obtain consistent country i price indexes \( P_i(p,y) \) by dividing the country i nominal private product, \( p^i \cdot y^i \), by the share function, \( S_i(p,y) \); i.e.,

\[
(55) \quad P_i(p,y) = \frac{p^i \cdot y^i}{S_i(p,y)}, \; i=1, \ldots, I.
\]

The resulting system of price index and share functions will have the following property:

\[
\sum_{i=1}^{I} P_i(p,y)S_i(p,y) = \sum_{i=1}^{I} p^i \cdot y^i = \text{nominal world private product}.
\]

We now leave the realm of production theory and turn our attention to consumer theory approaches to multilateral comparisons.


Consider household \( h \) in country \( i \). Let \( F^i_h(z,x) \) denote the preference or utility function of this household over nonnegative combinations of market goods \( x = (x_1, \ldots, x_N) \geq 0_N \) and other variables \( z = (z_1, \ldots, z_L) \gg 0_L \). It should be understood that the \( N \) used in this section is in general not equal to the \( N \) of previous sections. The \( z \) variables could be demographic variables or consumptions of public goods. We assume that \( x^i_{ih} > 0_N \) is a solution to the following expenditure minimization problem: for \( i=1, \ldots, I \) and \( h=1, \ldots, H_i \),

\[
(56) \quad \min_x \{w^i \cdot x : F^i_{h}(z^ih,x) \geq u^i_h, \; x \geq 0_N \} = c^i_{h}(u^i_h, z^ih, w^i) > 0
\]

where \( w^i \gg 0_N \) is the positive vector of consumer prices that each household
in country \( i \) faces and \( C_h \) is the expenditure function of household \( h \) in country \( i \). Note that we are assuming that there are \( I \) countries and \( H_i \) households in country \( i \).

The household \( h \) Konüs [1924] cost of living or price index of country \( i \) relative to country \( j \) can be defined as

\[
K_{ij}^h = \frac{c^i_h(u^i_h, z^{ih}, w^i)}{c^i_h(u^i_h, z^{ih}, w^j)}, \quad i, j = 1, \ldots, I, h = 1, \ldots, H_i
\]

\[
= w^i \cdot x^{ih}/c^i_h(u^i_h, z^{ih}, w^j)
\]

using (56).

In definition (57), we used the preferences and demographic variables of household \( h \) in country \( i \) as reference quantities. In the following definition of the household \( k \) cost of living or price index of country \( i \) relative to \( j \), we use the preferences and demographic variables of household \( k \) in country \( j \) as reference quantities:

\[
K_{ij}^k = \frac{c^j_k(u^j_k, z^{jk}, w^i)}{c^j_k(u^j_k, z^{jk}, w^j)}, \quad i, j = 1, \ldots, I, k = 1, \ldots, H_j
\]

\[
= c^j_k(u^j_k, z^{jk}, w^i)/w^j \cdot x^{jk}
\]

using (56).

The following Proposition shows that under certain restrictions on preferences (which do not seem to be too restrictive), we can compute the geometric mean of the theoretical price indexes \( K_{ij}^h \) and \( K_{ij}^k \) given only the observable price vectors in the two countries, \( w^i \) and \( w^j \), the consumption vector of household \( k \) in country \( i \), \( x^{ih} \), and the consumption vector of household \( k \) in country \( j \), \( x^{jk} \).

**Proposition 9** (Caves, Christensen and Diewert [1982b; 1410]): If the
expenditure functions for household $h$ in country $i$, $C^h_i$, and household $k$ in
country $j$, $C^j_k$, are translog\(\alpha\) with identical coefficients on the second order
terms in commodity prices, then

\[
(K_{h k}^i x_{hk}^i)^{1/2} = \prod_{n=1}^{N} (w_{n i}^i / w_{n j}^j)^{(1/2)} [(w_{n i}^i x_{i n}^j / w_{n j}^j x_{j n}^j) + (w_{n j}^j x_{j n}^j / w_{n j}^j x_{j n}^j)]
\]

\[
= P_T(w_{j}^j, w_{i}^i, x_{j k}^j, x_{i h}^i)
\]

where $P_T$ is the translog or Törnqvist price index of consumer prices in
country $i$ relative to $j$, using the consumption vectors of household $h$ in
country $i$ and household $k$ in country $j$ as quantity weighting vectors.

Proposition 9 gives us a reasonable approach for comparing prices in
country $i$ to those of country $j$ through the preferences of single households in
each country. At this stage, it seems reasonable to average over households
to obtain an average index of country $i$ consumer prices relative to those of
country $j$. Thus fix $i$ and $j$ and choose $H_i, H_j$ weights $a_{hk}^{ij}$ which satisfy

\[
a_{hk}^{ij} \geq 0, \sum_{h=1}^{H_i} \sum_{k=1}^{H_j} a_{hk}^{ij} = 1
\]

and form the following average index of country $i$ prices relative to those
of country $j$:

\[
K_{h k}^{ij} = \prod_{h=1}^{H_i} \prod_{k=1}^{H_j} (K_{h k}^i x_{hk}^i)^{(1/2)} a_{hk}^{ij}
\]

\[
= \prod_{h=1}^{H_i} \prod_{k=1}^{H_j} [P_T(w_{j}^j, w_{i}^i, x_{j k}^j, x_{i h}^i)] a_{hk}^{ij}
\]

where the equality (61) follows under the hypotheses of Proposition 9,
hypotheses which we assume for the remainder of this section.

Two special cases of the general weighting scheme defined by (60) are of
some interest.

**Case (i): Democratic Weighting.** In this case, each household in each country gets an equal weight, i.e., we have

\[ a_{hk}^{ij} = 1/H_i H_j. \]

In this case, the equality (61) may be rewritten as

\[ K_{ij} = [\prod_{h=1}^{H_i} (K_{hh}^{ij})^{1/H_i}]^{1/2} [\prod_{k=1}^{H_j} (K_{kk}^{*ij})^{1/H_j}]^{1/2} \]

\[ = \prod_{h=1}^{H_i} \prod_{k=1}^{H_j} \left[ p_T(w^{ji}, w^{ik}, x^{jk}, x^{ih}) \right]^{1/H_i H_j}. \]

Note that the right hand side of (63) may be evaluated empirically provided that we have information on the individual household consumption vectors, \( x^{ih} \) and \( x^{jk} \), in each country.

**Case (ii): Plutocratic Weighting.** In this case, each household gets a weight that is proportional to its share of country consumption; i.e., we have

\[ a_{hk}^{ij} = s_{h}^{i} s_{k}^{j} \; ; \; i,j=1,\ldots,I; \; h=1,\ldots,H_i; \; k=1,\ldots,H_j; \]

where \( s_{h}^{i} = w^{i} x^{ih} / \sum_{m=1}^{H_i} w^{i} x^{im} \) and \( s_{k}^{j} = w^{j} x^{jk} / \sum_{m=1}^{H_j} w^{j} x^{jm} \). In this case, the equality (61) becomes

\[ K_{ij} = [\prod_{h=1}^{H_i} (K_{hh}^{ij})^{s_{h}^{i}}]^{1/2} [\prod_{k=1}^{H_j} (K_{kk}^{*ij})^{s_{k}^{j}}]^{1/2} \]

\[ = p_T(p^{i}, p^{j}, x^{j}, x^{i}) \]

where the country \( i \) and \( j \) aggregate consumption vectors, \( x^{i} \) and \( x^{j} \), appear in the translog price index formula on the right hand side of (65); i.e., we have defined
Thus the advantage of the plutocratic weighting system is that we can evaluate the theoretical average Konüs price index defined by the left hand side of (65) using only aggregate data. Note that the theoretical index is a geometric mean of two terms. The first term represents a plutocratically weighted geometric mean of individual consumer h Konüs price indexes for country i relative to j prices where the average is taken over all households h in country i while the second term represents a similar plutocratically weighted geometric mean of consumer k price indexes, where the average is taken over all households k in country j.

Given that the $K^{ij}$ defined by (63) or (65) forms a satisfactory approximation to the level of consumer prices in country i relative to those in country j, we still have to address the issue of achieving a consistent multilateral ranking of consumer prices among all I countries. Fortunately, it is not necessary to engage in a lengthy discussion of this problem. All we have to do is reinterpret the previous analysis presented in section 3 in the following way: replace the old $P(p^j, p^i, y^j, y^i)$ which occurs in (22) by $K^{ij}$ defined by (63) or (65). The rest of the analysis in section 3 then carries through. We leave the details to the reader.

We turn now to the related problems involved in making interhousehold and intercountry comparisons of real consumption.

10. Bilateral and Multilateral Real Consumption Comparisons.

Suppose we wished to compare the real consumption of household h in
country $i$ with household $k$ in country $j$. Then a first approach to measuring this real consumption ratio would be to deflate the actual expenditure ratio, $w^i x^i_k / w^j x^j_k$, by one of the Konüs price deflators defined by (57) or (58).

In order for this procedure to give meaningful theoretical results, it is necessary to assume that each household in each country has the same preferences over market goods and the vectors of demographic variables can be ignored. Under these conditions, the cost functions $C^i_h(u^i_h, z^i_k, w^i)$ defined by (56) become $C(u^i_h, w^i)$. Now we may define the following Pollak [1971; 64] implicit real consumption indexes for household $h$ in country $i$ relative to household $k$ in country $j$ as

$$Q^i_{hk} = w^i x^i_h / w^j x^j_k$$

$$= C(u^i_h, w^i) / C(u^i_j, w^j) [C(u^i_h, w^i) / C(u^i_h, w^j)]$$ using (56) and (57)

$$= C(u^i_h, w^i) / C(u^j_k, w^j)$$

$$Q^*_{hk} = w^i x^i_h / w^j x^j_k$$

$$= C(u^i_h, w^i) / C(u^j_k, w^j)$$ using (56) and (58).

The extreme right hand sides of (67) and (68) are Allen [1949; 199] quantity indexes, and the reader is referred to his article for their properties.

The following Proposition is a straightforward consequence of Proposition 9.

**Proposition 10:** Under the hypotheses of Proposition 9 plus the additional hypothesis that each household in each country has the same preferences, then a geometric mean of the Allen quantity indexes defined by (67) and (68) is
equal to an implicit translog price index; i.e., for \(i,j=1,\ldots,I; h=1,\ldots,H_i; k=1,\ldots,H_j:\)

\[
(Q_{hk}^{iij} / h_{jk})^{1/2} = w^i \cdot x^{ih} / w^j \cdot x^{jk} p_T(w^j, w^i, x^{jk}, x^{ih})
\]

\[
= \tilde{Q}_T(w^j, w^i, x^{jk}, x^{ih})
\]

where \(p_T\) is the translog price index defined in (59).

We defined the right hand side of (69) to be \(\tilde{Q}_T(w^j, w^i, x^{jk}, x^{ih})\), the implicit translog index of consumption of household \(h\) in country \(i\) to the consumption of household \(k\) in country \(j\). Note that if the number of consumption goods equals one (\(N=1\)), then the right hand side of (69) reduces to the "right" answer, \(x^{ih}_i / x^{jk}_j\).

Proposition 10 allows us to make bilateral real consumption comparisons between any pair of households in any two countries. If individual household consumption data \(x^{ih}\) are available, then Proposition 10 provides a solution to the bilateral comparison problem. Solutions to the problem of making consistent multilateral comparisons in a symmetric way can now be obtained by adapting the techniques outlined in section 6.

The own share method for making multilateral consumption comparisons may be defined as follows. First define, for \(i=1,\ldots,I\) and \(h=1,\ldots,H_i:\)

\[
\alpha_i^h = [\sum_{j=1}^I \sum_{k=1}^{H_j} (\tilde{Q}_T(w^j, w^i, x^{jk}, x^{ih}))]^{-1} - 1.
\]

The share of household \(h\) in country \(i\) of real world consumption is defined as

\[
s_i^h = \alpha_i^h / \sum_{j=1}^I \sum_{k=1}^{H_j} \alpha_j^k, \quad i=1,\ldots,I; h=1,\ldots,H_i.
\]

The democratic share method\(^{33}\) for making multilateral consumption
comparisons requires the following definitions for \( i=1,\ldots,I; h=1,\ldots,H_j \):

\[
\sigma_{ih}^{jk} = \hat{q}_T(w^j, w^i, x^k, x^{ih}) / \sum_{m=1}^{H_j} \sum_{n=1}^{H_i} \hat{q}_T(w^j, w^m, x^k, x^{mn}).
\]

\( \sigma_{ih}^{jk} \) is the share of household \( h \) in country \( i \) of world consumption, using the preferences of household \( k \) in country \( j \) as a metric. The democratic share of world consumption of household \( h \) in country \( i \) is obtained by averaging the \( \sigma_{ih}^{jk} \) over all households \( j \):

\[
\sigma_{ih} = \sum_{j=1}^{I} \sum_{k=1}^{H_j} \sigma_{ih}^{jk} / (H_1 + H_2 + \ldots + H_i); \quad i=1,\ldots,I; \quad h=1,\ldots,H_i.
\]

We now prefer a democratic share method to the corresponding plutocratic share method because it seems fair to let each household count equally when forming the averages in (73); i.e., we now have a natural indivisible measure of size (the household) which was missing when we were making output comparisons.

In both the own share and democratic share method for making multilateral consumption comparisons, we essentially treated each household in each country as a separate comparison unit. To obtain country shares of world consumption, simply sum over households in the country; e.g., \( s_i = \sum_{h=1}^{H_i} s_{ih} \) and \( \sigma_i = \sum_{h=1}^{H_i} \sigma_{ih} \). However, in practise, it will be very difficult if not impossible to implement these methods empirically due to the unavailability of the individual household consumption data.\(^{34} \) Hence we will develop a method in the remainder of this section that requires less empirical information to implement, (but at the same time, it is not quite as satisfactory from a theoretical point of view).

Let the preferences of household \( h \) in country \( i \) be represented by the nondecreasing, continuous from above utility function \( F_{ih}(x) \), defined for \( x \geq \)
0_N. (We have absorbed any demographic or public good variables into F_ih, which can differ arbitrarily across households). For i=1,...,I and h=1,...,H_i, define the household h in country i deflation function D_{ih}(u,x) for x \geq 0_N and u belonging to the range of F_{ih} by

\[ D_{ih}(u,x) = \max_{\delta} \{ \delta : F_{ih}(x/\delta) \geq u, \delta > 0 \} . \]

Then for any reference utility level u belonging to the range of F_{ih}, the Malmquist [1953] quantity index of x^* \geq 0_N relative to x \geq 0_N is defined by

\[ Q_{ih}(u,x,x^*) = D_{ih}(u,x^*)/D_{ih}(u,x) . \]

Let the observed household h in country i in consumption vector be x^{ih} > 0_N and define the corresponding utility level by

\[ u_{ih} = F(x^{ih}), \quad i=1,...,I; \quad h=1,...,H_i . \]

Now define the following index of average household consumption in country i relative to country j:

\[ Q_{ij} = \Sigma_{h=1}^{H_i} H_i^{-1} Q_{ih}(u_{ih}, \Sigma_{k=1}^{H_j} H_j^{-1} x^{jk}, \Sigma_{k=1}^{H_j} x^{ih}) , \quad i,j=1,...,I . \]

To explain the meaning of (77), note that Q_{ih}(u_{ih}, \Sigma_{h=1}^{H_i} H_i^{-1} x^{ih}, x^{ih}) is the household h in country i Malmquist quantity index which compares the household ih consumption vector x^{ih} to the average per household consumption vector in country j, \Sigma_{k=1}^{H_j} H_j^{-1} x^{jk}, where the household ih indifference surface through the household ih consumption vector is used as the reference indifference surface. Q_{ij} is the average (over all households in country i)
of these individual \( i \)th Malmquist quantity indexes \( Q_{ih} \) just described.

Let us assume expenditure minimizing behavior on the part of each household; i.e., assume for \( i=1, \ldots, I \) and \( h=1, \ldots, H_i \):

\[
(78) \quad w_i \cdot x_i^h = \min_{x} \{w_i \cdot x : F_i(x) \geq u_{ih}, x \geq 0, \} = C_{ih}(u_{ih}, w_i).
\]

**Proposition 11:** Assume: (i) \( w_i \gg 0 \), \( i=1, \ldots, I \); (ii) \( x_1^h > 0 \) for \( i=1, \ldots, I \) and \( h=1, \ldots, H_i \), (iii) (76) and (78). Then \( Q_{ij} \) defined by (77) has the following lower bound:

\[
(79) \quad Q_{ij}^{ij} \geq w_i \cdot x_i^{H_i} / \sum_{h=1}^{H_i} x_i^{H_i} x_j^{H_j} = w_i \cdot x_i / w_i \cdot x_j
\]

where we have defined the per household average consumption vectors by \( x_i = \sum_{h=1}^{H_i} x_i^h / H_i \).

It can be seen that \((Q_{ij}^{ij})^{-1}\) is also an index of the average consumption of households in country \( i \) relative to the average consumption of households in country \( j \). In fact, if \( N=1 \), we have

\[
(80) \quad Q_{ij} = [Q_{ij}^{ij}]^{-1} = \sum_{h=1}^{H_i} x_i^{H_i} x_j^{H_j} / \sum_{k=1}^{H_j} x_k^{H_j} = x_i / x_j
\]

where the right hand side of (80) is the average country \( i \) consumption of the good divided by the average country \( j \) consumption.

Using (79) for \( i=j \) and \( j=i \), we derive

\[
(81) \quad [Q_{ij}^{ij}]^{-1} \leq w_i \cdot x_i^{H_i} / \sum_{h=1}^{H_i} x_i^{H_i} x_j^{H_j} = w_i \cdot x_i / w_i \cdot x_j.
\]

Having established the bounds (79) and (81), we can prove the following counterpart to Proposition 2, using the same technique of proof.
Proposition 12: Assume the hypotheses of Proposition 11. Then for each pair of countries $i$ and $j$ there exists a $\lambda^*_ij$ such that $0 \leq \lambda^*_ij \leq 1$ and $\lambda^*_ij Q^{ij} + (1-\lambda^*_ij) [Q^{ij}]^{-1}$ lies between the average household Paasche and Laspeyres quantity indexes for country $i$ relative to $j$, $w^i \cdot \bar{x}^i / \tilde{w}^i \cdot \tilde{x}^j$ and $w^j \cdot \bar{x}^j / \tilde{w}^j \cdot \tilde{x}^j$, respectively.

Proposition 12 suggests that we approximate the theoretical index $\lambda^*_ij Q^{ij} + (1-\lambda^*_ij) [Q^{ij}]^{-1}$ by a symmetric average of the Paasche and Laspeyres quantity indexes such as the Fisher index $Q_F$ defined by

$$Q_F(w^j, w^i, \bar{x}^j, \bar{x}^i) = [w^i \cdot \bar{x}^i / \tilde{w}^i \cdot \tilde{x}^j]^{1/2} [w^j \cdot \bar{x}^j / \tilde{w}^j \cdot \tilde{x}^j]^{1/2}$$

(82)

It is important to note that the consumption index defined by (82) should not be interpreted as a welfare index for country $i$ relative to $j$; rather it approximates an average of the theoretical per household or per capita real consumption of country $i$ relative to country $j$ indexes defined by (78) and (80). We have not taken into account any inequality in the distribution of consumption in each country.

At this point, we could largely duplicate the material presented in section 6: simply replace our old $Q(p^j, p^k, y^j, y^k)$ by the new relative total quantity index, $Q_F(w^j, w^i, H^j, \bar{x}^j, H^i, \bar{x}^i)$, which converts the per capita or per household relative consumption index defined by (82) into an index of total consumption in country $i$ relative to total consumption in country $j$. With these changes, the 6 methods for making multilateral quantity comparisons described in sections 6 and 8 can be repeated.

We can summarize our analysis up to this point as follows. Bilateral international or interregional quantity comparisons can be made either on the
basis of producer theory or consumer theory. Each approach leads to one or two "ideal" bilateral index number formulae. The problem of aggregating up the bilateral comparison information to yield a consistent multilateral ranking of real outputs or real consumptions can then be approached in a common axiomatic manner. We have considered six different multilateral systems and discussed their relative advantages and disadvantages.

Before concluding the paper, we devote the next section to an exposition of two multilateral systems that are not based on averaging over bilateral indexes.

11. Multilateral Systems that are Not Based on Bilateral Formulae.

Let us revert to the multilateral output comparison problem discussed above in sections 6 to 8.\textsuperscript{35} We use the definitions and notation explained there.

The first new multilateral system we wish to consider is the Geary [1958] Khamis [1970][1972] or GK system. Consider the following two sets of equations:

\[ (83) \quad \Pi_n = \sum_{i=1}^{I} \frac{p_i^i \Pi_n p_i^{-1}}{\Sigma_i^i \Pi_n} ; n=1,\ldots,N \text{ and} \]

\[ (84) \quad p_i^{-1} = \sum_{n=1}^{N} \frac{\Pi_n y_n \Pi_n}{\Sigma_{n=1}^{N} \Pi_n y_n} ; i=1,\ldots,I \]

Equations (83) and (84) are to be regarded as \(N+I\) simultaneous equations in the \(N\) unknown "international prices" \(\Pi_n\) and the \(I\) purchasing power parity (or country \(i\) price index) functions \(P_i = P_i(p,y)\). It is easy to show that the system of equations defined by (83) is homogeneous and linearly dependent in
the unknowns $\Pi_1, \ldots, \Pi_N, P_1, \ldots, P_I$. Hence we may drop any one of the equations and we may impose any normalization on the $P_i$. Khamis [1970][1972] shows that the resulting system of equations has a unique strictly positive solution (up to a positive scalar multiple) provided that the price and quantity data, $p_n^i, y_n^i, i=1, \ldots, I, n=1, \ldots, N$, are all positive. Given a set of solution functions, $P_1(p,y), \ldots, P_I(p,y)$, we shall choose to normalize these functions so that the share functions $S_i(p,y)$ defined by

$$ (85) \quad S_i(p,y) = p^i \cdot y^i / P_i(p,y), \quad i=1, \ldots, I, $$

sum up to unity. Remember that $p = [p^1, \ldots, p^I]$ and $y = [y^1, \ldots, y^I]$ are $N$ by $I$ price and quantity matrices.

**Proposition 13:** The GK system for making multilateral real output comparisons defined by (83)-(85) satisfies all multilateral tests except MT1, MT3 and MT10.

The GK system fails MT1 because in general one cannot guarantee a positive solution to (83) and (84) when some quantities $y_n^i$ are negative. Khamis [1984; 204] recognized this problem but did not exhibit a solution to it. However, it turns out to be relatively straightforward to solve this problem by adapting a technique used by Van Ijzeren [1983; 43]. First, substitute equations (83) into (84). It is then easy to show that the resulting system of equations in $[p_1^{-1}, \ldots, p_I^{-1}]^T = P^{-1}$ is equivalent to the following system of equations,

$$ (86) \quad (M-I_I)P^{-1} = 0_I $$

where $I_I$ is an $I$ by $I$ identity matrix and the $i,j$th element of the $I$ by $I$ matrix $M$ is defined by
If the matrix $M$ has all elements positive (or even weaker, if $M$ is nonnegative but irreducible), then by the Theorem of Frobenius in matrix algebra, there is a unique (up to a positive scale factor) positive solution $P^{-1} \gg 0_i$ to (86). This is the solution we are looking for. Hence even if some quantities $y^i_n$ are negative, as long as the $m_{ij}$ defined by (87) are positive, the GK system will satisfy MT1.

Necessary and sufficient conditions for the GK shares to satisfy MT3 are that the GK price functions $P_j(p,y)$ be homogeneous of degree zero in their $y^i$ arguments; i.e., for $\lambda > 0$, $P_j(p,y^i,\ldots,y^{i+1},\ldots) = P_j(p,y)$ for $j=1,\ldots,I$ and for $i=1,\ldots,I$. In words, these homogeneity properties imply that the country price indexes are invariant to scale changes in country quantity vectors. Drechsler [1973; 26] noted that the GK price indexes do not have these desirable invariance properties. Moreover, Drechsler also observed that the GK system fails MT3 in a relatively spectacular manner:

"If an infinitely great country is compared with an infinitely small country, the GK quantity index will be the same as the index using the big country's prices."

(Drechsler [1973; 26]).

Another way of expressing the idea behind the above quotation is to replace the country $i$ quantity vector $y^i$ by $\lambda y^i$ for $\lambda > 0$ in (83) and (84) and then let $\lambda$ tend to infinity. The limiting system of GK relative price parities turns out to be
which is equal to the Paasche price index for country \( j \) relative to country \( i \) divided by the Paasche price index for country \( 1 \) relative to country \( i \). This point was originally made by Geary [1958].

We mentioned earlier that the own share system defined by (44) and (45) also failed to pass the multilateral test MT3. However, its failure is not nearly as "bad" as the failure of the GK system exhibited in (88). Again replacing \( y^i \) by \( \lambda y^i \) and taking limits as \( \lambda \) tends to infinity yields the following limiting system of own share price parities, provided the underlying bilateral formula \( Q \) satisfies BT1, BT3 and BT6:

\[
\frac{p_j}{p_1} = \frac{[p^j \cdot y^j / Q(p^i, p^j, y^i, y^j)]}{[p^1 \cdot y^1 / Q(p^i, p^1, y^i, y^1)]} , \quad j = 2, \ldots, I.
\]

The results given by (89) are quite reasonable from the viewpoint of economic theory if \( Q \) is chosen to be \( Q_F \) or \( Q_T \).

Of course, it is not surprising that the GK system fails the strong dependence on a bilateral formula test, MT10, since the GK system has no bilateral foundation. How important is it for a multilateral system to have a bilateral foundation? While there can be no definitive answer to this question, it seems worthy of some discussion. Our test MT10 is one way of formalizing a property that Drechsler [1973] termed "characteristicity." It is perhaps best to let Drechsler explain this concept in his own words:

"In general, this requirement means that the weights used for any index computations should be characteristic of the given two
countries. To use Indian weights in a Netherlands-Belgium comparison would be considered wrong by everybody just as if in an Indian-Pakistan comparison Dutch weights were used. In the latter cases, the weights would be very uncharacteristic; their use would amount to the same as if in the case of the computation of a 1971-1970 inter-temporal index 1920 (or 2020) prices were used."

(Drechsler [1973; 19]).

Gerardi [1985] criticizes the GK method for its lack of characteristicity; he feels that the method is equivalent to the use of the price structure of a recent year in order to evaluate the quantities of all years of the century.

The first part of our test MT10 imposes a certain amount of characteristicity on the multilateral system: the country shares are to depend only on the "best" bilateral quantity indexes, each of which has a maximal degree of characteristicity. The second part of MT10 says that as the world economy shrinks to a two country economy, the relative share of the two nonshrinking economies tends to the "best" bilateral quantity index between the two countries. Thus MT10 can be interpreted as a characteristicity test.

Turning now to other criticisms of the GK system, Marris [1984; 52] notes that in the consumer theory context, the GK quantity index will be biased upwards for countries with price structures far from the \( \Pi_n \) international prices (the bias will go in the opposite direction in the producer context). This is perhaps an obvious point, since the derivation of the GK system does not draw on either consumer or producer theory. Thus from the viewpoint of being consistent with economic theory, methods (i)-(vi) described earlier in
section 6 have a clear advantage over the GK method.\textsuperscript{39}

A final criticism of the GK method is the following one due to Gerardi [1985]: it is not stable with respect to the entry and withdrawal of countries; i.e., suppose we implement the method for I countries and then drop country I and redo the method. Then the relative output shares of the first I-1 countries can change markedly with the deletion of the last country. Empirical work cited by Gerardi has shown that the GK method can be quite unstable. The reason for this instability can be seen by considering the system of I equations defined by (86). Small changes in the country quantity vectors $y^i$ can generate relatively large changes in the $M$ matrix and these changes in turn can generate even larger changes in the $p^{-1}$ vector.

It is obvious that the star system will not suffer from this kind of instability; in fact the method will be completely stable with respect to the entry and withdrawal of nonnumeraire countries. While methods (ii)-(vi) explained in section 6 are not completely stable, they are all much more stable than the GK method, because they all involve taking averages of bilateral quantity indexes of the form $Q(p^i, p^j, y^i, y^j)$ in a symmetric manner. As an additional country is added, these averages will not change by very much.

From a narrow point of view, the GK system is dominated by the own share system, since the latter satisfies MT1 and MT10 whereas the former satisfies neither. However, proponents of the GK system would probably be correct in asserting that prices $P_i$ and shares $S_i$ will be positive in real life situations, so that in practice, MT1 will be satisfied. Proponents could also dismiss the failure of MT10 since it is a test that is biased in terms of
weighting up bilateral formulae.

From a broader point of view, it also seems that the use of the GK system is not warranted for three reasons discussed above: (i) it fails MT3 in a somewhat disastrous manner, (ii) it has no reasonable producer or consumer theory interpretation and (iii) it is not stable with respect to the addition or deletion of countries.

The final multilateral method, which is not based on aggregating bilateral indexes, that we wish to consider is Van Yzeren's [1956; 25] balanced method. It is similar to the GK method in that we first define unique (up to a proportional factor) purchasing power parities, $P_1, \ldots, P_I$, and then define the corresponding share functions by (85). The parities $P_i$ may be found by iterative substitution into the following system of equations:

\[(90) \quad P_i = \left[ \sum_{j=1}^{I} (p^i_j \cdot y^j_j / p^i \cdot y^j) P_j \right]^{1/2} \left[ \sum_{j=1}^{I} (p^j_j \cdot y^i_j / p^j \cdot y^i) P^{-1}_j \right]^{-1/2}, \quad i=1, \ldots, I.\]

If the $I^2$ inner products $p^i \cdot y^j$ are all positive, Van Yzeren [1956; 25-26] shows that there is a unique positive solution ray to (90); his proof is based on showing that the solution to (90), subject to a positive normalization, minimizes the function $f(P_1, \ldots, P_I)$ over the positive orthant, subject to the same positive normalization, where $f$ is defined by

\[(91) \quad f(P_1, \ldots, P_I) = \sum_{i=1}^{I} \sum_{j=1}^{I} p^{-1}_i (p^i \cdot y^j / p^i \cdot y^j) P_j.\]

Thus the balanced method is what Diewvert [1981; 179] calls a neostatistical approach to the construction of indexes: one tries to obtain price and quantity indexes, $P_i$ and $Y_i$ say for $i=1, \ldots, I$, which satisfy the equations
\[ p^i y^j = p_i y_j + e_{ij}, \quad i, j = 1, \] for some errors \( e_{ij} \) which are minimal in some norm.

The following Proposition sums up the mathematical properties of the balanced method due to Van Yzeren.

**Proposition 14:** The balanced method for making multilateral output comparisons defined by (90) and (85) satisfies all of the multilateral tests considered in section 8 except MT8, MT9, and MT10, provided that the inner products \( p^i y^j \) are all positive.

It is of course, not surprising that the balanced method fails MT10, since that test is biased in favor of aggregative bilateral methods over genuinely multilateral methods such as the GK or balanced methods. The failure of the two country weighting tests, MT8 and MT9, is more troublesome.

From a narrow point of view, the balanced method is dominated by the EKSCCD system since it satisfies more multilateral tests. From the viewpoint of consistency with microeconomic theory, the EKSCCD system also seems preferable.

12. **Extensions and Conclusion**

It is straightforward to rework our material on output indexes into corresponding indexes for inputs. Thus if \( w^i \) denotes a positive vector of input prices for country or region \( i \) and \( x^i \) the corresponding nonnegative, nonzero input vector, we may rework our analysis and obtain a system of input share functions, \( s_i(w^1, \ldots, w^I, x^1, \ldots, x^I) = s_i(w, x) > 0 \) for \( i = 1, \ldots, I \) such that \( \sum_{i=1}^{I} s_i(w, x) = 1 \). The six multilateral methods discussed in section 6 may now be reinterpreted as multilateral real input indexes.

Given a system of multilateral real output share functions \( S_i(p, y) \) and a
corresponding system of multilateral real input share functions \( s_i(w,x) \), it is natural to define the following system of multilateral productivity functions, \( \Pi_i \), by

\[
\Pi_i(p,y,w,x) = \frac{S_i(p,y)}{S_i(w,x)}, \quad i=1,\ldots,I.
\]

Such a system of productivity functions should be very useful in the context of cross sectional industry or plant data which are to be treated in a symmetric manner.

Another extension of our analysis can be made to situations where it is desirable to achieve consistency over time and space. If there are output data for \( I \) countries and \( T \) time periods, then consistent relative outputs over time and space can be achieved by treating the country data for each time period as separate data for an artificial country. Thus there will be \( IT \) separate multilateral share functions. However, in practice, I would not recommend this method. From the viewpoint of economic theory, the most accurate way of determining the output of country \( i \) in period \( t \) relative to its output in period 1 is to use the best bilateral quantity index available and the chain principle; i.e., the desired quantity index would be

\[
Q(p^{i1},p^{i2},y^{i1},y^{i2})Q(p^{i2},p^{i3},y^{i2},y^{i3})\ldots Q(p^{i(T-1)},p^{iT},y^{i(T-1)},y^{iT})
\]

where \( p^{it} \) and \( y^{it} \) denote the price and quantity vectors for country \( i \) in period \( t \) and \( Q \) is the "best" bilateral quantity index. Thus in practice, I would recommend constructing a system of multilateral indexes for the first time period and then using individual country bilateral indexes for a number of time periods until a new multilateral system of relative country outputs could be constructed using the data for a single period.
The multilateral methods we have developed above in this paper may be used to construct subaggregates of aggregate output. We should not expect these multilateral subaggregates to add up to the corresponding multilateral aggregate. As a limiting case, consider a system of subaggregates that was so disaggregated, that each subaggregate consisted of a single homogeneous good. No one would expect these subaggregates to simply add up to aggregate utility or output. The point is that subaggregates cannot simply be added up to equal an aggregate -- an index number formula must be used in order to combine the subaggregates into an aggregate.

We would like to stress that we have developed our best multilateral methods (the star, the EKSCCD, and the own share systems) by aggregating up bilateral indexes. The bilateral indexes ($Q_F$ and $Q_T$) were chosen for their consistency with microeconomic theory. It is this emphasis on microeconomic theory that distinguishes our work from the pioneering work of others.

Microeconomic theory enables us to provide at least tentative answers to a number of vexing problems. For example, should government expenditures, investment expenditures, stocks of consumer durables, exports and or imports be included in the international or interregional comparison? If we are doing an international comparison of real consumption, then obviously the nonconsumption items should not be included. Moreover, existing stocks of consumer durables should be added to new purchases of consumer durables and the total stock of durable should be priced out at its rental price, not its (stock) purchase price. If we are doing an international comparison of real outputs, then new sales of consumer goods (including consumer durables), exports and production of investment goods by private (i.e., nongovernmental)
producers should be included as outputs. Import purchases by nongovernmental agencies should be included as negative outputs. All activities by non profit maximizing governmental agencies should be excluded from the comparisons.  

Note that in the output context, consumer stock prices are used rather than rental prices.

Another vexing problem is: should prices be before or after tax prices? If we are making real consumption comparisons, the prices should be the after tax prices that consumers face. If we are making real output comparisons, the prices should be before tax prices in the case of outputs and after tax (and tariff) prices in the case of intermediate inputs and imports; i.e., the prices should be the prices that producers face. In the case of subsidized goods (e.g., health care and education in many countries), the prices consumers face are the lower after subsidy prices, while producers face the higher before subsidy prices.

We conclude by noting some additional novel features of our analysis: (i) aggregation over consumer issues are not ignored in our analysis, (ii) our best bilateral quantity indexes, $Q_F$ and $Q_T$, do not satisfy some traditional tests such as the mean value test due to the existence of negative quantities and (iii) we have devised some new multilateral tests which should prove to be useful to researchers whether they share our microeconomic approach or not.
13. **Appendix: Proofs of Propositions:**

**Proof of Proposition 1:**

\[ p^i_j(p^i_j, p^i_j) = g^j_i(p^j_i, v^j_i)/g^j(p^j_i, v^j_i) \]

using definition (3)

\[ = \max_y \{p^i_j \cdot y : (y, v^j_i) \in S^j_i / p^j_i \cdot y^j \} \]

using (1) and (4)

\[ \geq p^i_j \cdot y^j / p^j_i \cdot y^j \]

since \( y^j \) is feasible for the maximization problem. The proof of (6) is similar.

**Proof of Proposition 2:** Let \( i \) and \( j \) be given and define the function \( h(\lambda) = P^i_j(p^i_j, p^i_j) \). Then definitions (10) and (12) and assumption (v) imply that \( h(\lambda) \) is a continuous function for \( 0 \leq \lambda \leq 1 \). Note that \( h(0) = g^i_j(p^i_j, 0) / g^i_j(p^i_j, 0) = g^i(p^i_j, v^i_j)/g^i(p^j_i, v^j_i) = P^i_j(p^j_i, p^i_j) \) and \( h(1) = g^i_j(p^i_j, 1) / g^i_j(p^i_j, 1) = g^i(p^i_j, v^i_j)/g^i(p^j_i, v^j_i) = P^i_j(p^j_i, p^i_j) \). There are 24 possible a priori inequalities between the 4 numbers \( h(0), h(1), P^i_j, P^i_j \). However, assumptions (i)-(iv) imply that the inequalities (5) and (7) hold, which may be rewritten as follows:

\[ P^i_j = p^i_j \cdot y^i / p^j_i \cdot y^i \geq P^i(p^j_i, p^i_j) = h(1) \] and \( P^i_j = p^i_j \cdot y^i / p^j_i \cdot y^j \leq P^i_j(p^j_i, p^i_j) = h(0) \). This means that only the following 6 inequalities are possible between the 4 numbers: (1) \( h(0) \geq P^i_j \geq P^i_j \geq h(1) \), (2) \( h(0) \geq P^i_j \geq P^i_j \geq h(1) \), (3) \( h(0) \geq P^i_j \geq h(1) \geq P^i_j \), (4) \( P^i_j \geq h(0) \geq P^i_j \geq h(1) \), (5) \( P^i_j \geq h(1) \geq h(0) \geq P^i_j \) and (6) \( P^i_j \geq h(0) \geq h(1) \geq P^i_j \). Since \( h(\lambda) \) is continuous over \( 0 \leq \lambda \leq 1 \), it assumes all intermediate values and hence there
exists $0 \leq \lambda^* \leq 1$ such that

$$(A1) \quad p^i_{ij} \leq p^i \cdot y^i/p^j \cdot y^j \leq h(\lambda^*) \leq p^i_{\lambda^*}(p^j, p^i) \leq p^i \cdot y^j/p^j \cdot y^j = p^i_L$$

is true if case (1) occurs or such that

$$(A2) \quad p^i_{ij} \leq p^i_{\lambda^*}(p^j, p^i) \leq p^i_p$$

is true if cases (2)-(6) occur. Our method of proof is due to Konüs [1939].

**Proof of Proposition 3:**

$$[P_F(p^j, p^i, y^j, y^i)]^2 = p^i \cdot y^i p^i \cdot y^j/p^j \cdot y^i p^j \cdot y^j$$

using (13)

$$= g^i(p^i, v^i) p^i \cdot y^i/p^j \cdot y^i g^j(p^j, v^j)$$

using (4)

$$= (p^i \cdot Bp^i)^{1/2} h^i(v^i) p^i \cdot y^i/p^j \cdot y^i (p^j \cdot Bp^j)^{1/2} h^j(v^j)$$

using (14)

$$= (p^i \cdot Bp^i)^{1/2} h^i(v^i) p^i \cdot Bp^j (p^j \cdot Bp^j)^{-1/2} h^j(v^j)$$

using (17) and (14)

$$= p^i \cdot Bp^i / p^j \cdot Bp^j$$

using $p^i \cdot Bp^j = p^j \cdot Bp^i$

$$= [(p^i \cdot Bp^i)^{1/2} h^i(v)/(p^j \cdot Bp^j)^{1/2} h^i(v)]^2$$

for any reference $v$

$$= [P^i(p^j, p^i)]^2$$

for $v = v^i$. 
Proof of Proposition 4:

\((1/2) \ln P_i(p^j, p^i) + (1/2) \ln P_j(p^j, p^i)\)

\[= (1/2) \ln \left[ g_i(p^i, v^i)/g_i(p^j, v^j) \right] + (1/2) \ln \left[ g_j(p^i, v^j)/g_j(p^j, v^j) \right] \]

using (2) and (3)

\[= (1/2) \sum_{n=1}^{N} \left[ p_n \ln g_i(p^i, v^i)/ap_n + p_n \ln g_j(p^j, v^j)/ap_n \right] \ln p_n^i - \ln p_n^j \]

using the translog identity in Caves, Christensen and Diewert [1982b, 1412]

\[= (1/2) \sum_{n=1}^{N} \left[ (p_n y^i/p_n y^j) + (p_n y^j/p_n y^j) \right] \ln p_n^i - \ln p_n^j \] using (17) and (4)

\[= \ln P_T(p^j, p^i, y^j, y^i) \]

using definition (19).

Proof of Proposition 5:

\[\Pi_i^I \delta_i = \Pi_i^I [\Pi_j^I P(p^j, p^i, y^j, y^i)^{S_j S_i}] \]

using (22) and (i)

\[= \Pi_i^I \Pi_j^I [P(p^j, p^i, y^j, y^i)]^{S_i S_j} \]

\[= \Pi_1 \leq i < j \leq I [P(p^j, p^i, y^j, y^i)P(p^j, p^i, y^i, y^j)]^{S_i S_j} \]

\[= \Pi_1 \leq i < j \leq I [P(p^j, p^i, y^j, y^i)/P(p^j, p^i, y^j, y^i)]^{S_i S_j} \] using (iii)

\[= 1.\]

Proof of Proposition 6: Assumption (ii) implies that \((y^i, v^i)\) is on the frontier of \(S^i\) and hence (32) will hold. Thus \(Q^i_M(y^j, y^i) = d^j(y^i, v^j) = \min_{\delta} \{\delta : (y^i/\delta, v^j) \in S^j, \delta > 0\} = \delta^i_{j1} > 0\) by (iii). Thus \((y^i/\delta^i_{j1}, v^j) \in S^j\) and i
so $y^i / \delta_{ji}$ is feasible for the maximization problem which follows:

$$p^j \cdot y^j = \max_y \{ p^j \cdot y : (y, v^j) \in S^j \} \geq p^j \cdot y^i / \delta_{ji}.$$ 

Therefore, $\delta_{ji} \geq p^j \cdot y^i / p^j \cdot y^j$ using (i) which is (35). (36) follows in an analogous manner.

**Proof of Proposition 7**: Define $h(\lambda) = Q_{\lambda}^{ij}(y^j, y^i)$. Using (42), (35) and (36), we have $h(0) = Q_{H}^{ij}(y^j, y^i) \leq Q_{p}(p^j, p^i, y^j, y^i)$ and $h(1) = Q_{H}^{ij}(y^j, y^i) \geq Q_{L}(p^j, p^i, y^j, y^i)$. This reduces the 24 a priori possible inequalities between the 4 numbers $h(0)$, $h(1)$, $Q_{p}$ and $Q_{L}$ to six inequalities. The remainder of the proof follows the proof of Proposition 2.

**Proof of Proposition 8**: Omitted due to its length and straightforward nature.

**Proof of Proposition 11**: Adapting a method of proof due to Malmquist [1953; 231] shows that

$$Q_{ih}(u_{ih}, i_{jk} \cdot x^j, x^i_{ih}) \geq w^i \cdot x^i_{ih} / w^i \cdot i_{jk} = H_{i}^{-1} x^j$$

for $h = 1, \ldots, H_i$. Now sum these inequalities, divide both sides by $H_{i}^{-1}$ and obtain (79).

**Proof of Proposition 13**: Proof of MT2: Suppose $p \gg 0$, $p \cdot y > 0$, $a_i > 0$, ...
\( \beta_i > 0 \) for \( i = 1, \ldots, I \), \( \sum_{i=1}^{I} \beta_i = 1 \), \( p = a_i p \), \( y = \beta_i y \) for \( i = 1, \ldots, I \). We show that \( S_i(p, y) = \beta_i \) for \( i = 1, \ldots, I \) by showing that \( \Pi_n = p_n \), \( n = 1, \ldots, N \), \( P_i = a_i, i = 1, \ldots, I \) and \( S_i = \beta_i, i = 1, \ldots, I \) satisfy (83), (84), and (85) subject to the normalization \( \sum_{i=1}^{I} S_i = 1 \). If in addition, \( y_n \neq 0 \) for \( n = 1, \ldots, N \), then the above solution is the unique solution.

The proofs of the other tests are made in a similar manner. In each case, conditions sufficient to imply the uniqueness and positivity of the \( P_i \), \( S_i \) part of the solution to (83)-(85) must be made.

**Proof of Proposition 14:** Straightforward computations. In this case, the conditions \( p_i \cdot y_j > 0 \) are sufficient to guarantee the uniqueness and positivity of the solution to (90) and (85), subject to the normalization \( \sum_{i=1}^{I} S_i = 1 \).
Footnotes

1. Note that since this theoretical purchasing power parity function requires the assumption of competitive (i.e., price taking) revenue maximizing behavior on the part of producers, the government sector in each country should be excluded in the international comparison. Moreover, appropriate marginal prices must be estimated for any noncompetitive sectors in the two countries. Thus the more traditional national income accounting approaches to international comparisons as outlined in Kravis [1984] perhaps have some advantages over the producer theory approaches explained in this paper, since government sectors are included and noncompetitive behavior is ignored.

2. Ruggles [1967; 189-190] constructed bilateral consumer price indexes for 19 Latin American countries for the year 1961. He found that Paasche and Laspeyres indexes differed by about 50 percent per observation (in constructing a Paasche and a Laspeyres index between two countries, the role of the base and comparison country are interchanged).

3. Notation: \( \mathbf{0}_M \) denotes an M dimensional vector of zeros, \( \mathbf{v} \geq \mathbf{0}_M \) means the vector \( \mathbf{v} \) is nonnegative, \( \mathbf{v} > \mathbf{0}_M \) means \( \mathbf{v} \) is nonnegative with at least one component positive and \( \mathbf{v} \gg \mathbf{0}_M \) means each component of \( \mathbf{v} \) is positive.

4. If country \( i \) is not producing the \( n \)th net output (or utilizing it as an input) in its private production sector, then we must estimate a positive shadow price for this good, \( p^i_n > 0 \), which would just induce the private production sector in country \( i \) to produce a zero amount of this \( n \)th good. Proposition 4 requires that all prices be positive.

5. We shall assume that the private national product functions \( g^i \) defined by (1) are well defined as maximums (rather than supremums) for the relevant net
output price vectors $p >> 0_N$.

6. On the other hand, sectors of country $i$ that behave monopolistically on (net) output markets should be excluded from $g^i$ (or appropriate marginal prices should be used).

7. See also Samuelson and Swamy [1974; 588-592], Archibald [1977; 60-61] and Diewert [1980; 461] [1983b; 1055].

8. If the sets $S^i$ are convex, then the functions $g^i(p,v)$ are concave and (normally) continuous functions in $v$. For the formal mathematical properties that the national product functions $g^i$ will have under various assumptions about the sets $S^i$, see Diewert [1973] or McFadden [1978].

9. See Diewert [1983b; 1060]. It should also be noted that our present Proposition 2 seems to be somewhat controversial in the literature on international comparisons. Consider the following quotation from Khamis [1984; 195]: "Also, in spite of the many reminders in the literature on index number methodology calling attention to the misconception that the Laspeyres and Paasche indices provide respectively upper and lower bounds to the 'true' index (if amenable to measurement), many proponents of the ideal Fisher formula and/or its extensions still commit this error."

10. For example, see Fisher [1922; 489].

11. We also assume that the matrix $B$ is such that $(p \cdot Bp)^{1/2}$ is a positive, convex function of $p$ over the relevant range of net output price vectors.

12. See Diewert [1976; 130].

13. The $\alpha^i_n$, $\alpha^i_k$ and $\gamma^i_{nm}$ must also satisfy certain restrictions which will ensure that $g^i(p,v)$ is linearly homogeneous in $p$; see Diewert [1974; 139].

14. This corresponds to the terminology used in Christensen, Cummings and
Jorgenson [1980]. Other terms for it include the Törnqvist index and the Divisia index; see Diewert [1976; 120]. Note that the $s^i_n$ will be negative if the nth good is an imported good which is used by the private production sector in country i, since in this case $y^i_n < 0$.

15. Under these conditions, definitions (18) must be changed so that terms involving the zero $v^i_m$ are dropped from the summations.

16. Ruggles [1967; 189-190] showed that $P_F$ and $P_T$ differed by about 1 to 2 per cent per observation using his 1961 data for 19 Latin American countries. The Paasche and Laspeyres indexes differed by about 50 per cent per observation.

17. Diewert [1978; 888] proved (21) under the assumptions that $p^i=p^j=p^>>0_N$ and $y^i=y^j=y^>>0_N$. However, it can be verified that (21) is still valid provided that $p^i=p^j>>0_N$, $p^i \cdot y^i>0$, $p^j \cdot y^j>0$, $p^i \cdot y^j>0$ and $p^j \cdot y^i>0$; i.e., we now allow for possible negative components in the vectors $y^i$ and $y^j$.


19. However, it seems to be true empirically that the choice of the base country is relatively unimportant if the Fisher or translog functional form, $P_F$ or $P_T$, is used. In the Ruggles [1967; 189-190] paper, changing the base country when using the Fisher (translog) formula changed the numerical results by about 2 (3) per cent per observation. In the Kravis, Kenesssey, Heston and Summers [1975; 52] book, changing the base country when using the Fisher formula changed the numerical results by less than 1 per cent.

21. The terms democratic and plutocratic are due to Prais [1959]. See also Diewert [1983a; 188-190].

22. In this case, a straightforward computation shows that the country $i$ parity $\delta_i$ defined by (22) is numerically equal to a (world) translog price index which compares the world price and quantity vectors $p^1, p^2, ..., p^I$ and $y^1, y^2, ..., y^I$ respectively to an artificial country $i$ economy which has the country $i$ price vectors replicated $I$ times, i.e., $p^i, p^i, ..., p^i$, and the quantity vector $s_1 y^i, s_2 y^i, ..., s_I y^i$ where $s_j = p^j, y^j / \sum_{k=1}^{I} p^k, y^k$ for $j=1,2, ..., I$. Thus we have $\delta_i = P_T (p^1, ..., p^I; p^i, ..., p^i; y^1, ..., y^I; s_1 y^i, ..., s_I y^i)$.


24. Diewert [1978; 896] also shows that $Q_F (p^j, p^i, y^j, y^i)$ and $Q_T (p^j, p^i, y^j, y^i)$ differentially approximate each other to the second order around any point where the two price vectors are positive and equal and where the two quantity vectors $y^i$ and $y^j$ are positive and equal. The positivity restrictions on the $y^i$ and $y^j$ may be replaced by $y^i = y^j \equiv y$ and $p \cdot y > 0$ where $p = p^i = p^j > 0$.

25. Much of the analysis in this section is adapted from Diewert [1983b; 1064-1075], who assumed that output quantities were always positive. In our present context, we must allow for the possibility that certain outputs can be positive in country $i$ and negative in country $j$. This seemingly minor change complicates the analysis considerably.

26. The minimum may have to be replaced by an infimum in certain cases.

Also, if there is no feasible solution for the minimization problem in (30), define $d^i(y,v) = +\infty$.

27. Malmquist [1953] developed the basic idea in the consumer theory context. It has been used in the producer theory context by Bergson [1961], Moorsteen
[1961], Samuelson and Swamy [1974; 590-591], Hicks [1981; 256], Caves, Christensen and Diewert [1982b; 1399-1401] and Diewert [1983b; 1064-1077].

28. See Diewert [1983b; 1072].

29. The direct translog quantity index, \( Q_T(p^j, p^i, y^j, y^i) = P_T(y^j, y^i, p^j, p^i) \) (the role of prices and quantities has been interchanged) cannot be used in the present circumstances because the \( y^i \) and \( y^j \) vectors will generally have negative components, and so the usual justification (see Caves, Christensen and Diewert [1982b; 14011]) for this index is not valid.

30. Van Ijzeren [1983] and Hill [1984; 131] informally discuss this consistency in aggregation property when they ask whether averages of bilateral indexes should be weighted according to the "importance" of the country.

31. In what follows, when we speak of \( Q(p^1, p^2, y^1, y^2) \), we shall always assume that the \( p^i \) and \( y^i \) satisfy the domain restrictions listed in BT1. When we specify that \( Q \) is \( Q_F \), we assume that the \( p^i \) and \( y^i \) satisfy the domain restrictions listed in BT1'.

32. \( C_h^i \) is defined in a manner analogous to the definition of \( g^i \) in (18) except that \( w \) replaces \( p \) and \( z \) replaces \( v \).

33. It is possible to use the direct translog quantity index \( Q_T \) in place of \( \tilde{Q}_T \) in definitions (70) and (72), provided that the individual household consumption vectors \( x_{ih} \) are positive in each component (i.e., \( x_{ih} > 0 \) for all \( i \) and \( h \)). Under these circumstances, it can be shown that (see Diewert [1976; 123-124]) \( Q_T \) equals a certain Malmquist quantity index, provided that preferences can be represented by a translog distance function. We have not developed this result in detail, since for disaggregated consumption data, it
is highly unlikely that the positivity restrictions $x^i h \geq 0_N$ would be satisfied.

34. We were able to suggest a practical procedure for forming multilateral consumer price comparisons; recall formula (65) which just involves aggregate data. However, that technique worked because we were able to assume that prices were constant over individuals within a country. We cannot assume that consumption vectors are identical over individuals within a country or region without distorting empirical facts considerably.

35. Problems involving the definitions of multilateral indexes are more complicated in the producer context due to the existence of negative quantities.

36. These positivity restrictions can be relaxed somewhat to nonnegativity restrictions.

37. Moreover, since the own share system closely approximates the corresponding EKSCCD system which satisfies MT3, the own share system will be approximately consistent with MT3.

38. Since these indexes satisfy the circularity property to a high degree of approximation, (89) will approximately equal $p^j y^j / p^i y^i Q(p^i, p^j, y^i, y^j)$, which is the "correct" bilateral answer.

39. Incidentally, Khamis [1984; 195] makes some unwarranted attacks on the use of the Fisher index; Propositions 3, 7 and 12 in the present paper present strong economic justifications for its use in the bilateral context.

40. This means that our methods are not strictly applicable to output and input comparisons between socialist countries, since profit maximizing or cost minimizing behavior cannot be assumed. However, if correct shadow or marginal product prices could be estimated for each quantity, then our methods could be
applied. Our methods could be applied to real consumption comparisons between socialist countries, provided that rationing and queuing problems are not too severe.
References


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