The Measurement of Business Capital, Income and Performance

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I. The Measurement of Capital: Traditional User Cost Approaches

1. Introduction
2. Inflation and the Measurement of Economic Activity
4. Relationships between Depreciation, Asset Prices and User Costs
5. The Empirical Determination of Interest Rates and Asset Inflation Rates
6. The Empirical Determination of Depreciation Rates
7. Time Series versus Cross Sectional Depreciation
8. Aggregation over Vintages of a Capital Good
9. The Production Function Framework
   9.1 Introduction
   9.2 The Austrian Production Function
   9.3 The Fisher-Hicks Intertemporal Production Function
   9.4 The Traditional Production Function
10. The Treatment of Business Income Taxes

Appendix A: Alternative Models of Depreciation

A1. The One Hoss Shay Model of Efficiency and Depreciation
A2. The Straight Line Depreciation Model
A3. The Declining Balance or Geometric Depreciation Model
A4. The Linear Efficiency Decline Model
A5. The Linearly Increasing Maintenance Expenditures Model

1. Introduction

1 The author is indebted to Kevin Fox, Emili Grifell, Peter Hill, Ning Huang, Ulrich Kohli, Knox Lovell, Alice Nakamura, Paul Schreyer and Frank Wykoff for helpful comments in developing the material in this chapter. Much of the material in this chapter is taken from Diewert (2001) (2005a).
“Capital (I am not the first to discover) is a very large subject, with many aspects; wherever one starts, it is hard to bring more than a few of them into view. It is just as if one were making pictures of a building; though it is the same building, it looks quite different from different angles.” John Hicks (1973; v).

“Perhaps a more realistic motive for reading earlier writers is not to rediscover forgotten truths, but to gain a perspective of how present day ideas have evolved and, perhaps, by reading the original statements of important ideas, to see them more vividly and understand them more clearly.” Geoffrey Whittington (1980; 240).

In this chapter, we discuss some of the problems involved in constructing price and quantity series for both capital stocks and the associated flows of services when there is general (and specific) price change in the economy. This chapter will show how rental prices, stock prices and depreciation rates for capital assets used in production are all related under somewhat strong assumptions. Some of these assumptions will be relaxed in chapter VI below.\(^2\) In addition, the assumptions underlying the user cost formulae developed in this chapter will be examined again in chapter VII when we discuss income concepts.

Our general purpose in this chapter is to show how approximations to market rental prices for various types of capital services can be formed, using data on purchases of new assets of the same type and information on the prices of second hand assets. Thus firms typically purchase capital inputs and use their services over many periods. As outside observers, all we can see is the initial purchase price of the asset. But for many purposes, we require information on how that initial purchase cost is allocated across time so that period by period estimates of the value of capital input can be formed.\(^3\) Obviously, if firms rented or leased all of their capital input period by period, we would not need to discuss the above problem of intertemporal cost allocation. Thus our primary purpose in this chapter is to suggest methods for forming approximations to period by period market rents for the use of capital assets in production when the corresponding rental markets do not exist.

Before getting into the algebra of capital, we first discuss some of the problems that occur when an economy is experiencing very high inflation. Under these conditions, it will be

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\(^2\) Basically, in this chapter, we take what might be called the *separable opportunity cost approach* to the measurement of capital services; i.e., we assume that at the beginning of each period of production, the firm (or higher aggregates of firms such as industries or even entire economies) can find relevant market prices for its stocks of used capital equipment or it can find market prices for renting the services of the used stocks of machinery, equipment, structures, land and inventories that it holds for the coming period, so that opportunity costs for the services of its capital stock components exist. We say this is a separable approach because we assume that these market prices for either the stocks or flows of the assets held by the firm at the beginning of each production period exist independently of the actions of the firm on output markets and other input markets. In chapter VI, we consider *nonseparable approaches* where some of these assumptions are relaxed.

\(^3\) Some of these purposes are: (i) the estimation of firm or industry one period production functions or their dual representations; (ii) the estimation of output supply elasticities or input demand elasticities; (iii) the estimation of firm or industry cost functions, particularly in a regulatory context and (iv) the estimation of periodic industry capital cost so that industry information on Gross Operating Surplus in the System of National Accounts can be decomposed into depreciation, tax and return to capital components. We will look at some of the issues surrounding this last point in more detail in chapter VII below.
necessary for the national price statistician to shorten the accounting period (or give up price measurement altogether).

Section 3 is a key section; it presents the basic equations relating stocks and flows of capital assuming that data on the prices of vintages of a homogeneous capital good are available. Our goal here is to show how information on used asset prices, combined with information on the relevant opportunity cost of capital and on expected future price movements in asset prices, can enable us to form rental prices for the assets used by a firm or industry, even if these assets are not actually rented. Once these rental prices have been estimated, they can be used to provide period costs for the assets and the contributions of these assets to period by period production can be evaluated. The framework presented in this chapter is not applicable under all circumstances but it is a framework that will allow us to disentangle the effects of general price change, asset specific price change and depreciation.

Section 4 continues the theoretical framework that was introduced in section 3. We show how information on vintage asset prices, vintage rental prices and vintage depreciation rates are all equivalent under certain assumptions; i.e., knowledge of any one of these three sequences or profiles is sufficient to determine the other two.

The previous two sections relied on assumptions about the production unit’s nominal interest rate and its expectations about the future course of asset prices. But how are national income accountants and applied economists to determine these variables as outsiders? Section 5 discusses the problems involved in making these choices. Section 6 discusses the problems associated with the empirical determination of depreciation rates.

Section 7 discusses a topic that is of great interest to national income accountants: namely, what is the “correct” concept for depreciation that should be entered into the System of National Accounts (SNA)? In particular, we discuss whether anticipated asset price decline should be an element of depreciation as understood by national income accountants.

Section 8 discusses the problems involved in aggregating over vintages of capital, both in forming capital stocks and capital services. Instead of the usual perpetual inventory method for aggregating over vintages, which assumes perfectly substitutable vintages of the same stock, we follow Diewert and Lawrence (2000) and suggest the use of a superlative index number formula to do the aggregation.

Section 9 looks at the consistency of our suggested treatment of capital with various production function concepts. This section is quite important because it shows how the general Hicksian (1946) intertemporal production function can be built up using conceptually simpler “Austrian” one period production functions, where goods in process

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4 Most notably, the framework laid out in this chapter cannot deal with unique or one of a kind assets, which by definition, do not have vintages. In chapter III, we will attempt to deal with this type of problem.

5 This point is due to Jorgenson (1989) and Hulten (1990; 127-129) (1996; 152-160).
are distinguished both as inputs (at the beginning of the period) and outputs (at the end of the period).

Section 10 shows how our basic user cost formulae can be adapted to deal with the business income tax.

Chapter I concludes with an Appendix which develops the material presented in the main text in more detail. Thus Appendix A shows how the general algebra presented in sections 3 and 4 can be adapted to deal with four specific models of depreciation. The four models considered are the one hoss shay model, the straight line depreciation model, the geometric model of depreciation and the linear efficiency decline model. The final section of Appendix A considers a fifth type of depreciation model, one that is based on the assumption that each vintage of the asset has a specific maintenance and operating cost requirement associated with it. We show that this type of model can lead to the linear efficiency decline model studied earlier in Appendix A. However, the main use of the analysis presented this section of Appendix A is to suggest a reason why accelerated depreciation assumptions are quite reasonable and likely to occur empirically.

2. Inflation and the Measurement of Economic Activity

Our goal in this chapter is twofold:

- To measure the price and quantity of the stock of reproducible capital held by a production unit (an establishment, a firm, an industry or an entire economy) at a point in time and
- To measure the price and quantity of the flow of reproducible capital services utilized by a production unit over a period of time.

In particular, we want to extend the procedures for measuring capital stocks and flows to cover situations where there is general price level change or inflation. In this section, we shall review some of the general measurement problems that arise when inflation is high.

When capital flows are measured, the normal period of time is either a year or a quarter. Under conditions of high inflation, the aggregation of homogeneous commodity flows within a quarter or a year is complicated by the fact that the within period transactions are valued at very different prices. The recent national income accounting literature explains the problem as follows:

“Conventional index number theory is mostly concerned with comparisons between points of time whereas, in national accounts, price and quantity comparisons have to be made between discrete periods of time. Significant changes in price and quantity flows may occur not only between different periods but also within a single accounting period, especially one as long as a year. Indeed, the central problem of accounting under high inflation is that prices are much higher at the end of the accounting period than at the beginning.” Peter Hill (1996; 11).

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6 An accelerated depreciation schedule for an asset means that depreciation increases as the asset ages.
“The underlying problem is not a traditional index number problem. It stems from the use of current value data as inputs into the calculation of indirect price or quantity measures under high inflation. Current accounts permit identical quantities of the same homogeneous product to be valued at very different prices during the course of the same year. Implicitly, quantities sold at higher prices later in the year are treated as if they were superior qualities when they are not.” Peter Hill (1996; 12).

“Under high inflation, the monetary value of flows of goods and services at different points of time within the same accounting period are not commensurate with each other because the unit of currency used as the numeraire is not stable. Adding together different quantities of the same good valued at different prices is equivalent, from a scientific point of view, to using different units of measurement for different sets of observations on the same variable. In the case of physical data, however, it is rather more obvious that adding quantities measured in grams to quantities measured in ounces is a futile procedure.” Peter Hill (1996; 32).

“Before the preparation of the 1993 SNA, issues connected with high or significant inflation had not been dealt with at all in international recommendations concerning national accounts. Uneasiness especially with the recording of nominal interest had been often expressed, for instance in Europe and North America at the time of two digit inflation and above all in countries, like in Latin America, experiencing high or hyper inflation. In relation with the latter situations, uneasiness extended to the whole set of accounts, because, due to the significant rate of inflation within each year, annual accounts in current values could no longer be deemed homogeneous as regards the level of prices in each year. They combine intra-annual flows that are valued at very different prices and are not, strictly speaking, additive. The effect of the intra-annual change in the general price level can be neglected for the sake of simplicity only when the rate of inflation is low. When it is high, the meaning of annual accounts in current values becomes fuzzy.” André Vanoli (1998).

“When inflation is high, the aggregation of flows from different periods becomes very much a case of ‘adding apples and bananas’— the flows at the end of the period will carry a much greater weight than the flows at the beginning of the period, so that the change on average will reflect development at the end of the period disproportionately. Annual national accounts at current prices become virtually meaningless and computation of national accounts at constant prices becomes very problematic.” Ezra Hadar and Soli Peleg (1998; 2).

Of course, concern over the effects of general price level change has a much longer history in the general cost accounting literature; see Baxter (1984), Tweedie and Whittington (1984) and Whittington (1992) for example.\footnote{The inflation accounting literature extends back to the accountant Middleditch: “Today’s dollar is, then, a totally different unit from the dollar of 1897. As the general price level fluctuates, the dollar is bound to become a unit of different magnitude. To mix these units is like mixing inches and centimeters or measuring a field with a rubber tape-line.” Livingston Middleditch (1918; 114-115).}

We now discuss in more detail the accounting problems caused by high inflation that are referred to in the above quotations. The basic problem is this: all discrete time economic theories and most of index number theory assumes that all of the transactions of a production unit with respect to a homogeneous commodity within the accounting period can be represented by a single price and a single quantity. It is natural to let the single quantity be the sum of the quantities sold (in the case of an output) or the sum of the quantities purchased (in the case of an input). But then, if we want the single price times the single quantity to equal the value of transactions for the commodity in the period, the single price must equal the value of transactions divided by the sum of quantities...
purchased or sold; i.e., the single price must equal a *unit value*. But when there is substantial inflation within the accounting period, unit values give a much higher weight to transactions that occur near the end of the period compared to transactions that occurred near the beginning; it is as if the end of period transactions are being *artificially quality adjusted* to be more valuable than the beginning of the period transactions.

The obvious solution to this artificial implicit weighting problem is to choose the accounting period to be small enough so that the general inflation within the period is small enough to be ignored. This is precisely the solution suggested by the index number theorist Fisher and the great measurement economist Hicks: the length of the accounting period should be the Hicksian “week”:

“I shall define a week as that period of time during which variations in price can be neglected.” John R. Hicks (1946; 122).

Thus it seems that there is a simple solution to the problem of constructing meaningful accounting period prices and quantities for homogeneous commodities when there is high inflation: simply shorten the accounting period!

Hill (1996) however notes that there are at least three classes of problems associated with the above solution:

“In order to keep these issues in perspective, it is useful to summarise the problems created by continually shortening the accounting period.

1. The compilation of accounts for shorter time periods requires more information about the times at which various transactions take place. Enquiries may have to be conducted more frequently thereby creating additional costs for the data collectors. More burdens are also placed on the respondents supplying the information. In many cases, they may be unable to supply the necessary information because their own internal records and accounts do not permit them to do so, especially when they traditionally report their accounts for longer time periods, such as a year.

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*8 The early index number theorists Walsh (1901; 96) (1921; 88), Fisher (1922; 318) and Davies (1924; 96) all suggested unit values as the prices that should be inserted into a bilateral index number formula. Walsh nicely sums up the case for unit values as follows: “Some nice questions arise as to whether only what is consumed in the country, or only what is produced in it, or both together are to be counted; and also there are difficulties as to the single price quotation that is to be given at each period to each commodity, since this, too, must be an average. Throughout the country during the period a commodity is not sold at one price, nor even at one wholesale price in its principle market. Various quantities of it are sold at different prices, and the full value is obtained by adding all the sums spent (at the same stage in its advance towards the consumer), and the average price is found by dividing the total sum (or the full value) by the total quantities.” Correa Moylan Walsh (1921; 88).

*9 “Essentially the same problem enters, however, whenever, as is usually the case, the data for prices and quantities with which we start are averages instead of being the original market quotations. Throughout this book, ‘the price’ of any commodity or ‘the quantity’ of it for any one year was assumed given. But what is such a price or quantity? Sometimes it is a single quotation for January 1 or July 1, but usually it is an average of several quotations scattered through the year. The question arises: On what principle should this average be constructed? The practical answer is any kind of average since, ordinarily, the variations during a year, so far, at least, as prices are concerned, are too little to make any perceptible difference in the result, whatever kind of average is used. Otherwise, there would be ground for subdividing the year into quarters or months until we reach a small enough period to be considered practically a point.” Irving Fisher (1922; 318).*
2. As production is a process which can extend over a considerable period of time, its measurement becomes progressively more difficult the shorter the accounting period. The problem is not confined to agriculture or forestry where many production processes take a year or more. The production of large fixed assets such as large ships, bridges, power stations, dams or the like can extend over several years. The output produced over shorter periods of time then has to be measured on the basis of work in progress completed each period. …

3. Because many transactions, especially large transactions, are not completed within the day, there are typically many receivables and payables outstanding at any given moment of time. They assume greater importance in relation to the flows as the accounting period is reduced. This makes it more difficult to reconcile the values of different flows in the accounts, especially if the two parties to the transaction perceive it as taking place at different times from each other and do not record it in the same way required by the system. … Peter Hill (1996; 34-35).

Thus shortening the accounting period leads to increased costs for the statistical agency and the businesses being surveyed. Moreover, firm accounting is geared to years and quarters and it may not be possible for production units to provide complete accounting information for periods shorter than a quarter. As the accounting period becomes shorter, it is less likely that production, shipment, billing and payment for the same commodity will all coincide within the accounting period. Also as the accounting period becomes shorter, work in progress will tend to become ever more important relative to final sales, creating difficult valuation problems.10 Put another way, more and more inputs will shift from being intermediate inputs (inputs that are used up within the accounting period) to being durable inputs (inputs whose contribution to production extends over more than one period). In addition to these difficulties, there are others. For example, as the accounting period becomes shorter, transactions tend to become more erratic and sporadic. Many goods will not be sold in a supermarket in a particular day or week. Normal index number theory breaks down under these conditions: it is difficult to compare a positive amount of a good sold in one period with a zero amount sold in the next period.11 A related difficulty is that many commodities are produced or demanded on a seasonal basis. If the accounting period is a year, then there are no seasonal commodity difficulties but as we shorten the period from a year, we will run into the problem of seasonal fluctuations in prices and quantities. In many cases, a seasonal commodity will not be available in all seasons and we again run into the problem of comparing positive values with zero values in the periods when the commodity is out of season. Even if the seasonal commodity does not disappear, the application of standard index number theory is not straightforward.12

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10 There are problems in constructing price indexes for work in progress! This is to be expected since there are very few transactions involving partially completed products. National income accountants solve this problem by using an appropriate construction price index or by construction an index of the materials and labour that are used to produce the work in progress. The problem with the latter strategy is that over time, the effects of technical progress in the industry cumulate and price indexes based on inputs used by the industry are unable to capture these effects.

11 We will attempt to deal with this problem in chapter III below.

12 Hadar and Peleg (1998; 5) comment on the importance of seasonal adjustment procedures in the context of high inflation: “As a by-product of the emphasis on quarterly estimates at constant prices the seasonal adjustment got large attention and many resources were spent to improve the adjustment.” Diewert (1996) (1998) (1999) reviews possible approaches to the problems involved in treating seasonal commodities (and suggests solutions) when there is high inflation.
Nevertheless, even in the face of the above difficulties, it seems that the only possible solution to the artificial implicit weighting problem that is generated by high inflation is to shorten the accounting period so that normal index number theory can be applied in order to construct meaningful economic aggregates.\textsuperscript{13}

In addition to the above general problems associated with economic measurement of flow variables under conditions of high inflation, there are some additional problems associated with the measurement of capital. These additional problems are associated with the stock and flow aspects of capital. We will conclude this section by explaining these problems.

Given an accounting period of some predetermined length, we can associate with it at least \textit{three separate points in time}:

- The beginning of the accounting period;
- The middle of the accounting period; and
- The end of the accounting period.

In interpreting the national accounts or the accounts of a business unit, we generally think of all flow variables as being concentrated in the middle of the period. If we follow this convention in the context of high inflation, then we require one (nominal) interest rate to index the value of money or financial capital going from the beginning of the period to the middle of the period and we require another (nominal) interest rate to index the value of money going from the middle of the period to the end of the period. Given these two interest rates, we could construct \textit{centered user costs of capital} for each type of reproducible capital, which would be the appropriate flow variables that would match up with the other flow variables in the production accounts of the production unit. However, in order to reduce the notational complexity of this annex, we do not construct centered user costs in what follows. Instead, for each type of asset, we construct either a \textit{beginning of the period user cost}, which measures the cost of using the asset for the period under consideration but under the assumption that payments and receipts for all flow variables are made at the beginning of the period, or an \textit{end of the period user cost}, which measures the cost of using the asset for the period under consideration but under the assumption that all inputs used and outputs produced are paid at the end of the period.\textsuperscript{14}

\textsuperscript{13} Our discussion in the previous paragraph indicates that this cannot be done if the economy is experiencing a hyperinflation. Thus meaningful economic measurement becomes impossible under very high inflation. This is a hidden cost of inflation that is not discussed very much in the literature on the costs of inflation.

\textsuperscript{14} Put another way, what we are doing is somewhat artificially moving the payments and receipts for the flow transactions of the production unit under consideration from the middle of the period (where they would occur on average if the revenue from the outputs sold was received as production was delivered and if the production unit paid inputs as they were used) to the beginning of the period or to the end of the period. Already, it can be seen that as soon as we have to deal with durable inputs, there are complications that do not arise with flow inputs, which are used up during the period of production. Hicks recognized that there were index number complications caused by the durable nature of capital that occurred \textit{within} the basic period: “In all probability these prices will have changed during the year, so that we have a kind of index number problem, parallel to the index number problem of comparing real income in different years.
In the following section, we explain the fundamental equations relating stocks and flows of capital.


Before we begin with our algebra, it seems appropriate to explain why accounting for the contribution of capital to production is more difficult than accounting for the contributions of labour or materials. The main problem is that when a reproducible capital input is purchased for use by a production unit at the beginning of an accounting period, we cannot simply charge the entire purchase cost to the period of purchase. Since the benefits of using the capital asset extend over more than one period, the initial purchase cost must be distributed somehow over the useful life of the asset. This is the fundamental problem of accounting. Hulten (1990) explained the consequences for accounting for the durability of capital as follows:

“Durability means that a capital good is productive for two or more time periods, and this, in turn, implies that a distinction must be made between the value of using or renting capital in any year and the value of owning the capital asset. This distinction would not necessarily lead to a measurement problem if the capital services used in any given year were paid for in that year; that is, if all capital were rented. In this case, transactions in the rental market would fix the price and quantity of capital in each time period, much as data on the price and quantity of labor services are derived from labor market transactions. But,
unfortunately, much capital is utilized by its owner and the transfer of capital services between owner and user results in an implicit rent typically not observed by the statistician. Market data are thus inadequate for the task of directly estimating the price and quantity of capital services, and this has led to the development of indirect procedures for inferring the quantity of capital, like the perpetual inventory method, or to the acceptance of flawed measures, like book value.” Charles R. Hulten (1990; 120-121).

Hicks made similar observations about the necessity for making imputations when attempting to value the contribution of a capital input to production during an arbitrarily chosen accounting period:

“Thus it seems true to say that while the valuation of income goods is characteristically a market valuation, the values of the goods which enter into the capital stock are characteristically imputed values. We cannot take over a market valuation for them; we have to set values upon them ourselves.” John R. Hicks (1961; 19).

The above quotations should alert us to the fact that the durability of capital inputs is going to lead to many measurement difficulties that are not present for goods and services whose contribution to production takes place within the accounting period.

Most economists agree that the value of an asset at the beginning of an accounting period is equal to the discounted stream of future rental payments that the asset is expected to yield. Thus the stock value of the asset is set equal to the discounted future service flows\textsuperscript{16} that the asset is expected to yield in future periods. Let the price of a new capital input purchased at the beginning of period t be \( P_0^t \). In a noninflationary environment, it is reasonable to assume that the (potentially observable) sequence of (cross sectional) rental prices classified by the age of the asset that prevails at the beginning of period t will prevail in future periods.\textsuperscript{17} Thus there is no need to have a separate notation for future expected rental prices for a new asset as it ages. However, in an inflationary environment, it is necessary to distinguish between the observable rental prices for the asset at different ages at the beginning of period t and future expected rental prices for assets of various ages.\textsuperscript{18} Thus let \( f_0^t \) be the (potentially observable) rental price of a new asset at the beginning of period t, let \( f_1^t \) be the (potentially observable) rental price of a one period old asset at the beginning of period t, let \( f_2^t \) be the (potentially observable) rental price of a 2 period old asset at the beginning of period t, etc. Then the fundamental equation relating the stock value of a new asset at the beginning of period t, \( P_0^t \), to the sequence of cross sectional rental prices by age prevailing at the beginning of period t, \( \{f_n^t : n = 0,1,2,\ldots\} \textsuperscript{19} \) is:

\[
(1) \quad P_0^t = f_0^t + \left[ \frac{(1+i_1^t)}{(1+r_1^t)} \right] f_1^t + \left[ \frac{(1+i_1^t)(1+i_2^t)}{(1+r_1^t)(1+r_2^t)} \right] f_2^t + \ldots
\]

\textsuperscript{16} Walras (1954) (first edition published in 1874) was one of the earliest economists to state that capital stocks are demanded because of the future flow of services that they render. Although he was perhaps the first economist to formally derive a user cost formula as we shall see, he did not work out the explicit discounting formula that Böhm-Bawerk (1891; 342) was able to derive.

\textsuperscript{17} This is an oversimplification because it neglects the obsolescence problem, which will be addressed in section 7 below and in chapters VI and VII.

\textsuperscript{18} Note that these future expected rental prices are not generally observable due to the lack of futures markets for these future rentals of the asset by the age of the asset.

\textsuperscript{19} The sequence of (cross sectional) rental prices at the beginning of period t for an asset that is n periods old, \( \{f_n^t : n = 0,1,2,\ldots\} \), is called the period t age efficiency profile of the asset.
In the above equation, \(1+i^t\) is the rental price escalation factor that is expected to apply to a one period old asset going from the beginning of period \(t\) to the end of period \(t\) (or equivalently, to the beginning of period \(t+1\)). \((1+i^1)(1+i^2)\) is the rental price escalation factor that is expected to apply to a 2 period old asset going from the beginning of period \(t\) to the beginning of period \(t+2\), etc. Thus the \(i^t\) are expected rental price by age inflation rates that are formed at the beginning of period \(t\). The term \(1+r^t\) is the discount factor that makes a dollar received at the beginning of period \(t\) equivalent to a dollar received at the beginning of period \(t+1\), the term \((1+r^t)(1+r^2)\) is the discount factor that makes a dollar received at the beginning of period \(t\) equivalent to a dollar received at the beginning of period \(t+2\), etc. Thus the \(r^t\) are one period nominal interest rates that represent the term structure of interest rates at the beginning of period \(t\).

Note that equation (1) assumes that the rentals are paid for at the beginning of each period. It should also be noted that Irving Fisher (1897; 365) seemed to be well aware of the complexities that are imbedded in equation (1): “There is not space here to discuss the theory in greater detail, nor to apply it to economic problems. A full treatment would take account of the various standards in which income is or may be expressed, of the case in which the rates of interest at different dates and for different periods does not remain constant, of the fact that the services of capital which are discounted in its value are only expected services, not those which actually materialise, and of the consequent discrepancy between income anticipated and income realised, of the propriety or impropriety of including man himself as a species of income-bearing capital, and so on.” Hicks (1939; 179) (1942; 177) and Hill and Hill (2003) provide additional discussions on the role of expectations in capital and income theory.

Peter Hill has noted a major problem with the use of equation (1) as the starting point of our discussion: namely, unique assets will by definition not be reproduced in future periods and so the cross sectional rental prices by age \(f_t^t\) will not exist for these assets! In this case, the \(f_t^t\) should be interpreted as expected future rentals that the unique asset is expected to generate at today’s prices. The \((1+i^t)\) terms then summarize expectations about the amount of asset specific price change that is expected to take place. Thus in this case, \((1+i^t)\) \(f_t^t\) is the period \(t+1\) nominal expected rental for a new unit of the durable that is expected to prevail at the beginning of period \(t+1\). This reinterpretation of equation (1) is more fundamental but we chose not to make it our starting point because it does not lead to a straightforward method for national statisticians to form reproducible estimates of these future rental payments. Hill (2000) works with this more general model and we will as well in chapter VI below.
These rental prices by age of asset prevailing at the beginning of period $t$, $f_0^t$, $f_1^t$, ... are potentially observable. These *cross sectional rental prices* reflect the relative efficiency of the various ages of the capital good under consideration at the beginning of period $t$. For now, we assume that these rentals are paid (explicitly or implicitly) by the users at the beginning of period $t$. Note that the sequence of used asset stock prices at the beginning of period $t$, $P_0^t$, $P_1^t$, ... is not affected by general inflation provided that the general inflation affects the expected asset inflation rates $i_n^t$ and the nominal interest rates $r_n^t$ in a proportional manner. We will return to this point later.

The physical productivity characteristics of a unit of capital of each age are determined by the sequence of cross sectional rental prices in our present model. Thus a brand new asset is characterized by the vector of current rental prices by age, $f_0^t$, $f_1^t$, $f_2^t$, ..., which are interpreted as “physical” contributions to output that the new asset is expected to yield during the current period $t$ (this is $f_0^t$), the next period (this is $f_1^t$), and so on. An asset which is one period old at the start of period $t$ is characterized by the vector $f_1^t$, $f_2^t$, ..., etc.

We have not explained how the expected rental price inflation rates $i_n^t$ are to be estimated. We shall deal with this problem in section 5 below. However, it should be noted that there is no guarantee that our expectations about the future course of rental prices are correct.

At this point, we make some simplifying assumptions about the expected rental inflation rates $i_n^t$ and the interest rates $r_n^t$. We assume that these anticipated vintage rental inflation factors at the beginning of each period $t$ are all equal; i.e., we assume:

\[(3) \quad i_n^t = i^t; \quad n = 1, 2, \ldots\]

We also assume that the term structure of interest rates at the beginning of each period $t$ is constant; i.e., we assume:

\[(4) \quad r_n^t = r^t; \quad n = 1, 2, \ldots\]

However, note that as the period $t$ changes, $r^t$ and $i^t$ can change.

Using assumptions (3) and (4), we can rewrite the system of equations (2), which relate the sequence or profile of *stock prices by age* at the beginning of period $t$ \{$P_n^t$\} to the sequence or profile of (cross sectional) *rental prices by age* at the beginning of period $t$ \{$f_n^t$\}, as follows:

\[(5) \quad P_0^t = f_0^t + \left[\frac{(1+i^t)}{(1+r^t)}\right] f_1^t + \left[\frac{(1+i^t)}{(1+r^t)}\right]^2 f_2^t + \left[\frac{(1+i^t)}{(1+r^t)}\right]^3 f_3^t + \ldots\]

---

22 This is the main reason that we use the vintage approach to capital measurement rather than the more fundamental discounted future expected rentals approach advocated by Hill and many other economists.

23 If they are paid at the end of the period, then we must discount these payments by an appropriate nominal interest rate. We consider this case later.

24 Triplett (1996; 97) used this characterization for capital assets of various vintages.
\[
\begin{align*}
P_1^t &= f_1^t + [(1+i^t)/(1+r^t)] f_2^t + [(1+i^t)/(1+r^t)]^2 f_3^t + [(1+i^t)/(1+r^t)]^3 f_4^t + \ldots \\
P_2^t &= f_2^t + [(1+i^t)/(1+r^t)] f_3^t + [(1+i^t)/(1+r^t)]^2 f_4^t + [(1+i^t)/(1+r^t)]^3 f_5^t + \ldots \\
\vdots \\
P_n^t &= f_n^t + [(1+i^t)/(1+r^t)] f_{n+1}^t + [(1+i^t)/(1+r^t)]^2 f_{n+2}^t + [(1+i^t)/(1+r^t)]^3 f_{n+3}^t + \ldots 
\end{align*}
\]

On the left hand side of equations (5), we have the sequence of asset prices by age at the beginning of period \(t\) starting with the price of a new asset, \(P_0^1\), moving to the price of an asset that is one period old at the start of period \(t\), \(P_1^1\), then moving to the price of an asset that is 2 periods old at the start of period \(t\), \(P_2^1\), and so on. On the right hand side of equations (5), the first term in each equation is a member of the sequence of rental prices by age that prevails in the market at the beginning of period \(t\). Thus \(f_0^t\) is the rent for a new asset, \(f_1^t\) is the rent for an asset that is one period old at the beginning of period \(t\), \(f_2^t\) is the rent for an asset that is 2 periods old, and so on. This sequence of current market rental prices for the assets of various ages is then extrapolated out into the future using the anticipated price escalation rates \(1+i^t\), \((1+i^t)^2\), \((1+i^t)^3\), etc. and then these future expected rentals are discounted back to the beginning of period \(t\) using the discount factors \(1+r^t\), \((1+r^t)^2\), \((1+r^t)^3\), etc. Note that given the period \(t\) expected asset inflation rate \(i^t\) and the period \(t\) nominal discount rate \(r^t\), we can go from the (cross sectional) sequence of rental prices \(\{f_n^t\}\) to the sequence of asset prices \(\{P_n^t\}\) using equations (5).

We shall show below how this procedure can be reversed; i.e., we shall show how given the sequence of asset prices by age at the beginning of period \(t\), we can construct estimates for the sequence of rental prices by age at the beginning of period \(t\).

It seems that Böhm-Bawerk was the first economist to use the above method for relating the future service flows of a durable input to its stock price:

“If the services of the durable good be exhausted in a short space of time, the individual services, provided that they are of the same quality— which, for simplicity’s sake, we assume— are, as a rule, equal in value, and the value of the material good itself is obtained by multiplying the value of one service by the number of services of which the good is capable. But in the case of many durable goods, such as ships, machinery, furniture, land, the services rendered extend over long periods, and the result is that the later services cannot be rendered, or at least cannot be rendered in a normal economic way, before a long time has expired. As a consequence, the value of the more distant material services suffers the same fate as the value of future goods. A material service, which, technically, is exactly the same as a service of this year, but which cannot be rendered before next year, is worth a little less than this year’s service; another similar service, but obtainable only after two years, is, again, a little less valuable, and so on; the values of the remote services decreasing with the remoteness of the period at which they can be rendered. Say that this year’s service is worth 100, then next year’s service— assuming a difference of 5 % per annum— is worth in today’s valuation only 95.23; the third year’s service is worth only 90.70; the fourth year’s service, 86.38; the fifth, sixth and seventh year’s services, respectively, worth 82.27, 78.35, 74.62 of present money.

The value of the durable good in this case is not found by multiplying the value of the current service by the total number of services, but is represented by a sum of services decreasing in value.” Eugen von Böhm-Bawerk (1891; 342).

Thus Böhm-Bawerk considered a special case of (5) where all service flows \(f_n\) were equal to 100 for \(n = 0, 1, \ldots, 6\) and equal to 0 thereafter, where the asset inflation rate was expected to be 0 and where the interest rate \(r\) was equal to .05 or 5 %.\(^{25}\) This is a special case of equations (5).
case of what has come to be known as the one hoss shay model and we shall consider it in more detail in Appendix A of this chapter.26

Note that equations (5) can be rewritten as follows:27

\[
\begin{align*}
(6) \ P_0^i &= f_0^i + [(1+i)/(1+r^i)] P_1^i ; \\
\ P_1^i &= f_1^i + [(1+i)/(1+r^i)] P_2^i ; \\
\ P_2^i &= f_2^i + [(1+i)/(1+r^i)] P_3^i ; \\
& \vdots \\
\ P_n^i &= f_n^i + [(1+i)/(1+r^i)] P_{n+1}^i ; \\
\end{align*}
\]

The first equation in (6) says that the value of a new asset at the start of period \( t \), \( P_0^i \), is equal to the rental that the asset can earn in period \( t \), \( f_0^i \), plus the expected asset value of the capital good at the end of period \( t \), \( (1+i) P_1^i \), but this expected asset value must be divided by the discount factor, \( (1+r^i) \), in order to convert this future value into an equivalent beginning of period \( t \) value.28

Now it is straightforward to solve equations (6) for the sequence of period \( t \) rental prices by age \( n \), \( \{f_n^i\} \), in terms of the sequence of asset prices by age \( n \), \( \{P_n^i\} \):

\[
\begin{align*}
(7) \ f_0^i &= P_0^i - [(1+i)/(1+r^i)] P_1^i = (1+r^i)^{-1} [P_0^i (1+r^i) - (1+i) P_1^i] \\
\ f_1^i &= P_1^i - [(1+i)/(1+r^i)] P_2^i = (1+r^i)^{-1} [P_1^i (1+r^i) - (1+i) P_2^i] \\
\ f_2^i &= P_2^i - [(1+i)/(1+r^i)] P_3^i = (1+r^i)^{-1} [P_2^i (1+r^i) - (1+i) P_3^i] \\
& \vdots \\
\ f_n^i &= P_{n+1}^i - [(1+i)/(1+r^i)] P_{n+1}^i = (1+r^i)^{-1} [P_n^i (1+r^i) - (1+i) P_{n+1}^i] ; \\
\end{align*}
\]

Thus equations (5) allow us to go from the sequence of rental prices by age \( \{f_n^i\} \) to the sequence of asset prices by age \( \{P_n^i\} \) while equations (7) allow us to reverse the process.

Equations (7) can be derived from elementary economic considerations. Consider the first equation in (7). Think of a production unit as purchasing a unit of the new capital asset at the beginning of period \( t \) at a cost of \( P_0^i \) and then using the asset throughout period \( t \). However, at the end of period \( t \), the producer will have a depreciated asset that is

---

26 Solow, Tobin, von Weizsäcker and Yaari (1966; 81) used the term “one hoss shay”.

27 Christensen and Jorgenson (1969; 302) do this for the geometric depreciation model except that they assume that the rental is paid at the end of the period rather than the beginning. Variants of the system of equations (6) were derived by Christensen and Jorgenson (1973), Jorgenson (1989; 10), Hulten (1990; 128) and Diewert and Lawrence (2000; 276). Irving Fisher (1908; 32-33) derived these equations in words as follows: “Putting the principle in its most general form, we may say that for any arbitrary interval of time, the value of the capital at its beginning is the discounted value of two elements: (1) the actual income accruing within that interval, and (2) the value of the capital at the close of the period.”

28 Note that we are implicitly assuming that the rental is paid to the owner at the beginning of period \( t \).

29 Another way of interpreting say the first equation in (6) runs as follows: the purchase cost of a new asset \( P_0^i \) less the rental \( f_0^i \) (which is paid immediately at the beginning of period \( t \)) can be regarded as an investment, which must earn the going rate of return \( r^i \). Thus we must have \[P_0^i - f_0^i]/(1+r^i) = (1+i)P_1^i\], which is the (expected) value of the asset at the end of period \( t \). This line of reasoning can be traced back to Walras (1954; 267): “A man who buys a house for his own use must be resolved by us into two individuals, one making an investment and the other consuming directly the services of his capital.”
expected to be worth \((1+i^t) P_1^t\). Since this offset to the initial cost of the asset will only be received at the end of period \(t\), it must be divided by \((1+r^t)\) to express the benefit in terms of beginning of period \(t\) dollars. Thus the net cost of using the new asset for period \(t\) is \(P_0^t - \left(\frac{(1+i^t)}{(1+r^t)}\right) P_1^t\).

The above equations assume that the actual or implicit period \(t\) rental payments \(f_n^t\) for assets of different ages \(n\) are made at the beginning of period \(t\). It is sometimes convenient to assume that the rental payments are made at the end of each accounting period. Thus define \(u_n^t\) as the \textit{end of period} \(t\) \textit{rental price or user cost} for an asset that is \(n\) periods old at the beginning of period \(t\). In this case, it can be seen that we can rewrite the system of equations (5), which relate the sequence of stock prices by age at the beginning of period \(t\) \(\{P_n^t\}\) to the sequence of end of period rental prices by age at the beginning of period \(t\) \(\{u_n^t\}\), as follows:

\[
(8) \quad P_0^t = u_0^t/(1+r^t) + \left[\frac{(1+i^t)}{(1+r^t)^2}\right] u_1^t + \left[\frac{(1+i^t)^2}{(1+r^t)^3}\right] u_2^t + \ldots \\
P_1^t = u_1^t/(1+r^t) + \left[\frac{(1+i^t)}{(1+r^t)^2}\right] u_2^t + \left[\frac{(1+i^t)^2}{(1+r^t)^3}\right] u_3^t + \ldots \\
P_2^t = u_2^t/(1+r^t) + \left[\frac{(1+i^t)}{(1+r^t)^2}\right] u_3^t + \left[\frac{(1+i^t)^2}{(1+r^t)^3}\right] u_4^t + \ldots \\
\ldots \\
P_n^t = u_n^t/(1+r^t) + \left[\frac{(1+i^t)}{(1+r^t)^2}\right] u_{n+1}^t + \left[\frac{(1+i^t)^2}{(1+r^t)^3}\right] u_{n+2}^t + \ldots
\]

where \(r^t\) is the relevant period \(t\) nominal interest rate or opportunity cost of capital facing the production unit and \(i^t\) is the period \(t\) anticipated end of period rental price escalation factor.

It can be seen that equations (8) can be rewritten in the form (5) if we convert the end of period rental prices \(u_n^t\) into corresponding beginning of period \(t\) rental prices \(f_n^t\), by defining the \(f_n^t\) as follows:

\[
(9) \quad f_n^t = u_n^t/(1+r^t) \quad ; \quad n = 0,1,2,\ldots
\]

Thus if the rental payment \(u_n^t\) is made at the end of the period instead of the beginning, then the corresponding beginning of the period rental \(f_n^t\) is equal to \(u_n^t\) divided by the discount factor \((1+r^t)^{31}\).

---

30 This explains why the rental prices \(f_n^t\) are sometimes called \textit{user costs}. This derivation of a user cost was used by Diewert (1974; 504), (1980; 472-473), (1992a; 194) and by Hulten (1996; 155). Diewert based his derivation on the general intertemporal model of production due to Hicks (1939) but specialized to one period.

31 It is interesting that Böhm-Bawerk (1891; 343) carefully distinguished between rental payments made at the beginning or end of a period: “These figures are based on the assumption that the whole year’s utility is obtained at once, and, indeed, obtained in anticipation at the beginning of the year; e.g., by hiring the good at a year’s interest of 100 payable on each 1st January. If, on the other hand, the year’s use can only be had at the end of the year, a valuation undertaken at the beginning of the year will show figures not inconsiderably lower. … That the figures should alter according as the date of the valuation stands nearer or farther from the date of obtaining the utility, is an entirely natural thing, and one quite familiar in financial life.”
Inserting equations (9) into the second set of equations in (7), it can be seen that the sequence of end of period $t$ user costs by age $n$, $\{u_{n}^{t}\}$, can be defined in terms of the period $t$ sequence of asset prices by age, $\{P_{n}^{t}\}$, as follows:

\[
\begin{align*}
(10) \quad u_{0}^{t} &= P_{0}^{t} (1+r^{t}) - (1+i^{t}) P_{1}^{t} \\
u_{1}^{t} &= P_{1}^{t} (1+r^{t}) - (1+i^{t}) P_{2}^{t} \\
u_{2}^{t} &= P_{2}^{t} (1+r^{t}) - (1+i^{t}) P_{3}^{t} \\
\vdots \\
u_{n}^{t} &= P_{n}^{t} (1+r^{t}) - (1+i^{t}) P_{n+1}^{t}; \ldots
\end{align*}
\]

Equations (10) can also be given a direct economic interpretation. Consider the following explanation for the user cost for a new asset, $u_{0}^{t}$. At the end of period $t$, the business unit expects to have an asset worth $(1+i^{t}) P_{1}^{t}$. Offsetting this benefit is the beginning of the period asset purchase cost, $P_{0}^{t}$. However, in addition to this cost, the business must charge itself either the explicit interest cost that occurs if money is borrowed to purchase the asset or the implicit opportunity cost of the equity capital that is tied up in the purchase. Thus offsetting the end of the period benefit $(1+i^{t}) P_{1}^{t}$ is the initial purchase cost and opportunity interest cost of the asset purchase, $P_{0}^{t} (1+r^{t})$, leading to an end of period $t$ net cost of $P_{0}^{t} (1+r^{t}) - (1+i^{t}) P_{1}^{t}$ or $u_{0}^{t}$. In chapter VII, we will explore these relationships in more depth.\(^{32}\)

It is interesting to note that in both the accounting and financial management literature of the past century, there was a reluctance to treat the opportunity cost of equity capital tied up in capital inputs as a genuine cost of production.\(^{33}\) However, more recently, there is an acceptance of an imputed interest charge for equity capital as a genuine cost of production.\(^{34}\)

In the following section, we will relate the asset price profiles $\{P_{n}^{t}\}$ and the user cost profiles $\{u_{n}^{t}\}$ to depreciation profiles. However, before turning to the subject of depreciation, it is important to stress that the analysis presented in this section is based on a number of restrictive assumptions, particularly on future price expectations. Moreover, we have not explained how these asset price expectations are formed and we have not explained how the period $t$ nominal interest rate is to be estimated (we will address these topics in sections 5 and 6 below). We have not explained what should be done if the sequence of second hand asset prices $\{P_{n}^{t}\}$ is not available and the sequences of rental prices or user costs by age, $\{f_{n}^{t}\}$ or $\{u_{n}^{t}\}$, are also not available (we will address this problem in chapter VI). We have also assumed that asset values and user costs are independent of how intensively the assets are used and these asset values and user costs are independent of the firm’s decisions about producing outputs and using other inputs.

\(^{32}\) For national income accounting purposes, the end of period user costs are more useful as we shall see in chapter VII.

\(^{33}\) This literature is reviewed in Diewert and Fox (1999; 271-274) and in chapter II below.

\(^{34}\) Stern Stewart & Co. has popularized this concept of EVA, Economic Value Added. In a newspaper advertisement in the Financial Post in 1999, it described this “new” concept as follows: “EVA measures your company’s after tax profits from operations minus the cost of all the capital employed to produce those profits. What makes EVA so revealing is that it takes into account a factor no conventional measures include: the cost of the operation’s capital— not just the cost of debt but the cost of equity capital as well.”
Finally, we have not modeled uncertainty (about future prices and the useful lives of assets) and attitudes towards risk on the part of producers. Thus the analysis presented in this chapter is only a start on the difficult problems associated with measuring capital input.

4. Relationships between Depreciation, Asset Prices and User Costs

Recall that in the previous section, $P_n^t$ was defined to be the price of an asset that was $n$ periods old at the beginning of period $t$. Generally, the decline in asset value as we go from an asset of a particular age to the next oldest at the same point in time is called depreciation. More precisely, we define the cross sectional depreciation $D_n^t$ of an asset that is $n$ periods old at the beginning of period $t$ as

$$D_n^t = P_n^t - P_{n+1}^t; n = 0,1,2,...$$

Thus $D_n^t$ is the value of an asset that is $n$ periods old at the beginning of period $t$, $P_n^t$, minus the value of an asset that is $n+1$ periods old at the beginning of period $t$, $P_{n+1}^t$.

Obviously, given the sequence of period $t$ asset prices by age, $\{P_n^t\}$, we can use equations (11) to determine the period $t$ sequence of declines in asset values by age, $\{D_n^t\}$. Conversely, given the period $t$ cross sectional depreciation sequence or profile, $\{D_n^t\}$, we can determine the period $t$ asset prices by age by adding up amounts of depreciation:

$$P_0^t = D_0^t + D_1^t + D_2^t + \ldots$$

$$P_1^t = D_1^t + D_2^t + D_3^t + \ldots$$

$$\ldots$$

$$P_n^t = D_n^t + D_{n+1}^t + D_{n+2}^t + \ldots$$

Rather than working with first differences of asset prices by age, it is more convenient to reparameterize the pattern of cross sectional depreciation amounts by defining the

---

35 This terminology is due to Hill (1999) who distinguished the decline in second hand asset values due to aging (cross section depreciation) from the decline in an asset value over a period of time (time series depreciation). Triplett (1996: 98-99) uses the cross sectional definition of depreciation and shows that it is equal to the concept of capital consumption in the national accounts but he does this under the assumption of no expected real asset inflation. The early accounting literature also defined depreciation as time series depreciation with the implicit assumption that the general price level was constant. We will examine the relationship of cross sectional to time series depreciation in section 7 below as well as in chapter VII.

36 Of course, the objections to the use of second hand market data to determine depreciation rates are very old: “We readily agree that where a market is sufficiently large, generally accessible, and continuous over time, it serves to coordinate a large number of subjective estimates and thus may impart a moment of (social) objectivity to value relations based on prices forced on it. But it can hardly be said that the second-hand market for industrial equipment, which would be the proper place for the determination of the value of capital goods which have been in use, satisfies these requirements, and that its valuations are superior to intra-enterprise valuation.” L.M. Lachmann (1941; 376-377). “Criticism has also been voiced about the viability of used asset market price data as an indicator of in use asset values. One argument, drawing on the Ackerlof Lemons Model, is that assets resold in second hand markets are not representative of the underlying population of assets, because only poorer quality units are sold when used. Others express concerns about the thinness of resale markets, believing that it is sporadic in nature and is dominated by dealers who under-bid.” Charles R. Hulten and Frank C. Wykoff (1996; 17-18).
sequence of period $t$ depreciation rates $\delta_{n}^{t}$ for an asset that is $n$ periods old at the start of period $t$ as follows:

$$(13) \quad \delta_{n}^{t} = 1 - \frac{P_{n+1}^{t}}{P_{n}^{t}} = \frac{D_{n}^{t}}{P_{n}^{t}}; \quad n = 0,1,2,\ldots$$

In the above definitions, we require $n$ to be such that $P_{n}^{t}$ is positive.$^{37}$

Obviously, given the sequence of period $t$ asset prices by age, $\{P_{n}^{t}\}$, we can use equations (13) to determine the period $t$ sequence of cross sectional depreciation rates, $\{\delta_{n}^{t}\}$. Conversely, given the sequence of period $t$ depreciation rates, $\{\delta_{n}^{t}\}$, as well as the price of a new asset in period $t$, $P_{0}^{t}$, we can determine the period $t$ asset prices by age as follows:

$$(14) \quad P_{1}^{t} = (1 - \delta_{0}^{t})P_{0}^{t}$$

$$P_{2}^{t} = (1 - \delta_{0}^{t})(1 - \delta_{1}^{t})P_{0}^{t}$$

$$\ldots$$

$$P_{n}^{t} = (1 - \delta_{0}^{t})(1 - \delta_{1}^{t})\ldots(1 - \delta_{n-1}^{t})P_{0}^{t}; \ldots$$

The interpretation of equations (14) is straightforward. At the beginning of period $t$, a new capital good is worth $P_{0}^{t}$. An asset of the same type but which is one period older at the beginning of period $t$ is less valuable by the amount of depreciation $\delta_{0}^{t}P_{0}^{t}$ and hence is worth $(1 - \delta_{0}^{t})P_{0}^{t}$, which is equal to $P_{1}^{t}$. An asset which is two periods old at the beginning of period $t$ is less valuable than a one period old asset by the amount of depreciation $\delta_{1}^{t}P_{1}^{t}$ and hence is worth $P_{2}^{t} = (1 - \delta_{1}^{t})P_{1}^{t}$ which is equal to $(1 - \delta_{1}^{t})(1 - \delta_{0}^{t})P_{0}^{t}$ using the first equation in (14) and so on. Suppose $L - 1$ is the first integer which is such that $\delta_{L-1}^{t}$ is equal to one. Then $P_{n}^{t}$ equals zero for all $n \geq L$; i.e., at the end of $L$ periods of use, the asset no longer has a positive rental value. If $L = 1$, then a new asset of this type delivers all of its services in the first period of use and the asset is in fact a nondurable asset.

Now substitute equations (14) into equations (10) in order to obtain the following formulae for the sequence of the end of the period user costs by age $\{u_{n}^{t}\}$ in terms of the price of a new asset at the beginning of period $t$, $P_{0}^{t}$, and the sequence of cross section depreciation rates, $\{\delta_{n}^{t}\}$:

$^{37}$ This definition of depreciation dates back to Hicks (1939) at least and was used extensively by Hulten and Wykoff (1981a) (1981b), Diewert (1974; 504) and Hulten (1990; 128) (1996; 155): “If there is a perfect second hand market for the goods in question, so that a market value can be assessed for them with precision, corresponding to each particular degree of wear, then the value-loss due to consumption can be exactly measured...” John R. Hicks (1939; 176). Current cost accountants have also advocated the use of second hand market data (when available) to calculate “objective” depreciation rates: “But as a practical matter the quantification and valuation of asset services used is not a simple matter and we must fall back on estimated patterns as a basis for current cost as well as historic cost depreciation. For those fixed assets which have active second hand markets the problem is not overly difficult. A pattern of service values can be obtained at any time by comparing the market values of different ages or degrees of use. The differences so obtained, when related to the value of a new asset, yield the proportions of asset value which are normally used up or foregone in the various stages of asset life.” Edgar O. Edwards and Philip W. Bell (1961; 175).
(15) \[ u^i_0 = [(1+r^i_0) - (1+i^i_0)(1-\delta^i_0)] P^i_0 \]
\[ u^i_1 = (1 - \delta^i_0)[(1+r^i_0) - (1+i^i_0)(1-\delta^i_1)] P^i_0 \]
\[ \vdots \]
\[ u^i_n = (1 - \delta^i_0) \cdots (1 - \delta^i_{n-1})[(1+r^i_n) - (1+i^i_0)(1-\delta^i_n)] P^i_0 \]

Thus given \( P^i_0 \) (the beginning of period t price of a new asset), \( i^i_0 \) (the new asset inflation rate that is expected at the beginning of period t), \( r^i_0 \) (the one period nominal interest rate that the business unit faces at the beginning of period t) and given the sequence of cross section vintage depreciation rates prevailing at the beginning of period t (the \( \delta^i_n \)), then we can use equations (15) to calculate the sequence of end of the period user costs by age of asset for period t, the \( u^i_n \). Of course, given the \( u^i_n \), we can use equations (8) to calculate the beginning of the period asset prices \( P^i_n \) and finally, given the \( P^i_n \), we can use equations (13) in order to calculate the sequence of depreciation rates by age of asset, the \( \delta^i_n \). Thus given any one of these sequences or profiles, all of the other sequences are completely determined. This means that assumptions about depreciation rates, the pattern of user costs by age or the pattern of asset prices by age cannot be made independently of each other.\(^{38}\)

It is useful to look more closely at the first equation in (15), which expresses the user cost or rental price of a new asset at the beginning of period t (but payment for the asset service is received at the end of period t), \( u^i_0 \), in terms of the depreciation rate \( \delta^i_0 \), the one period nominal interest rate \( r^i_0 \), the new asset inflation rate \( i^i_0 \) that is expected to prevail at the beginning of period t and the beginning of period t price for a new asset, \( P^i_0 \):\

\[ (16) \quad u^i_0 = [(1+r^i_0) - (1+i^i_0)(1-\delta^i_0)] P^i_0 = [r^i_0 - i^i_0 + (1+i^i_0)\delta^i_0] P^i_0 \]

Thus the user cost of a new asset \( u^i_0 \) that is purchased at the beginning of period t (and the actual or imputed rental payment is made at the end of the period) is equal to \( r^i_0 - i^i_0 \) (a nominal interest rate minus an asset inflation rate which can be loosely interpreted\(^{39}\) as a real interest rate) times the initial asset cost \( P^i_0 \) plus \( (1+i^i_0)\delta^i_0 P^i_0 \) which is depreciation on the asset at beginning of the period prices, \( \delta^i_0 P^i_0 \), times the inflation escalation factor, \((1+i^i_0)\).\(^{40}\) If we further assume that the expected asset inflation rate is 0, then (16) further simplifies to:

---

\(^{38}\) This point was first made explicitly by Jorgenson and Griliches (1967; 257): “An almost universal conceptual error in the measurement of capital input is to confuse the aggregation of capital stock with the aggregation of capital service.” See also Jorgenson and Griliches (1972; 81-87). Much of the above algebra for switching from one method of representing capital inputs by age of asset to another was first developed by Christensen and Jorgenson (1969; 302-305) (1973) for the geometrically declining depreciation model. The general framework for an internally consistent treatment of capital services and capital stocks in a set of vintage accounts was set out by Jorgenson (1989) and Hulten (1990; 127-129) (1996; 152-160).

\(^{39}\) We will provide a more precise definition of a real interest rate later.

\(^{40}\) This formula was obtained by Christensen and Jorgenson (1969; 302) for the geometric model of depreciation but it is valid for any depreciation model. Griliches (1963; 120) also came very close to deriving this formula in words: “In a perfectly competitive world the annual rent of a machine would equal the marginal product of its services. The rent itself would be determined by the interest costs on the
(17) \( \omega_t = [r^t + \delta_0^t] P_0^t \).

Under these assumptions, the user cost of a new asset is equal to the interest rate plus the depreciation rate times the initial purchase price.\(^{41}\) This is essentially the user cost formula that was obtained by Walras in 1874:

> “Let \( P \) be the price of a capital good. Let \( p \) be its gross income, that is, the price of its service inclusive of both the depreciation charge and the insurance premium. Let \( \mu P \) be the portion of this income representing the depreciation charge and \( \nu P \) the portion representing the insurance premium. What remains of the gross income after both charges have been deducted, \( \pi = p - (\mu + \nu)P \), is the net income. We are now able to explain the differences in gross incomes derived from various capital goods having the same value, or conversely, the differences in values of various capital goods yielding the same gross incomes. It is, however, readily seen that the values of capital goods are rigorously proportional to their net incomes. At least this would have to be so under certain normal and ideal conditions when the market for capital goods is in equilibrium. Under equilibrium conditions the ratio \([p - (\mu + \nu)P]/P\), or the rate of net income, is the same for all capital goods. Let \( i \) be this common ratio. When we determine \( i \), we also determine the prices of all landed capital, personal capital and capital goods proper by virtue of the equation \( p - (\mu + \nu)P = iP \) or \( P = p/[i + \mu + \nu] \).” Léon Walras (1954; 268-269).

However, the basic idea that a durable input should be charged a period price that is equal to a depreciation term plus a term that would cover the cost of financial capital goes back much further\(^{42}\). For example, consider the following quotation from Babbage:

> “Machines are, in some trades, let out to hire, and a certain sum is paid for their use, in the manner of rent. This is the case amongst the frame-work knitters: and Mr. Hensen, in speaking of the rate of payment for the use of their frames, states, that the proprietor receives such a rent that, besides paying the full interest for his capital, he clears the value of his frame in nine years. When the rapidity with which improvements succeed each other is considered, this rent does not appear exorbitant. Some of these frames have been worked for thirteen years with little or no repair.” Charles Babbage (1835; 287).

Babbage did not proceed further with the user cost idea. Walras seems to have been the first economist who formalized the idea of a user cost into a mathematical formula. However, the early industrial engineering literature also independently came up with the user cost idea; Church described how the use of a machine should be charged as follows:

> “No sophistry is needed to assume that these charges are in the nature of rents, for it might easily happen that in a certain building a number of separate little shops were established, each containing one machine, all making some particular part or working on some particular operation of the same class of goods, but each shop occupied, not by a wage earner, but by an independent mechanic, who rented his space, power and machinery, and sold the finished product to the lessor. Now in such a case, what would be the shop charges of these mechanics? Clearly they would comprise as their chief if not their only item, just the rent paid. And this rent would be made up of: (1) Interest. (2) Depreciation. (3) Insurance. (4) Profit on the capital involved in the building, machine and power-transmitting and generating plant. There would also

\(^{41}\) Using equations (14) and (15) and the assumption that the asset inflation rate \( i^t = 0 \), it can be shown that the user cost of an asset that is \( n \) periods old at the start of period \( t \) can be written as \( \omega_n^t = (r^t + \delta_n^t)P_n^t \) where \( P_n^t \) is the beginning of period \( t \) second hand market price for the asset.

\(^{42}\) Solomons (1968; 9-17) indicates that interest was regarded as a cost for a durable input in much of the nineteenth century accounting literature. The influential book by Garcke and Fells (1893) changed this.
most probably be a separate charge for power according to the quantity consumed. Exclude the item of profit, which is not included in the case of a shop charge, and we find that we have approached most closely to the new plan of reducing any shop into its constituent production centres. No one would pretend that there was any insuperable difficulty involved in fixing a just rent for little shops let out in this plan.” A. Hamilton Church (1901; 907-908).

“A production centre is, of course, either a mechanic, or a bench at which a hand craftsman works. Each of these is in the position of a little shop carrying on one little special industry, paying rent for the floor space occupied, interest for the capital involved, depreciation for the wear and tear, and so on, quite independently of what may be paid by other production centres in the same shop.” A. Hamilton Church (1901; 734).

Church was well aware of the importance of determining the “right” rate to be charged for the use of a machine in a multiproduct enterprise. This information is required not only to price products appropriately but to determine whether an enterprise should make or purchase a particular commodity. Babbage and Canning were also aware of the importance of determining the right machine rate charge.43

43 Under moderate inflation, the difficulties with traditional cost accounting based on historical cost and no proper allowance for the opportunity of capital, the proper pricing of products becomes very difficult. Diewert and Fox (1999; 271-274) argued that this factor contributed to the great productivity slowdown that started around 1973 and persisted to the early 1990’s. The traditional method of cost accounting can be traced back to a book first published in 1887 by the English accountants, Garcke and Fells, who suggested allocating the “indirect costs” of producing a good proportionally to the amount of labour and materials costs used to make the item: “In some establishments the direct expenditures in wages and materials only is considered to constitute the cost; and no attempt is made to allocate to the various working or stock orders any portion of the indirect expenses. Under this system the difference between the sum of the wages and materials expended on the articles and their selling price constitutes the gross profit, which is carried in the aggregate to the credit of profit and loss, the indirect factory expenses already referred to, together with the establishment expenses and depreciation, being particularised on the debit side of that account. This method has certainly simplicity in its favour, but a more efficient check upon the indirect expenses would be obtained by establishing a relation between them and the direct expenses. This may be done by distributing all the indirect expenses, such as wages of foremen, rent of factory, fuel, lighting, heating, and cleaning, etc. (but not the salaries of clerks, office rent, stationery and other establishment charges to be referred to later), over the various jobs, as a percentage, either upon the wages expended upon the jobs respectively, or upon the cost of both wages and materials.” Emile Garcke and John Manger Fells (1893; 70-71). Compare this rather crude approach to cost accounting to the masterful analysis of Church! Garcke and Fells endorsed the idea that depreciation was an admissible item of cost that should be allocated in proportion to the prime cost (i.e., labour and materials cost) of manufacturing an article but they explicitly ruled out interest as a cost: “The item of Depreciation may, for the purpose of taking out the cost, simply be included in the category of the indirect expenses of the factory, and be distributed over the various enterprises in the same way as those expenses may be allocated; or it may be dealt with separately and more correctly in the manner already alluded to and hereafter to be fully described. The establishment expenses and interest on capital should not, however, in any case form part of the cost of production. There is no advantage in distributing these items over the various transactions or articles produced. They do not vary proportionately with the volume of business. … The establishment charges are, in the aggregate, more or less constant, while the manufacturing costs fluctuate with the cost of labour and the price of material. To distribute the charges over the articles manufactured would, therefore, have the effect of disproportionately reducing the cost of production with every increase, and the reverse with every diminution, of business. Such a result is greatly to be deprecated, as tending to neither economy of management nor to accuracy in estimating for contracts. The principles of a business can always judge what percentage of gross profit upon cost is necessary to cover fixed establishment charges and interest on capital.” Emile Garcke and John Manger Fells (1893; 72-73). The aversion of accountants to include interest as a cost can be traced back to this quotation.
“The great competition introduced by machinery, and the application of the principle of the subdivision of labour, render it necessary for each producer to be continually on the watch, to discover improved methods by which the cost of the article he manufactures may be reduced; and, with this view, it is of great importance to know the precise expense of every process, as well as of the wear and tear of machinery which is due to it.” Charles Babbage (1835; 203).

“The question of ‘adequate’ rates of depreciation, in the sense that they will ultimately adjust the valuations to the realities, is often discussed as though it had no effect upon ultimate profit at all. Of some modes of valuing, it is said that they tend to overvalue some assets and to undervalue others, but the aggregate of book values found is nearly right. If the management pay no attention at all to the unit costs implied in such valuations, no harm is done. But if the cost accountant gives effect to these individually bad valuations through a machine-rate burden charge, and if the selling policy has regard for apparent unit profits, the valuation may lead to the worst rather than to the best possible policy.” John B. Canning (1929; 259-260).

The above equations relating asset prices by age \( P_n \), beginning of the period user costs \( f_n \), end of the period user costs \( u_n \) and the (cross sectional) depreciation rates by age of asset \( \delta_n \) are the fundamental ones that we will specialize in Appendix A below in order to measure both capital stocks and capital services under conditions of inflation. In the following section, we shall consider several options that could be used in order to determine empirically the interest rates \( r_t \) and the asset inflation rates \( i_t \) that appear in these user cost formulae.

Note that all of the algebra developed above can be applied not only to reproducible capital stock components (equipment and machinery and structures) but also to stocks of inventories and land that are used by the production unit. Typically, we set the depreciation rates for inventory and land components equal to zero. This is only approximately correct, since theft and spoilage of some inventory components can give rise to positive depreciation rates and environmental degradation could be regarded as a depreciation component for some land stocks.

5. The Empirical Determination of Interest Rates and Asset Inflation Rates

What is the “correct” nominal interest rate \( r_t \) that should be used in the various user cost formulae that were developed in the previous section? We consider eight theoretical approaches that might be used to answer this question.

If the production unit raises financial capital by a combination of debt and equity financing, then it would seem to be appropriate to choose the reference nominal interest rate \( r_t \) for a particular period \( t \) to be a weighted average of its anticipated period cost of debt and equity for that period. Since determining the average interest rate for debt would seem to be a reasonably straightforward exercise,\(^4\) in the first two approaches, we will focus on various alternative approaches that have been suggested in the literature for the determination of the equity opportunity cost of capital. In the subsequent approaches,

\(^4\) If all bonds were one period bonds, then there would be no major problems. However, when some debt is floated using multiperiod bonds, there are some difficult problems involved in determining the appropriate period by period interest rate when there are changes in the market price of the bond over time.
we look at methods that have been suggested to determine a relevant opportunity cost of capital for the entire stock of nonfinancial capital held by the firm.

**Approach 1: Discounted Cash Flow**

Suppose that a company’s current period dividends $D_t$ are expected to grow at the constant real rate $g$ for the indefinite future and that the expected inflation rate for the indefinite future is $\rho$. The company's current share price $S_t$ should equal the discounted future expected dividends. The discount rate should be the long run cost of equity capital $r_t$ minus the anticipated inflation rate $\rho$. Under these assumptions, we should have the following relationship between the company's current share price $S_t$ and current dividend rate $D_t$:

$$S_t = D_t/(r_t - g - \rho).$$

Formula (18) can be rearranged to give the following formula for the cost of equity capital:

$$(19) \ r_t = (D_t/S_t) + g + \rho.$$

This method for determining the opportunity cost of equity capital is due to Williams (1938) and Gordon and Shapiro (1956). According to Myers (1992; 489), this method is widely used to determine allowed equity rates of return for regulated utilities in the United States.

There are many problems with this method. The determination of the anticipated future inflation rate $\rho$ will be problematical given that past inflation rates have been very variable during the past three decades. Dividend growth rates are also variable over the business cycle. Finally, dividend price ratios of the form $D_t/S_t$ are also tremendously variable and moreover, this method is not suitable for the determination of an economy wide equity cost of capital since many businesses are not incorporated and many incorporated businesses do not have publicly traded shares.

**Approach 2: The Capital Asset Pricing Model**

Under certain assumptions, the *capital asset pricing model* of Sharpe (1964), Lintner (1965) and Mossin (1966) yields the following relationship between the expected cost of capital for a company $r_t^e$, a safe or riskless interest rate $r_s^t$ and the expected return on a market portfolio of assets $r_m^t$:

$$r_e^t = r_s^t + \beta (r_m^t - r_s^t),$$

where $\beta$ is the covariance between the company’s equity rate of return and the market portfolio rate of return divided by the variance of the market portfolio rate of return. Given a time series of ex post company rates of return $r_t^e$, market rates of return $r_m^t$ and the safe rate of return $r_s^t$, ex post returns can be substituted into (20) in place of the
anticipated returns $r_c^t$ and $\beta$ can be estimated in a regression model.\footnote{For an example of this approach, see Nagorniak (1972; 351).} Alternatively, an estimate for $\beta$ can be constructed by taking an average of past covariances $\text{Cov}(r_c^t; r_m^t)$ divided by $\text{Var}(r_m^t)$. Given this estimator for $\beta$, an ex ante $r_c^t$ can be calculated as the right hand side of (20) where $r_m^t$ is a forecast for the period $t$ market rate of return.

Some of the assumptions that are required to derive (20) are: (i) each investor is a von Neumann and Morgenstern (1947) expected utility maximizer with the same preferences over current period consumption and end of the period wealth; (ii) a riskless one period asset actually exists; (iii) all investors have preferences over the same set of risky assets and the common riskless asset; (iv) all investors have the same expectations about the returns, variances and covariances of the risky assets and (v) there are no transactions costs. All of these assumptions are somewhat suspect from the empirical point of view. Machina (1992; 860-862) documents some of the empirical evidence which contradicts the expected utility model. In particular, Mehra and Prescott (1985) show that the equity premium over the safe asset seems to be too large for generally agreed upon values of relative risk aversion. Epstein and Zinn (1990) explain this equity premium by a generalization of the usual expected utility model that allows for first order risk aversion.\footnote{For more recent discussions of the equity premium puzzle, see Benninga and Protopapadakis (1992; 770), Burnside and McCurdy (1992) and Diewert (1993a; 427-432).} Assumption (ii), the assumption that a perfectly safe one period asset exists, is also problematical: nominal government bonds are not risk free due to inflation risk. Assumption (iii) is also dubious: what is the relevant set of risky assets facing any investor? Should we include housing or foreign stock markets? Our $\beta$ estimates will generally change as we change our definition of the “market” for risky assets. Assumption (iv) is also problematical: what will happen to our estimate for $\beta$ as we include or exclude data for 1987, the year of the great worldwide stock market crash? Finally, assumption (v) is also far from being satisfied. Although the capital asset pricing model could be used to estimate the cost of equity capital for some companies whose shares are traded in a stock market, it cannot be used to estimate the cost of equity capital for many companies and for the economy as a whole since a large proportion of private sector companies are not listed on any stock exchange.

**Approach 3: The Ex Post Return Method**

To illustrate this approach, we suppose for simplicity that the production unit uses only $K_0^t$ units of a new capital asset at the beginning of period $t$ and $u_0^t$ defined by (16) above is the corresponding user cost concept that we wish to use. Suppose further that we have somehow estimated the relevant depreciation rate $\delta_0^t$ as well as the expected rate of asset price inflation $i^t$. Let CF$^t$ be the period $t$ cash flow for the production unit.\footnote{Cash flow for period $t$ is defined as the value of goods and services sold during period $t$ less the value of intermediate inputs used in period $t$ less the gross value of labour costs. In national income accounting terminology, cash flow is called gross operating surplus.} The period $t$ ex post return to capital for this production unit, $r_c^t$, can be obtained by setting cash flow equal to the value of capital services and then solving for the balancing $r_c^t$; i.e., we solve the following equation for $r_c^t$:  

\footnote{For an example of this approach, see Nagorniak (1972; 351).}
\[(21) \text{CF}^t = [r^t - i^t + (1 + i^t)\delta_0] P_0^t K_0^t.\]

The resulting \(r^t\) is called the *ex post nominal rate of return to capital employed* and it could be used as the period \(t\) reference opportunity cost of capital for this production unit.

The ex post cost of capital method for determining the opportunity cost of capital that is based on solving equation (21) (and its generalizations to the case of many assets) for \(r^t\) is due to Jorgenson and Griliches (1967) (1972) and Christensen and Jorgenson (1969).\(^{48}\) This method has been used frequently in the regulatory context, starting with Christensen, Schoech and Meitzen (1995). In addition to the simplicity of this method, Christensen, Schoech and Meitzen (1995; 10) note that this method can be applied in a symmetric manner to both a single enterprise as well as to the economy as a whole. National statistical agencies that have programs that measure the productivity of market sector industries generally use this method.\(^{49}\) From a national income accounting perspective, this method has the great advantage for statistical agencies that it preserves the present structure of the System of National Accounts 1993; i.e., the resulting user costs just sum to the present Gross Operating Surplus that is already in SNA 1993. Thus this method can be viewed as a straightforward elaboration of the present system of accounts which does not change its basic structure; it only provides a decomposition of Gross Operating Surplus or Cash Flow into more basic components; see (21) above.\(^{50}\)

The problem with the ex post return method for estimating \(r^t\) is that it does not correspond to a true opportunity cost of capital for the business unit; instead, it corresponds to an ex post measure of period \(t\) *performance* for the business unit.\(^{51}\) In the present chapter, what we are attempting to do is to find a way to construct an approximation to a market rental price for an asset (using information on used asset prices) when information on market rental prices is not available. Put another way, what we are doing is looking at the factors that would assist an owner of an asset to decide what price to charge for the services of the asset in the rental market. Hence, our measure of user cost should be *forward looking* or an *ex ante measure* rather than an ex post measure of how things actually turned out during the period. Hicks had some relevant thoughts on the issue of ex ante versus ex post measures:

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\(^{48}\) These authors set their expected period \(t\) asset inflation rates \(i^t\) equal to the ex post asset inflation rate that occurred over period \(t\). This use of ex post asset inflation rates will generally lead to very volatile user costs and even negative user costs in many cases which means that these user costs are not suitable for some purposes. Some official productivity programs replace negative user costs by tiny positive ones. We discuss these issues below.

\(^{49}\) The Bureau of Labor Statistics (1983) in the U.S. was the first country to introduce an official program to measure Multi Factor Productivity or Total Factor Productivity; see Dean and Harper (2001) Other countries with TFP programs now include Canada, Australia, the UK and New Zealand.

\(^{50}\) However, this advantage of this ex post method for determining an endogenous balancing rate of return is not decisive: if an exogenous \(r^t\) were used in the user cost formula, we could simply add another line to the list of primary inputs called “pure profits or losses” and have a reconciling item that would allow gross output less intermediate input to equal the sum of primary input payments.

\(^{51}\) See chapter VIII below.
“Ex post calculations of capital accumulation have their place in economic and statistical history; they are a useful measuring rod for economic progress; but they are of no use to theoretical economists, who are trying to find out how the economic system works, because they have no significance for conduct. The income ex post of any particular week cannot be calculated until the end of the week, and then it involves a comparison between present values and values which belong wholly to the past. On the general principle of ‘bygones are bygones’, it can have no relevance to present decisions.” J.R. Hicks (1939; 179).

Thus in our present context where we are trying to construct counterparts to market rental rates for durable inputs, ex ante measures are preferred to ex post measures (which could be useful in evaluating ex post economic performance—a different context).

**Approach 4: The Average of Past Ex Post Nominal Returns**

Instead of using the ex post returns to capital employed method outlined in Approach 3 above, we could switch to a forecasting framework, using some sort of an average of past ex post returns to capital to forecast a current period opportunity cost of capital. The problem with methods of this type is their arbitrariness: which ex post approach to the determination of the opportunity cost of capital should be used? Which forecasting method should be used? How far back in time should we go? It may be difficult to reach agreement on what is the most reasonable specific method in this general class of methods. Statistical agencies will be reluctant to use this method because of its lack of reproducibility: different statistical agencies would no doubt use somewhat different forecasting or smoothing methods and obtain different results—perhaps leading to a lack of international comparability in the System of National Accounts. Moreover, as we argued at the end of Approach 3 above, ex post returns incorporate pure profits (or losses) and hence are not true ex ante opportunity costs for equity capital. Nevertheless, over long periods of time, averages of ex post rates of return should approximate an ex ante rate of return that is appropriate to use in a user cost formula, at least when general inflation is not too variable.

**Approach 5: The Use of An Exogenous Nominal Market Interest Rate**

In this method, a relevant market interest rate is used as a proxy for the equity opportunity cost of capital. This market interest rate could be: (i) the prime lending rate that banks or other financial institutions charge borrowers in “similar” lines of business; or (ii) an index of ex ante interest rates constructed by some reputable private or public agency. As an example of (ii), Christensen, Schoech and Meitzen (1995) used the Moody’s public utility bond as a proxy for the cost of capital for a regulated utility. These authors noted that this method has the advantages that the Moody bond yield is publicly available and is updated annually.

This method has some attractions for both regulators and national statistical agencies since it is simple and (somewhat) reproducible.

**Approach 6: The Use of An Official Nominal Rate of Return**
In this approach, a government or regulatory agency would set an “official” interest rate that could be used to approximate a business unit’s cost of equity capital. For example, the official rate might be: (i) the interest rate that is used by the taxation authorities to assess late payment of income taxes; (ii) an equity interest rate that is recommended by the country’s accounting standards board or (iii) the midpoint of a regulator’s range of acceptable returns to equity capital for a regulated firm. A problem with this method is that there is no guarantee that the official rate set by a taxation authority, accounting standards board or regulator will be “reasonable”; i.e., this method gives no guidance on how the authority will in fact determine the “official” rate. In practice, official rates determined by the tax authorities are probably based on Approach 5(ii) outlined above.

**Approach 7: The Use of an Official Real Rate of Return**

This approach kills two birds with one stone; i.e., it determines not only the reference nominal interest rate $r^i$ but it also determines the anticipated asset inflation rate $i^t$ that appears in the user cost formulae developed above.

This approach works as follows. Let the consumer price index for the economy at the beginning of period $t$ be $c^t$ say. Then the ex post general consumer inflation rate for period $t$ is $\rho^t$ defined as:

\[
1 + \rho^t = \frac{c^{t+1}}{c^t}.
\]

We assume that the production unit under consideration faces the constant real interest rate $r^*$. Then by the Fisher (1896) effect$^{52}$, the relevant period $t$ nominal interest rate that the producer faces should be approximately equal to $r^i$ defined as follows:

\[
1 + r^i = (1+r^*)(1+\rho^t) - 1.
\]

The Australian Bureau of Statistics assumes that producers face an annual real interest rate of 4%, so that $r^* = .04$. This is consistent with long run observed economy wide real rates of return for most OECD countries which fall in the 2 to 5 per cent range.

The final assumption made in order to implement this approach is to assume that the producer anticipates that each asset inflation rate is equal to the general inflation rate $\rho^t$ defined by (22); i.e., we assume:

\[
i^t = \rho^t.
\]

With this assumption, the equations relating stock and flow prices simplify dramatically. Thus if we replace $1+i^t$ by $1+\rho^t$ and $1+r^t$ by $(1+r^*)(1+\rho^t)$, equations (5), which relate the vintage asset prices $P^n_t$ to the vintage rental prices $f^n_t$, become:

\[52\] The Fisher effect was independently derived by several economists: “Thornton, Marshall, Wicksell, Fisher, Keynes and others have known that the own rate and money rate of interest must diverge by a term equal to the percentage price of the good in terms of which the own rate is measured.” Paul A Samuelson (1961; 40).
(25) \( P_0^t = f_0^t + \frac{1}{1/(1+r^*)} f_1^t + \frac{1}{1/(1+r^*)} f_2^t + \frac{1}{1/(1+r^*)} f_3^t + \ldots \)
\( P_1^t = f_1^t + \frac{1}{1/(1+r^*)} f_2^t + \frac{1}{1/(1+r^*)} f_3^t + \frac{1}{1/(1+r^*)} f_4^t + \ldots \)
\[ \vdots \]
\( P_n^t = f_n^t + \frac{1}{1/(1+r^*)} f_{n+1}^t + \frac{1}{1/(1+r^*)} f_{n+2}^t + \frac{1}{1/(1+r^*)} f_{n+3}^t + \ldots \)

Note that only the constant real interest rate \( r^* \) appears in these equations.

If we substitute equations (14) into equations (7) and replace \( 1+i^t \) by \( 1+\rho^t \) and \( 1+r^t \) by \( (1+r^*)(1+\rho^t) \) in the resulting equations, we obtain the following equations, which relate the beginning of period user costs by age \( f_n^t \) to the price of a new asset \( P_0^t \), the real interest rate \( r^* \) and the asset depreciation rates by age \( \delta_n^t \):

(26) \( f_0^t = (1+r^*)^{-1}[r^* + \delta_0^t] P_0^t \)
\( f_1^t = (1+r^*)^{-1}(1-\delta_0^t)[r^* + \delta_1^t] P_0^t \)
\[ \vdots \]
\( f_n^t = (1+r^*)^{-1}(1-\delta_0^t)\ldots(1-\delta_{n-1}^t) \left[ r^* + \delta_n^t \right] P_0^t ; \ldots \)

Note that only the constant real interest rate \( r^* \) appears in equations (26) and the anticipated asset inflation rates \( \rho^t \) have disappeared.

Now replace \( 1+i^t \) by \( 1+\rho^t \) and \( 1+r^t \) by \( (1+r^*)(1+\rho^t) \) in equations (15) in order to obtain the following formulae for the sequence of the end of period user costs by age \( \{u_n^t\} \) in terms of the real interest rate \( r^* \), the price of a new asset at the beginning of period \( t \), \( P_0^t \), the sequence of cross section depreciation rates, \( \{\delta_n^t\} \) and the period \( t \) general inflation rate \( \rho^t \):

(27) \( u_0^t = (1+\rho^t) \left[ r^* + \delta_0^t \right] P_0^t \)
\( u_1^t = (1+\rho^t)(1-\delta_0^t)[r^* + \delta_1^t] P_0^t \)
\[ \vdots \]
\( u_n^t = (1+\rho^t)(1-\delta_0^t)\ldots(1-\delta_{n-1}^t) [r^* + \delta_n^t] P_0^t ; \ldots \)

Comparing (26) and (27), we see that it is slightly easier to work with the \( f_n^t \) rather than the \( u_n^t \), since the former user costs do not contain the nuisance variable \( \rho^t \), the general inflation rate.\(^{53}\)

Although the assumptions that were made to derive equations (26) and (27) were quite strong, the end result is very attractive to statistical agencies that may want a simple reproducible approach to the determination of user costs. The user costs defined by (26) are particularly attractive because they will always be positive and they will be relatively smooth; the main source of intertemporal variation is the variation in the beginning of the period stock price of a new asset, \( P_0^t \).

**Approach 8: The Use of a Long Run Average Ex Post Real Rate of Return**

\(^{53}\) However, for most practical accounting purposes, the end of period user costs will generally be preferred.
Approach 7 can be combined with Approach 4 to give us approach 8. Make the same simplifying assumptions that we made in Approach 3 except redefine the period t cash flow $CF^t$ of the business unit to be realized at the beginning of period t. Then determine the period t balancing ex post real rate of return for the business unit, $r^{*t}$, to be the real interest rate that solves the following counterpart to (21):

\[
(28) \quad CF^t = f_0^t K_0^{t} = (1+r^{*t})^{-1} [r^{*t} + \delta_0^t] P_0^t K_0^{t} \text{ or }
\]

\[
(29) \quad CF^t (1+r^{*t}) = [r^{*t} + \delta_0^t] P_0^t K_0^{t}.
\]

Note that equation (29) is linear in the balancing real rate of return, $r^{*t}$, and so it is easy to determine this period t balancing $r^{*t}$. Now obtain a time series of these balancing ex post real rates of return for the business unit, take the average of these rates (or use some other prediction or smoothing method) and use this average as the reference real rate.

Any of the Approaches 3-8 could be used in empirical applications. The relative simplicity and reproducibility of Approaches 7 and 8 is appealing.

We turn now to a discussion of possible methods for determining the expected asset inflation rates, the $i^t$ rates, which appear in the $f_n^t$ user costs defined by (7) and (14) and the $u_n^t$ user costs defined by (15). There are four possible methods that could be used.

**Method 1: Assume All Anticipated Asset Inflation Rates are Zero**

This is one very simple reproducible method for determining the $i^t$ that could be used if the general rate of inflation is low. In addition to simplicity and reproducibility, it has the advantage of leading to always positive user costs. However, if general inflation is somewhat variable and greater than zero, then the resulting user costs will be consistently above market rental prices for the services of the asset and the variability in the general inflation rate will generally lead to variability in the nominal reference interest rate $r^t$ via the Fisher (1896) effect, leading to unduly variable user costs. Thus this method can only be recommended if general inflation is close to zero.

**Method 2: Use the Actual Ex Post Asset Inflation Rates**

Thus the ex ante anticipated asset inflation rate $i^t$ for a new asset is approximated by the actual ex post asset inflation rate for the new asset that materialized over period t. This method also has the advantages of simplicity and reproducibility but it has two very big disadvantages:

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54 Recall that we want our user costs to be a close approximation to the one period rental rate for the asset, if such rental markets existed. A negative user cost can usually not be a very close approximation to a market rental price because if an asset had a negative rental price, then the seller of the service is paying renters of the service to use it! Under these conditions, unless transactions costs were very high, the negative rental price would soon be bid up to zero.
• The resulting user costs can often turn out to be negative and as we pointed out earlier, a negative user cost is not a good approximation to a market rental rate.
• The resulting time series of user costs is invariably very volatile, suggesting again that this type of user cost is not usually going to be a very close approximation to a market rental rate, which will usually be much smoother.

Thus this method cannot be recommended if we want our user costs to be a reasonably close approximation to market rental rates.

**Method 3: Use the Ex Post Asset Inflation Rates to Forecast the Ex Ante Rate**

This method follows up on Method 2 in that past ex post asset inflation rates are used to predict what the current period asset inflation rate will be. A variant of this method is to simply smooth the time series of ex post asset inflation rates. This method has much to recommend it in that it probably captures the actual dynamics of market rentals (if the “right” forecasting method were used). It will generally lead to considerably smoother user costs than the user costs generated by Method 2. However, this method suffers from two problems:

• The method will sometimes generate negative user costs, leading to somewhat ad hoc methods for solving this problem.
• The method is not completely reproducible; i.e., different forecasting or smoothing methods will generally lead to different estimates for the anticipated asset inflation rates.

In spite of the above difficulties, this method can be recommended for assets which are “generally” known to be consistently increasing in price (land in some cases although it is easy to run into the negative user cost problem in this case) or consistently decreasing in price over time (any equipment that uses computer chips in a predominant manner).

**Method 4: Expected Asset Inflation Rates are Equal to the General Inflation Rate**

This is assumption (24) above; i.e., the assumption that the expected specific asset inflation rate $i_t$ is equal to the general inflation rate $\rho_t$. Approaches 7 and 8 explained above showed how this assumption led to fairly sensible, smooth and reproducible user costs so this material does not have to be repeated here.

To sum up: Approaches 7 and 8 and Methods 3 and 4 are our favored methods for determining the $r_t$ and $i_t$ terms that appeared in our user cost formula.\(^{55}\)

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\(^{55}\) See Harper, Berndt and Wood (1989) and Diewert (2001) for empirical evidence on this point. Harper, Berndt and Wood (1989; 351) noted that the use of time series techniques to smooth ex post asset inflation rates and the use of such estimates as anticipated price change dates back to Epstein (1977).

\(^{56}\) Other methods for determining the appropriate interest rates that should be inserted into user cost formulae are discussed by Harper, Berndt and Wood (1989) and Schreyer (2001). Harper, Berndt and Wood (1989) evaluated empirically 5 alternative rental price formulae using geometric depreciation but making different assumptions about the interest rate and the treatment of asset price change. They showed
The present system of national accounts, SNA 1993, does not recognize interest as an explicit cost of production in the production accounts. Also the accounting profession has generally been unwilling to have imputed interest on equity capital appear as a cost in historical accounting. Thus although it seems fairly obvious to most economists that interest is a cost of production, there is still some controversy on this point in the accounting profession. We will explore this topic more fully in Chapter II below. However, over the years, there have been many accountants who agree with including imputed interest on equity capital as a cost and we conclude this section with some quotes that illustrate this point of view.

“Once again, the basic reason why interest on the use of total capital should be recorded as a cost is that interest is a cost ... . A company has not performed satisfactorily, either for its shareholders or for society, if it has not generated enough revenue to cover all its costs, including the cost of using capital. The current income statement does not show whether or not the company has met this fundamental test. Its implication is that any earnings above the cost of debt interest are a ‘plus’.” Robert N. Anthony (1973; 96).

“The argument in favor of including interest as an element of cost is twofold. From the viewpoint of the business as a whole, it helps to point out an important fact to the managers of any enterprise which persistently fails to return a normal current rate of interest on the investment. From the more detailed cost accounting viewpoint, it is said to make an important cost distinction between those manufacturing departments using costly machinery and those using inexpensive machinery or none at all.” Stephen Gilman (1939; 322).

6. The Empirical Determination of Depreciation Rates

The empirical determination of asset depreciation rates is not an easy task. The OECD (1993) conducted a survey of average asset lives used by national statisticians for various asset classes in 14 OECD countries. For machinery and equipment (excluding vehicles) used in manufacturing activities, the average life ranged from 11 years for Japan to 26 years for the United Kingdom. For vehicles, the average service lives ranged from 2 years for passenger cars in Sweden to 14 years in Iceland and for road freight vehicles, the average life ranged from 3 years in Sweden to 14 years in Iceland. For buildings, the average service lives ranged from 15 years (for petroleum and gas buildings in the U.S.) to 80 years for railway buildings in Sweden. Faced with this wide range of possible lives, Angus Madison (1993) simply assumed an average service life of 14 years for machinery and equipment and 39 years for nonresidential structures.

In addition to just guessing average asset lives and converting this information into depreciation rates, there are four possible evidence based methods for determining depreciation rates, which we will now outline.

Method 1: Approaches Based on Used Asset Prices

“If there is a perfect second hand market for the goods in question, so that a market value can be assessed for them with precision, corresponding to each particular degree of wear, then the value-loss due to consumption can be exactly measured ...”. John R. Hicks (1946; 176).

that the choice of formula matters. Diewert (2001) (2005a) reached a similar conclusion in his comparison of various user cost formulae.
“Some depreciation patterns have very little economic justification (except accounting convenience), but most of them at least purport to approximate the decline in the economic value of the remaining services (i.e., market value). Of the various possible depreciation schemes (net stock measures), two measures seem to be of the most interest: (a) a net stock concept based on a purely physical deterioration depreciation scheme, and (b) the market value of the existing stock of capital. The latter figure can be approximated by the use of depreciation rates derived from studies of used machinery prices.” Zvi Griliches (1963; 120).

Economists (like Hicks and Griliches) and accountants (like Bell and Edwards)\(^{57}\) have long realized that a possible method for estimating the decline in value of a durable input due to its use over an accounting period is to use information on the market prices of used assets at a point in time and to compare differences in price as a function of the age of the input. This is the method we used in section 4 above. Thus given market data on the prices of used assets at any point in time, period by period depreciation rates \(\delta_n\) can be obtained by using equations (13).\(^{58}\) If we have information on used asset prices for many different time periods \(t\) and we are willing to make the assumption that depreciation rates are stable over time, so that \(\delta_n\) equals \(\delta_{in}\), then a stochastic specification of a variant of (13) can be made and econometric techniques can be used to estimate the sequence of one period depreciation rates. If the market data on used asset prices is sparse, then instead of estimating a completely general pattern of period to period depreciation rates, various restrictions on these parameters can be imposed.\(^{59}\) The simplest such restriction is that \(\delta_t\) be constant from period to period. Empirical studies of depreciation rates using second hand asset prices have been made by the accountant Beidleman (1973) (1976)\(^{60}\) and the economists Hall (1971), Hulten and Wykoff (1981a) (1981b) and Oliner (1996). The literature on this used asset approach is ably reviewed by Hulten and Wykoff (1996)\(^{61}\) and Jorgenson (1996a).

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\(^{57}\) “But as a practical matter the quantification and valuation of asset services used is not a simple matter and we must fall back on estimated patterns as a basis for current cost as well as historic cost depreciation. For those fixed assets which have active second hand markets the problem is not overly difficult. A pattern of service values can be obtained at any time by comparing the market values of assets of different ages or degrees of use. The differences so obtained, when related to the value of a new asset, yield the proportions of asset value which are normally used up or foregone in the various stages of asset life.” Edgar O. Edwards and Philip W. Bell (1961; 175).

\(^{58}\) Beidleman (1973) (1976) and Hulten and Wykoff (1981a) (1996; 22) showed that equations (13) must be adjusted to correct for early retirement of assets; i.e., equations (13) assume that all units of the asset are retired at the same time. Schmalenbach (1959; 91) noted that neglect of the survival problem leads to serious errors in the estimation of depreciation rates.

\(^{59}\) See Jorgenson (1996a; 27-28) for a nice summary of the methods that have been used to date.

\(^{60}\) “The findings of this chapter provide extensive evidence regarding the predominant role of age in the decline in value of certain fixed assets and the relative unimportance of valuation parameters other than age. The initial rapid decline in second hand value calculated for the regression models supports the use of accelerated depreciation techniques and the approach to finite scrap value favors declining balance methods of depreciation. The range of possible asset lives endorses the need for probability life depreciation.” Carl R. Beidleman (1973; 51-52).

\(^{61}\) “We have used this approach to study the depreciation patterns of a variety of fixed business assets in the United States . . . . The straight-line and concave patterns are strongly rejected; geometric is also rejected, but the estimated patterns are extremely close to (though steeper than) the geometric form, even for structures. Although it is rejected statistically, the geometric pattern is far closer to the estimated pattern than either of the other two candidates. This leads us to accept the geometric pattern as a reasonable approximation for broad groups of assets . . . .” Charles R. Hulten and Frank C. Wykoff (1996; 16).
Many economists and accountants have objected to the use of second hand data to estimate depreciation rates for a variety of reasons:

“We readily agree that where a market is sufficiently large, generally accessible, and continuous over time, it serves to coordinate a large number of subjective estimates and thus may impart a moment of (social) objectivity to value relations based on prices formed on it. But it can hardly be said that the second-hand market for industrial equipment, which would be the proper place for the determination of the value of capital goods which have been in use, satisfies these requirements, and that its valuations are superior to intra-enterprise valuation.” L.M. Lachmann (1941; 376-377).

“But why, if market values are the key to asset values, does not the accountant find depreciation by direct reference to market quotations for assets of different ages, and abandon his formulae? Various answers suggest themselves ... Second-hand markets tend to be small and scrubby, so that quotations may be hard to find and harder to trust. Many assets are built specifically for the one firm, and therefore worn replicas do not exist. An owner usually regards his own worn assets as different from, and better than, replicas in the market, because he knows their history, condition and foibles.” William T. Baxter (1971; 31).

“One argument, drawing on the Akerlof Lemons Model, is that assets resold in second hand markets are not representative of the underlying population of assets, because only poorer quality units are sold when used. Others express concerns about the thinness of resale markets, believing that it is sporadic in nature and is dominated by dealers who under-bid.” Charles R. Hulten and Frank C. Wykoff (1996; 17-18).

In spite of the above objections to the use of the second hand market method for estimating depreciation rates, this method seems more “objective” than simply guessing at the appropriate rates.

A more serious objection to the above model of depreciation rate determination is that the method includes only length of asset service or time in use as an explanatory variable and thus the method neglects variations in the intensity of use of the durable input. There are at least two ways of meeting this criticism:

- We can follow the advice of Edwards and Bell and estimate separate sequences of depreciation rates that pertain to assets that are used with approximately the same intensity and have similar maintenance policies or

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62 This criticism of depreciation theory dates back to Saliers (1922; 172-174) at least. Many other authors noted this problem: “The question of charging depreciation as a function of output rather than of time has been discussed of late. It is more natural to consider the depreciation of an automobile in terms of miles than of years.” Harold Hotelling (1925; 352). “While there is much to be said in favor of depreciating automobiles and trucks on a mileage basis, rather than by the number of years of use, and while a similar expedient may well be adopted in distributing the depreciation of other machinery and equipment, it must be observed that depreciation is seldom a sole function of use or time. Generally it is a combination of the two, and it is often desirable to check one method by applying the other.” Stephen Gilman (1939; 345-346). “The two main defects of depreciation data are found to be that they ignore variations in the degree of utilisation, and that they are largely based on original rather than reproduction cost.” L.M. Lachmann (1941; 375).

63 “A truck used to haul logs in timber country is not likely to yield the same pattern of services as one used to haul produce over superhighways. Physically identical assets having sharply different uses should be placed in separate categories and treated as different assets, for example, logging trucks and produce trucks. How fine a distinction should be drawn is a matter of practicality.” Edgar O. Edwards and Philip W. Bell (1961; 174).
• We can incorporate utilization and maintenance variables as explanatory variables in stochastic versions of (13).

Jorgenson (1996a, 27-29) reviews the literature on these extensions of the basic used asset model of depreciation rate determination.

Method 2: Approaches Based on Rental Prices

Suppose that a rental market for a durable input exists so that we can observe in period \( t \) the beginning of the period rental prices for assets of age \( n, f_{n,t} \), which appear in the user cost formulae (7) or with (14) substituted into (7). Then given information on the firm’s cost of capital \( r^t \), on the rental price for a new asset \( f_{0,t} \), and on the expected rate of price inflation for a new asset \( i^t \), we can use the first equation in (7) and the first equation in (14) to solve for the period \( t \) depreciation rate for a new asset, \( \delta_{0,t} \). Once \( \delta_{0,t} \) has been determined, then given information on the rental price for a one period old asset \( f_{1,t} \), then we can use the second equation in (7) and the second equation in (14) to solve for the period \( t \) depreciation rate for a one period old asset, \( \delta_{1,t} \), etc. Thus given that rental markets exist for durable inputs being used by a business unit, these rental prices can be equated to the corresponding user costs and depreciation rates can be derived from the resulting equations.

Of course, if rental markets do not exist (and for the most part they will not), then this method will not work, an obvious point made by Hulten and Wykoff (1996;15). Jorgenson (1996a; 32) reviewed the few studies that have used this method. Even when rental markets exist, this method may not generate very accurate depreciation rates, because it is very sensitive to the assumptions made about the “correct” nominal opportunity cost of capital \( r^t \) and the “correct” expected asset inflation rate \( i^t \). Also the transactions costs in the rental and leasing markets may be very high, creating additional complications and measurement errors. However, in some markets (perhaps the leasing of structures market), the method may work well. Finally, even though the rental price method is unlikely to be a useful method for the empirical determination of depreciation rates, rental prices are useful when they exist since they can be used as period \( t \) (opportunity) costs for the use of the corresponding durable inputs during period \( t \).

Method 3: Approaches Based on Production Function Estimation

Suppose a durable input is used by a business unit and it purchases \( I^t \) units of it at the beginning of period \( t \) in order to produce \( y^t \) units of output using the vector \( x^t \) of

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64 The first approach could be viewed as a special case of the second if we allowed discrete classification variables in place of continuous ones. In his study of used automobile prices, Hall (1971) used the first approach, which Jorgenson (196a; 27) termed the “analysis of variance approach”. Beidleman (1973) (1976) used the second approach, which Jorgenson (1996c; 28) termed the “hedonic approach”.

65 Diewert (1983a; 212) (1983b; 1100-1102) discussed these expectational difficulties with ex ante user cost formulae and noted that given the user cost formula for \( u^t \) and information on \( r^t \), \( P^t \) and as estimate for the depreciation rate \( \delta^t \), then the first equation in (15) could be used to estimate the producer’s anticipated rate of asset price inflation \( i^t \).
nondurable inputs\textsuperscript{66} during period t as well as the services of past purchases of the durable input. Suppose further that the durable input lasts N periods and that after adjusting for physical loss of efficiency, all unretired units of the durable input are perfect substitutes in production. The production function which relates output flow to inputs used during a period is \( F \) and if there is no technological progress, we have the following relationship between output produced and inputs used during period t:

\[
(30) \quad y^t = F[x^t; I^t + (1-\delta_0)I^{t-1} + (1-\delta_0)(1-\delta_1)I^{t-2} + \ldots + (1-\delta_0)(1-\delta_1)\ldots(1-\delta_{N-1})I^{t-N}] 
\]

where \( \delta_0, \delta_1, \ldots, \delta_{N-1} \) are the one period depreciation rates for a new asset, a one period old asset, etc. Note that we are assuming that these depreciation rates are constant across time periods t.

The \textit{production function method} for determining depreciation rates works as follows:

- Collect data on output produced during period t, \( y^t \), nondurable inputs used, \( x^t \), and durable input purchases, \( I^t \), for a number of periods t.
- Assume a functional form for the production function \( F \).
- Add a stochastic specification to equations (30) for \( t = 0, 1, \ldots, T \).
- Use econometric techniques to simultaneously estimate the unknown parameters which appear in the production function \( F \) as well as the depreciation rates \( \delta_0, \delta_1, \ldots, \delta_{N-1} \).

Variants of this basic method include:

- Restricting the depreciation parameters \( \delta_0, \delta_1, \ldots, \delta_{N-1} \) in some a priori fashion (e.g., make them all equal to a common \( \delta \)).
- Using the assumption of short run profit maximizing or cost minimizing behavior on the part of the business unit in order to add extra estimating equations involving period t prices to the single estimating equation (30).
- Instead of estimating the production function \( F \), estimate the unknown parameters in the dual cost or profit function.\textsuperscript{67}

Empirical studies using this approach to the estimation of depreciation rates include Epstein and Denny (1980), Pakes and Griliches (1984), Nadiri and Prucha (1996) and Doms (1996). It should be noted that the depreciation rates which are estimated using this production function approach may be different from the estimates that result from the used asset approach studied above. The latter approach incorporates the effects of exhaustion, deterioration and obsolescence (to use Griliches’ terminology), while the production function approach typically incorporates only the effects of physical deterioration and exhaustion.\textsuperscript{68}

\textsuperscript{66} If the business unit produces more than one output, the additional outputs can be absorbed into the \( x^t \) vector as negative inputs.

\textsuperscript{67} For a review of duality theory and the associated functional form problems, see Diewert (1993b).

\textsuperscript{68} Pigou (1935; 238) clearly distinguished exhaustion (or decline in useful life) from physical deterioration: “A distinction should be drawn between physical changes which, while leaving the element as productive
There are some problems with the production function approach:

- The approach will only work in a highly aggregated model with a small number of outputs, nondurable inputs and durable inputs due to the difficulties involved in estimating the parameters pertaining to a general production function when there are numerous inputs and outputs.\(^{69}\)
- The assumption that the different vintages of capital can be combined together in the additive capital aggregate that appears as the last term on the right hand side of (30) is restrictive; i.e., it is assumed that the depreciation adjusted different vintages of capital are perfect substitutes in production, an assumption which may or may not be true.
- The estimates of the depreciation rates may not be invariant to the degree of disaggregation of the other inputs and outputs used by the business unit.
- Econometric estimates are often sensitive to the stochastic specification of the model and the method of estimation. In other words, econometrically based estimates are often not reproducible: different econometricians using the same data base will often come up with very different answers, particularly in models with many parameters.

Thus as a good general method for the empirical determination of depreciation rates, the production function method is not entirely satisfactory. However, it can provide a check on whether other assumption based estimates for depreciation rates are consistent with the data; i.e., the production function method is at least evidence based.

**Method 4: Approaches Based on Insured Values or Other Expert Appraisals**

Since most businesses insure their structures against accidental loss, insurance appraisals on the value of structures (and other property) provide an objective source of information.\(^{70}\) Engineers and other experts may be able to provide reasonably accurate estimates for the value of machinery and equipment components.

The main problem with insurance based value information is that it is not readily made available to the outside observer.

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\(^{69}\) If we assume a flexible functional form for the production function F, the number of parameters to be estimated will grow approximately as the square of the number of inputs and outputs that are distinguished in the model. If we do not assume a flexible functional form for F, then our a priori restrictive assumptions on the substitution possibilities for the technology will generally lead to biased estimates for the depreciation rates. These difficulties with the production function approach are discussed in more detail by Diewert (1992a; 177).

\(^{70}\) Insurance companies have an incentive to avoid appraised values that are too high and the owner of the asset being insured has an incentive to avoid appraised values that are too low. Hence appraised values for insurance purposes are likely to be more reproducible than other expert appraisals.
Our overall conclusion in evaluating different methods for the empirical determination of depreciation rates is that the used asset price method seems best when the relevant second hand markets actually exist. However, for firm specific assets that are not traded in second hand markets, it appears that depreciation will have to be determined on the basis of engineering estimates or other expert appraisals.

In the next section, we discuss a topic of some current interest: namely the interaction of (foreseen) obsolescence and depreciation. We also discuss cross sectional versus time series depreciation.

7. Time Series versus Cross Sectional Depreciation

We begin this section with a definition of the time series depreciation of an asset. Define the ex post time series depreciation of an asset that is n periods old at the beginning of period t, \( E_n^t \), to be its second hand market price at the beginning of period t, \( P_n^t \), less the price of an asset that is one period older at the beginning of period t+1, \( P_{n+1}^{t+1} \); i.e.,

\[
(31) \quad E_n^t = P_n^t - P_{n+1}^{t+1} \quad ; \quad n = 0, 1, 2, \ldots
\]

Definitions (31) should be contrasted with our earlier definitions (11), which defined the cross section amounts of depreciation for the same assets at the beginning of period t, \( D_n^t \equiv P_n^t - P_{n+1}^t \).

We can now explain why we preferred to work with the cross sectional definition of depreciation, (11), over the time series definition, (31). The problem with (31) is that time series depreciation captures the effects of changes in two things: changes in time (this is the change in t to t+1)\(^{71}\) and changes in the age of the asset (this is the change in n to n+1).\(^{72}\) Thus time series depreciation aggregates together two effects: the asset specific price change that occurred between time t and time t+1 and the effects of asset aging (depreciation). Thus the time series definition of depreciation combines together two distinct effects. At first sight, the fact that time series depreciation combines two effects does not seem to be a problem. But there is always a potential problem when we compare values at two different time periods: we must be aware that general changes in the purchasing power of money can make comparisons across time very misleading. Thus as a general rule, when comparing prices or values across time, in order to achieve comparability, the two values must either be adjusted for general price level change or the later value should be deflated by 1 plus the relevant nominal interest rate.\(^{73}\)

The above definition of ex post time series depreciation is the original definition of depreciation and it extends back to the very early beginnings of accounting theory:

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\(^{71}\) This change could be captured by either \( P_n^t - P_{n+1}^{t+1} \) or \( P_{n+1}^t - P_{n+1}^{t+1} \).

\(^{72}\) This change could be captured by either \( P_n^t - P_{n+1}^t \) or \( P_{n+1}^{t+1} - P_{n+1}^{t+1} \).

\(^{73}\) Of course, a nominal interest rate should contain within it a general price change component; i.e., it contains the Fisher effect.
“[There are] various methods of estimating the Depreciation of a Factory, and of recording alteration in value, but it may be said in regard to any of them that the object in view is, so to treat the nominal capital in the books of account that it shall always represent as nearly as possible the real value. Theoretically, the most effectual method of securing this would be, if it were feasible, to revalue everything at stated intervals, and to write off whatever loss such valuations might reveal without regard to any prescribed rate… The plan of valuing every year instead of adopting a depreciation rate, though it might appear the more perfect, is too tedious and expensive to be adopted … The next best plan, which is that generally followed … is to establish average rates which can without much trouble be written off every year, to check the result by complete or partial valuation at longer intervals, and to adjust the depreciation rate if required.” Ewing Matheson (1910; 35).

Hotelling, in the first mathematical treatment of depreciation in continuous time, defined time series depreciation in a similar manner:

“Depreciation is defined simply as rate of decrease of value.” Harold Hotelling (1925; 341).

However, what has to be kept in mind that these early authors who used the concept of time series depreciation were implicitly or explicitly assuming that prices were stable across time, in which case, time series and cross sectional depreciation will normally coincide.

Hill (2000) recently argued that a form of time series depreciation was to be preferred over cross section depreciation for national accounts purposes:

“The basic cost of using an asset over a certain period of time consists of depreciation, the decline in the value of that asset, plus the associated financial, or capital cost. An alternative definition of depreciation has been proposed in recent years in what may be described as the vintage accounting approach to depreciation. In the context of vintage accounting, depreciation is defined as the difference between the value of an asset of age k and one of age k+1 at the same point of time, the two assets being identical except for their ages. This concept, although superficially the same as the traditional concept, is in fact radically different because it effectively rules out obsolescence from depreciation by definition.

The issue is not a factual one about whether obsolescence does or does not cause the value of assets to decline over time. The question is how should such a decline be interpreted and classified. Whereas the traditional concept of depreciation treats expected obsolescence as an integral part of depreciation, in the vintage approach it is treated as a separate item, a revaluation, which has to be treated as real holding loss in the SNA. Reclassifying part of what has always been treated as depreciation in both business and economic accounting as a holding loss would reduce depreciation and increase every balancing item in the SNA from Net National Product and Income to net saving.” Peter Hill (2000; 6).

Thus Hill argued that cross sectional depreciation does not capture the effects of obsolescence and hence it is not the “right” concept for business and national accounts depreciation since obsolescence charges are equivalent to the effects of wear and tear and hence should be regarded as depreciation. Hill’s observations on this topic date back to the controversy between Pigou and Hayek in the 1940’s. Pigou argued that depreciation should be measured relative to a concept that maintained capital intact from a physical point of view:

“I accept too the view that, if maintaining capital intact has to be defined in such a way that capital need not be maintained intact even though every item in its physical inventory is unaltered, the concept is worthless. But the inference I draw is, not that we should abandon the concept; rather that we should try to define it in
such a way that, when the physical inventory of goods in the capital stock is unaltered, capital is maintained intact; more generally, in such a way that, not indeed the quantity of capital—which, with heterogeneous items, can only a conventionalised number—is independent of the equilibrating process, but changes in its quantity are independent of changes in that process." A.C. Pigou (1941; 273).

Pigou (1941; 274) went on to suggest that the Paasche quantity index for capital could be used to determine whether capital was maintained intact between two points in time; i.e., the price weights of the second point in time should be used to value the two capital stocks. Hence if the two capital stocks were unchanged in each component, the Paasche quantity index would be equal to unity, correctly indicating that there was no physical change in the capital stock between the two points in time. However, Hayek responded, correctly, that Pigou’s concept of maintaining capital intact would neglect forecast obsolescence:

“Professor Pigou’s answer to the question of what is meant by ‘maintaining capital intact’ consists in effect of the suggestion that for this purpose we should disregard obsolescence and require merely that such losses of value of the existing stock of capital goods be made good as are due to physical wear and tear. ... If Professor Pigou’s criterion is to be of any help, it would have to mean that we have to disregard all obsolescence, whether it is due to foreseen or foreseeable causes, or whether it is brought about by entirely unpredictable causes, such as the ‘acts of God or the King’s enemy’, which alone he wanted to exclude in an earlier discussion of this problem.” F.A. v. Hayek (1941; 276).

Hayek went on to give a clear example of where Pigou’s point of view would lead to a mismeasurement of depreciation and income:

“Assume three entrepreneurs, X, Y, and Z, to invest at the same time in equipment of different kinds but of the same cost and the same potential physical duration, say ten years. X expects to be able to use his machine continuously throughout the period of its physical ‘life’. Y, who produces some fashion article, knows that at the end of one year his machine will have no more than its scrap value. Z undertakes a very risky venture in which the changes of employing the machine continuously so long as it lasts and having to scrap it almost as soon as it starts to produce are about even. According to Professor Pigou the three entrepreneurs will have to order their investments in such a way that during the first year they can expect to earn the same gross receipts: since the wear and tear of their respective machines during the first years will be the same, the amount they will have to put aside during the first year to ‘maintain their capital intact’ will also be the same, and this procedure will therefore lead to their earning during that year the same ‘net’ income from the same amount of capital. Yet it is clear that the foreseen result of such dispositions would be that at the end of the year X would still possess the original capital, Y one tenth of it, while Z would have an even chance of either having lost it all or just having preserved it. ... To treat all receipts except what is required to make good physical wear and tear as net income for income tax purposes would evidently discriminate heavily against industries where the rate of obsolescence is high and reduce investment in these industries below what is desirable.” F.A. v. Hayek (1941; 276-277).

Since the depreciation rates $\delta_n^1$ defined by (13) are cross sectional depreciation rates (and thus seemingly reflect only wear and tear depreciation) and since they play a key role in

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74 The accounting literature has been wrestling with the appropriate treatment of expected obsolescence for a long time as well: “In a number of industries development has been so rapid and revolutionary over a period of years that the capitalization of losses due to so-called premature retirements would have led to an absurd inflation of asset values. In such situations, the use of higher depreciation rates, rather than capitalization of losses, is indicated.” W.A. Paton (1931; 93). This issue was important in Paton’s time with respect to the regulation of public utilities and it is still important today.
the definitions of the beginning of period t user costs $f_n^t$ defined by (7) and (14) and the end of period user costs $u_n^t$ defined by (15), it is necessary to clarify their use in the context of Hayek and Hill’s point that simple wear and tear depreciation rates should not be used to measure depreciation in the national accounts (or in business accounts), since they neglect anticipated obsolescence.

Our response to the Hayek and Hill critique is threefold:

- Our user costs $f_n^t$ and $u_n^t$ that are based on cross sectional depreciation rates $\delta_n^t$ are affected by anticipated obsolescence in principle but
- Hill is correct in arguing that cross section depreciation will not generally equal ex post time series depreciation or anticipated time series depreciation and
- Hill is correct in arguing that our cross sectional depreciation based user costs defined by $f_n^t$ and $u_n^t$ will not capture obsolescence effects that cause asset lives to shrink\(^75\) (but $f_n^t$ and $u_n^t$ will capture obsolescence effects that work by reducing the real price of the asset in future periods).

Let us consider the first point above. Provisionally, we define anticipated obsolescence as a situation where the expected new asset inflation rate (adjusted for quality change) $i^t$ is negative.\(^76\) For example, everyone anticipates that the quality adjusted price for a new computer next quarter will be considerably lower than it is this quarter.\(^77\) Now turn back to equations (5) above, which define the sequence of asset prices by age $P_n^t$ at the start of period t. It is clear that the negative $i^t$ plays a role in defining the sequence of asset prices by age as does the sequence of rental prices by age that is observed at the beginning of period t, the $f_n^t$. Thus in this sense, cross sectional depreciation rates certainly embody assumptions about anticipated obsolescence.

Zvi Griliches had a nice verbal description of the factors which explain the pattern of used asset prices, which will cast some light on our present subject:

"The net stock concept is motivated by the observed fact that the value of a capital good declines with age (and/or use). This decline is due to several factors, the main ones being the decline in the life expectancy of the asset (it has fewer work years left), the decline in the physical productivity of the asset (it has poorer work years left) and the decline in the relative market return for the productivity of this asset due to the

\(^75\) Ahmad, Aspden and Schreyer (2005) make this point as well in a recent paper. We further illustrate this point in chapter VI below.

\(^76\) Paul Schreyer and Peter Hill noted a problem with this provisional definition of anticipated obsolescence as a negative value of the expected asset inflation rate: it will not work in a high inflation environment. In a high inflation environment, the nominal asset inflation rate $i^t$ will generally be positive but we will require this nominal rate to be less than general inflation in order to have anticipated obsolescence. Thus our final definition of anticipated obsolescence is that the real asset inflation rate $i^t$ defined later by (37) be negative; see the discussion just above equation (39) below.

\(^77\) Our analysis assumes that the various vintages of capital are adjusted for quality change (if any occurs) as they come on the market. We also need to assume that the form of quality change affects all future efficiency factors (i.e., the $f_n^t$) in a proportional manner. Thus we are assuming that technical change is of the capital augmenting type. This is obviously only a rough approximation to reality: technical change may increase the durability of a capital input or it may decrease the amount of maintenance or fuel that is required to operate the asset. These changes can lead to nonproportional changes in the $f_n^t$. We will consider more general models of technical change in chapter VI.
availability of better machines and other relative price changes (its remaining work years are worth less). One may label there three major forces as exhaustion, deterioration and obsolescence.” Zvi Griliches (1963; 119).

Thus for an asset that has a finite life, as we move down the rows of equations (5), the number of discounted rental terms decline and hence asset value declines, which is Griliches’ concept of exhaustion. If the cross sectional rental prices are monotonically declining with age (due to their declining efficiency), then this corresponds Griliches’ concept of deterioration. Finally, a negative anticipated asset inflation rate will cause all future period rentals to be discounted more heavily, which could be interpreted as Griliches’ concept of obsolescence. Thus all of these explanatory factors are imbedded in equations (5).

Now let us consider the second point: that cross section depreciation is not really adequate to measure time series depreciation in some sense to be determined.

Define the ex ante time series depreciation of an asset that is n periods old at the beginning of period t, $\Delta_n^t$, to be its second hand market price at the beginning of period t, $P_n^t$, less the anticipated price of an asset that is one period older at the beginning of period t+1, $(1+i^t)P_{n+1}^t$; i.e.,

\[(32) \Delta_n^t = P_n^t - (1+i^t)P_{n+1}^t ; n = 0,1,2,...\]

Thus ex ante or anticipated time series depreciation for an asset that is t periods old at the start of period t, $\Delta_n^t$, differs from the corresponding cross section depreciation defined by (11), $D_n^t = P_n^t - P_{n+1}^t$, in that the anticipated new asset inflation rate, $i^t$, is missing from $D_n^t$. However, note that the two forms of depreciation will coincide if the expected asset inflation rate $i^t$ is zero.

78 However, our present model does not capture obsolescence that may be caused by a future abrupt decline in the rents earned by the asset due to say rising real wage rates or anticipated changes in tastes (recall Hayek’s example of the Y entrepreneur); i.e., our present model has the weakness that it projects the present pattern of rents earned by the various vintages of the asset into the future using a single geometric price escalation factor, $i^t$. Thus our present model cannot deal with cases where at some future date, all of the assets are expected to earn zero rent due to technological progress or shifts in demand. See chapter VI for examples of this type of phenomenon. Our original model defined by equations (2) is flexible enough to deal with Hayek’s example of the Y entrepreneur but it is too general to be used in empirical applications without further information on how expectations are formed.

79 “Normal wear-and-tear in the course of production is clearly a reason why the value of a capital instrument should be greater at the beginning of a year than at the end, even if the final value was foreseen accurately. Normal wear-and-tear is therefore an element of true depreciation. So is exceptional wear-and-tear, due to exceptionally heavy usage; if the exceptionally heavy usage had been foreseen, the gap between the beginning-value and the end-value would have been larger. On the other hand, any deterioration which the machine undergoes outside its utilisation does not give rise to true depreciation; if such deterioration had been foreseen, the initial capital value would have been written down in consequence; the deterioration which it undergoes is therefore not depreciation, but a capital loss.” John R. Hicks (1942; 178). In our view, foreseen price declines in future rentals are reflected in initial capital values.
We can use equations (13) and (14) in order to define the ex ante depreciation amounts \( \Delta_n^t \) in terms of the cross section depreciation rates \( \delta_n^t \). Thus using definitions (32), we have:

\[
\begin{align*}
(33) \quad & \Delta_n^t = P_n^t - (1+i^t) P_{n+1}^t \\
& = P_n^t - (1+i^t)(1-\delta_n^t) P_n^t \\
& = [1 - (1+i^t)(1-\delta_n^t)] P_n^t \\
& = (1-\delta_1^t)(1-\delta_2^t) \ldots (1-\delta_{n-1}^t)[1 - (1+i^t)(1-\delta_n^t)] P_0^t \\
& = (1-\delta_1^t)(1-\delta_2^t) \ldots (1-\delta_{n-1}^t) \delta_n^t P_0^t.
\end{align*}
\]

We can compare the above sequence of ex ante time series depreciation amounts \( \Delta_n^t \) with the corresponding sequence of cross sectional depreciation amounts:

\[
\begin{align*}
(34) \quad & D_n^t = P_n^t - P_{n+1}^t \\
& = P_n^t - (1-\delta_n^t) P_n^t \\
& = [1 - (1-\delta_n^t)] P_n^t \\
& = (1-\delta_1^t)(1-\delta_2^t) \ldots (1-\delta_{n-1}^t) \delta_n^t P_0^t \\
& = (1-\delta_1^t)(1-\delta_2^t) \ldots (1-\delta_{n-1}^t) \delta_n^t P_0^t.
\end{align*}
\]

Of course, if the anticipated asset inflation rate \( i^t \) is zero, then (33) and (34) coincide and ex ante time series depreciation equals cross sectional depreciation. If we are in the provisional expected obsolescence case with \( i^t \) negative, then it can be seen comparing (33) and (34) that

\[
(35) \quad \Delta_n^t > D_n^t \quad \text{for all } n \text{ such that } D_n^t > 0;
\]

i.e., if \( i^t \) is negative (and \( 0 < \delta_n^t < 1 \)), then ex ante time series depreciation exceeds cross section depreciation over all in use vintages of the asset. If \( i^t \) is positive so that the rental price of each vintage is expected to rise in the future, then ex ante time series depreciation is less than the corresponding cross section depreciation for all assets that have a positive price at the end of period \( t \). This corresponds to the usual result in the user cost literature where capital gains or an ex post price increase for a new asset leads to a negative term in the user cost formula (plus a revaluation of the cross section depreciation rate). Here we are restricting ourselves to anticipated capital gains rather than the actual ex post capital gains and we are focusing on depreciation concepts rather than the full user cost.

This is not quite the end of the story in the high inflation context. National income accountants often readjust asset values at either the beginning or end of the accounting period to take into account general price level change. At the same time, they also want to decompose nominal interest payments into a real interest component and another component that compensates lenders for general price change.

Recall (22), which defined the general period \( t \) inflation rate \( \rho^t \) and (23), which related the period \( t \) nominal interest rate \( r^t \) to a constant real rate \( r^* \) and the inflation rate \( \rho^t \). We
rewrite now generalize (23) by allowing the \textit{period t real interest rate} $r^t$ to vary over time as follows:

\[(36) \quad 1 + r^*t \equiv (1 + r^t)/(1 + \rho^t).\]

In a similar manner, we define the \textit{period t anticipated real asset inflation rate} $i^*t$ as follows:

\[(37) \quad 1 + i^*t \equiv (1 + i^t)/(1 + \rho^t).\]

Recall definition (32), which defined the \textit{ex ante time series depreciation} of an asset that is $n$ periods old at the beginning of period $t$, $\Delta_n^t$. The first term in this definition reflects the price level at the beginning of period $t$ while the second term in this definition reflects the price level at the end of period $t$. We now express the second term in terms of the beginning of period $t$ price level. Thus we define the \textit{ex ante real time series depreciation} of an asset that is $n$ periods old at the beginning of period $t$, $\Pi_n^t$, as follows:

\[(38) \quad \Pi_n^t = P_n^t - (1+i^t) \frac{P_{n+1}^t}{(1+\rho^t)} \quad n = 0,1,2,\ldots\]

\[= P_n^t - (1+i^t)(1-\delta_n^t) \frac{P_n^t}{(1+\rho^t)} \quad \text{using (13)}\]

\[= 1 - (1+i^t)(1+\rho^t)(1-\delta_n^t) \frac{P_n^t}{(1+\rho^t)} \quad \text{using (37)}\]

\[= (1-\delta_0^t)(1-\delta_1^t) \ldots (1-\delta_{n-1}^t) \left[1 - (1+i^t)(1-\delta_n^t)\right] P_0^t \quad \text{using (14)}\]

\[= (1-\delta_0^t)(1-\delta_1^t) \ldots (1-\delta_{n-1}^t) \left[ \delta_n^t - i^t/(1-\delta_n^t) \right] P_0^t.\]

The ex ante real time series depreciation amount $\Pi_n^t$ defined by (38) can be compared to its cross section counterpart $D_n^t$, defined by (34) above. Of course, if the real anticipated asset inflation rate $i^*t$ is zero, then (38) and (34) coincide and real ex ante time series depreciation equals cross section depreciation.\(^{80}\)

We are now in a position to provide a more satisfactory definition of expected obsolescence, particularly in the context of high inflation. We now define \textit{expected obsolescence} to be the situation where the \textit{real asset inflation rate} $i^*t$ is negative. If the real asset inflation rate is negative, then it can be seen comparing (38) and (34) that

\[(39) \quad \Pi_n^t > D_n^t \quad \text{for all $n$ such that $D_n^t > 0$};\]

i.e., real anticipated time series depreciation exceeds the corresponding cross sectional depreciation provided that $i^*t$ is negative.\(^{81}\)

Thus the general user cost formulae that we have developed from the vintage accounts point of view can be reconciled to reflect the point of view of national income

\(^{80}\)The ex ante real time series depreciation defined by (38) is defined in terms of the perspective of discounting from the beginning of the period. In chapter VII, we will rework our analysis from the more useful perspective of the end of the period.

\(^{81}\)What happens if $i^*t$ is positive instead of negative? In this case, real time series depreciation will be less than the corresponding cross sectional depreciation. In chapter VII, we will argue that real time series depreciation is still an appropriate depreciation concept for accounting purposes even if $i^*t$ is positive.
accountants. We agree with Hill’s point of view that cross sectional depreciation is not really adequate to measure time series depreciation as national income accountants have defined it since Pigou:

“All allowance must be made for such part of capital depletion as may fairly be called ‘normal’; and the practical test of normality is that the depletion is sufficiently regular to be foreseen, if not in detail, at least in the large. This test brings under the head of depreciation all ordinary forms of wear and tear, whether due to the actual working of machines or to mere passage of time— rust, rodents and so on— and all ordinary obsolescence, whether due to technical advance or to changes of taste. It brings in too the consequences of all ordinary accidents, such as shipwreck and fire, in short of all accidents against which it is customary to insure. But it leaves out capital depletion that springs from the act of God or the King’s enemies, or from such a miracle as a decision tomorrow to forbid the manufacture of whisky or beer. These sorts of capital depletion constitute, not depreciation to be made good before current net income is reckoned, but capital losses that are irrelevant to current net income.” A.C. Pigou (1935; 240-241).

Pigou (1924) in an earlier work had a more complete discussion of the obsolescence problem and the problems involved in defining time series depreciation in an inflationary environment. Pigou first pointed out that the national dividend or net annual income (or in modern terms, real net output) should subtract depreciation or capital consumption:

“For the dividend may be conceived in two sharply contrasted ways: as the flow of goods and services which is produced during the year, or as the flow which is consumed during the year. Dr. Marshall adopts the former of these alternatives …. Naturally, since in every year plant and equipment wear out and decay, what is produced must mean what is produced on the whole when allowance has been made for this process of attrition. … In concrete terms, his conception of the dividend includes an inventory of all the new things that are made [i.e., gross investment], accompanied as a negative element, by an inventory of all the decay and demolition of old things [i.e., capital consumption]. A.C. Pigou (1924; 34-35).

Pigou then went on to discuss the roles of obsolescence and general price change in measuring depreciation:

“The concrete content of the dividend is, indeed, unambiguous— the inventory of things made and (double counting being eliminated) and services rendered, minus, as a negative element, the inventory of things worn out during the year. But how are we to value this negative element? For example, if a machine originally costing £1000 wears out and, owing to a rise in the general price level, can only be replaced at a cost of £1500, is £1000 or £1500 the proper allowance? Nor is this the only, or, indeed, the principle difficulty. For depreciation is measured not merely by the physical process of wearing out, and capital is not therefore maintained intact when provision has been made to replace what is thus worn out. Machinery that has become obsolete because of the development of improved forms is not really left intact, however excellent its physical condition; and the same thing is true of machinery for whose products popular taste has declined. If, however, in deference to these considerations, we decide to make an allowance for obsolescence, this concession implies that the value, and not the physical efficiency, of instrumental goods [i.e., durable capital inputs] is the object to be maintained intact. But, it is then argued, the value of instrumental goods, being the present value of the services which they are expected to render in the future, necessarily varies with variations in the rate of interest. Is it really a rational procedure to evaluate the national dividend by a method which makes its value in relation to that of the aggregated net product of the country’s industry depend on an incident of that kind? If that method is adopted, and a great war, by raising the rate of interest, depreciates greatly the value of existing capital, we shall probably be compelled to put, for the value of the national dividend in the first year of that war, a very large negative figure. This absurdity must be avoided at all costs, and we are therefore compelled, when we are engaged in evaluating the national dividend, to leave out of account any change in the value of the country’s capital equipment that may have been brought about by broad general causes. This decision is arbitrary and unsatisfactory, but it is one which it is impossible to avoid. During the period of the war, a similar difficulty was created
by the general rise, for many businesses, in the value of the normal and necessary holding of materials and stocks, which was associated with the general rise of prices. On our principles, this increase of value ought not to be reckoned as an addition to the income of the firms affected, or, consequently, to the value of the national dividend.” A.C. Pigou (1924; 39-41).

The above quotation indicates that Pigou was responsible for many of the conventions of national income accounting that persist down to the present day. He essentially argued that (unanticipated) capital gains or losses be excluded from income and that the effects of general price level change be excluded from estimates of depreciation. He also argued for a physical maintenance of capital concept and ignoring obsolescence although he was not happy about neglecting obsolescence in the depreciation accounts. Unfortunately, he did not spell out exactly how all of this could be done in the accounts. Our algebra above can be regarded as an attempt to deal with some of the complications that Pigou raised when measuring depreciation.

It should be noted that the early industrial engineering literature also stressed that the possibility of obsolescence meant that depreciation allowances should be larger than those implied by mere wear and tear:

“Machinery for producing any commodity in great demand, seldom actually wears out; new improvements, by which the same operations can be executed either more quickly or better, generally supersedes it long before that period arrives: indeed, to make such an improved machine profitable, it is usually reckoned that in five years it ought to have paid itself, and in ten to be superceded by a better.” Charles Babbage (1835; 285).

“The possibility of New Inventions, processes, or machines coming into use, which may supercede or render an existing plant Obsolete, is a contingency that presses on most manufacturing trades, principally those which have long established, but sometimes also in new concerns where old methods have been adopted or imitated just as they were being superceded elsewhere.” Ewing Matheson (1910; 38).

“A reserve beyond the ordinary depreciation above described may then become necessary, because the original plant, when once superceded by such inventions, may prove unsaleable as second-hand plant, except in so far as it may have a piecemeal or scrap value. … This risk sometimes arises, not from improvements in the machinery, but from alterations in the kind of product, rendering new machines necessary to suit new patterns or types. Contingencies such as these should encourage an ample reduction of nominal value in the early years of working, so as to bring down the book value of the plant to a point which will allow even of dismantling without serious loss. In such trades, profits should be large enough to allow for a liberal and rapid writing off of capital value, which is in effect the establishment of a reserve-fund as distinct from depreciation for wear and tear.” Ewing Matheson (1910; 39-40).

Thus Matheson considered obsolescence that could arise not only from new inventions but also from shifts in demand.

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82 Colin Clark (1940; 31) echoed Pigou’s recommendations: “The appreciation in value of capital assets and land must not be treated as an element in national income. Depreciation due to physical wear and tear and obsolescence must be treated as a charge against current income, but not the depreciation of the money value of an asset which has remained physically unchanged. Appreciation and depreciation of capital were included in the American statistics of national income prior to 1929, but now virtually the same convention has been adopted in all countries.”
We will end this section by pointing out another important use for the concept of real anticipated time series depreciation. However, before doing this, it is useful to rewrite equations (5), which define the beginning of period $t$ asset prices by age $P_n^t$ in terms of the beginning of period $t$ rental prices by age of asset $f_n^t$ and equations (7), which define the user costs $f_n^t$ in terms of the asset prices $P_n^t$, using definitions (36) and (37), which define the period $t$ real interest rate $r^*$ and expected asset inflation rate $i^*$ respectively in terms of the corresponding nominal rates $r^i$ and $i^i$ and the general inflation rate $\rho^i$. Substituting (36) and (37) into (5) yields the following system of equations:

\[
(40) \quad P_0^t = f_0^t + \left[\frac{1+i^*}{1+r^*}\right] f_1^t + \left[\frac{1+i^*}{1+r^*}\right]^2 f_2^t + \left[\frac{1+i^*}{1+r^*}\right]^3 f_3^t + \ldots \\
P_1^t = f_1^t + \left[\frac{1+i^*}{1+r^*}\right] f_2^t + \left[\frac{1+i^*}{1+r^*}\right]^2 f_3^t + \left[\frac{1+i^*}{1+r^*}\right]^3 f_4^t + \ldots \\
\ldots \\
P_n^t = f_n^t + \left[\frac{1+i^*}{1+r^*}\right] f_{n+1}^t + \left[\frac{1+i^*}{1+r^*}\right]^2 f_{n+2}^t + \left[\frac{1+i^*}{1+r^*}\right]^3 f_{n+3}^t + \ldots 
\]

Similarly, substituting (36) and (37) into (7) yields the following system of equations:

\[
(41) \quad f_0^t = P_0^t - \left[\frac{1+i^*}{1+r^*}\right] P_1^t = (1+r^*)^{-1} \left[ P_0^t (1+r^*) - (1+i^*) P_1^t \right] \\
f_1^t = P_1^t - \left[\frac{1+i^*}{1+r^*}\right] P_2^t = (1+r^*)^{-1} \left[ P_1^t (1+r^*) - (1+i^*) P_2^t \right] \\
\ldots \\
f_n^t = P_n^t - \left[\frac{1+i^*}{1+r^*}\right] P_{n+1}^t = (1+r^*)^{-1} \left[ P_n^t (1+r^*) - (1+i^*) P_{n+1}^t \right] ; \ldots 
\]

Note that the nominal interest and inflation rates have entirely disappeared from the above equations. In particular, the beginning of the period user costs by age $f_n^t$ can be defined in terms of real variables using equations (41) if this is desired. On the other hand, entirely equivalent formulae for the user costs can be obtained using the initial set of equations (7), which used only nominal variables. Which set of equations is used in practice can be left up to the judgment of the statistical agency or the user.\(^83\) The point is that the careful and consistent use of discounting should eliminate the effects of general inflation from our price variables; discounting makes comparable cash flows received or paid out at different points of time.

Recall definition (38), which defined $\Pi_n^t$ as the \textit{ex ante real time series depreciation} of an asset that is $n$ periods old at the beginning of period $t$. It is convenient to convert this amount of depreciation into a \textit{percentage} of the initial price of the asset at the beginning of period $t$, $P_n^t$. Thus we define the \textit{ex ante time series depreciation rate} for an asset that is $n$ periods old at the start of period $t$, $\pi_n^t$, as follows:\(^84\)

\(^83\) In particular, it is not necessary for the statistical agency to convert all nominal prices into real prices as a preliminary step before “real” user costs are calculated. The above algebra shows that our nominal user costs $f_n^t$ can also be interpreted as “real” user costs that are expressed in terms of the value of money prevailing at the beginning of period $t$. However, note that typically, real interest and asset specific inflation rates are likely to be more stable than the corresponding nominal rates.

\(^84\) To see that there can be a very large difference between the cross sectional depreciation rate $\delta_n^t$ and the corresponding ex ante time series depreciation rate $\pi_n^t$, consider the case of an asset whose vintages yield exactly the same service for each period in perpetuity. In this case, all of the asset prices by age $P_n^t$ would be identical and the cross sectional depreciation rates $\delta_n^t$ would all be zero. Now suppose a marvelous new invention is scheduled to come on the market next period which would effectively drive the price of this class of assets down to zero. In this case, $i^{*t}$ would be $-1$ and substituting this expected measure of price
Now substitute definition (13) for the cross sectional depreciation rate \( \delta_n^t \) into the nth equation of (41) and we obtain the following expression for the beginning of period \( t \) user cost of an asset that is \( n \) periods old at the start of period \( t \):

\[
(43) \quad f_n^t = (1 + r^{*t})^{-1} [P_n^t (1 + r^{*t}) - (1 + i^{*t}) P_{n+1}^t] \\
= (1 + r^{*t})^{-1} [(1 + r^{*t}) - (1 + i^{*t}) (1 - \delta_n^t)] P_n^t \\
= (1 + r^{*t})^{-1} [r^{*t} + \pi_n^t] P_n^t \\
\]

Thus the period \( t \) user cost for an asset that is \( n \) periods old at the start of period \( t \), \( f_n^t \), can be decomposed into the sum of two terms.\(^{85}\) Ignoring the discount factor, \( (1 + r^{*t})^{-1} \), the first term is \( r^{*t} P_n^t \), which represents the (per unit capital) real interest cost of the financial capital that is tied up in the asset, and the second term is \( \pi_n^t P_n^t = \Pi_n^t \), which represents a concept of national accounts depreciation.

The last line of (43) is important if at some stage statistical agencies decide to switch from measures of gross domestic product to measures of net domestic product as their featured output measure. If this change were to occur, then the user cost for each age of capital, \( f_n^t \), should be split up into two terms as in (43). The first term, \( (1 + r^{*t})^{-1} r^{*t} P_n^t \) times the number of units of that age of capital in use, (the real opportunity cost of financial capital) could remain as a primary input charge while the second term, \( (1 + r^{*t})^{-1} \pi_n^t P_n^t \) times the number of units of that age of capital in use, (real national accounts depreciation) could be treated as an intermediate input charge (similar to the present treatment of imports). The second term would be an offset to gross investment.\(^{86}\)

This completes our discussion of the obsolescence problem.\(^{87}\) In the next section, we turn our attention to the problem of aggregating across vintages of the same capital good.

\(^{85}\) Alternatively, we could decompose \( r^{*t} + \pi_n^t \) into the three terms \( r^{*t} + \delta_n^t + i^t(1-\delta_n^t) \) which is equal to a real interest rate term plus a cross sectional depreciation term plus a revaluation term.

\(^{86}\) Using this methodology, we would say that capital is being maintained intact for the economy if the value of gross investments made during the period (discounted to the beginning of the period) is equal to or greater than the sum of the real national accounts depreciation terms over all assets. This is a maintenance of financial capital concept as opposed to Pigou’s maintenance of physical capital concept: “Net income consists of the whole of the annual output minus what is needed to maintain the stock of capital intact; and this stock is kept intact provided that its physical state is held constant.” A.C. Pigou (1935; 235). We will discuss these points in more detail in chapter VII.

\(^{87}\) It should be noted that our discussion of the obsolescence issue only provides an introduction to the many thorny issues that make this area of inquiry quite controversial. For further discussion, see Oulton (1995),
8. Aggregation over Ages of a Capital Good

In previous sections, we have discussed the beginning of period $t$ stock price $P_n^t$ of an asset that is $n$ periods old and the corresponding beginning and end of period user costs, $f_n^t$ and $u_n^t$. The stock prices are relevant for the construction of real wealth measures of capital and the user costs are relevant for the construction of capital services measures. We now address the problems involved in obtaining quantity series that will match up with these prices.

Let the period $t-1$ investment in a homogeneous asset for the sector of the economy under consideration be $I^{t-1}$. We assume that the starting capital stock for a new unit of capital stock at the beginning of period $t$ is $K_0^t$ and this stock is equal to the new investment in the asset in the previous period; i.e., we assume:

\[(44) \quad K_0^t = I^{t-1}.
\]

Essentially, we are assuming that the length of the period is short enough so that we can neglect any contribution of investment to current production; a new capital good becomes productive only in the period immediately following its construction. In a similar manner, we assume that the capital stock available of an asset that is $n$ periods old at the start of period $t$ is $K_n^t$ and this stock is equal to the gross investment in this asset class during period $t-n-1$; i.e., we assume:

\[(45) \quad K_n^t = I^{t-n-1}; \quad n = 0, 1, 2, \ldots
\]

Given these definitions, the value of the capital stock in the given asset class for the sector of the economy under consideration (the wealth capital stock) at the start of period $t$ is

\[(46) \quad W_t^i = P_0^t K_0^t + P_1^t K_1^t + P_2^t K_2^t + \ldots
\]

\[= P_0^t I^{t-1} + P_1^t I^{t-2} + P_2^t I^{t-3} + \ldots \quad \text{using (45).}
\]

Turning now to the capital services quantity, we assume that the quantity of services that an asset of a particular vintage at a point in time is proportional (or more precisely, is equal) to the corresponding stock. Thus we assume that the quantity of services provided in period $t$ by a unit of the capital stock that is $n$ periods old at the start of period $t$ is $K_n^t$ defined by (45) above. Given these definitions, the value of capital services for all ages of the asset in the given asset class for the sector of the economy under consideration (the productive services capital stock) during period $t$ using the end of period user costs $u_n^t$ defined by equations (15) above is

\[(47) \quad S_t^i = u_0^t K_0^t + u_1^t K_1^t + u_2^t K_2^t + \ldots
\]

Scott (1995) and Triplett (1996) and the references in these papers. We will take up the discussion of obsolescence again when we discuss the Solow Harper model of technical change in chapter VI.
\[ u^t I^{t-1} + u_1^t I^{t-2} + u_2^t I^{t-3} + \ldots \] using (45).

Now we are faced with the problem of decomposing the value aggregates \( W^t \) and \( S^t \) defined by (46) and (47) into separate price and quantity components. If we assume that each new unit of capital lasts only a finite number of periods, \( L \) say, then we can solve this value decomposition problem using normal index number theory. Thus define the period \( t \) vintage stock price and quantity vectors, \( P^t \) and \( K^t \) respectively, as follows:

\[
(48) \quad P^t = [P_0^t, P_1^t, \ldots, P_{L-1}^t] \quad ; \quad K^t = [K_0^t, K_1^t, \ldots, K_{L-1}^t] = [I^t_{L-1}, I^t_{L-2}, \ldots, I^t_{1-L-1}] \quad ; \quad t = 0,1,\ldots,T.
\]

Fixed base or chain indexes may be used to decompose value ratios into price change and quantity change components. In the empirical work which follows, we have used the chain principle.\(^{88}\) Thus the value of the capital stock in period \( t \), \( W^t \), relative to its value in the preceding period, \( W^{t-1} \), has the following index number decomposition:

\[
(49) \quad \frac{W^t}{W^{t-1}} = P(u^{t-1}, u^t, K^{t-1}, K^t) Q(u^{t-1}, u^t, K^{t-1}, K^t) ; \quad t = 1,2,\ldots,T
\]

where \( P \) and \( Q \) are \textit{bilateral price and quantity indexes} respectively.

In a similar manner, we define the period \( t \) vintage end of the period user cost price and quantity vectors, \( u^t \) and \( K^t \) respectively, as follows:

\[
(50) \quad u^t = [u_0^t, u_1^t, \ldots, u_{L-1}^t] \quad ; \quad K^t = [K_0^t, K_1^t, \ldots, K_{L-1}^t] = [I^t_{L-1}, I^t_{L-2}, \ldots, I^t_{1-L-1}] \quad ; \quad t = 0,1,\ldots,T.
\]

We ask that the value of capital services in period \( t \), \( S^t \), relative to its value in the preceding period, \( S^{t-1} \), has the following index number decomposition:

\[
(51) \quad \frac{S^t}{S^{t-1}} = P(u^{t-1}, u^t, K^{t-1}, K^t) Q(u^{t-1}, u^t, K^{t-1}, K^t) ; \quad t = 1,2,\ldots,T
\]

where again \( P \) and \( Q \) are \textit{bilateral price and quantity indexes} respectively.

There is now the problem of choosing the functional form for either the price index \( P \) or the quantity index \( Q \).\(^{89}\) For empirical work, we recommend the \textit{Fisher (1922) ideal price and quantity indexes}. These indexes appear to be “best” from the axiomatic viewpoint\(^{90}\) and can also be given strong economic justifications.\(^{91}\)

It should be noted that our use of an index number formula to aggregate both stocks by age and services by age is more general than the usual aggregation over age procedures, which essentially assume that the different ages of the same capital good are perfectly

\(^{88}\) Given smoothly trending price and quantity data, the use of chain indexes will tend to reduce the differences between Paasche and Laspeyres indexes compared to the corresponding fixed base indexes and so chain indexes are generally preferred; see Diewert (1978; 895) for a discussion.

\(^{89}\) Obviously, given one of these functional forms, we may use (40) to determine the other.

\(^{90}\) See Diewert (1992b; 214-223).

\(^{91}\) See Diewert (1976; 129-134).
substitutable so that linear aggregation techniques can be used. However, as we shall see in the Appendix, the more general method of aggregation suggested here frequently reduces to the traditional linear method of aggregation provided that the vintage asset prices all move in strict proportion over time.

Many researchers and statistical agencies relax the assumption that an asset lasts only a fixed number of periods, L say, and make assumptions about the distribution of retirements around the average service life, L. In Appendix A below that considers different assumptions about the form of cross sectional depreciation, for simplicity, we will stick to the sudden death assumption; i.e., that all assets in the given asset class are retired at age L. However, this simultaneous retirement assumption can readily be relaxed (at the cost of much additional computational complexity) using the following methodology suggested by Hulten:

“We have thus far taken the date of retirement T to be the same for all assets in a given cohort (all assets put in place in a given year). However, there is no reason for this to be true, and the theory is readily extended to allow for different retirement dates. A given cohort can be broken into components, or subcohorts, according to date of retirement and a separate T assigned to each. Each subcohort can then be characterized by its own efficiency sequence, which depends among other things on the subcohort’s useful life T.” Charles R. Hulten (1990; 125).

We now have all of the pieces that are required in order to decompose the capital stock of an asset class and the corresponding capital services into price and quantity components. However, in order to construct price and quantity components for capital services, we need information on the relative efficiencies \( f_n \) of the various vintages of the capital input or equivalently, we need information on cross sectional vintage depreciation rates \( \delta_n \) in order to use (49) and (51) above. The problem is that we do not have accurate information on either of these series so in Appendix A below, we will assume a standard asset life L and make additional assumptions on the either the pattern of vintage efficiencies or depreciation rates. Thus in a sense, we are following the same somewhat mechanical strategy that was used by the early cost accountants:

“The function of depreciation is recognized by most accountants as the provision of a means for spreading equitably the cost of comparatively long lived assets. Thus if a building will be of use during twenty years of operations, its cost should be recognized as operating expense, not of the first year, nor the last, but of all twenty years. Various methods may be proper in so allocating cost. The method used, however, is unimportant in this connection. The important matter is that at the time of abandonment, the cost of the asset shall as nearly as possible have been charged off as expense, under some systematic method.” M.B. Daniels (1933; 303).

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92 This more general form of aggregation was first suggested by Diewert and Lawrence (2000). For descriptions of the more traditional linear method of aggregation, see Jorgenson (1989; 4) or Hulten (1990; 121-127) (1996; 152-165).
93 Recall that Edwards and Bell (1961; 174) made a similar suggestion.
94 Canning (1929; 204) criticized this strategy as follows: “The interminable argument that has been carried on by the text writers and others about the relative merits of the many formulas for measuring depreciation has failed, not only to produce the real merits of the several methods, but, more significantly, it has failed to produce a rational set of criteria of excellence whereby to test the aptness of any formula for any sub-class of fixed assets.”
However, our mechanical strategy is more complex than that used by early accountants in that we translate assumptions about the pattern of cross sectional depreciation rates into implications for the pattern of vintage rental prices and asset prices, taking into account the complications induced by discounting and expected future asset price changes.

In the following section, we discuss the relationship of our suggested procedures with assumptions about the form of the production unit’s production function.

9. The Production Function Framework

9.1 Introduction

“Thus far, however, we have left out of consideration the fact that commodities are products which result from the combination of productive factors such as land, men and capital goods.” Leon Walras (1954; 211).

“Almost all of our theorizing about investment and the desired stock of capital rests implicitly on some technological considerations and is derived from some kind of general production function. As long as we stick to the production function framework, it is clear that quantity rather than value is the relevant dimension, since the production function is defined as a relationship between the quantity of output and the quantity of various inputs.” Zvi Griliches (1963; 118).

In order to measure the contribution of capital to the production of outputs, it is useful to have an idealized model of how capital inputs interact with other flow inputs to produce outputs. The idealized models that economists utilize are based on production functions, or more specifically, on production possibilities sets which are technologically feasible sets of inputs and outputs that can be produced by a specified business unit in a specified time period. There are a number of different production function concepts that can be distinguished. Thus in section 9.2, we discuss the short run production function which distinguishes capital as an input at the beginning of the accounting period and (depreciated) capital as an output at the end of an accounting period. In section 9.3, we consider an intertemporal production function which relates inputs to outputs over many accounting periods. In this production function concept, the capital stocks the firm has available at the start of the first accounting period are distinguished as inputs and the (depreciated) capital stocks at the end of the last accounting period (when the assets of the firm are sold) are distinguished as outputs, but there is no apparent necessity to keep track of used capital inputs in intermediate accounting periods in this framework (unless they are sold before the final period). Purchases of new capital inputs over intermediate periods are distinguished in this framework. In section 9.3, we also attempt to reconcile this intertemporal production function concept with the one period “Austrian” production function concept in section 9.2. In section 9.4, we indicate how the usual one period production function that treats capital just as an input in each accounting period can be extracted from the Austrian production function framework explained in section 9.2.

9.2 The Austrian Production Function

“We must look at the production process during a period of time, with a beginning and an end. It starts, at the commencement of the Period, with an Initial Capital Stock; to this there is applied a Flow Input of
labour, and from it there emerges a Flow Output called Consumption; then there is a Closing Stock of Capital left over at the end. If Inputs are the things that are put in, the Outputs are the things that are got out, and the production of the Period is considered in isolation, then the Initial Capital Stock is an Input. A Stock Input to the Flow Input of labour; and further (what is less well recognized in the tradition, but is equally clear when we are strict with translation), the Closing Capital Stock is an Output, a Stock Output to match the Flow Output of Consumption Goods. Both input and output have stock and flow components; capital appears both as input and as output” John R. Hicks (1961; 23).

“The business firm can be viewed as a receptacle into which factors of production, or inputs, flow and out of which outputs flow...The total of the inputs with which the firm can work within the time period specified includes those inherited from the previous period and those acquired during the current period. The total of the outputs of the business firm in the same period includes the amounts of outputs currently sold and the amounts of inputs which are bequeathed to the firm in its succeeding period of activity.” Edgar O. Edwards and Philip W. Bell (1961; 71-72).

Hicks and Edwards and Bell obviously had the same model of production in mind: in each accounting period, the business or production unit combines the capital stocks and goods in process that it has inherited from the previous period with “flow” inputs purchased in the current period (such as labour, materials, services and additional durable inputs) to produce current period “flow” outputs as well as end of the period depreciated capital stock components, which are regarded as outputs from the perspective of the current period (but will be regarded as inputs from the perspective of the next period). The model could be viewed as an Austrian model of production in honour of the Austrian economist Böhm-Bawerk (1891) who viewed production as an activity which used raw materials and labour to further process partly finished goods into finally demanded goods.96

It should be noted that the neo-Austrian model of Hicks (1973) is different from the model that we are describing in this section. Hicks (1973; 7) interpreted Böhm-Bawerk’s production model as follows:

“Like Böhm-Bawerk (or Hayek) I think of the general productive process as being composed of a number (presumably a large number) of separable elementary processes ... . The elementary process, of the older Austrian theory, was of a simple, too simple, type. There was associated with a unit of output, forthcoming at a particular date, a sequence of units of input at particular previous dates. The sequence of inputs, and the single output, constituted the process.”

Hicks (1973; 8) then asserted that Böhm-Bawerk’s production model was inconsistent with the existence of fixed capital inputs:

95 For more on this model of production and additional references to the literature, see Malinvaud (1953) and the Appendices in Diewert (1977) (1980). Diewert derived the Malinvaud (1953) Hicks (1961) and Edwards and Bell (1961) model of production by specializing the general intertemporal production model of Hicks (1939) to the case of only one period.
96 “They (entrepreneurs) buy goods of remoter rank, such as raw materials, tools, machines, the use of land, and, above all, labour, and, by the various processes of production, transform them into goods of first rank, finished products ready for consumption. ... Goods of remoter rank ... are incapable of satisfying human want; they require first to be changed into consumption goods; and since this process, naturally, takes time, they can only render their services to the wants of a future period—at the earliest, that period distant by the time which the productive process necessarily takes to change them into consumption goods.” Eugen von Böhm-Bawerk (1891; 299-300).
“For the only kind of capital-using production which will fit into the old Austrian scheme is production without fixed capital, production that uses working capital (or circulating capital) only. Fixed capital (plant and machinery) will not fit in. For fixed capital goods are ‘durable-use goods’; their essential characteristic is that they contribute, not just to one unit of output, at one date, but to a sequence of units of output, at a sequence of dates.”

However, Böhm-Bawerk (1891; 299-300) certainly mentioned various durable capital inputs such as “tools”, “machines” and “agricultural implements” as inputs in his model of production; what he did not explain explicitly is how time and use (i.e., depreciation) would transform these inputs into less valuable outputs at the end of a production period. What Böhm-Bawerk emphasized was the transformation of partly finished goods into more valuable partly finished goods and final products.  

It will be useful in what follows to develop some notation to describe the one period Austrian model of production of this section. We suppose that there are M durable inputs that the business unit is using at the beginning of period 0. These durable inputs include machines, transportation equipment, other equipment, computers, plant structures, office buildings, tools, office furnishings and furniture, etc. These fixed capital stock components are classified into discrete categories according to their age and other relevant physical characteristics. The list of durable inputs also includes circulating capital stock components: inventories of raw materials, finished goods and partly finished goods (goods in process). Finally, we include in the business unit’s list of initial capital stock components any patents or other marketable knowledge products as well as any holdings of land or other natural resources that it might possess. We denote the business unit’s beginning of period 0 holdings of durable capital inputs by the nonnegative vector k^0 = [k_1^0,k_2^0,...,k_M^0] where k_m^0 ≥ 0 denotes the initial stock of durable input m for m = 1,....,M. We also suppose that P_m^0 > 0 is the beginning of period 0 market opportunity cost for a unit of durable input m for m = 1,...,M and the vector of these initial market values is P^0 = [P_1^0,P_2^0,...,P_M^0].

Next, we suppose that there are N outputs or inputs that the business unit can sell or purchase in the marketplace during period 0. The vector of average market prices that the business unit faces for the N commodities in period 0 is p^0 = [p_1^0,p_2^0,...,p_N^0] where p_n^0 > 0 is the average market price for commodity n in period 0 for n = 1,...,N. The vector of net outputs that the business unit produces during period 0 is denoted by y^0 = [y_1^0,y_2^0,...,y_N^0]. If y_n^0 > 0, then y_n^0 units of commodity n are being produced by the business unit during period 0 while if y_n^0 < 0, then −y_n^0 > 0 units of commodity n are

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97 Hicks (1973; 5-6) (1965; 238-250) attributed the Austrian model of this section (i.e., the model of Hicks (1961) and Edwards and Bell (1961)) to von Neumann (1937) and Malinvaud (1953), but von Neumann’s model is at best only a special case of our Austrian model. Diewert (1977; 108-111) (1980; 472-475) made extensive use of this Austrian model of production but he regarded it as a special case of the intertemporal production model of Hicks (1946; 193-194) to be studied in the following section.

98 In chapter III below, we will consider various alternative market opportunity costs concepts that might be used. However for now, we think of P_m^0 as being the net realizable value at the beginning of period 0 for a unit of durable input m if no additional units of m are purchased by the business unit during period 0 (this will typically be the case for fixed capital stock components) or P_m^0 is the beginning of period 0 purchase price for a unit of durable input m if additional units of m are purchased during period 0 (this will typically be the case for circulating capital stock components).
being used as inputs during period 0. The list of period 0 “flow” inputs includes: various types of labour services, including both establishment employees and contracted professional services, purchases of electricity, heating fuels and telecommunications services. In principle, the entire list of durable capital stock components could be included in the list of flow inputs, since the business unit could purchase additional units of capital during period 0 to add to its initial stocks. The list of “flow” outputs will include the usual outputs that the business unit produces, classified as finely as seems necessary for the purpose at hand.\textsuperscript{99} In principle, all of the initial capital stock components held by the firm at the start of period 0 could appear in the list of flow outputs, since these stocks 	extit{could} be sold in the marketplace during period 0.\textsuperscript{100} Note that we are distinguishing as separate flow commodities sales of initial capital stock components from additional purchases of capital stock components during period 0 for two reasons: (i) the selling price of an asset will usually differ from the purchase price of a similar unit of the asset due to transactions costs and (ii) the technological impact of the sale of a fixed capital stock component is often quite different from the purchase of an additional unit due to internal transactions costs (such as installation and training costs for purchases and dismantling and renovation costs for sales). Thus the dimensionality of the “flow” commodity space N will generally be much greater than the dimensionality of the initial durable input “stock” space M.

Finally, at the end of period 0, the business unit will have at its disposal a vector \( k^1 \) of durable inputs which can be valued at the end of period 0 (or beginning of period 1) nonnegative market opportunity costs vector \( P^1 = [P_1^1, P_2^1, \ldots, P_M^1] \). The components of \( k^1 \) consist of depreciated units of the business unit’s beginning of period 0 vector of capital stocks \( k^0 \) that were not sold during period 0 plus any additional units of capital that might have been purchased during period 0.\textsuperscript{101}

We now turn our attention to the definition of the period 0 Austrian production function or more generally, the period 0 Austrian production possibilities set for the production unit under consideration. Several definitions are possible. The broadest definition for the period 0 production possibilities set \( S^0 \) is to define \( S^0 \) as the set of all technologically feasible vectors of the form \((k^0, y^0, k^1)\) where \( k^0 \geq 0_M \)\textsuperscript{102} is a nonnegative beginning of period 0 vector of capital inputs, \( y^0 \) is an N dimensional vector of period 0 net outputs that can be produced given \( k^0 \) and \( k^1 \geq 0_M \) is a nonnegative vector of capital stocks that are left over at the end of period 0. In this broadest definition of the period 0 production possibilities set, we allow the production unit to choose its initial vector of capital stocks \( k^0 \). In our next definition of the period 0 production possibilities set, we restrict the

\textsuperscript{99} In some cases, it may be necessary to distinguish the same “physical” commodity by its time of production within the accounting period; e.g., electricity produced during the peak time of a day is more valuable than off-peak electricity; strawberries supplied during off seasons are more valuable than strawberries supplied during the local growing season, etc.

\textsuperscript{100} The underlying accounting framework for inventory items is explained in detail in Diewert and Smith (1994).

\textsuperscript{101} These new purchases would show up as (negative) components of the vector \( y^0 \). Note that we have assumed that the number of components of \( k^0 \) is equal to the number of components of \( k^1 \); M. Since we allow components of \( k^0 \) and \( k^1 \) to be zero, this restriction involves no real loss of generality.

\textsuperscript{102} Notation: \( k^0 \geq 0_M \) means that each component of the M dimensional vector \( k^0 \) is nonnegative.
production unit's choices for the initial capital stock vector \( k^0 \) to be an observed vector of capital stocks \( k^0 \) for the production unit under consideration. In this case, the period 0 production possibilities set for the business unit can be described as a feasible set of net outputs and end of the period capital stocks \( y^0, k^1 \) that could be produced using the observed initial vector of capital stocks \( k^0 \), i.e., the technology set now has the form

\[
(52) \{(y^0,k^1) : (k^0,y^0,k^1) \in S^0\}.
\]

The production possibilities set defined by (52) is a smaller set than the entire set \( S^0 \) where \( k^0 \) is also freely chosen.

The business unit's competitive profit maximization problem that corresponds to the broadest definition of the period 0 production possibilities set \( S^0 \) can be formalized as:

\[
(53) \max_{k^0,y^0,k^1} \{-P^0 \cdot k^0 + (1+r^0)^{-1} p^0 \cdot y^0 + (1+r^0)^{-1} P^1 \cdot k^1 : (k^0,y^0,k^1) \in S^0\}.
\]

where \( P^0 \cdot k^0 = \sum_{m=1}^{M} p^0_m k^0_m \), \( r^0 \) is the period 0 nominal interest rate or opportunity cost of capital prevailing at the beginning of the period and \( S^0 \) is the period 0 production possibilities set for the business unit. Note that we have divided the net “flow” revenues for period 0, \( p^0 \cdot y^0 \), and the market value of the business unit’s end of the period holdings of capital stocks, \( P^1 \cdot k^1 \), by one plus the interest rate, \( (1+r^0) \). Thus we are assuming that period 0 “flow” revenues and costs \( p^0 \cdot y^0 \) are “realized” at the end of period 0 along with the end of period 0 value of the business unit’s capital stocks, \( P^1 \cdot k^1 \). These end of period 0 capital stocks are discounted to make them equivalent to beginning of period 0 values, as is traditional in economics.

However, from the perspective of accounting theory, it is more natural to express all values in terms of end of the period values and thus from this perspective, the business unit's period 0 profit maximization problem becomes:

\[
(54) \max_{k^0,y^0,k^1} \{- (1+r^0) P^0 \cdot k^0 + p^0 \cdot y^0 + P^1 \cdot k^1 : (k^0,y^0,k^1) \in S^0\}.
\]

Note that the objective function in (54) is \( (1+r^0) \) times the objective function in (53) so that the maximization problems (53) and (54) have the same solution sets.

Now let us shift our focus to the business unit’s profit maximization problem in period 1. Let \( S^1 = \{(k^1,y^1,k^2)\} \) denote the business unit’s unrestricted period 1 production possibilities set, which consists of feasible vectors of starting capital stocks \( k^1 \), period 1 “flow” inputs and outputs \( y^1 \) and end of period 1 finishing capital stock vectors \( k^2 \). If there is no technological progress or managerial improvement in the organization of production, \( S^1 \) will equal \( S^0 \); i.e., the period 1 and period 0 production possibilities sets will be the same (but typically, there is technical progress so that the set \( S^1 \) is bigger than the set \( S^0 \)).

The period 1 counterpart to the period 0 unrestricted profit maximization problem (54) is:
(55) \[ \max_{k_1', y_1', k_2'} \{- (1+r^1)p^1 \cdot k_1' + p^1 \cdot y_1' + p^2 \cdot k_2' : (k_1', y_1', k_2') \in S^1 \} \]

where \( r^1 \) is the beginning of period 1 nominal opportunity cost of capital; \( p^1 \) is the vector of beginning of period 1 opportunity costs for capital stock components; \( p^2 \) is the vector of end of period 1 opportunity costs for capital stock components; \( p^1 \) is the vector of period 1 average prices for units of outputs and inputs and \( y_1 \) is a period 1 net output vector (positive components of \( y_1 \) denote outputs, negative components denote inputs).

Obviously, one period profit maximization problems that are analogous to (54) and (55) can be defined for each accounting period \( t \) that the business unit is in operation.

We can also define a one period profit maximization problem that has the same structure as (54) except that the restricted production possibilities set defined by (52) is used in place of \( S^0 \). This restricted period 0 profit maximization problem (with \( k^0 \) restricted to equal the fixed initial capital stock vector \( k^0 \)) is:

(56) \[ \max_{y_0', k_1} \{- (1+r^0)p^0 \cdot k_0 + p^0 \cdot y_0 + p^1 \cdot k_1 : (k_0, y_0, k_1) \in S^0 \} \]

Suppose \( y_0^{*0} \) and \( k_1^{*0} \) solves (56). Then the end of period 0 capital stock vector \( k_1^{*0} \) can serve as a vector of fixed starting capital stocks for the business unit’s period 1 restricted profit maximization problem which is analogous to (55) except that \( k^1 \) is fixed at \( k_1^{*0} \):

(57) \[ \max_{y_1', k_2} \{- (1+r^1)p^1 \cdot k_1^{*0} + p^1 \cdot y_1 + p^2 \cdot k_2 : (k_1^{*0}, y_1', k_2) \in S^1 \} \]

The differences between the two period 0 profit maximization problems (54) and (56) can be explained as follows: in (54), the business unit is allowed to sell its initial holdings of capital (the components of the vector \( k^0 \)) or buy additional units of capital at the beginning of period 0 at the prices \( P^0 \); in (56), the business unit is stuck with its initial holdings of capital \( k^0 \) at the beginning of period 0 (but still values these initial holdings at the prices \( P^0 \)). Thus the different period 0 profit maximization problems reflect different assumptions about what options are open to the business unit at the beginning of period 0. However, for each of the problems, the Austrian production possibilities set \( S^0 \) plays a crucial role.

In the following subsection, we no longer assume that the business unit’s decision horizon is limited to a sequence of single periods; we will allow the business unit to make production plans that extend over a number of periods.

### 9.3 The Fisher–Hicks Intertemporal Production Function

“An option is any possible income stream open to an individual by utilizing his resources, capital, labor, land, money, to produce or secure said income stream. An investment opportunity is the opportunity to shift from one such option, or optional income stream, to another ... Some of the optional income streams, however, would never be chosen, because none of their respective present values could possibly be the maximum.” Irving Fisher (1930; 151).
“The problem of the firm, dynamically considered, is to find that stream of outputs, capable of being produced from the initial equipment, which shall have the maximum capital value ... If we write \( x_0, x_1, x_{21}, \ldots, x_{tv} \) for the [net] outputs of \( x \), planned to be sold in successive ‘weeks’ from the present, then the production function takes the form \( f(x_{10}, x_{20}, \ldots, x_{0}; x_{11}, x_{21}, \ldots, x_{01}; x_{12}, x_{22}, \ldots, x_{02}; \ldots; x_{1v}, x_{2v}, \ldots, x_{0v}) = 0 \) assuming that the plan extends forward for \( v \) weeks. The capitalized value of the plan is \( C = \sum_{n=1}^{\infty} \sum_{t=0}^{T} (\beta^t \cdot x_n) \) where \( \beta_t = 1/(1+i_t) \) and \( i_t \) is the rate of interest per week for loans of \( t \) weeks; \( p_{n0} \) is the current price of \( x_0 \) and \( p_n \) is the price the entrepreneur expects to rule in the week beginning \( t \) weeks hence.” John R. Hicks (1946; 326).

In this section, we will utilize the intertemporal production function concepts developed by Fisher (1930; 151) and Hicks (1946; 136). As in section 9.2, we assume that there are \( M \) types of durable capital equipment and that the business unit's initial holdings of capital stock components at the beginning of period 0 is \( k^0 \equiv [k_1^0, k_2^0, \ldots, k_M^0] \) where \( k_m^0 \geq 0 \) denotes the initial stock of durable input \( m \) for \( m = 1, \ldots, M \). We now assume that the business unit's time horizon extends over \( T \) periods. Denote a vector of planned net outputs for period \( t \) by \( y^t \equiv [y_{1t}, y_{2t}, \ldots, y_{Nt}] \) for \( t = 0, 1, 2, \ldots, T-1 \) and denote the corresponding vector of anticipated average prices for period \( t \) by \( p^t \equiv [p_{1t}, \ldots, p_{Nt}] \geq 0_N \) for \( t = 0, 1, 2, \ldots, T-1 \). At the end of period \( T-1 \) (or equivalently, at the beginning of period \( T \)), we assume that the business unit is sold, i.e., the components of its beginning of period \( T \) capital stock vector \( k^T \equiv [k_1^T, k_2^T, \ldots, k_M^T] \geq 0_M \). The production unit’s intertemporal production possibilities set \( S = \left\{ (k_0^0, y_1^0, y_1^1, \ldots, y_T^1, k^T) \right\} \) is the feasible set of net output vectors \( y_0^0, y_1^1, \ldots, y_T^1 \) for periods 0, 1, \ldots, \( T-1 \) and beginning of period \( T \) (depreciated) capital stock components \( k^T \) that can be produced by an initial vector of capital stocks \( k^0 \) and existing technology (and technology that can be anticipated to exist over the time horizon of the business unit).

Let \( r^t \) be the interest rate or opportunity cost of financial capital that is relevant to the business unit at the beginning of period \( t \) for \( t = 0, 1, 2, \ldots, T-1 \) and let \( p^0 = [p_1^0, p_2^0, \ldots, p_M^0] \geq 0_M \) be the vector of opportunity costs for capital stock components at the start of period 0. Then assuming that per period \( t \) cash flow or net revenues from variable inputs and outputs, \( p^t y^t = \sum_{n=1}^{N} p_n y_n^t \), are “realized” at the end of period \( t \), the business unit’s intertemporal planned profit maximization problem can be written as follows:

\begin{align*}
\text{(58) max}_{k^0, y^0, y^1, \ldots, y^{T-1}, k^T} \quad & \left\{ -p_0^0 k_0^0 + (1+r_0^0)^{-1} p_0^0 y_0^0 + (1+r_0^0)^{-1} (1+r_1^0)^{-1} p_1^1 y_1^1 \right. \\
& \left. \quad + (1+r_0^0)^{-1} (1+r_1^0)^{-1} (1+r_2^0)^{-1} p_2^2 y_2^2 + \ldots + (1+r_0^0)^{-1} (1+r_1^0)^{-1} p_T^T y_T^T \right. \\
& \left. \quad + (1+r_0^0)^{-1} (1+r_1^0)^{-1} \cdots (1+r_T^0)^{-1} p_T^T k_T : (k_0^0, y_0^0, y_1^1, \ldots, y_T^T, k_T) \in S \right\}
\end{align*}

Note that all values in the objective function of (58) that are realized after the beginning of period 0 are discounted by interest rate terms \((1+r^t)\). Thus all values are expressed in beginning of period 0 equivalent values. Note also that the intertemporal profit maximization problem (58) reduces to the single period Austrian profit maximization problem (53) if the business unit's time horizon is only one period; i.e., if \( T = 1 \). Note also that in both (53) and (58), we allowed the initial vector of beginning of period 0 capital

\[ \text{This assumption was used by Hicks (1946; 193-194)}. \]
stocks $k^0$ to be variable. A counterpart to (58) which freezes $k^0$ to equal the business unit’s historically determined capital stocks is:

$$\begin{align*}
\text{(59) } \max_{y^0, y^1, \ldots, y^T} & \{ -P^0 k^0 + (1+r^0)^{-1} p^0 y^0 + (1+r^0)^{-1}(1+r^1)^{-1} p^1 y^1 \\
+ (1+r^0)^{-1}(1+r^1)^{-1} p^2 y^2 + \ldots + (1+r^0)^{-1}(1+r^1)^{-1} \ldots (1+r^{T-1})^{-1} p^{T-1} y^{T-1} \\
+ (1+r^0)^{-1}(1+r^1)^{-1} \ldots (1+r^T)^{-1} p^T y^T \\
& \text{subject to } (k^0, y^0, y^1, \ldots, y^T, k^T) \in S \}. \end{align*}$$

If we divide the objective function in (56) by $(1+r^0)$, it can be seen that the resulting version of (56) is the same problem as (59) if $T = 1$; i.e., the intertemporal profit maximization problem (59) is equivalent to our restricted one period Austrian profit maximization problem (56) when the business unit’s time horizon is only a single period.

In the case where the business unit has a multiperiod planning horizon (i.e., the case where $T > 1$), it is possible to relate the one period technology sets of section 9.2 to the intertemporal technology set $S$ of the present section. Suppose that the one period Austrian technology sets $S^0$, $S^1$, $\ldots$, $S^{T-1}$ are given for periods 0, 1, $\ldots$, $T-1$. Then these Austrian technology sets can be used to define a Hicksian intertemporal technology set $S$ as follows:

$$\begin{align*}
\text{(60) } S &= \{(k^0, y^0, y^1, \ldots, y^{T-1}, k^T) : (k^0, y^0, k^1) \in S^0, (k^1, y^1, k^2) \in S^1, \ldots, (k^{T-1}, y^{T-1}, k^T) \in S^{T-1} \}; \end{align*}$$

i.e., in defining (60), we simply force the end of period $t$ capital stocks $k^{t+1}$ to be equal to the starting capital stocks for period $t+1$ for $t = 0, 1, 2, \ldots, T-1$. Thus we do not allow the business unit to sell or purchase any units of capital at the very end of each period $t$ in definition (60) for each time period.$^{105}$

We conclude section 9.3 by indicating that under certain conditions, solutions to the Hicksian intertemporal profit maximization problem (59) are also solutions to a sequence of Austrian single period profit maximization problems, provided that the period by period capital stock valuation vectors $P^1$, $P^2$, $\ldots$, $P^{T-1}$ that appear in the Austrian problems (but do not appear in (59)) are chosen appropriately. In order to minimize notational complexity, we will demonstrate the above assertion for the case of a two period intertemporal technology; i.e., we will assume $T = 2$ in (59).$^{106}$

Suppose that $y^{0*}$, $y^{1*}$, $k^{2*}$ solves (59) when $T = 2$. Under certain conditions, we can define end of period 0 or beginning of period 1 capital stock price and quantity vectors $P^{1*}$ and $k^{1*}$ such that: (i) $y^{0*}$ and $k^{1*}$ solve the period 0 profit maximization problem (56) provided that $P^1 = P^{1*}$ and (ii) $(k^{1*}, y^{1*}, k^{2*})$ solves the period 1 profit maximization problem (55) provided that the $P^1$ which appears in (55) is equal to $P^{1*}$.

The translation of the last rather technical sentence is this: period by period “Austrian” profit maximization can be consistent with the intertemporal Hicksian profit maximization

$^{104}$ This is the intertemporal profit maximization problem that Hicks (1946; 326) considered.

$^{105}$ Of course units of capital can be bought or sold during period $t$; these purchases or sales are components of $y^t$.

$^{106}$ The reader who is not interested in the technical details of our demonstration can skip to the end of this section.
model (59) provided that the correct “economic” capital stock prices \( P^t \) are used in the single period profit maximization problems.

In order to establish the above assertion, it is necessary to introduce the period \( t \) variable profit function \( \pi^t \) that is dual to the Austrian technology set \( S^t \) for \( t = 0, 1 \).\(^{107}\)

\[
(61) \pi^t(p^i, k^i, k^{i+1}) = \max_y \{p^i'y : (k^i', y, k^{i+1}) \in S^t\}; \quad t = 0, 1.
\]

Using definition (61), the single period constrained profit maximization problem (56) can be rewritten as the following unconstrained profit maximization problem involving only the components of \( k^1 \):

\[
(62) \max_{k^1} \{-(1+t^0)P_0^0 \cdot k^0 + \pi^0(p^0, k^0, k^1) + P^1 \cdot k^1\}.
\]

Suppose that \( k^{1*} \) is a solution to (62) and each component of \( k^{1*} \) is positive; i.e., \( k^{1*} \gg 0_M \). If \( \pi^t(p^t, k^t, k^{t+1}) \) is differentiable with respect to the components of \( k^1 \) at \( k^1 = k^{1*} \), then the vector of first order partial derivatives of \( \pi^0(p^0, k^0, k^{1*}) \) with respect to the components of \( k^1 \),

\[
\nabla_{k^1} \pi^t(p^0, k^0, k^{1*}) = \left[ \frac{\partial \pi^0(p^0, k^0, k^{1*})}{\partial k^1_1}, \ldots, \frac{\partial \pi^0(p^0, k^0, k^{1*})}{\partial k^1_M} \right]
\]

will satisfy the following first order necessary conditions to solve (62).\(^{108}\)

\[
(63) \nabla_{k^1} \pi^0(p^0, k^0, k^{1*}) + P^1 = 0_M.
\]

Now use definition (61) for \( t = 1 \) and rewrite the period 1 constrained profit maximization problem (55) as the following unconstrained profit maximization problem involving the vector of beginning of period 1 capital stocks \( k^1 \) and the vector of end of period 1 capital stocks \( k^2 \):

\[
(64) \max_{k^1, k^2} \{-(1+t^1)P^1 \cdot k^1 + \pi^1(p^1, k^1, k^2) + P^2 \cdot k^2\}.
\]

Suppose that \( k^{1*} \gg 0_M \) and \( k^{2*} \gg 0_M \) are solution vectors for (64) and that \( \pi^1(p^1, k^1, k^2) \) is differentiable with respect to the components of \( k^1 \) and \( k^2 \) at \( (k^1, k^2) = (k^{1*}, k^{2*}) \). Then the vector of first order partial derivatives of \( \pi^1 \) with respect to the components of \( k^1 \),

\[
\nabla_{k^1} \pi^1(p^1, k^{1*}, k^{2*})
\]

and the vector of first order partial derivatives of

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\(^{107}\) We assume that the technology sets \( S^t \) are nonempty closed convex sets subject to free disposal. This will imply that \( \pi^t(p^t, k^t, k^{t+1}) \) will be jointly concave in the components of \( k^t \) and \( k^{t+1} \), nondecreasing in the components of \( k^t \) and nonincreasing in the components of \( k^{t+1} \); see Diewert and Lewis (1982; 303) or Diewert (1985; 226). For duality theorems between \( S^t \) and \( \pi^t \) and references to the literature, see Diewert (1973) (1993b; 165-168).

\(^{108}\) The assumption that \( S^0 \) is a convex set will imply that \( \pi^0(p^0, k^0, k^1) \) is a concave function in the components of \( k^1 \); see Diewert (1973) (1985; 226). Thus conditions (63) are also sufficient to imply that \( k^{1*} \) solves (62).
\( \pi^1 \) with respect to the components of \( k^2 \), \( \nabla_{k^1} \pi^1(p^1,k^{1**},k^{2**}) \) will satisfy the following first order necessary conditions to solve (64):\(^{109}\)

\[
\begin{align*}
(65) \quad & \nabla_{k^1} \pi^1(p^1,k^{1**},k^{2**}) + P^2 = 0_M; \\
(66) \quad & \nabla_{k^1} \pi^1(p^1,k^{1**},k^{2**}) - (1 + r^1)P^1 = 0_M.
\end{align*}
\]

Now assume that the intertemporal production possibilities set \( S \) is constructed using the one period technology sets \( S^0 \) and \( S^1 \) and definition (60) when \( T = 2 \). Using definitions (61), we can rewrite the constrained intertemporal profit maximization problem (59) (when \( T = 2 \)) as the following unconstrained profit maximization problem involving the beginning and end of period 1 capital stock vectors \( k^1 \) and \( k^2 \) as decision variables:\(^{110}\)

\[
(67) \quad \max_{k^1,k^2} \{- (1+r^0)P^0k^0 + \pi^0(p^0,k^0,k^1) + (1+r^1)^{-1}\pi^1(p^1,k^1,k^2) + (1+r^1)^{-1}P^2k^2\}.
\]

Assume that \( k^{1*} \gg 0_M \) and \( k^{2*} \gg 0_M \) solves (67) and that \( \pi^0 \) and \( \pi^1 \) are differentiable with respect to the components of \( k^1 \) and \( k^2 \) when \( (k^1, k^2) = (k^{1*}, k^{2*}) \). Then \( k^{1*} \) and \( k^{2*} \) will satisfy the following first order necessary conditions for solving (67):

\[
\begin{align*}
(68) \quad & \nabla_{k^1} \pi^0(p^0,k^0,k^{1*}) + (1+r^1)^{-1}\nabla_{k^1} \pi^1(p^1,k^{1*},k^{2*}) = 0_M; \\
(69) \quad & (1+r^1)^{-1}\nabla_{k^2} \pi^1(p^1,k^{1**},k^{2**}) + (1+r^1)^{-1}P^2 = 0_M.
\end{align*}
\]

We use the vector of partial derivatives \( \nabla_{k^1} \pi^0(p^0,k^0,k^{1*}) \) in order to define a vector of end of period 0 “economic prices” or shadow prices of capital \( P^{1*}\):\(^{111}\)

\[
(70) \quad P^{1*} = - \nabla_{k^1} \pi^0(p^0,k^0,k^{1*}).
\]

Problems:

1. Let \( k^{1*} \gg 0_M \) and \( k^{2*} \gg 0_M \) solve (67) in the differentiable case. Replace the \( P^1 \) which occurs in the Austrian maximization problems (62) and (64) by the \( P^{1*} \) defined by (70).
   (a) Show that \( k^{1*} \) also solves (62).
   (b) Show that \( k^{1*} \) and \( k^{2*} \) also solve (64).
2. Derive counterparts to the results in problem 1 when \( T \) is increased from 2 to 3.

The thrust of the above algebra is this: under some regularity conditions,\(^{112}\) single period Austrian profit maximization is perfectly consistent with the long run intertemporal

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\(^{109}\) The convexity of \( S^1 \) implies that conditions (65) and (66) are sufficient for \( k^{1**} \gg 0_M \) and \( k^{2**} \gg 0_M \) to solve (64).

\(^{110}\) Intertemporal profit maximization problems of this type are studied in much greater detail in Diewert and Lewis (1982) and Diewert (1985; 225-228).

\(^{111}\) Since \( \pi^0(p^0,k^0,k^1) \) is nonincreasing in the components of \( k^1 \), \( P^{1*} \equiv 0_M \).

\(^{112}\) These regularity conditions are not insignificant. In particular, the following assumptions may not be satisfied: (i) convexity of the one period technology sets \( S^1 \); (ii) differentiability of the variable profit functions \( \pi^1 \) and (iii) the assumption of interior solutions to (67); i.e., that \( k^{1*} \gg 0_M \) for each \( t \).
maximization of profits, provided that the business unit uses “economic” prices to value its end of period capital stock components. The problem with this result is that it is usually difficult to determine these economic prices as outside observers of the business unit (or even as insiders); i.e., at the end of period 0, how can we determine $P^1*$ defined by (70)? In the earlier part of this chapter, we implicitly assumed that these shadow prices could be adequately be approximated by user costs based on information about resale prices for used assets but this approximation may not be adequate. The difficulties involved in the practical determination of “economic prices” explain why most accountants dismiss the use of economic prices as a practical alternative for the valuation of a business unit’s end of the period capital stocks. However, as we shall see in the next two chapters, accountants’ methods for valuing the assets of the firm are often even more arbitrary than those used by economists.

Traditional production function models do not distinguish capital as an input at the beginning of a period and capital as an output at the end of the same period as was done in the Austrian production function. In the following section, we indicate how a traditional production function can be derived from an Austrian production function.

9.4 The Traditional Production Function

“I belong to the party which is still looking to find, at the end of its journey, a rehabilitation of the so-called ‘Production Function’ $P = f(L,C)$ [where $P$ is product, $L$ is labour input and $C$ is capital input] in some form or other; what I am looking for is a concept of capital which will ultimately allow us to think, more or less, in those terms.” John R. Hicks (1961; 18).

“In the context of the Hicksian model, it is clear that we can construct several capital aggregates that must be carefully distinguished: (a) a current period capital stock aggregate (an input from the viewpoint of the current period) using current period capital stock prices as weights in the aggregation procedure; (b) a (depreciated) following period capital stock aggregate (an output from the viewpoint of the current period) using discounted expected following period capital stock prices as weights; (c) a current period investment aggregate (an output) using current period investment goods prices as weights in the aggregation procedure; and (d) a capital aggregate that is an aggregate of (a) and (b) where capital as an input and capital as an output are oppositely signed in the index number formula that is used.” W. Erwin Diewert (1980; 474-475).

We return to the Austrian model of section 9.2 and note that there is an easy way of simplifying the model so that we do not have to distinguish each durable commodity as both an input and an output: we need only use Leontief’s (1936; 54-57) Aggregation Theorem. This result says that if two commodities are always used or produced in fixed proportions by a production unit in each period $t$, then the two commodities can be aggregated into a single composite commodity. More specifically, let $x_1^t$ and $x_2^t$ denote the quantities of say two inputs used during period $t$ and let $p_1^t > 0$ and $p_2^t > 0$ denote the period $t$ average price for each commodity. If $x_1^t = \alpha x_2^t$ for all periods $t$ under consideration, then the two commodities can be aggregated into a composite commodity with period $t$ aggregate input $X^t$ equal to the quantity of input 1 during period $t$; $x_1^t$ and with period $t$ composite price $P^t$ equal to the period $t$ value of the two commodities divided by $x_1^t$; i.e.,
(71) \( X^t = x_1^t \); \( P^t = \left[ p_1^t x_1^t + p_2^t x_2^t \right] / x_1^t = p_1^t + \alpha p_2^t. \)

Definitions (71) can still be used to aggregate the two commodities even if say commodity 1 is an input and commodity 2 is an output; in this case, \( x_1^t \) and \( x_2^t \) have opposite signs and \( \alpha \), the factor of proportionality, is negative.

Now consider the case of a single durable input that lasts 2 or more periods and whose productivity declines only with age (and not use). Suppose that \( k_0^0 > 0 \) units of the (new) durable input were purchased at the start of period 0 at price \( p_0^0 > 0 \), and suppose that the end of period 0 price for depreciated units is \( p_1^1 > 0 \). Then from the perspective of the end of period 0, the net cost of using \( k_0^0 \) units of the durable input during period 0 is

\[
(72) \ (1 + r_0^0) p_0^0 k_0^0 - p_1^1 k_0^0 = u_0^0 k_0^0
\]

where \( u_0^0 \) is an end of period 0 user cost similar to those defined by (10) above. Now define \( x_1^0 = k_0^0 \), \( x_2^0 = -k_0^0 \); \( p_1^0 = (1 + r_0^0) p_0^0 \) and \( p_2^0 = p_1^1 \) and apply Leontief’s Aggregation Theorem. It can be seen that \( X^0 = k_0^0 \) and \( P^0 = u_0^0 \); i.e., the ex post (ex ante if \( p_1^1 \) is an expected price) end of period 0 user cost \( u_0^0 \) can be viewed as the period 0 price for the use of one unit of an aggregate of capital where the two capitals are capital input at the beginning of period 0 and capital output at the end of period 0. The resulting aggregate capital can be viewed as the capital input which appears in a “traditional” production function and a user cost is the price which is associated with the capital aggregate.

Obviously, the above aggregation technique can be applied to all vintages of a capital input provided that declines in value over the period are independent of use. If declines in value are not independent of use, then we need to distinguish different end of period prices that depend on the intensity of use of the durable input over the accounting period. This disaggregation of each type of beginning of the period capital input into separate categories depending on period 0 use can be carried out as finely as seems empirically necessary.\(^{113}\) Thus Leontief’s Aggregation Theorem can be applied to aggregate capital inputs in an Austrian production function even if the value of the assets declines with use as well as with age.

The above aggregation technique will not work for assets that lose their identity during the period 0 production process; e.g., a computer chip on hand at the beginning of the period emerges as part of a computer at the end of the period or a concrete foundation at the beginning of the period becomes part of a building at the end of the period, etc. We will look at the treatment of goods in process and other inventory items in chapter V.

The production theory framework explained in this section will be helpful in addressing some specific measurement issues in subsequent chapters (such as the measurement of income and inventory change).

10. The Treatment of Business Income Taxes

\(^{113}\) Again, this observation is due to Edwards and Bell (1961; 174).
There are a number of possible methods for incorporating business income taxes into a user cost formula. We consider three approaches in this section.

The first approach is the simplest. Assume that the firm sets up a leasing unit that purchases a capital input at price \( P^0 \) at the beginning of the period, rents out the services of the capital to the firm during the period and then “sells” the depreciated asset at the end of the period at the price \( P^1(1-\delta^0) \). The problem for the leasing unit is to determine an appropriate rental price \( p^0_K \) for the use of the durable input during the period. We need a few more assumptions in order to accomplish this task. Thus we assume that the leasing firm is financed by equity capital and the appropriate (nominal) beginning of the period opportunity cost of capital is \( r^0 \). We assume that the rental is paid at the end of the period as are (possible) property taxes (or other specific taxes on the asset) and the business income tax. We assume that the per unit capital property tax is \( \tau^0 P^0 \) (where \( \tau^0 \) is the appropriate specific tax rate) and the business income tax rate is \( t^0 \). Taxable income per unit of capital employed is defined as rental income, \( p^0_K \), less allowable depreciation expense, \( \alpha^0 \delta^0 P^0 \), where economic depreciation is \( \delta^0 P^0 \) and \( \alpha^0 \) is the proportion of economic depreciation that the tax authorities allow the firm to deduct from taxable income in period 0, less property taxes, \( \tau^0 P^0 \); i.e., we have:

\[
\text{(73) Period 0 taxable income per unit capital employed} = p^0_K - \alpha^0 \delta^0 P^0 - \tau^0 P^0.
\]

Now the leasing unit can determine the appropriate rental price for a unit of capital, \( p^0_K \), by solving the following equation for \( p^0_K \): the initial beginning period purchase price \( P^0 \) times one plus the opportunity cost of capital should equal the depreciated end of period value \( P^1(1-\delta^0) \) plus the rental income \( p^0_K \) less the specific taxes payable \( \tau^0 P^0 \) less the business income tax rate \( t^0 \) times taxable income; i.e., solve the following equation for \( p^0_K \):

\[
\text{(74) } P^0(1+r^0) = P^1(1-\delta^0) + p^0_K - \tau^0 P^0 - t^0 [p^0_K - \alpha^0 \delta^0 P^0 - \tau^0 P^0].
\]

The solution to (74) is:

\[
\text{(75) } p^0_K = \frac{(1-t^0)^{-1} \{P^0(1+r^0) - P^1(1-\delta^0) - t^0 \alpha^0 \delta^0 P^0\} + \tau^0 P^0}{1}.
\]

If there is no asset price inflation during the course of period 0 so that \( P^0 = P^1 \) and if the tax authorities set the allowed depreciation equal to economic depreciation so that \( \alpha^0 = 1 \), then (75) simplifies dramatically to the following formula for the rental price:

\[
\text{(76) } p^0_K = [\delta^0 + (1-t^0)^{-1} r^0 + \tau^0] P^0.
\]

\(^{114}\) The “sale” can be back to the leasing unit.

\(^{115}\) This approach to determining the user cost of capital in the context of a business income tax was explained in Diewert (1980; 470-471). The original approaches to incorporating the business income tax into a user cost formula (using a continuous time formulation) are due to Jorgenson (1963) and Hall and Jorgenson (1967).
Thus in this case, the rental price is equal to the asset purchase price \( P^0 \) times the economic depreciation rate \( \delta^0 \) plus the tax augmented opportunity cost of capital \( (1-t^0)^{-1}r^0 \) plus the asset specific tax rate \( \tau^0 \). In this simple case, the intent of the business income tax is made clear: the intent is to tax the real return to capital. Unfortunately, in the general case when there is asset price inflation during the period or when the tax authorities do not know what the economic depreciation rate is (or set tax depreciation rates artificially low or high for other purposes), the original purpose of the business income tax is lost and a considerable amount of deadweight loss to the economy can result.\(^{116}\)

We turn now to our second approach for deriving a user cost formula when there is a business income tax. In this second approach, we introduce the production function, \( Y = F(L,K) \) into the model, where \( F \) is the production function, \( Y \) is the output that can be produced by \( L \) units of variable nondurable inputs (labour say) and \( K \) units of durable capital. We make the same assumptions as above with respect to capital and the tax regime and in addition assume that the end of period price for output and labour is \( p_Y^1 \) and \( w^1 \) respectively. Taxable income for the producer is now defined (as functions of \( Y, L \) and \( K \)) as follows:

\[
(77) \text{Taxable income} \equiv p_Y^1 Y^1 - w^1 L^1 - \left[ \alpha^0 \delta^0 P^0 + \tau^0 P^0 \right] K. 
\]

The net cost of buying one unit of capital at the beginning of the period, using it during the period and selling it (possibly to itself) at the end of the period, ignoring income tax, is

\[
(78) \text{End of period user cost ignoring income tax} \equiv P_0^0 (1+r^0) - P^1 (1-\delta^0) + \tau^0 P^0 
\]

Using (77) and (78), the firm’s period 0 profit maximization problem can be written as follows:

\[
(78) \max_{Y,L,K} \{ p_Y^1 Y^1 - w^1 L^1 - \left[ P^0_0 (1+r^0) - P^1 (1-\delta^0) + \tau^0 P^0 \right] K 
- t^0 \left[ p_Y^1 Y^1 - w^1 L^1 - \left[ \alpha^0 \delta^0 P^0 + \tau^0 P^0 \right] K \right] : Y = F(L,K) \}
\]

\[
(79) = (1-t^0) \max_{Y,L,K} \{ p_Y^1 Y^1 - w^1 L^1 - p_K^0 K : Y = F(L,K) \}
\]

where (79) follows from (78) by substituting the tax adjusted rental price of capital \( p_K^0 \) defined by (75) into (78). Thus our second approach is equivalent to our first approach.\(^{117}\)

**Problems**

3. Modify the above two approaches by assuming that the firm is financed entirely by debt and interest is tax deductible.

\(^{116}\) This is another large topic which we will not examine here.

\(^{117}\) This equivalence result may be found in Diewert (1980; 471).
4. Modify the above two approaches by assuming that capital purchases are financed partly by debt and partly by equity.

Obviously the above two approaches to dealing with capital taxation in a user cost context become much more complicated as we model in more detail the intricacies of the business income tax in most countries. Hence a third approach is sometimes the only feasible one and that is to assume that business income taxes fall on the rate of return to capital and to simply treat them as specific taxes on each capital stock component; i.e., we treat business income taxes in much the same manner as we treated specific property taxes in the above two approaches. This approach is not as theoretically sound as the first two approaches but sometimes data limitations will force us to adopt it.

Appendix A: Alternative Models of Depreciation

A1. The One Hoss Shay Model of Efficiency and Depreciation

In section 3 above, we noted that Böhm-Bawerk (1891; 342) postulated that an asset would yield a constant level of services throughout its useful life of $L$ years and then collapse in a heap to yield no services thereafter. This has come to be known as the one hoss shay or light bulb model of depreciation. Hulten noted that this pattern of relative efficiencies has the most intuitive appeal:

“Of these patterns, the one hoss shay pattern commands the most intuitive appeal. Casual experience with commonly used assets suggests that most assets have pretty much the same level of efficiency regardless of their age— a one year old chair does the same job as a 20 year old chair, and so on.” Charles R. Hulten (1990; 124).

Thus the basic assumptions of this model are that the period $t$ efficiencies and hence cross sectional rental prices $f_n$ are all equal to say $f$ for vintages $n$ that are less than $L$ periods old and for older vintages, the efficiencies fall to zero. Thus we have:

$$\begin{align*}
(A1) \quad f_n &= f \quad \text{for } n = 0, 1, 2, \ldots, L-1; \\
&= 0 \quad \text{for } n = L, L+1, L+2, \ldots
\end{align*}$$

Now substitute (A1) into the first equation in (5) and get the following formula\textsuperscript{118} for the rental price $f$ in terms of the price of a new asset at the beginning of year $t$, $P_0$:

$$\begin{align*}
(A2) \quad f &= P_0^\gamma \left[ 1 + (\gamma^\prime)^t + (\gamma^\prime)^{2t} + \ldots + (\gamma^\prime)^{(L-1)t} \right]
\end{align*}$$

where the period $t$ discount factor $\gamma^\prime$ is defined in terms of the period $t$ nominal interest rate $r^\prime$ and the period $t$ expected asset inflation rate $i^\prime$ as follows:

\textsuperscript{118} This formula simplifies to $P_0^\gamma [1-(\gamma^\prime)^t]/[1-\gamma^\prime]$ provided that $\gamma^\prime$ is less than 1 in magnitude. However, this restriction on the magnitude of $\gamma^\prime$ does not always hold empirically. However, (A2) is still valid under these conditions.
(A3) \( \gamma^t \equiv (1 + i^t)/(1 + r^t) \).

Now that the period \( t \) rental price \( f^t \) for an unretired asset has been determined, substitute equations (A1) into equations (5) and determine the sequence of period \( t \) vintage asset prices, \( P_{n}^t \).119

\[
(A4) P_{n}^t = f^t \left[ 1 + (\gamma^t) + (\gamma^t)^2 + \ldots + (\gamma^t)^{L-1-n} \right] \quad \text{for } n = 0, 1, 2, \ldots, L-1
\]

\[
= 0 \quad \text{for } n = L, L+1, L+2, \ldots.
\]

Finally, use equations (9) to determine the end of period \( t \) rental prices, \( u_{n}^t \), in terms of the corresponding beginning of period \( t \) rental prices, \( f_{n}^t \):

\[
(A5) u_{n}^t = (1 + r^t)f_{n}^t; \quad n = 0, 1, 2, \ldots
\]

Given the vintage asset prices defined by (A4), we could use equations (13) above to determine the corresponding vintage cross section depreciation rates \( \delta_{n}^t \).

**A2. The Straight Line Depreciation Model**

The straight line method of depreciation is very simple in a world without price change: one simply makes an estimate of the most probable length of life for a new asset, \( L \) periods say, and then the original purchase price \( P_{0}^t \) is divided by \( L \) to yield as estimate of period by period depreciation for the next \( L \) periods. In a way, this is the simplest possible model of depreciation, just as the one hoss shay model was the simplest possible model of efficiency decline.120 The accountant Canning summarizes the straight line depreciation model as follows:

“Straight Line Formula ... In general, only two primary estimates are required to be made, viz., scrap value at the end of \( n \) periods and the numerical value of \( n \). ... Obviously the number of periods of contemplated use of an asset can seldom be intelligently estimated without reference to the anticipated conditions of use. I the formula is to be respectable at all, the value of \( n \) must be the most probable number of periods that will yield the most economical use.” John B. Canning (1929; 265-266).

The following quotations indicate that the use of straight line depreciation dates back to the 1800’s at least:

“Sometimes an equal installment is written off every year from the original value of the plant; sometimes each machine or item of plant is considered separately; but it is more usual to write off a percentage, not of

119 Note that all of the asset prices \( P_{n}^t \) will vary in strict proportion over time provided that the discount factor \( \gamma^t \) is constant over time, which will be the case if we can assume that the real interest rate \( r^t \) is constant over time. In this constant real interest rate case, we can apply Hicks’ (1946; 312-313) Aggregation Theorem in order to form a capital stock aggregate over vintages. The Theorem says that if all prices in the aggregate move in strict proportion over time, then any one of these prices can be taken as the price of the aggregate. The corresponding quantity aggregate is equal to the value aggregate divided by the chosen price.

120 In fact, it can be verified that if the nominal interest rate \( r^t \) and the nominal asset inflation rate \( i^t \) are both zero, then the one hoss shay efficiency model will be entirely equivalent to the straight line depreciation model.
the original value, but from the balance of the plant account of the preceding year.” Ewing Matheson (1910; 55).

“In some instances the amount charged to revenue account for depreciation is a fixed sum, or an arbitrary percentage on the book value.” Emile Garcke and John Manger Fells (1893; 98).

The last two quotations indicate that the declining balance or geometric depreciation model (to be considered in the next section) also dates back to the 1800’s as a popular method for calculating depreciation.

We now set out the equations which describe the straight line model of depreciation in the general case when the anticipated asset inflation rate $i_t$ is nonzero. Assuming that the asset has a life of $L$ periods and that the cross sectional amounts of depreciation $D_{n,t} = P_n - P_{n+1}$ defined by (11) above are all equal for the assets in use, then it can be seen that the beginning of period $t$ vintage asset prices $P_n$ will decline linearly for $L$ periods and then remain at zero; i.e., the $P_n$ will satisfy the following restrictions:

\[(A6) \quad P_n^t = P_0^t \frac{[L - n]}{L} \quad \text{for } n = 0,1,2,...,L \]
\[= 0 \quad \text{for } n = L+1,L+2,... \]

Recall definition (13) above, which defined the cross sectional depreciation rate for an asset that is $n$ periods old at the beginning of period $t$, $\delta_n^i$. Using (A6) and the $n$th equation in (13), we have:

\[(A7) \quad (1 - \delta_0^i)(1 - \delta_1^i)\ldots(1 - \delta_{n-1}^i) = P_n^i / P_0^i = 1 - (n/L) \quad \text{for } n = 1,2,...,L. \]

Using (A7) for $n$ and $n+1$, it can be shown that

\[(A8) \quad (1 - \delta_n^i) = \frac{[L - (n+1)]}{[L - n]} \quad \text{for } n = 0,1,2,...,L -1. \]

Now substitute (A7) and (A8) into the general user cost formula (15) in order to obtain the period $t$ end of the period straight line user costs, $u_n^i$:

\[(A9) \quad u_n^i = (1 - \delta_0^i)\ldots(1 - \delta_{n-1}^i)[(1+r^i) - (1+i^t)(1 - \delta_n^i)] P_0^i \quad \text{for } n = 0,1,2,...,L -1 \]
\[= [1 - (n/L)][(1+r^i) - (1+i^t)((L - (n+1))/[L - n])] P_0^i. \]

Equations (A6) give us the sequence of asset prices by age that are required to calculate the wealth capital stock while equations (A9) give us the user costs by age that are required to calculate capital services for the asset. It should be noted that if the anticipated asset inflation rate $i_t$ is large enough compared to the nominal interest rate $r^i$, then the user cost $u_n^i$ can be negative. This means that the corresponding asset becomes an output rather than an input for period $t$.\footnote{The user costs for $n = L, L+1,L+2,...$ are all zero.}

\footnote{However, one is led to wonder if the model is reasonable if some vintages of the asset have negative user costs while other vintages have positive one. As we noted before, it is not really reasonable to have ex ante negative user costs (but it is quite reasonable to have negative ex post user costs).}
A3. The Declining Balance or Geometric Depreciation Model

The declining balance method of depreciation dates back to Matheson (1910; 55). In terms of the algebra presented in section 4 above, the method is very simple: all of the cross sectional vintage depreciation rates \( \delta_n \) defined by (13) are assumed to be equal to the same rate \( \delta \), where \( \delta \) a positive number less than one; i.e., we have for all time periods \( t \):

\[
(A10) \quad \delta_n = \delta; \quad n = 0,1,2,\ldots.
\]

Substitution of (A10) into (15) leads to the following formula for the sequence of period \( t \) user costs by age:

\[
(A11) \quad u_n^t = (1 - \delta)^{n-1} \left[ (1+r^t) - (1+i^t)(1-\delta) \right] P_0^1; \quad n = 0,1,2,\ldots
\]

The second set of equations in (A11) says that all of the vintage user costs are proportional to the user cost for a new asset. This proportionality means that we do not have to use an index number formula to aggregate over assets by age to form a capital services aggregate. To see this, using (A11), the period \( t \) services aggregate \( S^t \) defined earlier by (47) can be rewritten as follows:

\[
(A12) \quad S^t \equiv u_0^t K_0^t + u_1^t K_1^t + u_2^t K_2^t + \ldots
\]

where the period \( t \) capital aggregate \( K_A^t \) is defined as

\[
(A13) \quad K_A^t \equiv K_0^t + (1-\delta) K_1^t + (1-\delta)^2 K_2^t + \ldots
\]

If the depreciation rate \( \delta \) and the vintage capital stocks are known, then \( K_A^t \) can readily be calculated using (A13). Then using the last line of (A12), we see that the value of capital services (over all vintages), \( S^t \), decomposes into the price term \( u_0^t \) times the quantity term \( K_A^t \). Hence, it is not necessary to use an index number formula to aggregate over vintages using this depreciation model.

A similar simplification occurs when calculating the wealth stock using this depreciation model. Substitution of (A10) into (14) leads to the following formula for the sequence of period \( t \) vintage asset prices:

\[
(A14) \quad P_n^t = (1-\delta)^{n-1} P_0^t; \quad n = 1,2,\ldots
\]

\[123\] Matheson (1910; 91) used the term “diminishing value” to describe the method. Hotelling (1925; 350) used the term “the reducing balance method” while Canning (1929; 276) used the term the “declining balance formula”.

Equations (A14) say that all of the vintage asset prices are proportional to the price of a new asset. This proportionality means that again, we do not have to use an index number formula to aggregate over vintages to form a capital stock aggregate. To see this, using (A14), the period t wealth aggregate \( W_t \) defined earlier by (46) can be rewritten as follows:

\[
(A15) \quad W_t = P_0^t K_0^t + P_1^t K_1^t + P_2^t K_2^t + \ldots = P_0^t [K_0^t + (1 - \delta) K_1^t + (1 - \delta)^2 K_2^t + \ldots] = P_0^t K_A^t
\]

where \( K_A^t \) was defined by (A13). Thus \( K_A^t \) can serve as both a capital stock aggregate or a flow of services aggregate, which is a major advantage of this model.\(^{124}\)

There is a further simplification of the model which is useful in applications. If we compare equation (55) for period \( t+1 \) and period \( t \), we see that the following formula results using equations (39):

\[
(A16) \quad K_A^{t+1} = K_0^{t+1} + (1 - \delta) K_A^t.
\]

Thus the period \( t+1 \) aggregate capital stock, \( K_A^{t+1} \), is equal to the investment in new assets that took place in period \( t \), which is \( K_0^{t+1} \), plus \( 1 - \delta \) times the period \( t \) aggregate capital stock, \( K_A^t \). This means that given a starting value for the capital stock, we can readily update it just using the depreciation rate \( \delta \) and the new investment in the asset during the prior period.

We now look at the ancient accounting and engineering literature for some hints on how to determine the geometric depreciation rate \( \delta \) for a particular asset class. Matheson was perhaps the first engineer to address this problem. On the basis of his experience, he simply postulated some approximate rates that could be applied:

“In most [brick or stone] factories an average of 3 per cent for buildings will generally be found appropriate, if due attention is paid to repairs. Such a rate will bring down a value of £ 1000 to £ 400 in thirty years.” Ewing Matheson (1910; 69).

“Buildings of wood or iron would require a higher rate, ranging from 5 to 10 per cent, according to the design and solidity of the buildings, the climate, the care and the regularity of the painting, and according also, to the usage they are subjected to.” Ewing Matheson (1910; 69).

“Contractors’ locomotives working on imperfect railroads soon wear out, and a rate of 20 per cent is generally required, bringing down the value of an engine costing £ 1000 to £ 328 in five years.” Ewing Matheson (1910; 86).

“In engineering factories, where the work is of a moderate kind which does not strain the machines severely, and where the hours of working do not average more than fifty per week, 5 per cent written off each year from the diminishing value will generally suffice for the wear-and-tear of machinery, cranes and fixed plant of all kinds, if steam engines and boilers be excluded.” Ewing Matheson (1910; 82).

\(^{124}\) This advantage of the model has been pointed out by Jorgenson (1989) (1996b) and his coworkers. Its early application dates back to Jorgenson and Griliches (1967) and Christensen and Jorgenson (1969) (1973).
“The high speed of the new turbo generators introduced since 1900, and their very exact fitting, render them liable to certain risks from variations in temperature and other causes. Several changes in regard to speed and methods of blading have occurred since their first introduction and if these generators are taken separately, only after some longer experience has been acquired can it be said that a depreciation rate of 10 per cent on the diminishing value will be too much for maintaining a book-figure appropriate to their condition. Such a rate will reduce £ 1000 to £ 349 in ten years.” Ewing Matheson (1910; 91).

How did Matheson arrive at his estimated depreciation rates? He gave some general guidance as follows:

“The main factors in arriving at a fair rate of depreciation are:

1. The Original value.
2. The probable working Life.
3. The Ultimate value when worn out or superceded.

Therefore, in deciding upon an appropriate rate of depreciation which will in a term of years provide for the estimated loss, it is not the original value or cost which has to be so provided for, but that cost less the ultimate or scrap value.” Ewing Matheson (1910; 76).

The algebra corresponding to Matheson’s method for determining $\delta$ was explicitly described by the accountant Canning (1929; 276). Let the initial value of the asset be $V_0$ and let its scrap value $n$ years later be $V_n$. Then $V_0$, $V_n$ and the depreciation rate $\delta$ are related by the following equation:

\[(A17) \quad V_n = (1 - \delta)^n V_0.\]

Canning goes on to explain that $1 - \delta$ may be determined by solving the following equation:

\[(A18) \quad \log (1 - \delta) = \frac{\log V_n - \log V_0}{n}.\]

It is clear that Matheson used this framework to determine depreciation rates even though he did not lay out formally the above straightforward algebra.

However, Canning had a very valid criticism of the above method:

“This method can be summarily rejected for a reason quite independent or the deficiencies of formulas 1 and 2 above [(A17) and (A18) above]. Overwhelming weight is given to $V_n$ in determining book values. ... Thus the least important constant in reality is given the greatest effect in the formula.” John B. Canning (1929; 276).

Thus Canning pointed out that the scrap value, $V_n$, which is not determined very accurately from an a priori point of view, is the tail that is wagging the dog; i.e., this poorly determined value plays a crucial role in the determination of the depreciation rate.\(^{125}\)

\(^{125}\)“There may be cases in which the formula fits the facts, but ... the chance of its being a formula of close fit is remote indeed. Its chief usefulness seems to be to furnish drill in the use of logarithms for students in accounting.” John B. Canning (1929; 277).
An effective response to Canning’s criticism of the declining balance method of depreciation did not emerge until relatively recently when Hall (1971), Beidelman (1973) (1976) and Hulten and Wykoff (1981a) (1981b) used an entire array of used asset prices at point in time in order to determine the geometric depreciation rate which best matched up with the data. As we noted in section 6 above, another theoretical possibility would be to use information on vintage rental prices in order to deduce the depreciation rate. Hulten and Wykoff summarize their experience in estimating depreciation rates from used asset prices by concluding that the assumption of geometric or declining balance depreciation described their data relatively well:

“We have used the approach to study the depreciation patterns of a variety of fixed business assets in the United States (e.g., machine tools, construction equipment, autos and trucks, office equipment, office buildings, factories, warehouses, and other buildings). The straight line and concave patterns [i.e., one hoss shay patterns] are strongly rejected; geometric is also rejected, but the estimated patterns are extremely lose to (though steeper than) the geometric form, even for structures. Although it is rejected statistically, the geometric pattern is far closer than either of the other two candidates. This leads us to accept the geometric pattern as a reasonable approximation for broad groups of assets, and to extend our results to assets for which no resale markets exist by imputing depreciation rates based on an assumption relating the rate of geometric decline to the useful lives of assets.” Charles C. Hulten and Frank C. Wykoff (1996; 16).

This brings us to our next problem: how can we convert our asset lives expressed in years until retirement into geometric rates?

One possible method for converting an average asset life, L periods say, into a comparable geometric depreciation rate is to argue as follows. Suppose that we believe that the straight line model of depreciation is the correct one and the asset under consideration has a useful life of L periods. Suppose further that investment in this type of asset is constant over time at one unit per period and asset prices are constant over time. Under these conditions, the long run equilibrium capital stock for this asset would be:

\[(A19) \quad 1 + [(L-1)/L] + [(L-2)/L] + ... + [2/L] + [1/L] = L(L+1)/2L = (L+1)/2.\]

---

126 Jorgenson (1996a) has a nice review of most of the empirical studies of depreciation. It should be noted that Beidelman (1973) (1976) and Hulten and Wykoff (1981a) (1996; 22) showed that equation (A17) must be adjusted to correct for the early retirement of assets. The accountant Schmalenbach (1959; 91) (the first German edition was published in 1919) also noticed this problem: “The mistake should not be made, however, of drawing conclusions about useful life from those veteran machines which are to be seen in most businesses. Those which one sees are but the rare survivors; the many dead have long lain buried. This can be the source of serious errors.”

127 This possibility is mentioned by Hulten and Wykoff (1996; 15): “In other words, if there were active rental markets for capital services as there are for labor services, the observed prices could be used to estimate the marginal products. And the rest of the framework would follow from these estimates. But, again, there is bad news: most capital is owner utilized, like much of the stock of single family houses. This means that owners of capital, in effect, rent it to themselves, leaving no data track for the analyst to observe.”

128 Recall equations (A6), which imply that the vintage asset prices are proportional. Hence Hicks’ Aggregation Theorem will imply that the capital aggregate will be the simple sum on the left hand side of (A19).
Under the same conditions, the long run equilibrium geometric depreciation capital stock would be equal to the following sum:

\[(A20) \ 1 + (1-\delta) + (1-\delta)^2 + ... = 1/[1-(1-\delta)] = 1/\delta.\]

Now find the depreciation rate \(\delta\) which will make the two capital stocks equal; i.e., equate (61) to (62) and solve for \(\delta\). The resulting \(\delta\) is:

\[(A21) \ \delta = 2/(L+1).\]

Obviously, there are a number of problematical assumptions that were made in order to derive the depreciation rate \(\delta\) that corresponds to the length of life \(L\), but (A21) gives us at least a definite method of conversion from one model to the other.

As an example of how the conversion formula (A21) might work, consider the case of nonresidential construction. Maddison (1993) assumed that the average length of life for nonresidential construction \(L\) was equal to 39 years. Applying the conversion formula (A21) in this case implies that the corresponding geometric depreciation rate equals .05. Similarly, Maddison’s assumed life of 14 years for machinery and equipment translates into a geometric depreciation rate \(\delta\) equal to a 13 1/3% for this asset class.

There is one remaining problem to deal with in the context of the geometric depreciation model that is not present in the other models which assume finite lives for the assets. The problem is this: since the geometric model implies that the effects of past investments linger on forever, no finite set of investment data can give us a completely accurate starting value for geometric capital stock components. To solve this problem, we outline an approximate method used by Kohli (1982).

Suppose that our investment data on an asset class start at year 0 and end at year \(T+1\). Calculate the average geometric rate of growth in (real) investment for this asset class over the sample time period, \(g\) as:

\[(A22) \ 1+g \equiv (I_T/I_0)^{(1/T)}.\]

Now assume that investments in this asset class prior to time period 0 grew at the same rate as the geometric rate of growth within the sample period. Thus investment in period

\[129\ \text{The two assumptions that are the least justified are: (1) the assumption that the straight line depreciation model is the correct model to do the conversion and (2) the assumption that investment has been constant back to minus infinity. Hulten and Wykoff (1996; 16) made the following suggestions for converting an} \ L \ \text{into a} \ \delta: \ \text{“Information is available on the average service life,} \ L, \ \text{from several sources. The rate of depreciation for non-marketed assets can be estimated using a two step procedure based on the ‘declining balance’ formula} \ \delta = X/L. \ \text{Under the ‘double declining balance’ formula,} \ X = 2. \ \text{The value of} \ X \ \text{can be estimated using the formula} \ X = \delta L \ \text{for those assets for which these estimates are available. In the Hulten-Wykoff studies, the average value for} \ X \ \text{for producer’s durable equipment was found to be 1.65 (later revised to 1.86). For nonresidential structures,} \ X \ \text{was found to be 0.91. Once} \ X \ \text{is fixed,} \ \delta \ \text{follows for other assets whose average service life is available.”} \]

\[130\ \text{See also Fox and Kohli (1998).}\]
\(-1\) is assumed to be \(I^0/(1+g)\), in year \(-2\) is assumed to be \(I^0/(1+g)^2\) and so on. Assuming that the geometric depreciation rate for the asset class under consideration is \(\delta\), we see that the capital stock \(K^0\) at the start of period 0 should be equal to the following sum under our assumptions:

\[
\text{(A23)} \quad K^0 = I^{-1} + (1-\delta) I^{-2} + (1-\delta)^2 I^{-3} + ... \\
= (1+g)^{-1}I^0 + (1-\delta)(1+g)^{-2}I^0 + (1-\delta)^2(1+g)^{-3}I^0 + ... \\
= (1+g)^{-1}I^0 [1 + (1-\delta)(1+g)^{-1} + (1-\delta)^2(1+g)^{-2} + ... ] \\
= (1+g)^{-1}I^0 \left[ 1 - \{(1-\delta)(1+g)^{-1}\}^{-1} \right] \text{ assuming } (1-\delta)(1+g)^{-1} \text{ is less than 1} \\
= (1+g)^{-1}I^0 (1+g)/(\delta+g) \\
= I^0/(\delta+g).
\]

A4. The Linear Efficiency Decline Model

Recall that our first class of models (the one hoss shay models) assumed that the efficiency (or cross section user cost) of the asset remained constant over the useful life of the asset. In our second class of models (the straight line depreciation models), we assumed that the cross sectional depreciation of the asset declined at a linear rate. In our third class of models (the geometric depreciation models), we assumed that cross sectional depreciation declined at a geometric rate. Comparing the third class with the second class of models, it can be seen that geometric depreciation is more accelerated than straight line depreciation; i.e., depreciation is relatively large for new vintages compared to older ones. In this section, we will consider another class of models that gives rise to an accelerated pattern of depreciation: the class of models that exhibit a linear decline in efficiency.

It is relatively easy to develop the mathematics of this model. Let \(f^t_0\) be the period \(t\) rental price for an asset that is new at the beginning of period \(t\). If the useful life of the asset is \(L\) years and the efficiency decline is linear, then the sequence of period \(t\) cross sectional user costs \(f^t_n\) is defined as follows:

\[
\text{(A24)} \quad f^t_n = f^t_0 \left[ L - n \right]/L ; \quad n = 0,1,2,...,L - 1 ; \\
= 0 \quad ; \quad n = L,L+1,L+2, ... .
\]

Now substitute (A24) into the first equation in (5) and get the following formula for the rental price \(f^t_0\) in terms of the price of a new asset at the beginning of year \(t\), \(P^t_0\):

\[
\text{(A25)} \quad f^t_0 = LP^t_0 \left[ L + (L-1)(\gamma^t) + (L-2)(\gamma^t)^2 + ... + 1(\gamma^t)^{L-1} \right]
\]

where the period \(t\) discount factor \(\gamma^t\) is defined in terms of the period \(t\) nominal interest rate \(r^t\) and the period \(t\) expected asset inflation rate \(i^t\) in the usual way:

\[
\text{(A26)} \quad \gamma^t \equiv (1 + i^t)/(1 + r^t).
\]
Now that $f_0^t$ has been determined, substitute (A25) into (A24) and substitute the resulting equations into equations (5) and determine the sequence of period $t$ vintage asset prices, $P_n^t$:

\[
(A27) \quad P_n^t = P_0^t \frac{[(L-n) + (L-n-1)(\gamma^t) + ... + 1(\gamma^{L-1-n})]/[L + (L-1)(\gamma^t) + ... + 1(\gamma^{L-1})]}{L(L+1)/2},
\]

for $n = 0, 1, 2, ..., L-1$

for $n = L, L+1, L+2, ...$.

Finally, use equations (9) to determine the end of period $t$ rental prices, $u_n^t$, in terms of the corresponding beginning of period $t$ rental prices, $f_n^t$:

\[
(A28) \quad u_n^t = (1 + r^t)f_n^t; \quad n = 0, 1, 2, ...
\]

Given the vintage asset prices defined by (A27), we could use equations (13) above to determine the corresponding vintage cross section depreciation rates $\delta_n^t$. We will not table these depreciation rates since our focus is on constructing measures of the capital stock and of the flow of services that the stocks yield.

There is a relationship between the linear efficiency decline model and another model of depreciation that appears in the tax literature. Assume that the nominal interest rate $r^t$ and the nominal asset inflation rate $i^t$ are both zero, then using (A27), it can be shown that

\[
(A29) \quad D_n^t = P_n^t - P_{n+1}^t = P_0^t \frac{L - n}{L(L+1)/2}, \quad n = 0, 1, 2, ..., L;
\]

i.e., when $r^t = i^t = 0$, depreciation declines at a linear rate for the linear efficiency decline model. When depreciation declines at a linear rate, the resulting formula for depreciation is called the sum of the year digits formula.\(^{131}\) Thus just as the one hoss shay and straight line depreciation models coincide when $r^t = i^t = 0$, so too do the linear efficiency decline and sum of the digits depreciation models coincide.

In the final section of this Appendix, we indicate why it is reasonable to expect depreciation to be accelerated even when, at first glance, it appears that the asset is of the one hoss shay type.

**A5. The Linearly Increasing Maintenance Expenditures Model**

Many years ago, the accountant Canning raised the following interesting problem that bears on our topic.\(^{132}\)

\(^{131}\) Canning (1929; 277) describes the method in some detail so it was already in common use by that time.

\(^{132}\) Matheson (1910; 76-77) raises the same sort of issues in a less focused manner: “But this principle has to be applied with considerable qualification where repairs really renew the life of a machine and prolong greatly its period of useful work. For instance, a locomotive during its life may have its wheel tires renewed four times, its boiler three times, and be painted seven times, so that before the framework, the wheels and other more durable parts fail, and the engine is broken up, much more than its original cost will have been expended on it. The value of any such serious renewals of this kind should be duly credited in a proper system of depreciation. Another course, followed more often in the United States than in Great...
“By spending enough for parts replacements (repairs), it is possible to keep any machine running for an indefinitely great length of time, but it does not pay to do so. Query: How does one know just when a machine is worn out?” John B. Canning (1929; 251).

In other words, Canning notes that the choice of when to retire an asset is really an endogenous decision\(^{133}\) rather than an exogenous one as we have assumed up to now. In this section, we attempt to model the retirement decision in a preliminary way using the concept of a maintenance profile.

For most new machines and new structures, engineers are able to devise a maintenance schedule that will ensure that the asset delivers its services during the period under consideration. Thus in the Queensland Competition Authority (2000; Chapter 13),\(^{134}\) a schedule of costs per kilometer of rail track that is required to keep the rails in working order as a function of the age of the track is laid out. These maintenance expenditures will enable the track to deliver transportation services over its lifetime. This schedule has a fixed cost aspect to it and then as the track ages, the maintenance expenditures increase linearly up to a certain point and then flatten out. Similarly, a new truck will have a schedule of recommended maintenance operations that the owner is urged to follow. In addition to these maintenance expenditures, we could also include operating costs like fuel and driver inputs since these inputs are necessary to deliver ton miles of output. Finally, an office building will also have maintenance expenditures associated with it and some operating expenditures such as heat since the renters of offices typically want square meters of space maintained at a comfortable temperature. In any case, we assume that at the beginning of period \(t\), we know the period \(t\) maintenance and operating expenditures necessary to operate an asset that is \(n\) periods old at the beginning of period \(t\), \(m_{n}^{t}\), \(n = 0,1,2,...\). We say that \(\{m_{n}^{t}\}\) is the period \(t\) (cross section) maintenance profile.

We now have to distinguish between the gross and net rental prices of an asset that is \(n\) periods old at the beginning of period \(t\), \(g_{n}^{t}\) and \(f_{n}^{t}\), respectively. An office or an apartment is typically rented on a gross basis; i.e., a tenant rents an office that is \(t\) periods old at the beginning of period \(t\) and pays the gross rent \(g_{n}^{t}\) at the beginning of the period and the landlord is responsible for the period \(t\) maintenance costs \(m_{n}^{t}\).\(^{135}\) On the other hand, a truck (on a long term lease) is usually rented on a net basis; i.e., the user of the truck is responsible for operating costs and maintenance. In any case, the relationship between the gross and net rental prices is:

\[
(A30) \quad g_{n}^{t} = f_{n}^{t} + m_{n}^{t}; \quad n = 0,1,2,...
\]

---

\(^{133}\) We consider some additional models of endogenous retirement later in this course. In this section, we are relaxing the strong separability assumptions that we have made up to now; i.e., up to now, we have assumed that used asset prices are independent of the actions of the firm and other prices in the economy.

\(^{134}\) Euan Morton of the Queensland Competition Authority brought this work to the author’s attention.

\(^{135}\) We assume that these costs are converted to beginning of period \(t\) costs using present values if necessary.
Thus our present notation is consistent with our previous notation where we valued an asset by the discounted stream of its net rentals; i.e., previously, we used the cross sectional profile of net rentals \( f_n \) (extrapolated to future periods) in order to value assets by age.

Our new asset valuation by age equation is the following equation, which gives the value of a new asset at the beginning of period \( t \), assuming that the asset will be retired after \( L \) periods of use:

\[
(A31) \ P_0^t(L) = g_0^t + (\gamma^t) g_1^t + (\gamma^t)^2 g_2^t + ... + (\gamma^t)^{L-1} g_{L-1}^t \\
- \left[ m_0^t + (\beta^t) m_1^t + (\beta^t)^2 m_2^t + ... + (\beta^t)^{L-1} m_{L-1}^t \right]
\]

where the discount factors \( \gamma^t \) and \( \beta^t \) are defined as follows:

\[
(A32) \ \gamma^t = (1+i^t)/(1+r^t) ; \ \beta^t = (1+\alpha^t)/(1+r^t).
\]

As in the previous sections, \( i^t \) is the one period anticipated inflation rate for the services of the asset at the beginning of period \( t \), \( r^t \) is the period \( t \) nominal interest rate and hence \( \gamma^t \) is the same discount factor that has appeared in previous sections. The new parameter is \( \alpha^t \), which is the one period anticipated inflation rate for maintenance (and operating cost) services and so \( \beta^t \) is the counterpart to \( \gamma^t \) except that \( \beta^t \) is the discount factor that applies to future anticipated operating and maintenance costs while \( \gamma^t \) is the discount factor that applies to future anticipated gross revenues.

The interpretation of equation (A31) is straightforward: a new asset that is to be used for \( L \) periods is equal to the discounted stream of the gross rentals that it is expected to yield minus the discounted stream of expected maintenance and operating costs.

We now evaluate equation (A31) for \( L = 1,2,... \), and pick the \( L \) which gives the highest value of \( P_0^t(L) \). We call this optimal value \( L^* \). Once the optimal \( L^* \) has been determined, then if used asset markets are in equilibrium, the sequence of period \( t \) asset prices by age \( n \), \( P_n^t \), the sequence of period \( t \) gross rental prices by age \( n \), \( g_n^t \), and the sequence of period \( t \) cross sectional maintenance costs by age of asset \( n \), \( m_n^t \), should satisfy the following system of equations:

\[
(A33) \ P_0^t = g_0^t + (\gamma^t) g_1^t + (\gamma^t)^2 g_2^t + ... + (\gamma^t)^{L^*-1} g_{L^*-1}^t \\
- \left[ m_0^t + (\beta^t) m_1^t + (\beta^t)^2 m_2^t + ... + (\beta^t)^{L^*-1} m_{L^*-1}^t \right]
\]

\[
P_1^t = g_1^t + (\gamma^t) g_2^t + (\gamma^t)^2 g_3^t + ... + (\gamma^t)^{L^*-1} g_{L^*-2}^t \\
- \left[ m_1^t + (\beta^t) m_2^t + (\beta^t)^2 m_3^t + ... + (\beta^t)^{L^*-1} m_{L^*-2}^t \right]
\]

...
\[ P_{L^*+2}^t = g_{L^*+2}^t + (\gamma^t) g_{L^*+1}^t - [m_{L^*+2}^t + (\beta^t) m_{L^*+1}^t] \]
\[ P_{L^*+1}^t = g_{L^*+1}^t - m_{L^*+1}^t. \]

Equations (A33) are the counterparts to our earlier system of equations (5). Given the cross sectional gross rental prices \( g_n^t \) and the cross sectional maintenance costs \( m_n^t \) (and the discount factors \( \gamma^t \) and \( \beta^t \)), we can determine the period \( t \) asset prices by age \( n \), \( P_n^t \), using equations (A33). Given the asset prices \( P_n^t \) and the cross sectional maintenance costs \( m_n^t \), we can also use equations (A33) to determine the period \( t \) gross rental prices by age \( n \), \( g_n^t \): start with the last equation in (A33) and determine \( g_{L^*+1}^t \); then move up to the second last equation and determine \( g_{L^*+2}^t \); etc. Of course, once the \( g_n^t \) have been determined, then we may use equations (A30) to determine the net rental prices (or user costs) \( f_n^t \) and then we can use these \( f_n^t \) as weights for the period \( t \) capital stocks by age \( n \) and construct a measure of capital services as in the previous sections. Thus equations (A33) are indeed the key equations in this section.

Unfortunately, in general, we cannot derive counterparts to equations (6)\(^{137} \) using equations (A33). To see why this is so, look at the first equation in (A33) and try to convert it into a counterpart to the first equation in (6):

\[(A34) \quad P_0^t = g_0^t - m_0^t + \gamma^t [g_1^t + \gamma^t g_2^t + ... + (\gamma^t)^{L-2} g_{L^*+1}^t] - \beta^t [m_1^t + \beta^t m_2^t + ... + (\beta^t)^{L-2} m_{L^*+1}^t]
= g_0^t - m_0^t + \gamma^t P_1^t.\]

It is also the case that we no longer have the simple formula for anticipated time series depreciation that we derived in equations (38) above. Put another way, suppose all expectations held at the beginning of period \( t \) turned out to be true. Under this assumption, we can derive the following relationship between the price of a one period old asset at the beginning of period \( t \), \( P_1^t \), and the price of a one period old asset at the beginning of period \( t+1 \), \( P_{t+1}^t \), as follows:

\[(A35) \quad P_{t+1}^t = (1+i^t) [g_{t+1}^t + \gamma^t g_{t+2}^t + ... + (\gamma^t)^{L-2} g_{L^*+1}^t] - (1+\alpha^t) [m_{t+1}^t + \beta^t m_{t+2}^t + ... + (\beta^t)^{L-2} m_{L^*+1}^t]
= (1+i^t) P_1^t.\]

However, if we assume that the period \( t \) anticipated gross rental price escalation factor \( 1+i^t \) is equal to the period \( t \) anticipated operating and maintenance cost escalation factor \( 1+\alpha^t \) so that \( \gamma^t \) is equal to \( \beta^t \), then the two inequalities (A34) and (A35) become equalities. Hence, to make further progress\(^{138} \), we make the following simplifying assumption:

\[(A36) \quad \alpha^t = i^t \quad \text{or} \quad \gamma^t = \beta^t. \]

---

\(^{137}\) It was equations (6) that allowed us to deduce the pattern of period \( t \) user costs from the pattern of period \( t \) used asset prices.

\(^{138}\) We do this in order to obtain a simpler set of relations between \( g_n^t \), \( m_n^t \) and \( P_n^t \) than the rather complex system of relations defined by equations (A33). However, for most industries, assumption (A36) will not be warranted; i.e., operating and maintenance costs in the industry will generally increase at rates that differ from the rate of increase in gross leasing prices (or gross output prices) for that industry. In this case, we are stuck with the general setup represented by equations (A30)-(A33).
Using assumption (A36), we can rewrite equations (7A33) as follows:

\[(A37)\quad P_0^t = g_0^t - m_0^t + \gamma^t P_1^t\]
\[P_1^t = g_1^t - m_1^t + \gamma^t P_2^t\]
\[\vdots\]
\[P_{L^*}^t - 2 = g_{L^*}^t - m_{L^*}^t + 0\]
\[= f_{L^*}^t + \left[\frac{1+i^t}{1+r^t}\right]P_{L^*}^t - 1\]

where the second set of equations follows using equations (A30). Note that equations (A37) are exact counterparts to equations (6) in section 3 above.

Obviously, equations (A37) can be rewritten to give us explicit formulae for the gross rental prices \(g_n^t\) in terms of the period \(t\) asset prices \(P_n^t\) and the period \(t\) maintenance costs by age \(n\), \(m_n^t\):

\[(A38)\quad g_n^t = m_n^t + P_n^t - \left[\frac{1+i^t}{1+r^t}\right]P_{n+1}^t ; \quad n = 0,1,2,\ldots,L^*-1.\]

Equations (A38) are exact counterparts to our earlier system of equations (7) for the period \(t\) user costs, \(f_n^t\).

In order to get a useful, explicit depreciation model, we make some further assumptions:

\[(A39)\quad \gamma^t = \frac{1+i^t}{1+r^t} = \gamma \quad \text{for all periods} \; t;\]
\[(A40)\quad g_n^t = \lambda^t g \quad \text{for all periods} \; t \text{ and } n = 0,1,2,\ldots;\]
\[(A41)\quad m_n^t = \lambda^t [b+nc] \quad \text{for all periods} \; t \text{ and } n = 0,1,2,\ldots;\]

where \(g\), \(b\) and \(c\) are positive parameters with \(g > b\); i.e., the gross rental must be greater than the fixed maintenance cost. We now explain the meaning of assumptions (A39)-(A41). Assumption (A39) means that the real interest rate is constant over all periods. Assumption (A40) is a one hoss shay type assumption except that it is applied to the gross output of the asset; i.e., (A40) means that the gross services yielded by a properly maintained asset of any age in period \(t\) is the same across all ages. Assumption (A41) says that the period \(t\) maintenance costs by age have a fixed cost component that is the same across all ages of the asset, \(\lambda^t b\), plus another component that increases linearly in the age of the asset, \(\lambda^t nc\) for an asset that is \(n\) periods old at the start of period \(t\). The presence of the scalar factor \(\lambda^t\) in both (A40) and (A41) means that we are assuming that period \(t\) rental prices \(g_n^t\) and maintenance costs \(m_n^t\) are essentially constant except for a common period \(t\) inflation factor \(\lambda^t\).

Now substitute assumptions (A36) and (A39)-(A41) into (A31) and obtain the following expression for the function \(P_0^t(L)\), which gives the anticipated asset value of a new asset as a function of the number of periods \(L\) that it is used:

\[\text{By examining (A42), it can be seen that as \(b\) and \(c\) increase (so that either the fixed cost component or the rate of increase in maintenance costs increases), then the optimal age of retirement \(L^*\) will decrease.}\]
\[
\begin{align*}
(A42) \quad P_0^t(L) &= g_0^i + (\gamma^i) g_1^i + (\gamma^i)^2 g_2^i + \ldots + (\gamma^i)^{L-1} g_{L-1}^i \\
&\quad \quad - [m_0^i + (\beta^i) m_1^i + (\beta^i)^2 m_2^i + \ldots + (\beta^i)^{L-1} m_{L-1}^i] \\
&= \lambda^i [g - b][1 + \gamma + \gamma^2 + \ldots + \gamma^{L-1}] - \lambda^i [1 + 2\gamma + 3\gamma^2 + \ldots + (L-1)\gamma^{L-2}] \\
\end{align*}
\]

Now reparameterize the positive parameter \(c\) as follows:

\[
(A43) \quad c = [g - b]d
\]

where \(d\) is another positive parameter. Substitute (A43) into (A42) and the resulting equation can be rewritten as follows:

\[
(A44) \quad P_0^t(L) = \lambda^i [g - b]h(L)
\]

where the function \(h(L)\) is defined as

\[
(A45) \quad h(L) = [1 + \gamma + \gamma^2 + \ldots + \gamma^{L-1}] - d [1 + 2\gamma + 3\gamma^2 + \ldots + (L-1)\gamma^{L-2}] \\
\]

Provided that \(d\) is small enough and the real interest rate escalation factor \(\gamma\) is close to one, a positive integer \(L^*\) that maximizes \(h(L)\) will exist. This exercise determines the optimal age of retirement of a new asset.

However, we now reverse the argument: given an \(L^*\), we look for a positive parameter \(d\) such that \(h(L)\) will be at a maximum when \(L = L^*\). This can be done numerically. For \(\gamma = 1/(1.04)\) (this corresponds to a real interest rate of 4\%), it can be verified that the \(d\) that corresponds to Maddison’s assumed life for nonresidential structures, namely \(L^* = 39\), is approximately equal to 0.002. This in turn corresponds to the assumption that maintenance costs for nonresidential structures are rising at the rate of 0.2 percentage points per year. The \(d\) that corresponds to Maddison’s (1993) assumed life for machinery and equipment, \(L^* = 14\), is approximately equal to 0.012. This in turn corresponds to the assumption that maintenance costs for machinery and equipment are rising at the rate of 1.2 percentage points per year.

Once the \(d^*\) that corresponds to the desired asset life \(L^*\) has been found, then the function \(h(L)\) defined by (A45) is known. Now set the right hand side of equation (A44) (evaluated at \(L = L^*\)) equal to the price of a new asset at the beginning of period \(t\), \(P_0^t\), and solve the resulting equation for \(\lambda^i [g - b]\). The solution is:

\[
(A46) \quad \lambda^i [g - b] = P_0^t/h(L^*).
\]

We now have enough information to evaluate the sequence of period \(t\) net rental prices, \(f_n^t\), as follows:\(^{140}\)

\[\text{Conversely, as } g \text{ increases (so that the gross revenue yielded by the asset exogenously increases), then the optimal age of retirement will increase.}\]

\(^{140}\) We do not have enough information to obtain the sequence of gross rental prices, \(g_n^t\), but, fortunately, it is the net rental prices that we need in order to calculate a capital services aggregate.
(A47) $f_n^t = g_n^t - m_n^t$

\[f_n^t = \lambda^t (g - \lambda^t [b + nc])\]

\[= \lambda^t [g - b] - \lambda^t n[g - b]d\]

\[= \lambda^t [g - b] (1 - nd)\]

\[= P_0^t (1 - nd)/h(L^*)\]

If the price $P_0^t$ of a new asset is known, then everything on the right hand side of (A47) is known and the sequence of net rental prices $f_n^t$ can be calculated. Once the $f_n^t$ are known, then the second set of equations in (A37) can be used in order to obtain the sequence of vintage asset prices, $P_n^t$. Given $P_0^t$ and $f_0^t$, use the first equation in (A37) to determine $P_1^t$; then use $f_1^t$ and the second equation to determine $P_2^t$ and so on.

It turns out that this special case of the linearly increasing maintenance expenditures model is virtually equivalent to the linear efficiency decline model explained in section A4 above. The explanation for this result is contained in equations (A47): these equations show that the (net) user costs decline linearly as older assets are used. This is precisely the assumption made in the model presented in the previous section, when we assumed constant real interest rates.

The importance of the model presented in this section is that it casts some light on the conditions under which we might expect net rental prices to decline in a linear fashion even though we know the asset is of the gross one hoss shay type; i.e., an older truck can deliver the same ton miles in a period as a younger one provided that we spend enough on maintenance. Thus the simplified model presented at the end of this section provides a justification for assuming a quite accelerated form of depreciation, even though the asset essentially delivers one hoss shay type services. Put another way, in the context of assets that are capable of delivering the same services as they age, then if maintenance costs rise as the asset ages, accelerated depreciation is inevitable. \(^{141}\) The more general model presented at the beginning of this session could also be used in regulatory contexts where maintenance schedules often exist and the determination of “economic” depreciation is a matter of some importance.

The simple model presented at the end of this section may also help to explain why there is tremendous diversity in the ages at which identical assets are retired in different countries. For example, if maintenance costs are higher or are expected to rise more quickly in a particular country, then the model implies that identical assets in that country will be retired at an earlier age. This observation can help to explain why well maintained assets in developing countries are used much longer than in developed countries. Conversely, assets employed in a country enjoying a boom so that gross rental prices are relatively high will be retired at a later age than assets employed in a country

\(^{141}\) The results in this section also enable us to reinterpret the geometric depreciation model, which is often interpreted as an asset evaporation model; i.e., each period, a fraction of the existing stock of assets simply “evaporates”. However, now we see that the asset may in fact be delivering a constant amount of gross services but a certain pattern of increasing maintenance costs is in fact causing used asset prices to have the profile implied by geometric depreciation (up to some limiting age).
experiencing relatively low rental rates, other things being equal. If future asset rental rates are expected to decline or increase less rapidly than future maintenance costs (i.e., $i'$ increases less than $\alpha'$), then the expected future gross revenues will decline or grow less rapidly than expected future operating costs and the asset will be retired earlier. Thus the models presented in this section can cast some light on why the same asset is retired at different ages across countries and uses.

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