain of definition of \( f \) defined over \( S \) can be extended to the entire nonnegative orthant in such a way so that the extension of \( f \) satisfies Conditions II. Using property 3-3, the continuity of \( f \) and the fact that \( S \) is a closed set, bounded from below by \( \mathbf{0}_N \), we can show that \( \gamma \equiv \min_x \{ f(x) : x \in S \} \) exists and is a positive number. For \( x \geq \mathbf{0}_N \), define the set \( S(x) \) as the intersection of the set \( S \) and \( \{ z : \mathbf{0}_N \leq z \leq x \} \). For \( x > \mathbf{0}_N \), define the extension of \( f \) by:

\[
3-4 \quad f^*(x) = \begin{cases} 
\max_z \{ f(z) : z \in S(x) \} & \text{if } S(x) \text{ is not empty} \\
\gamma & \text{if } S(x) \text{ is empty} 
\end{cases}
\]

It can be verified that \( f^*(x) = f(x) \) if \( x \in S \) and \( f^* \) satisfies Conditions II over the nonnegative orthant.

Suppose that \( f \) is defined by 3-2 where the nonnegativity restrictions \( a_{ij} \geq 0 \) are not imposed, but we have a point \( x^* > > \mathbf{0}_N \) such that:

\[
3-5 \quad \begin{align*}
(i) & \quad f(x^*) > 0 \\
(ii) & \quad \nabla f(x^*) > > \mathbf{0}_N \text{ and} \\
(iii) & \quad d^T \nabla^2 f(x^*) d < 0 \text{ for every } d \text{ such that } d^T d = 1 \text{ and } d^T \nabla f(x^*) = 0
\end{align*}
\]

where \( \nabla f(x^*) \) is the vector of first order partial derivatives of \( f \) evaluated at \( x^* \) and \( \nabla^2 f(x^*) \) is the matrix of second order partial
derivatives of $f$ evaluated at $x^*$. It can be verified that if (3-5) holds for some $x^* \gg O_N$, then there will exist a closed convex subset of the positive orthant $S$ such that $x^*$ belongs to the interior of $S$ and conditions (3-3) are satisfied for this $S$. (To check that (3-5)(iii) implies 3-3(iii), see Arrow and Enthoven [2; 797-798] and Debreu [11] or the appendix to this paper). Thus conditions (3-5) are sufficient conditions for a twice continuously differentiable function $f$ (such as that defined by (3-2)) to have an extension $f^*$ which satisfies Conditions II and such that $f(x) = f^*(x)$ for all $x$'s belonging to a neighbourhood of $x^* \gg O_N$.

It turns out that the Generalized Cobb-Douglas production function defined by (3-2) is capable of providing a good local approximation to an arbitrary twice continuously differentiable production function $f^*$ (consistent with Conditions II) at a given input point $x^* \gg O_N$; that is, we can prove an analogue to Theorem 2-5. We omit the details.

If we can observe output $z = f(x)$, then taking the logarithm of both sides of (3-2) yields the following equation which is linear in the parameters $a_{ij}$ (and thus linear regression techniques can be applied):

$$3.6 \quad \ln z = a_0 + \sum_{i=1}^{N} a_{i} \ln x_i + 2 \sum_{1 \leq i < j \leq N} a_{ij} \ln \left(\frac{1}{2}x_i + \frac{1}{2}x_j\right) + 2 \sum_{i=1}^{N} a_{0i} \ln \left(\frac{1}{2} + \frac{1}{2}x_i\right)$$

where $a_0 \equiv \ln k$. Note that $f$ will be a homothetic production function if and only if $0 = a_{0i}$ for $i = 1, 2, \ldots, N$. If in addition, we have
1 = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}, \text{ then } f \text{ will be homogeneous of degree one. If in addition, } a_{ij} = 0 \text{ for all } i \neq j, \text{ then } f \text{ defined by 3-2 reduces to the Cobb-Douglas production function. Note that all of the above restrictions are linear, and thus it is easy to test for the validity of them using standard econometric techniques, under the appropriate stochastic specification.}

4. A Generalization of the Cobb-Douglas Inverse Indirect Utility Function

Consider the following functional form for an inverse indirect production (or utility) function:

\begin{align*}
4-1 \quad h(v_1, v_2, \ldots, v_N) & = k^* \left[ \prod_{i=1}^{N} \prod_{j=1}^{N} \left( \frac{1}{2} v_i + \frac{1}{2} v_j \right)^{a_{ij}} \right] \left[ \prod_{k=1}^{N} \left( \frac{1}{2} + \frac{1}{2} v \right)^{2a_{0k}} \right] \\
\end{align*}

where \( v_i \) is the \( i^{th} \) normalized price for \( i = 1, 2, \ldots, N \), \( k^* > 0 \) and \( a_{ij} = a_{ji} \) for \( 1 \leq i, j \leq N \).

The functional form defined by 4-1 is the same as that defined by 3-2 and thus we can largely repeat the analysis of the previous section. If all \( a_{ij} \geq 0 \) for \( 0 \leq i, j \leq N \), then \( h \) defined by 4-1 will satisfy Conditions II for all \( v \geq 0_N \). If not all \( a_{ij} \geq 0 \), then we can look for a region of \( v \)'s where the regularity conditions 3-3 are satisfied (where \( h \) replaces \( f \)) and then extend \( h \) to the entire nonnegative orthant by means of definition 3-4 (where \( f \) and \( f^* \) are replaced by \( h \) and \( h^* \) and \( x \) is replaced by \( v \)). Such a region of \( v \)'s will exist if there exists a \( v^* \gg 0_N \) such that:
4-2 (i) \( h(v^*) > 0 \),

(ii) \( \nabla h(v^*) \gg O_N \) and

(iii) \( d^T v^2 h(v^*) d < 0 \) for every \( d \) such that \( d^T d = 1 \) and 
\[ d^T v^* f(v^*) = 0 . \]

If we could observe output or utility \( z \), then we could apply linear regression techniques to the equation \( \ln z^{-1} = \ln h(v) \) where \( h(v) \) is defined by 4-1. We will pursue this line of inquiry at the end of this section, but at present, we assume that \( z \) is not observable.

Let us assume that \( h \) is defined by 4-1 and that \( v \gg O_N \). If conditions 4-2 are satisfied\(^25\) for \( v \), then the vector of input demand functions \( x(v) \) can be obtained by applying the version of Roy's Identity given by 1-8; i.e., \( x(v) \equiv \nabla h(v)/v^T \nabla h(v) \). If we define the \( i^{th} \) factor share as \( s_i(v) \equiv v^T x_i(v) \), we find that:

\[ s_i = \frac{\sum_{j=1}^{N} a_{ij} v_i \left( \frac{1}{2} v_i + \frac{1}{2} v_j \right)^{-1} + \alpha_{0i} v_i \left( \frac{1}{2} + \frac{1}{2} v_i \right)^{-1}}{\sum_{k=1}^{N} \sum_{l=1}^{N} a_{kl} v_k \left( \frac{1}{2} v_k + \frac{1}{2} v_l \right)^{-1} + \sum_{k=1}^{N} \alpha_{0k} v_k \left( \frac{1}{2} + \frac{1}{2} v_k \right)^{-1}} \]

Note that the parameter \( k^* \) which appeared in 4-1 does not appear in 4-3. Thus we cannot uniquely determine a cardinal version of the direct micro function \( f \) which corresponds to the inverse indirect micro function \( h \) defined by 4-1, unless we have data on the micro output \( z \). Similarly, we note that the factor share equations 4-3 are homogeneous of degree zero in the parameters \( a_{ij} \) and thus in order to econometrically estimate the
parameters $\alpha_{ij}$ using (4-3), we will have to impose a normalization on the
$\alpha_{ij}$. A convenient normalization is:

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} + 2 \sum_{k=1}^{N} \alpha_{0k} = 1. \]

The system of derived demand equations given by (4-3) subject to
the normalization (4-4) is not linear in the unknown $\alpha_{ij}$ parameters and
thus in order to avoid problems of nonlinear estimation, we will not
make the stochastic specification for (4-3) which would be analogous to the
stochastic specification we make for 2-2. Furthermore, we will not impose
\textit{a priori} the Slutsky-Samuelson [33; 107] symmetry conditions, $\alpha_{ij} = \alpha_{ji}$
for $i \neq j$.

Let us define the right-hand side of (4-3) as $\hat{s}_i$, the predicted
$i^{th}$ factor share, and distinguish it from the observed $i^{th}$ factor share,
s$_i$. It is reasonable to suppose that $s_i$ and $\hat{s}_i$ differ by an error
term. Let us assume that we have $K$ observations on input shares and
on normalized input prices. Let $s^k = (s_{1}^k, s_{2}^k, \ldots, s_{N}^k)^T$ be the vector of
observed shares in period $k$, $\hat{s}^k = (\hat{s}_{1}^k, \hat{s}_{2}^k, \ldots, \hat{s}_{N}^k)^T$ be the vector of pre-
dicted factor shares in period $k$, $\varepsilon^k = (\varepsilon_{1}^k, \varepsilon_{2}^k, \ldots, \varepsilon_{N}^k)$ be a vector of
unobserved errors in period $k$ and let $\alpha = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1N}; \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2N};
\vdots; \alpha_{N1}, \alpha_{N2}, \ldots, \alpha_{NN}; \alpha_{01}, \alpha_{02}, \ldots, \alpha_{0N})^T$ be the $N^2+N$ vector of $\alpha_{ij}$'s.

Let $v^k = (v_1^k, v_2^k, \ldots, v_N^k)^T \gg 0$ be the vector of normalized input prices
observed during period $k$ for $k = 1, 2, \ldots, K$. We assume that for
$i = 1, 2, \ldots, N$ and $k = 1, 2, \ldots, K$: 

4-5 \quad y_i^k = 0 = \left[ \sum_{m=1}^{N} \sum_{l=1}^{N} \alpha_{mk} v_i^{k,1} \left( \frac{1}{2} \right)_m m \left( \frac{1}{2} \right)_k - 1 \sum_{m=1}^{N} \alpha_{0m} v_i^{k,1} \left( \frac{1}{2} \right)_m m \left( \frac{1}{2} \right)_k - 1 \right] s_i^k - s_i^k + \epsilon_i^k \\
= \sum_{j=1}^{N} \alpha_{ij} v_i^{k,1} \left( \frac{1}{2} \right)_j m \left( \frac{1}{2} \right)_i - 1 s_i^j - 1 + \alpha_{0i} v_i^{k,1} \left( \frac{1}{2} \right)_i m \left( \frac{1}{2} \right)_i - 1 s_i^0 \\
+ \sum_{m=1}^{N} \sum_{l=1}^{N} \alpha_{mk} v_i^{k,1} \left( \frac{1}{2} \right)_m m \left( \frac{1}{2} \right)_k - 1 s_i^k - 1 + \alpha_{0m} v_i^{k,1} \left( \frac{1}{2} \right)_m m \left( \frac{1}{2} \right)_m - 1 s_i^0 + \epsilon_i^k.

Let us define \( y^k = (y_1^k, y_2^k, \ldots, y_N^k)^T \) and rewrite 4-5 as:

4-6 \quad y^k = \frac{1}{N} X^k + \epsilon^k \quad ; \quad k = 1, 2, \ldots, K

where \( X^k \) is an \( N \times N \) matrix whose elements depend on the normalized prices \( v^k \) and the observed factor shares vector \( s^k \). Since \( \frac{1}{N} (s^k - s^k) \equiv 0 \), we see that \( \frac{1}{N} X^k \equiv 0 \) for every vector \( \alpha \). Thus we must have \( \frac{1}{N} \epsilon^k \equiv 0 \) for \( k = 1, 2, \ldots, K \), and thus the covariance matrix of \( \epsilon^k \) must be singular, just as in section 2. Again, define the \( N \times N-1 \) matrix \( F \) as \( F = [f_1, f_2, \ldots, f_{N-1}] \) where the \( N \) dimensional vectors \( f_i \) satisfy \( f_i^T f_j = 0 \) if \( i \neq j \), \( f_i^T f_i = 1 \) and \( f_i^T f_{N-1} = 0 \). We now assume that \( y^k \) is a vector of random variables such that \( E y^k = X^k \alpha \) and \( E[(y^k - X^k \alpha)(y^k - X^k \alpha)^T] = F F^T \sigma^2 \) where \( \sigma^2 > 0 \) and the expectations are conditional on \( s^k \) and \( v^k \) for \( k = 1, 2, \ldots, K \). Assuming that the vectors \( y^k \) are independently distributed over time, our stochastic assumptions are equivalent to assuming that:

4-7 \quad \epsilon^k \equiv F \eta^k \quad ; \quad E \eta^k \equiv 0_{N-1} \quad for \quad k = 1, 2, \ldots, K \quad ; \quad E \eta^T \eta = \delta_{ij} \sigma^2 \quad for \quad i, j = 1, 2, \ldots, N-1
where \( E \) denotes the (conditional on \( x^k \)) expectation operator, \( \eta^k \) is an \( N-1 \) vector of disturbances which are distributed independently over time and \( \delta_{ij} \) is the Kronecker delta. Now premultiply both sides of 4-6 by \( F^T \) and we obtain the following system of equations:

\[ 4-8 \quad F^T y^k = O_{N-1} = F^T x^k \alpha + \eta^k ; \quad k = 1, 2, \ldots, K. \]

Let us combine observations and rewrite 4-8 as:

\[ 4-9 \quad y = O_{(N-1)} = X \alpha + \eta ; \quad E \eta = O_{(N-1)K} ; \quad E \eta^T = \sigma^2 I_{(N-1)K} \]

where \( X \) is an \( (N-1)K \) by \( N^2+N \) matrix and \( \eta \) is an \( (N-1)K \) vector of residuals. Let us rewrite the normalization 4-4 as:

\[ 4-10 \quad r^T \alpha = 1 \]

where \( r^T = \left[ \frac{1}{N^2}, 2 \frac{1}{N} \right]^T \) where \( \frac{1}{N^2} \) is an \( N^2 \) vector of ones.

The constrained least squares estimator \( \hat{\alpha} \) for the \( \alpha \) which appears in 4-9, subject to the constraint 4-10, is:

\[ 4-11 \quad \hat{\alpha} = (r^T (X^T X)^{-1} r)^{-1} (X^T X)^{-1} r. \]

As in section 2, we can verify that Theil's regularity conditions are satisfied and we can show that \( \hat{\alpha} \) defined by 4-11 is the best linear unbiased estimator of \( \alpha \) subject to the constraint 4-10, where linear means linear in the "observations" \( y^k \) defined by 4-6 and a constant. Note also that our estimator \( \alpha \) treats all \( N \) of the original derived
demand equations in a symmetric manner. Furthermore, if the residual vector $\eta$ is normally distributed, then we may test the validity of the symmetry conditions $a_{ij} = a_{ji}$ for $i \neq j$ using a standard $F$ test, without having to appeal to asymptotic theory.

If we wish to replace the assumption that $E\eta_i^2 = \delta_{ij} \sigma_i^2$ by the less restrictive assumption $E\eta_i^2 = \delta_{ij} \Sigma$ where $\Sigma$ is an $N-1$ by $N-1$ positive definite symmetric matrix, then we may apply a two stage procedure analogous to the two stage procedure outlined in section 2.

Note that $h$ defined by 4-1 will be homothetic if and only if $a_{01} = a_{02} = \ldots = a_{0N} = 0$, in which case the system of factor share equations 4-3 reduces to the system given by 2-2 (if 4-10 holds and $k^* = 1$). Thus the model of this section generalizes the model of section 2 in that the micro function $f$ is no longer restricted to be homogeneous of degree one. Note also that if $a_{0i} = 0$ for $i = 1, 2, \ldots, N$, $a_{ij} = 0$ for all $i \neq j$, and $\sum_{i=1}^{N} a_{ii} = 1$, then the Generalized Cobb-Douglas Inverse Indirect Utility function defined by 4-1 reduces to the Cobb-Douglas case:

$$h(v) = k^* \prod_{i=1}^{N} v_{i}^{a_{ii}} = k^* \prod_{i=1}^{N} p_{i}^{a_{ii}} / Y \quad \text{where} \quad Y = \sum_{i=1}^{N} p_{i} x_{i} \quad \text{and} \quad v_{i} = p_{i} / Y.$$ 

We conclude this section by noting a possible estimation procedure when micro output in period $k$, $z_{k}$ say, is observable. Since $z_{k}^{-1} = h(v_{1}^{k}, v_{2}^{k}, \ldots, v_{N}^{k})$, we may take the logarithms of both sides of 4-1 for $k = 1, 2, \ldots, K$: 
4-12 \quad \ln z_k^{-1} = a_0 + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} \ln \left( \frac{1}{2} v_i^k + \frac{1}{2} v_j^k \right) + 2 \sum_{i=1}^{N} a_{0i} \ln \left( \frac{1}{2} + \frac{1}{2} v_i^k \right)

where \( a_0 \equiv \ln k^* \). We now add an error term to 4-12, \( \eta_0^k \) say, combine the resulting \( K \) observations with the \((N-1)K\) observations given by 4-8 and then estimate the \( a_i \)'s via a two stage least squares procedure, the first stage being used in order to obtain an estimate of the \( N \) by \( N \) variance covariance matrix of \((\eta_0^k, \eta^k)\). We omit the details. \( ^{30} \) Note that we no longer impose the "cardinalizing" normalization 4-10 since output \( z \) can be observed.

5. Conclusions

If the micro function \( f \) satisfies Conditions I and ouptut \( z \) is not observable and we wish to impose the Slutsky-Samuelson symmetry conditions, then the procedure outlined in section 2 based on the Generalized Cobb-Douglas unit cost function defined by 2-1 will enable us to estimate a second order approximation to a normalization of the true micro function \( f \).

If the micro function \( f \) satisfies Conditions II (i.e., \( f \) is not necessarily linear homogeneous or homothetic) and output \( z \) is not observable but we wish to test the validity of the symmetry conditions, then the procedure outlined in section 4 based on the Nonhomothetic Generalized Cobb-Douglas Inverse Indirect (production) function defined by 4-1 is a possible approach to the problem. Even if \( f \) satisfies Conditions I (i.e., \( f \) is linear homogeneous) and output is not observable but we wish to test the validity of the symmetry conditions \( a_{ij} = a_{ji} \),
then the model outlined in section 4 could be used—we need only add the linear restrictions $a_{0i} = 0$ for $i = 1, 2, \ldots, N$ to the linear restriction 4-10 and modify the constrained least squares estimator defined by 4-11 appropriately.

Our emphasis throughout this paper has been on postulating functional forms which give rise to systems of derived demand equations which are linear in the unknown parameters and thus linear regression techniques may be used in order to estimate the parameters. Another generalization of the Cobb-Douglas unit cost function which gives rise to factor share equations which are linear in the parameters is:

$$5-1 \quad c(p_1, p_2, \ldots, p_N) = k^* \prod_{i=1}^{N} \left( \frac{1}{2} p_i^r + \frac{1}{2} p_j^r \right)^{a_{ij}/r}$$

where $p_i$ is the price of the $i^{th}$ input, $k^* > 0$, $a_{ij} = a_{ji}$, $\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} = 1$ and $r < 1$, $r \neq 0$. Since $\left( \frac{1}{2} p_i^r + \frac{1}{2} p_j^r \right)^{1/r}$ is a mean of order $r$ and is a concave function in $p (p_1, p_2, \ldots, p_N)^T$ for $p >> 0_N$ for $r \leq 1$, if all $a_{ij} > 0$, then 5-1 is a geometric mean in means of order $r$ of the prices taken two at a time and is a concave function for $p >> 0_N$.

An application of Shephard's Lemma 1-4 yields the following system of factor share equations, where $s_i = p_i x_i / \sum_{k=1}^{N} p_k x_k$:

$$5-2 \quad s_i = \sum_{j=1}^{N} \frac{a_{ij} p_i^r}{\frac{1}{2} p_i^r + \frac{1}{2} p_j^r} \quad ; \quad i = 1, 2, \ldots, N$$
We note that 5-2 is not valid if \( r = 0 \), since in this case 5-1 reduces to
\[
c(p) = k \prod_{i=1}^{N} \prod_{j=1}^{N} (p_i^{1/2} p_j^{1/2})^{\alpha_{ij}}
\]
which is just an ordinary Cobb-Douglas unit cost function. If \( r \) is taken to be any number less than or equal to one but not equal to zero, then 5-2 is a system of equations which is linear in the unknown \( \alpha_{ij} \) parameters. In this paper, we have carried the analysis through for \( r = 1 \); we could have equally well chosen \( r = -1 \).

The functional form defined by 5-1 should be contrasted with the following functional form for a unit cost function:

\[
5-3 \quad c(p_1, p_2, \ldots, p_N) = \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} \left( \frac{1}{2} p_i^r + \frac{1}{2} p_j^r \right)
\]

where \( r \neq 0, r \leq 1, b_{ij} = b_{ji} > 0 \). Since \( c(p) \) defined by 5-3 is a nonnegative sum of linear homogeneous concave functions, it is a linear homogeneous concave function. If in addition, at least one \( b_{ij} > 0 \), we see that \( c(p) \) satisfies Conditions I. Note that the limit of 5-3 as \( r \) approaches zero is given by
\[
c(p) = \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} p_i^{1/2} p_j^{1/2},
\]
which is the Generalized Leontief unit cost function defined in Diewert [12]. For a fixed \( r \leq 1, 33 \) the system of derived input demand functions which 5-3 generates via Shephard's Lemma 1-4 is linear in the \( b_{ij} \) parameters; however the system of factor share functions is not linear in the \( b_{ij} \)'s. Note that 5-3 reduces to a Leontief unit cost function if all \( b_{ij} = 0 \) for \( i \neq j \).
Thus 5-1 is a constant times a geometric mean of means of order r, while 5-3 is a constant times an arithmetic mean of means of order r.

6. **Appendix: Conditions for Quasiconcavity in the Twice Differentiable Case**

For background material on quasiconcave functions, see Arrow and Enthoven [2], Mangasarian [28] and Greenberg and Pierskalla [15].

Let $S$ be an open convex subset of Euclidean $N$ dimensional space and let $f$ be a twice continuously differentiable function defined on $S$. We wish to find necessary and sufficient conditions for $f$ to be a quasiconvex function over $S$.

**6-1 Definition:** $f$ is a quasiconcave function over $S$ if $x^1, x^2 \in S$, $0 \leq \lambda \leq 1$, $f(x^1) \leq f(x^2)$ implies $f(\lambda x^1 + (1-\lambda)x^2)$.

Let $g$ be a function of one variable defined over a nonempty (possibly infinite) open interval $(a, b) \equiv \{t: a < t < b\}$. We require the following concept of a semi-strict local minimum:

**6-2 Definition:** $g$ attains a semi-strict local minimum at $t_0 c(a,b)$ if there exists $\delta > 0, \delta_1 \geq 0, \delta_2 \geq 0$ such that $t_0 - \delta_1 - \delta \geq a$, $t_0 + \delta_2 + \delta \leq b$ and (i) $g(t) = g(t_0)$ for $t_0 - \delta_1 \leq t \leq t_0 + \delta_2$, (ii) $g(t_0) < g(t)$ for $t_0 - \delta_1 - \delta < t < t_0 - \delta_1$ and (iii) $g(t_0) < g(t)$ for $t_0 + \delta_2 < t < t_0 + \delta_2 + \delta$.

Thus a strict local minimum is a semi-strict local minimum (take $\delta_1 = \delta_2 = 0$ in definition 6-2), but if $g(t_0)$ is a local minimum (i.e.,
there exists $\delta > 0$ such that $g(t_0) \leq g(t)$ for $t_0 - \delta < t < t_0 + \delta$), it need not be a semi-strict local minimum. For example, $g$ could be constant for an interval around the point $t_0$, but then decrease on one side of this interval where $g$ is constant. Such a local minimum is not a semi-strict local minimum. Thus the concept defined by 6-2 is intermediate to the concepts of a strict local minimum and a local minimum.

The following theorem yields the usual second order sufficiency conditions for quasiconcavity as well as the usual second order necessary conditions.

6-3 Theorem: Let $S$ be an open convex subset of $\mathbb{E}^N$ and let $f$ be a twice continuously differentiable function over $S$. Then

(1) $f$ is quasiconcave over $S$ if and only if (2) for every $x^0 \in S$ and for every vector $d$ such that $d^T d = 1$ and $d^T \nabla f(x^0) = 0$, we have either: (i) $d^T \nabla^2 f(x^0) d < 0$ or (ii) $d^T \nabla^2 f(x^0) d = 0$ and the function of one variable $g(t) \equiv f(x^0 + td)$ does not attain a semi-strict local minimum at $t = 0$.

Proof: (1) implies (2): Let $x^0 \in S$, $d^T d = 1$ and $d^T \nabla f(x^0) = 0$. We assume that (2) does not hold. Define $g(t) \equiv f(x^0 + td)$. Then $dg(0)/dt = d^T \nabla f(x^0) = 0$ and $d^2 g(0)/dt^2 = d^T \nabla^2 f(x^0) d$ using the twice continuous differentiability of $f$. If 2(i) held, then $g(t)$ attains a strict local maximum at $t = 0$. Thus assuming that 2(i) and 2(ii) do not hold is equivalent to assuming that $g(t)$ attains a semi-strict
local minimum at \( t = 0 \). Therefore, there exists \( t_1 > 0 \) and \( t_2 > 0 \) such that \( (x^0 - t_1 d) \in S \) with \( g(0) < g(-t_1) \) and \( (x^0 + t_2 d) \in S \) with \( g(0) < g(t_2) \). Thus \( x^0 \) is on the line segment joining \( x^0 - t_1 d \) and \( x^0 + t_2 d \) with \( f(x^0) < \min \{ f(x^0 - t_1 d), f(x^0 + t_2 d) \} \) which contradicts the quasiconcavity of \( f \). (2) implies (1): Let \( x^1 \neq x^2 \in S \), \( 0 \leq \lambda \leq 1 \) and \( f(x^1) \leq f(x^2) \). Define \( L = [(x^2 - x^1)^T (x^2 - x^1)]^{1/2} > 0 \) and \( d = (x^2 - x^1)/L \). We wish to show that \( g(t) = f(x^1 + td) \geq f(x^1) \) for \( 0 \leq t \leq L \). Let us suppose that the minimum of \( g(t) \) over the interval \( 0 \leq t \leq L \) is attained at \( t = t_0 \). If \( g(t_0) \geq f(x^1) \) we are done so assume that \( g(t_0) < f(x^1) \leq f(x^2) \). Then \( 0 < t_0 < L \) and \( g(t) \) must attain a local minimum at \( t = t_0 \). Therefore \( \frac{dg(t_0)}{dt} = \frac{d^T v f(x^1 + t_0 d)}{dt} = 0 \) and \( \frac{d^2 g(t_0)}{dt^2} = \frac{d^T v^2 f(x^1 + t_0 d) d}{dt} > 0 \) and moreover, because \( g(t_0) < f(x^1) \leq f(x^2) \), \( g(t) \) must attain a semi-strict local minimum at \( t = t_0 \). This contradicts (2) and thus we must have \( g(t_0) \geq f(x^1) \). Q.E.D.

If \( f \) is twice continuously differentiable over the open convex set \( S \) and \( f(x^0) \gg \underline{0}_N \) for every \( x^0 \in S \), then \( f \) is quasiconcave over \( S \) if and only if for every \( x^0 \in S \) and every vector \( d \) such that \( d^T v f(x^0) = 0 \), we have \( d^T v^2 f(x^0) d \leq 0 \). (See Katzner [24; 211] for a proof).

Economists (e.g., Katzner [24; 210]) and operation researchers (e.g., Greenberg and Pierskalla [15; 1564]) have both defined the notion of a strictly quasiconcave function but the definitions do not coincide. We define the economists' version of strictly quasiconcave below.
Definition: \( f \) is a strongly strictly quasiconcave function over the convex set \( S \) if and only if for every \( x^1, x^2 \in S, \ x^1 \neq x^2, \ 0 < \lambda < 1, \ f(x^1) \leq f(x^2) \) implies \( f(x^1) < f(\lambda x^1 + (1-\lambda)x^2) \).

The operations research definition of strict quasiconcavity is 6-4 except that \( f(x^1) \leq f(x^2) \) is replaced by \( f(x^1) < f(x^2) \). Thus a strongly strictly quasiconcave function will also be strictly quasiconcave. Note that a strongly strictly quasiconcave function will have strictly convex "indifferent or preferred to" sets \( L(u) = \{ x: \ f(x) \geq u; \ x \in S \} \), which is not the case for a strictly quasiconcave function. However, both types of quasiconcave functions will satisfy a local nonsatiation condition; i.e., there will be no "thick" indifference surfaces.

The following theorem can be proven using the same method of proof as in Theorem 6-3.

Theorem: Let \( S \) be an open convex subset of \( \mathbb{E}^N \) and let \( f \) be a twice continuously differentiable function over \( S \). Then

(1) \( f \) is strongly strictly quasiconcave over \( S \) if and only

(2) for every \( x^0 \in S \) and for every vector \( d \) such that \( d^T d = 1 \) and \( d^T \nabla f(x^0) = 0 \), we have either: (i) \( d^T \nabla f(x^0) d < 0 \) or (ii) \( d^T \nabla f(x^0) d = 0 \) and the function of one variable \( g(t) = f(x^0 + td) \) does not attain a local minimum at \( t = 0 \).

Thus the usual sufficient conditions for quasiconcavity in the twice differentiable case (i.e., (2)(1) of 6-3) are also sufficient for strict quasiconcavity but are not necessary.
Footnotes


2. The $N$ input, constant returns to scale translog production function may be written as

$$\ln y = a_0 + \sum_{i=1}^{N} \alpha_i \ln x_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln x_i \ln x_j$$

where $y =$ output, $x_i =$ quantity of input $i$ for $i = 1,2,\ldots,N$ and the parameters satisfy the following restrictions: (i) $\sum_{i=1}^{N} \alpha_i = 1$, (ii) for $i = 1,2,\ldots,N$, $\sum_{j=1}^{N} \gamma_{ij} = 0$, (iii) $\gamma_{ij} = \gamma_{ji}$ for all $i, j$.

The terminology translog is due to Christensen, Jorgenson and Lau [8]. Independently, the translog form has been discussed by Griliches and Ringstad [16] and Sargan [35].

3. Notation: $0_N$ is the $N$ dimensional zero vector; $p \gg 0_N$ means that each component of the vector $p$ is positive; $p^T$ stands for the transpose of the column vector $p$.

4. If $x' \geq x''$, then $f(x') \geq f(x'')$.

5. If $\lambda > 0$, $x \gg 0_N$, then $f(\lambda x) = \lambda f(x)$.

6. Let $x' \gg 0_N$, $x'' \gg 0_N$, $0 \leq \lambda \leq 1$. Then $f(\lambda x' + (1-\lambda)x'') \geq \lambda f(x') + (1-\lambda)f(x'')$.

7. We extend the domain of definition of $f$ to the nonnegative orthant by continuity. See Rockafellar [31; 85].

8. The domain of definition of $c$ is also extended to the nonnegative orthant by continuity.

9. See Shephard [27] for a proof which assumed differentiability, Samuelson [35] for a statement of the theorem and Shephard [38; 94] for a proof which does not assume differentiability. The second definition of $f$ in terms of $c$ contained in 1-3 may be found in Diewert [13] as well as an alternative proof of the theorem.

10. See for example, Diewert [12; 495].

11. The function $f$ will not be uniquely determined by this procedure since we obtain the same system of equations 1-5 if we replace $c(p)$ by $kc(p)$ where $k > 0$, but $f$ defined by 1-3 would change.
12. A function of \( N \) variables \( f \) is quasiconcave if for every scalar \( k \), the set \( \{ x : f(x) > k \} \) is convex.

13. The concept of the indirect utility function dates back to Hotelling [22; 594]. Hanoch [19] and Shephard [38; 105-6] used the concept of the indirect production function, the latter author calling it the cost limited maximal output function.

14. See Hanoch [19] for a proof of the duality theorem between \( f \) and \( h \) under slightly different regularity conditions. See Newman [29; 138-172], Lau [25], Shephard [38; 105-111], and Weddepohl [45; 125] for closely related duality theorems.

15. Actually, Houthakker worked with the indirect utility function \( g(v) \equiv 1/h(v) \) instead of the inverse indirect utility function, in which case Roy's Identity yields the following counterpart to 1-8; \( x(v) \equiv \nabla g(v)/\nabla^T g(v) \). We have preferred to work with \( h \) instead of \( g \), since \( h \) and \( f \) satisfy the same regularity conditions (cf. Hanoch [19]).

16. This functional form was actually used by Wickless [46; 98,133] in the nineteenth century.

17. For example, see Fenchel [14; 61] for the proof of a similar proposition.

18. The translog unit cost function given by \( \ln c(p) = a_0 + \sum_{i=1}^{N} a_i \ln p_i \) 

\[ + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i \ln p_j \] 

where \( \sum_{i=1}^{N} a_i = 1, \sum_{j=1}^{N} \gamma_{ij} = 0 \) for \( i = 1, 2, \ldots, N \) and \( \gamma_{ij} = \gamma_{ji} \) has a similar property since the system of factor share equations is given by \( s_i = a_i + \sum_{j=1}^{N} \gamma_{ij} \ln p_j ; i = 1,2,\ldots,N \).

Thus if \( \gamma_{ij} = 0 \) for \( i \neq j \), then the translog unit cost function also reduces to the Cobb-Douglas unit cost function. We note that if any \( \gamma_{ij} \neq 0 \), then the translog unit cost function cannot satisfy Conditions I for all \( p \gg 0_N \). However, the translog unit cost function can provide a good local approximation to an arbitrary twice differentiable unit cost function.

19. \( Vc(p) > 0_N \) means \( Vc(p) \geq 0_N \) but \( Vc(p) \neq 0_N \). This condition will imply that \( c(p) \) is positive and nondecreasing for \( p \in S \). The
linear homogeneity of \( c(p) \) is implied by the restriction
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_{ij} = 1.
\]

20. This condition will imply that \( c(p) \) is a concave function for
\( p \in S \). See Fenchel [14; 87-8] or Rockafellar [31; 27].

21. See Allen [4; 504] for the original definition of the partial elas-
ticity of substitution in terms of the partial derivatives of the
production function.

22. Since \( \left( \frac{1}{2} \right)^{\alpha_{00}} = 1 \) for any \( \alpha_{00} \), we can choose \( \alpha_{00} = 0 \).

23. See Shephard [37; 41] and [38; 30] for definitions of homotheticity.

24. That is, \( v_i = \frac{p_i}{y} \) where \( p_i \) is the price of the \( i \)th
input and \( Y \) is expenditure on the \( N \) inputs.

25. Or if conditions 3-3 are satisfied for \( h \) for a closed convex set
\( S \) which contains \( v \) in its interior and \( \nabla h(v) \neq 0 \).

26. It is also convenient to measure the observed inputs \( x_i \) in units
which make the normalized prices \( v_i (\equiv \frac{p_i}{\sum_{k=1}^{N} p_k x_k} \) where \( p_i \)
the price of the \( i \)th input) equal to unity in a base period, since
this will make \( h(1) = 1 \) where \( h \) is defined by 4-1 with \( k^* = 1 \),
where \( 1 \) is a vector of ones. Thus indirect utility (or indirect
normalized production) \( z \equiv g(v) \equiv 1/h(v) \) will also equal unity in
the base period.

27. Provided that \( (X^T X)^{-1} \) exists and thus we need \( (N-1)K \geq N^2+N \).

28. From 4-6, we have \( 0 = \frac{1}{\lambda}^T y_k = \frac{1}{\lambda}^T X^T k + \frac{1}{\lambda}^T \varepsilon k = \frac{1}{\lambda}^T k \) since \( \frac{1}{\lambda}^T X^T k \equiv 0 \)
for any \( \lambda \). Thus the \( K \) constraints \( 0 = \frac{1}{\lambda}^T X^T k, k = 1, \ldots, K \) do
not impose any additional constraints on the vector of parameters \( \lambda \).

29. If \( h \) is homothetic, then \( g \equiv 1/h \), the indirect utility function,
will also be homothetic and thus by a result due to Lau [25], the
direct utility function \( f \) will also be homothetic.

30. It is not reasonable to assume that the variance covariance matrix
be \( \sigma^2 I_N \) because of heteroskedasticity. In the case of a two stage
procedure the matrix \( F = [f_1, f_2, \ldots, f_{N-1}] \), which occurs in 4-8, need not have orthonormal columns; the columns of \( F \) need only be linearly independent and orthogonal to \( 1_N \), the \( N \) dimensional vector of ones.

31. For \( N \) small, we can use nonlinear regression techniques to estimate parameters and if \( K \) is large, asymptotic sampling theory may be used and thus our emphasis on linear regression techniques is not warranted under these circumstances.

32. See Hardy, Littlewood and Polyá [30; 13 and 30].

33. If \( r = 1 \), then 5-3 collapses into a Leontief unit cost function. Thus two convenient choices of \( r \) are \( r = 0 \) (the limiting case) or \( r = -1 \). For these last two choices of \( r \), we can prove a version of Theorem 2-5.
References


