Productivity Measurement in the Public Sector

W. Erwin Diewert.[1] December 29, 2016.
Discussion Paper 16-08,
Vancouver School of Economics,
University of British Columbia,
Vancouver, B.C., Canada, V6T 1L4.
Email: erwin.diewert@ubc.ca

Abstract

In many sectors of the economy, governments either provide various goods and services at no cost or at highly subsidized prices. It is usually possible to measure the quantities of these government sector outputs and inputs as well as input prices but the problem is how to estimate the corresponding output prices. Once meaningful output prices have been estimated, the measurement of productivity growth using index numbers can proceed in the usual manner. This chapter suggests three possible general methods for measuring public sector output prices and quantities. If little or no information on the quantity of nonmarket outputs produced is available, then the method recommended in the *System of National Accounts 1993* must be used, where aggregate output growth is set equal to aggregate input growth. If information on nonmarket public sector outputs is available then the second general method sets the missing output prices equal to the unit costs of producing each output while the third general method uses purchaser's valuations to determine the missing output prices. Specific measurement issues in the health and education sectors are discussed. Similar output and productivity measurement issues arise in the regulated sectors of an economy since regulated producers are forced to provide services at prices that are not equal to marginal or average unit costs. Finally, the problems associated with measuring capital services are discussed. The focus of the chapter is on the use of index number methods to measure the Total Factor Productivity of production units in the public and regulated sectors.

Keywords

Measurement of output, input and productivity, nonmarket sector, health, education, regulated industries, cost functions, marginal cost prices, activity analysis, technical progress, quality adjustment, hedonic regressions, index number theory.

JEL Classification Numbers

C43, D24, D57, E23, H40, H51, H52, I10, I12, I20, O47

---

[1] W. Erwin Diewert: School of Economics, University of British Columbia, Vancouver B.C., Canada, V6T 1Z1 and the School of Economics, UNSW Australia, Sydney 2052. Email: erwin.diewert@ubc.ca. The author thanks Kevin Fox, Knox Lovell and Paul Schreyer for helpful comments and the SSHRC of Canada and the Australian Research Council (DP150100830) for financial support. None of the above are responsible for any opinions expressed in the paper.

1. Introduction: How Should Public Sector Nonmarket Outputs be Valued?

In order to measure the Total Factor Productivity of a public sector production unit (or establishment) using index number techniques, it is necessary to measure the prices and quantities of the outputs produced and the inputs used by that unit for two periods of time. Then Total Factor Productivity growth can be defined as a quantity index of outputs produced divided by a quantity index of inputs used by the establishment.[2] It is usually possible to measure the price and quantity of inputs in a fairly satisfactory manner[3] but there are problems in measuring the prices and quantities of public sector nonmarket outputs. Thus in this chapter, we will take a systematic look at possible methods for the valuation of nonmarket outputs produced by public sector production units.

In many cases, it is difficult to determine exactly what it is that a public sector production unit produces. In this case, we may have neither quantities or prices for the outputs of the government service provider. However, in many cases, we can measure at least the quantities of the outputs produced by the public sector unit but not the corresponding prices. Finally, in some cases, it may be possible to measure output quantities produced and to obtain estimates of purchaser's valuations for the missing nonmarket output prices. Thus the "best practice" methodology that can be used to form productivity estimates for the public sector unit will depend to a large extent on what information on prices and quantities is available.

From the perspective of measuring the effects on the *welfare of households* of public sector production, we suggest the following methods for valuing government outputs in the order of their desirability:

- *First best*: valuation at market prices or purchaser's valuations;
- *Second best*: valuations at producer's unit costs of production;
- *Third best*: output growth of the public sector production unit is set equal to real input growth and the corresponding output price growth is set equal to an index of input price growth.

Obviously, the third best option is the least desirable option. If it is used, then productivity growth for the public sector production unit will be nonexistent by construction. *If* there are competitive markets with no economies of scope and constant returns to scale in production, then the first and second best options will be roughly equivalent; i.e., the purchaser's price will be approximately equal to the long run marginal cost of producing a unit of the commodity. However, in a nonmarket setting, even with no economies of scope and constant returns to scale in production, there is

[2] This follows the approach pioneered by Jorgenson and Griliches (1967) (1972). For further developments of the index number approach to measuring TFP growth, see Diewert (1976) (1980) (1983) (1992a) (1992b) (2014), Caves, Christensen and Diewert (1982a) (1982b), Diewert and Morrison (1986), Kohli (1990), Fox and Kohli (1998), Balk (1998) (2003), Schreyer (2001), Diewert and Nakamura (2003) and Inklaar and Diewert (2016).

[3] However, there are some significant problems associated with the measurement of capital services inputs and we will spend some time dealing with these difficulties.

nothing to force the cost of producing a unit of public sector output to equal its value to recipients of the commodity.[4] In an extreme case, the production unit could be producing nothing of value. Thus from the viewpoint of welfare economics, valuation of public sector outputs at purchasers' valuations appears to be the preferred option.[5]

The task of this chapter is to suggest methods for measuring the *Total Factor Productivity* (TFP) of a public sector production unit which produces at least some outputs that are allocated to recipients at nonmarket prices; i.e., at zero prices or highly subsidized prices that do not cover their unit costs of production. In general, the TFP level of a production unit is defined as the real output produced by the production unit at a time period divided by the real input utilized by the production unit at the same time period. There is TFP growth if real output grows more rapidly than real input. The main drivers of TFP growth are: (i) technical progress; i.e., at outward shift of the unit's production possibilities set; (ii) increasing returns to scale combined with input growth and (iii) improvements in technical and allocative efficiency.[6] This volume outlines many methods for measuring TFP but in this chapter, we will concentrate on index number methods.[7] The use of index number methods means that we will not be able to decompose the TFP growth of a public sector production unit into the contributions of the above three main components of TFP growth; i.e., index number methods do not allow us to measure separately the effects of the above explanatory factors; in general, all three explanatory factors will be combined into the overall TFP growth measure. However, the use of index number methods requires a methodological framework for the determination of output prices for the nonmarket outputs produced by the public sector. If the public sector unit minimizes its cost of production in producing its nonmarket outputs, then from the viewpoint of the economic approach to productivity measurement, the "right" prices to use to value nonmarket outputs are the (long run) marginal costs of producing the nonmarket outputs. Diewert (2012; 222-228) provided a nonparametric methodological justification for this marginal cost method of output price valuation for nonmarket outputs.[8] Diewert also justified the use of the Fisher (1922) index number formula to aggregate inputs and outputs using his approach. We will summarize his approach in the Appendix to this chapter. Variants of this cost based method for valuing nonmarket outputs will be discussed in Section 5.

[4] For a more detailed discussion of valuation principles based on purchaser versus supplier valuations, see Hill (1975; 19-20), Atkinson (2005; 88) and Schreyer (2012a; 261-266).

[5] However, from the perspective of measuring the Total Factor Productivity growth of a public sector production unit, valuation of unpriced outputs by their marginal or unit costs is best as we shall see later.

[6] During recessions, outputs decrease more than inputs due to the short run fixity of many inputs. Thus production units may be in the interior of their production possibilities sets at times and so movements from a technically inefficient allocation of resources towards the production frontier will improve TFP.

[7] Other methods involve the estimation of cost or production functions but typically, statistical agencies do not have the resources to undertake the required econometric estimation. Nonparametric productivity measurement methods could also be used but unless price information is used in addition to quantity information on the inputs used and outputs produced by a production unit, these methods often do not generate reasonable estimates.

[8] Atkinson (2005; 88-90) advocated the use of marginal costs to value outputs in the nonmarket sector. Hill (1975; 19-21) advocated the use of (average) unit costs to value nonmarket outputs.

The third best option outlined above is the only option that can be used when there is little or no information on both the prices and quantities produced by a government establishment. This is the option which is recommended in the *System of National Accounts 1993* to value government production when direct information on the prices and quantities of government outputs is not available. The quantity or volume measure for establishment output that results from using this methodology can be interpreted as a *measure of real resources used* by that establishment and as such, it is an acceptable indicator of the output produced by a government unit. This third best option will be discussed in more detail in the following three sections. Section 2 introduces the method. Section 3 discusses how user costs should be used to value public sector capital stock inputs. Section 4 addresses the problems associated with choosing an index number formula and how TFP could be measured either as gross output TFP or as value added TFP. This section also shows that the choice of an index number formula to aggregate outputs and inputs does matter.

The second best option for valuing public sector outputs will be discussed in section 5 and the first best option in section 6. Section 7 provides a numerical example due to Schreyer (2010; 21) that illustrates how the first best option (from the viewpoint of welfare economics) could be used in order to value nonmarket outputs. This section also discusses some aspects of the problem of adjusting output quantities for quality change. Section 8 discusses some of the practical difficulties associated with the measurement of public sector outputs in selected industries. Section 9 concludes.

2. The Case where No Information on the Prices and Quantities of Nonmarket Outputs is Available

The third best option outlined above is the only option that can be used when there is little or no information on both the prices and quantities produced by a public sector production unit (or there is no agreement on how to measure the outputs of the unit). For example, usually there is little information on the price and quantity of educational services produced by the public sector.[9] As was mentioned earlier, the quantity or volume measure for establishment output that results from using this methodology can be interpreted as a *measure of real resources used* by the production unit and as such, it is an acceptable indicator of the output produced by the unit. Although this third best option is fairly straightforward in principle (and has been extensively discussed in the national income accounting literature), there are some aspects of the method that deserve some additional discussion.

The two aspects of the third method that we will discuss in more detail in the following two sections are as follows:

- How exactly should the contribution of durable inputs used in a public sector production unit to current period production be valued?

[9] In section 8.1 below, we will discuss various suggested methods for estimating output prices and quantities for this sector. It will be seen that there is no general agreement on these methods.

- How exactly should estimates for the aggregate real output produced by a nonmarket production unit be constructed? In particular, which index number formula should be chosen to perform the aggregation and does the choice of formula make a difference?

3. The Valuation of Durable Inputs Used in the Public Sector

The basic tool that economists use to value the contribution of a durable input (or a capital input) to production in an accounting period is the concept of a *user cost*.

We will first explain how to construct the user cost of a capital input[10] for a durable input for which market prices exist for the same input at different ages. Consider a production unit which purchases q^t units of a durable input at the beginning of accounting period t at the price P^t. After using the services of the capital input during period t, the production unit will have $q_u^t = (1-\delta)q^t$ units of used or depreciated capital (in constant quality units) on hand at the end of period t where δ is the one period depreciation rate for the capital good under consideration.[11] Finally, we assume that the production unit has a one period financial opportunity cost of capital at the beginning of period t (i.e., a beginning of the period nominal interest rate) equal to r^t. The gross cost of the capital input is the beginning of the period purchase cost of the capital inputs, P^tq^t. But this cost is offset by the revenue that could be raised by selling the depreciated capital stock value at its imputed market value at the end of the period, which is $P^{t+1}q_u^t = P^{t+1}(1-\delta)q^t$. But this imputed revenue is not equivalent to the cost outlay made at the beginning of the period. To make it equivalent, we need to take into account that money received at the end of the period is less valuable than money received at the beginning of the period and so the end of period market value should be discounted by $(1+r^t)$. Thus the *net cost of using the services of the capital input* during period t is U^t defined as follows:[12]

(1) $U^t \equiv P^t - (1+r^t)^{-1}(1-\delta)P^{t+1}$.

Define the *constant quality asset inflation rate* over period t, i^t, by the following equation:

(2) $1+ i^t \equiv P_K^{t+1}/P_K^t$.

---

[10] A capital input is an input which contributes to production for more than one accounting period. In practice, national income accountants treat capital inputs that last less than three years as nondurable inputs.

[11] For simplicity, we assume the geometric model of depreciation where the one period depreciation rate δ remains constant no matter what the age of the asset is at the beginning of the period. For more on the geometric model of depreciation, see Jorgenson (1989) (1996).

[12] This simple discrete time derivation of a user cost (as the net cost of purchasing the durable good at the beginning of the period and selling the depreciated good at an interest rate discounted price at the end of the accounting period) was developed by Diewert (1974a; 504), (1980; 472-473), (1992b; 194). Simplified user cost formulae (the relationship between the rental price of a durable input to its stock price) date back to Babbage (1835; 287) and to Walras (1954; 268-269). The original version of Walras in French was published in 1874. The early industrial engineer, Church (1901; 907-909) also developed a simplified user cost formula.

Substituting (2) into (1) leads to the following expression for the *ex post user cost* U^t defined by (1):

(3) $U^t \equiv P^t - (1+r^t)^{-1}(1-\delta)(1+i^t)P^t = (1+r^t)^{-1}[(1+r^t) - (1-\delta)(1+i^t)]P^t$.

Rather than discounting the end of period value of the capital stock to the beginning of the period, it is more convenient to anti-discount costs and benefits to the end of the period. Thus the *ex post end of period user cost of a capital input*, u^t, is defined as $(1+r^t)$ times U^t:[13]

(4) $u^t \equiv (1+r^t)U^t = [(1+r^t) - (1-\delta)(1+i^t)]P^t = [r^t - i^t + \delta(1+i^t)]P^t$.

Using this formula, we add the opportunity cost of tying up financial capital for one period, r^tP^t, to the beginning of the period purchase cost of the asset, P^t, to obtain a total cost of purchasing one unit of an asset and tying up financial capital for one period, which is $(1+r^t)P^t$. This total asset cost is offset by the end of period (imputed) benefit of the depreciated value of one unit of the purchased capital stock, $(1-\delta)(1+i^t)P^t$. Taking the difference of this cost and benefit leads to the user cost formula (4). This ex post formula for the user cost of capital defined was obtained by Christensen and Jorgenson (1969; 302) for the geometric model of depreciation.[14]

However, the derivation of the user cost of capital defined by (3) or (4) is not the end of the problems associated with valuing the cost of using a capital input over an accounting period. There are four additional issues that need to be discussed:

- Ex ante versus ex post user costs;
- How to treat specific taxes on some capital inputs such as property taxes;
- What to do if there are no market prices for used capital stocks and
- What should be done if inappropriate user costs of capital are used.

We will address each of these problems in turn.

The problem with the user cost formulae defined by (3) or (4) is that they use the ex post or actual asset inflation rate i^t defined by (2). For land assets in particular, ex post asset inflation rates can at times be so large that it makes the user costs defined by (3) or (4) negative. Basically, we would like the user cost of an asset to be approximately equal to the rental price for the asset (if rental markets for the asset exist). Rental prices will rarely be negative so obviously, a negative user cost will be a poor approximation to a rental

[13] It should be noted that the user costs that are anti-discounted to the end of the period are more consistent with commercial accounting conventions than the corresponding user costs that are discounted to the beginning of the period; see Peasnell (1981).

[14] Diewert (2005) (2010) and Diewert and Wei (2017) derive user cost formula for more general models of depreciation. In particular, one hoss shay or light bulb depreciation may be a more appropriate model of depreciation in valuing the contribution of structures and long lived infrastructure assets. Diewert (2005) also discussed in more detail user costs that are formed by discounting or anti-discounting costs and benefits to either the beginning or end of the accounting period.

price. A way around the negative user cost problem is to replace the ex post asset inflation rate by an *anticipated asset inflation rate*. Thus suppose that at the beginning of period t, the *anticipated end of period t price for an asset of the same quality* is P^{t+1*}. Use this price in order to define an *anticipated asset inflation rate* as $i^{t*} \equiv (P^{t+1*}/P^t) - 1$. Now replace the ex post asset inflation rate i^t which appears in the user cost formulae (3) and (4) by their anticipated counterparts i^{t*} and we obtain the corresponding *ex ante user costs*, U^{t*} and u^{t*}. Jorgenson (1989) (1996) and his coworkers[15] endorsed the use of ex post user costs, arguing that producers can perfectly anticipate future asset prices. On the other hand, Diewert (1980; 476) (2005; 492-493), Schreyer (2001) (2009) and Hill and Hill (2003) endorsed the ex ante version for most purposes, since these ex ante user costs will tend to be smoother than their ex post counterparts and they will generally be closer to a rental or leasing price for the asset.[16] Diewert and Fox (2016) used sectoral data on the US corporate and noncorporate financial sector to compute capital services aggregates and the resulting rates of TFP growth using both Jorgensonian and smoothed user costs that use predicted asset inflation rates.[17] They found that Jorgensonian ex ante user costs for land components were indeed negative for many years but the use of ex ante or predicted asset inflation rates cured the problem of negative user costs and indeed, led to much smoother user costs as could be expected.[18] There is another solution to the problem of negative user costs if rental prices for the asset are available: take the maximum of the user cost and the corresponding market rental price as the appropriate valuation for the services of the asset during the period under consideration. The user cost valuation for the services of the asset is essentially a financial opportunity cost of using the asset while the rental price is the opportunity cost of using the services of the asset for productive purposes rather than renting it out for the period. Both valuations are valid opportunity costs so the true opportunity cost of using the asset should be the maximum of these two costs. Diewert (2008) called this *opportunity cost method* for asset services valuation.[19]

We have ignored tax complications in deriving the user cost formulae (3) and (4). Any specific capital taxes (such as property taxes on real estate assets) should be added to the user cost formula for the relevant assets.[20] Business income taxes that fall on the gross

---

[15] See in particular Jorgenson and Griliches (1967) (1972) and Christensen and Jorgenson (1969).

[16] Of course, the problem with using ex ante user costs is that there are many methods that could be used to predict asset inflation rates and these different methods could generate very different user costs. For empirical evidence on this point, see Harper, Berndt and Wood (1989), Diewert (2005) and Schreyer (2012b).

[17] The predicted asset inflation rate was set equal to the ex post geometric average inflation rate for the asset over the past 25 years.

[18] While Jorgensonian user costs may not be the best for productivity measurement purposes, they are the "right" user cost concept to use when calculating sectoral ex post rates of return. Moreover, Diewert and Fox (2016) found that even though many Jorgensonian land user costs turned out to be negative, the resulting rates of Total Factor Productivity growth did not differ much from the corresponding TFP growth rates using ex ante or smoothed asset inflation rates.

[19] See also Diewert, Nakamura and Nakamura (2009).

[20] Thus the user cost formula (4) should be modified to $u^t \equiv (1+r^t)U^t = [(1+r^t) - (1-\delta)(1+i^t) + \tau^t]P^t = [r^t - i^t + \delta(1+i^t) + \tau^t]P^t$ where τ^t is the period t specific tax rate on one unit of the asset. This modified user cost formula assumes that the specific tax (such as a property tax in the case of a structure or a land plot) is paid at the end of the accounting period.

return to the asset base can be absorbed into the cost of capital, r^t, so that r^t can be interpreted as the before income tax gross return to the asset used by the production unit.[21]

The user costs (3) and (4) were derived under the assumption that there are market prices for used assets that can be used to value the asset at the end of the accounting period. However, for many *unique assets* that do not trade in each accounting period (such as real estate, intellectual property and mining assets and certain types of artistic asset like a movie), there are no end of period asset prices that are available. In these cases, estimates of the future discounted cash flows that the asset might generate have to be used in order to value the asset as it ages. It will usually be difficult to form these estimates.[22]

There is still a certain amount of controversy on how exactly to measure capital services in the international System of National Accounts.[23] A particular problem with the System of National Accounts 1993 and 2008 is that capital services in the general government sector are to be measured by depreciation only; i.e., there is no allowance for the opportunity cost of capital in government sector user costs[24] whereas as we have seen above, market sector user costs of capital include both depreciation and the opportunity cost of capital that is tied up in holding productive assets. This omission of imputed interest cost will lead to a substantial underestimate of public sector costs (from an opportunity cost perspective) and hence economy wide GDP will also be underestimated.[25] The Office of National Statistics (ONS) in the UK has made a substantial effort to measure productivity in the public sector and it recognized that the SNA recommended treatment of capital input in the public sector is not appropriate for productivity measurement purposes. Thus the ONS treatment of capital services costs in the public sector for productivity measurement purposes is different from its System of National Accounts treatment, which follows the international guidelines. This is unfortunate because ideally, we would like the official GDP measure to coincide with the GDP measure that is used for productivity measurement purposes.

The above material should alert the reader to the fact that the measurement of capital services input in the public sector is not a completely straightforward exercise. Thus it is

[21] For material on the construction of user costs for more complex systems of business income taxation, see Diewert (1992b) and Jorgenson (1996).

[22] For references to the literature on valuing fixed inputs as they age, see Diewert (2009), Cairns (2013) and Diewert and Fox (2016). See Diewert and Huang (2011) for a discussion on how to value intellectual property products. Finally, for references to the literature on decomposing property values into their land and structure components, see de Haan and Diewert (2011), Diewert, de Haan and Hendriks (2015) and Diewert and Shimizu (2015a) for residential property decompositions, Diewert and Shimizu (2017) for condominium property decompositions and Diewert and Shimizu (2015b) (2016) and Diewert, Fox and Shimizu (2016) for commercial property decompositions.

[23] See Diewert (1980; 475-486) and Schreyer (2001) (2009).

[24] The property tax component of the use cost of public sector property input is also omitted from the SNA user cost treatment.

[25] Some countries want to make their GDP as small as possible in order to minimize international transfer payments that are based on their per capita GDP. Thus the treatment of capital services for the general government sector in the System of National Accounts is at least partially a political issue rather than a pure measurement issue.

not that simple to measure the nonmarket output produced by a public sector production unit by its corresponding input measure due to the fact that it is not completely straightforward to measure capital services in both the private and public sector.

4. Measuring Output Growth by Input Growth: Gross Output versus Value Added

If it proves to be difficult or impossible to measure nonmarket output quantities, then as indicated above, economic statisticians have generally measured the value of nonmarket outputs by the value of inputs used and implicitly or explicitly set the price of nonmarket output equal to the corresponding nonmarket input price index. Atkinson (2005; 12) describes the situation in the UK prior to 1998 as follows:

"In many countries, and in the United Kingdom from the early 1960's to 1998, the output of the government sector has been measured by convention as the value equal to the total value of inputs; by extension the volume of output has been measured by the volume of inputs. This convention regarding the volume of government output is referred to below as the (output = input) convention, and is contrasted with direct measures of government output. The inputs taken into account in recent years in the United Kingdom are the compensation of employees, the procurement costs of goods and services and a charge for the consumption of fixed capital. In earlier years and in other countries, including the United States, the inputs were limited to employment."

As was noted in the previous section, the above conventions imply that capital services input for government owned capital will generally be less than the corresponding capital services input if the capital services were rented or leased. In the owned case, the government user cost of capital consists only of depreciation but in the leased case, the rental rate would cover the cost of depreciation plus the opportunity cost of the financial capital tied up in the capital input. Atkinson (2005; 49) makes the following recommendation on this issue (and we concur with his recommendation):

"We recommend that the appropriate measure of capital input for production and productivity analysis is the flow of *capital services* of an asset type. This involves adding to the capital consumption an interest charge, with an agreed interest rate, on the entire owned capital."

In addition to the above problem, there are some subtle problems associated with measuring public sector outputs by their real input utilization:

- How exactly should real input be measured; i.e., which index number formula should be used to aggregate inputs and does the choice of formula matter?
- Should the output aggregate for a public sector production unit be measured by its real primary input or by its real gross input; i.e., by primary plus intermediate inputs?

In order to address these questions, we will calculate alternative input aggregates for an artificial data set. In Table 1 below, we list the prices (w_1^t, w_2^t) and the corresponding quantities (x_1^t, x_2^t) for two primary inputs and the prices (p_1^t, p_2^t) and quantities (z_1^t, z_2^t) for two intermediate inputs for five periods, $t = 1,...,5$.

Table 1: Primary and Intermediate Input Prices (w_1^t,w_2^t,p_1^t,p_2^t) and the Corresponding Quantities (x_1^t,x_2^t,z_1^t,z_2^t)

Period t	w_1^t	w_2^t	p_1^t	p_2^t	x_1^t	x_2^t	z_1^t	z_2^t
1	1.00	1.00	1.00	1.00	60	40	50	50
2	1.02	0.95	0.99	1.10	59	42	52	48
3	1.04	0.90	0.95	1.20	58	50	55	45
4	1.05	0.82	0.92	1.30	57	57	57	43
5	1.06	0.75	0.90	1.40	56	65	60	40

It can be seen that the price of the first primary input (labour) is slowly trending upwards while the price of the second primary input (capital services) is trending downwards at a faster rate. The quantity of the first primary input is slowly trending downwards while the quantity of the second primary input is trending upward at a rapid rate. Thus normal substitution effects are taking place over time. The price of the first intermediate input (general imports) is trending down while the quantity trends up. Finally, the price of the second intermediate input (energy imports) is rising rapidly while the corresponding quantity falls. The trends in these prices and quantities are fairly smooth.

The most commonly used index number formula that are used to aggregate prices and quantities are the Laspeyres, Paasche, Fisher and Törnqvist price indexes. Once the sequence of aggregate prices has been formed using these formulae, the corresponding quantity indexes are formed by dividing the period t value for the aggregate by the corresponding price indexes. Let w^t and x^t denote the price and quantity vectors for primary inputs for t = 1,...,5. Then the period t *fixed base Laspeyres, Paasche, Fisher and Törnqvist price indexes* are defined as follows for t = 1,...,5:[26]

(5) $P_L^t \equiv P_L(w^1,w^t,x^1,x^t) \equiv w^t{\cdot}x^1/w^1{\cdot}x^1$;[27]
(6) $P_P^t \equiv P_P(w^1,w^t,x^1,x^t) \equiv w^t{\cdot}x^t/w^1{\cdot}x^t$;
(7) $P_F^t \equiv P_F(w^1,w^t,x^1,x^t) \equiv [P_L(w^1,w^t,x^1,x^t)P_F(w^1,w^t,x^1,x^t)]^{1/2}$;
(8) $P_T^t \equiv P_T(w^1,w^t,x^1,x^t)] \equiv \exp[\Sigma_{n=1}^N (1/2)(s_n^1+s_n^t)\ln(w_n^t/w_n^1)]$

where $s_n^t \equiv w_n^tx_n^t/w^t{\cdot}x^t$ is the nth primary input cost share in period t. The four fixed base primary input price indexes defined by (5)-(8) are listed in columns 2-5 of Table 2 below using the primary input data listed in Table 1 above. These period t fixed base indexes are denoted by P_{LX}^t, P_{PX}^t, P_{FX}^t and P_{TX}^t. It can be seen that P_{LX}^t is always greater than P_{PX}^t for t = 2,...,5 and the gap between the fixed base Laspeyres and Paasche price indexes gradually becomes greater as time marches on. Since the Fisher index P_{FX}^t is the

[26] For further discussion on all of these indexes and their properties, see Fisher (1922) and Diewert (1978) (1992a). The US Bureau of Economic Analysis uses chained Fisher indexes to aggregate over inputs and outputs. The use of Törnqvist price and quantity indexes in productivity analysis can be traced back to Jorgenson and Griliches (1967) (1972). Justifications for the use of Törnqvist indexes based on the economic approach to index number theory can be found in Diewert (1976) (1980), Caves, Christensen and Diewert (1982a) (1982b), Diewert and Morrison (1986), Kohli (1990) and Inklaar and Diewert (2016). Justifications for the use of Fisher indexes based on the economic approach to index number theory can be found in Diewert (1992a) (2012; 222-228).
[27] Notation: $w^1{\cdot}x^1 \equiv \Sigma_{n=1}^N w_n^1x_n^1$ denotes the inner product of the vectors w^1 and x^1.

geometric mean of $P_{LX}{}^{t}$ and $P_{PX}{}^{t}$, it lies between these two equally plausible fixed basket type price indexes. Note that the fixed base Törnqvist price index, $P_{TX}{}^{t}$, is quite close to its fixed base Fisher counterpart, $P_{TX}{}^{t}$.[28]

Table 2: Fixed Base and Chained Laspeyres, Paasche, Fisher and Törnqvist Primary Input Price Indexes

Period t	$P_{LX}{}^{t}$	$P_{PX}{}^{t}$	$P_{FX}{}^{t}$	$P_{TX}{}^{t}$	$P_{LX}{}^{t*}$	$P_{PX}{}^{t*}$	$P_{FX}{}^{t*}$	$P_{TX}{}^{t*}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.99200	0.99089	0.99145	0.99145	0.99200	0.99089	0.99145	0.99145
3	0.98400	0.97519	0.97958	0.97963	0.98288	0.97844	0.98066	0.98067
4	0.95800	0.93500	0.94643	0.94661	0.95096	0.94314	0.94704	0.94706
5	0.93600	0.89347	0.91449	0.91492	0.92045	0.90957	0.91499	0.91502

An alternative to the use of fixed base indexes is to use *chained indexes*. Consider how a chained Laspeyres price index, say $P_{LX}{}^{t*}$, is formed. For periods 1 and 2, the chained indexes coincide with their fixed base counterparts; i.e., $P_{LX}{}^{1*} \equiv 1$ and $P_{LX}{}^{2*} \equiv P_{LX}{}^{2} = P_L(w^1,w^2,x^1,x^2)$. The period 3 chained Laspeyres price index is defined as the period 2 chained index level, $P_{LX}{}^{2*}$, multiplied by $P_L(w^2,w^3,x^2,x^3)$, which is (one plus) the Laspeyres rate of change of input prices going from period 2 to period 3. In general, the chained Laspeyres price level in period t+1 is equal to the corresponding Laspeyres price level in period t times the Laspeyres chain link going from period t to t+1; i.e., $P_{LX}{}^{t+1*} \equiv P_{LX}{}^{t*}P_L(w^t,w^{t+1},x^t,x^{t+1})$. The chained Paasche, Fisher and Törnqvist input price indexes, $P_{PX}{}^{t*}$, $P_{FX}{}^{t*}$ and $P_{TX}{}^{t*}$ are formed in a similar fashion, except that the Paasche, Fisher and Törnqvist chain link formulae $P_P(w^t,w^{t+1},x^t,x^{t+1})$, $P_F(w^t,w^{t+1},x^t,x^{t+1})$ and $P_T(w^t,w^{t+1},x^t,x^{t+1})$ are used instead of the Laspeyres chain link formula $P_L(w^t,w^{t+1},x^t,x^{t+1})$. The chained Laspeyres, Paasche, Fisher and Törnqvist input price indexes, $P_{LX}{}^{t*}$, $P_{PX}{}^{t*}$, $P_{FX}{}^{t*}$ and $P_{TX}{}^{t*}$ are listed in the last four columns of Table 2.

It can be seen that the use of the chained indexes dramatically reduces the spread between these four types of indexes as compared to their fixed base counterparts. In period 5, the percentage difference between the Laspeyres and Paasche fixed base indexes is 4.8% but the difference between the chained Laspeyres and Paasche indexes is only 1.6%. In period 5, the percentage difference between the superlative Fisher and Törnqvist fixed base indexes is 0.047% but the difference between the chained Fisher and Törnqvist indexes is only 0.003%, which is negligible. Thus chaining has substantially reduced the spread between the four most popular index number formula that are used in empirical applications. This will generally happen if annual data are used so that trends in prices and quantities are fairly smooth.[29]

---

[28] Both of these indexes are *superlative indexes*; i.e., they are exact for flexible functional forms for an underlying flexible functional form for an economic aggregator function as defined by Diewert (1976). Diewert (1978) showed that these two functional forms for an index number formula approximated each other to the accuracy of a second order Taylor series approximation when the derivatives are evaluated at a point where the two price vectors are equal and where the two quantity vectors are equal.

[29] Diewert (1978) pointed this out many years ago. However, chaining does not always work well if the data are available on a subannual basis: seasonal fluctuations and price bouncing behavior can create a chain drift problem, which was pointed out by Szulc (1983). Ivancic, Diewert and Fox (2011) and de Haan and van der Grient (2011) suggested adapting multilateral indexes (used to make cross sectional

The period t primary input implicit quantity indexes that correspond to the input price indexes listed in Table 2 can be obtained by dividing the period t aggregate input value by the corresponding period t price index.[30] The resulting implicit quantity indexes are listed in Table 3.

Table 3: Fixed Base and Chained Implicit Laspeyres, Paasche, Fisher and Törnqvist Primary Input Quantity Indexes

Period t	Q_{LX}^{t}	Q_{PX}^{t}	Q_{FX}^{t}	Q_{TX}^{t}	Q_{LX}^{t*}	Q_{PX}^{t*}	Q_{FX}^{t*}	Q_{TX}^{t*}
1	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
2	100.887	101.000	100.944	100.943	100.887	101.000	100.944	100.943
3	107.033	108.000	107.515	107.510	107.154	107.640	107.397	107.396
4	111.263	114.000	112.623	112.602	112.086	113.016	112.550	112.548
5	115.502	121.000	118.219	118.164	117.453	118.859	118.154	118.150

Exactly the same methodology can be used to form eight alternative price indexes for intermediate inputs using the data listed in Table 1. Denote the Laspeyres, Paasche, Fisher and Törnqvist intermediate input price indexes for period t by P_{LZ}^{t}, P_{PZ}^{t}, P_{FZ}^{t} and P_{TZ}^{t} and their chained counterparts by P_{LZ}^{t*}, P_{PZ}^{t*}, P_{FZ}^{t*} and P_{TZ}^{t*} respectively. These indexes are listed in Table 4.

Table 4: Fixed Base and Chained Laspeyres, Paasche, Fisher and Törnqvist Intermediate Input Price Indexes

Period t	P_{LZ}^{t}	P_{PZ}^{t}	P_{FZ}^{t}	P_{TZ}^{t}	P_{LZ}^{t*}	P_{PZ}^{t*}	P_{FZ}^{t*}	P_{TZ}^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.04500	1.04280	1.04390	1.04390	1.04500	1.04280	1.04390	1.04390
3	1.07500	1.06250	1.06873	1.06874	1.07226	1.06587	1.06906	1.06906
4	1.11000	1.08340	1.09662	1.09664	1.10102	1.09198	1.09649	1.09649
5	1.15000	1.10000	1.12472	1.12475	1.13313	1.12050	1.12680	1.12680

Viewing the entries in Table 4, it can be seen that the fixed base Laspeyres and Paasche intermediate input price indexes differ by almost 5% in period 5 while the Fisher and Törnqvist fixed base indexes are virtually the same. The chained Laspeyres and Paasche intermediate input price indexes differ by about 1% in period 5 while the corresponding chained Fisher and Törnqvist indexes are exactly the same. Thus again, chaining reduces the spread between indexes for this data set.

---

comparisons) to the time series context in order to deal with the chain drift problem that generally arises when monthly or weekly data are aggregated. See also de Haan and Krsinich (2014).

[30] Thus the period t fixed base *implicit* quantity indexes that match up with the period t fixed base Laspeyres and Paasche price indexes are $Q_{LX}^{t} \equiv w^{t}\cdot x^{t}/P_{LX}^{t}$ and $Q_{PX}^{t} \equiv w^{t}\cdot x^{t}/P_{PX}^{t}$ and so on. It should be noted that our Q_{LX}^{t} is normally called the Paasche quantity index and our Q_{PX}^{t} is normally called the Laspeyres quantity index. Our notation is simplified if we label the implicit quantity index that matches up with a particular price index formula in the same way. Note that our Q_{LX}^{t} is less than our Q_{PX}^{t} for t = 2,...,5. This is a typical relationship between Laspeyres and Paasche price and quantity indexes (but it does not always hold).

Use the same methodology to form eight alternative aggregate input price indexes for both primary and intermediate inputs using the data listed in Table 1. Denote the Laspeyres, Paasche, Fisher and Törnqvist aggregate input price indexes for period t by P_{LY}^{t}, P_{PY}^{t}, P_{FY}^{t} and P_{TY}^{t} and their chained counterparts by P_{LY}^{t*}, P_{PY}^{t*}, P_{FY}^{t*} and P_{TY}^{t*} respectively. These indexes are listed in Table 5.

Table 5: Fixed Base and Chained Implicit Laspeyres, Paasche, Fisher and Törnqvist Aggregate Input Price Indexes

Period t	P_{LY}^{t}	P_{PY}^{t}	P_{FY}^{t}	P_{TY}^{t}	P_{LY}^{t*}	P_{PY}^{t*}	P_{FY}^{t*}	P_{TY}^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.01850	1.01672	1.01761	1.01761	1.01850	1.01672	1.01761	1.01761
3	1.02950	1.01716	1.02331	1.02331	1.02747	1.02135	1.02441	1.02440
4	1.03400	1.00435	1.01907	1.01919	1.02470	1.01474	1.01971	1.01972
5	1.04300	0.98692	1.01457	1.01490	1.02346	1.00923	1.01632	1.01635

Viewing the entries in Table 5, it can be seen that the fixed base Laspeyres and Paasche aggregate input price indexes differ by approximately 5% in period 5 while the Fisher and Törnqvist fixed base indexes differ by 0.0003. The chained Laspeyres and Paasche intermediate input price indexes differ by about 2% in period 5 while the corresponding chained Fisher and Törnqvist indexes differ by 0.00003. Thus again, chaining reduces the spread between indexes. Also, the superlative indexes (whether they are fixed base or chained) are much closer to each other than are the corresponding Laspeyres and Paasche indexes.[31]

Once the aggregate input price indexes have been calculated, corresponding aggregate implicit input quantity or volume measures can be calculated by deflating each period's total input costs by the appropriate aggregate input price index. Thus the period t fixed base implicit aggregate input quantity that corresponds to fixed base Laspeyres price aggregation, Q_{LY}^{t}, is defined as follows for t = 1,...,5:[32]

(9) $Q_{LY}^{t} \equiv (w^{t}\cdot x^{t} + p^{t}\cdot z^{t})/P_{LY}^{t}$.

The other seven aggregate implicit input quantity levels, Q_{PY}^{t}, Q_{FY}^{t}, Q_{TY}^{t}, Q_{lY}^{t*}, Q_{PY}^{t*}, Q_{FY}^{t*}, Q_{TY}^{t*}, are defined in an analogous manner. The eight aggregate implicit input quantity indexes are listed in Table 6. Note that the *value of gross output* produced during period t, say V_{Y}^{t}, is equal to the value of aggregate input used during period t; i.e., we have for t = 1,...,5:

[31] Diewert (1978) found the same results using Canadian annual national accounts data and thus he advocated the use of chained superlative indexes for national accounts purposes. This is probably good advice if the data are at an annual frequency but there is the possibility of some chain drift if quarterly data are used.

[32] As mentioned earlier, our implicit Laspeyres quantity index Q_{LY}^{t} corresponds to what is normally called a fixed base Paasche quantity index. Cost weighted input quantity indexes of the type defined by (9) have been used widely in the UK in recent years when constructing measures of nonmarket output quantity growth; see Atkinson (2005; 88).

(10) $V_Y^t \equiv w^t \cdot x^t + p^t \cdot z^t = P_{LY}^t Q_{LY}^t = P_{PY}^t Q_{PY}^t = ... = P_{TY}^{t*} Q_{TY}^{t*}$

Thus for each period t, the price of gross output times the corresponding quantity of gross output is equal to the nominal value of aggregate input for each of our eight pairs of gross output price and quantity indexes.

Table 6: Fixed Base and Chained Laspeyres, Paasche, Fisher and Törnqvist Aggregate Implicit Input Quantity Indexes

Period t	Q_{LY}^t	Q_{PY}^t	Q_{FY}^t	Q_{TY}^t	Q_{LY}^{t*}	Q_{PY}^{t*}	Q_{FY}^{t*}	Q_{TY}^{t*}
1	200.000	200.000	200.000	200.000	200.000	200.000	200.000	200.000
2	200.648	201.000	200.824	200.824	200.648	201.000	200.824	200.824
3	205.508	208.000	206.750	206.750	205.913	207.147	206.529	206.530
4	207.863	214.000	210.909	210.883	209.749	211.808	210.776	210.773
5	209.118	221.000	214.977	214.907	213.110	216.114	214.607	214.602

The differences between the Laspeyres and Paasche fixed base and chained aggregate implicit input indexes grow over time and are fairly substantial by period 5 but the other four superlative indexes are reasonably close to each other.

With no actual output prices and quantities available for our imaginary public sector production unit, we set the aggregate output price and quantity levels for period t to equal the corresponding aggregate input price and quantity levels that are listed in Tables 5 and 6. Thus if we choose to use the fixed base Laspeyres formula to form aggregate input prices, the period t output price is set equal to P_{LY}^t listed in Table 5 and the period t aggregate output quantity is set equal to Q_{LY}^t listed in Table 6. The *period t Gross Output Total Factor Productivity level* using the fixed base Laspeyres price index formula is defined as TFP_{LY}^t equal to period t gross output using Laspeyres fixed base price indexes divided by the period t aggregate input level using fixed base Laspeyres price indexes which leads to the result $TFP_{LY}^t \equiv Q_{LY}^t/Q_{LY}^t = 1$ for each t. Using the other seven methods for aggregating inputs similarly leads to *gross output productivity levels* that are identically unity; i.e., $TFP_{PY}^t \equiv Q_{PY}^t/Q_{PY}^t = 1$, $TFP_{FY}^t \equiv Q_{FY}^t/Q_{FY}^t = 1$, ..., $TFP_{TY}^{t*} \equiv Q_{TY}^{t*}/Q_{TY}^{t*} = 1$ for all t.

Instead of calculating gross output TFP levels, it is also useful to construct *value added TFP levels*.[33] Nominal value added for a production unit for period t is defined as the value of outputs produced during period t less the value of intermediate inputs used by the production unit during period t, where an intermediate input is an input that was produced by another domestic or foreign production unit. Thus for our numerical example, *period t nominal value added* V_O^t is defined as follows for t = 1,...,5:

$$\begin{aligned}(11)\ V_O^t &\equiv V_Y^t - p^t \cdot z^t \\ &= (w^t \cdot x^t + p^t \cdot z^t) - p^t \cdot z^t \\ &= w^t \cdot x^t\end{aligned}$$

[33] Generally speaking, gross output TFP growth will be smaller than value added TFP growth; see Schreyer (2001) and Diewert (2015) for explanations of this phenomenon.

where the second equality follows using (10). Thus period t nominal value added for our public sector production unit, V_O^t, is equal to period t value of primary inputs used in the unit, $w^t \cdot x^t$, for each period t. Thus period t nominal value added V_O^t is unambiguously defined. But how exactly should the *price* of real value added and the corresponding *quantity or volume* be defined? We will follow the methodology that was used in the *Producer Price Index Manual* [34] and simply apply normal index number theory to the components of value added, treating all prices and output quantities as positive numbers but changing the sign of intermediate input quantities from positive to negative.[35] Thus the *fixed base Laspeyres, Paasche, Fisher and Törnqvist price indexes for period t value added* are defined as follows for t = 1,...,5:

(12) $P_{LO}^t \equiv [P_{LY}^t Q_{LY}^1 - p^t \cdot z^1]/[P_{LY}^1 Q_{LY}^1 - p^1 \cdot z^1]$;
(13) $P_{PO}^t \equiv [P_{PY}^t Q_{PY}^t - p^t \cdot z^t]/[P_{PY}^1 Q_{PY}^t - p^1 \cdot z^t]$;
(14) $P_{FO}^t \equiv [P_{LO}^t P_{PO}^t]^{1/2}$;
(15) P_{TO}^t
$$\equiv \exp[(½)(s_{T1}^1+s_{T1}^t)\ln(P_{TY}^t/P_{TY}^1)+(½)(s_{T2}^1+s_{T2}^t)\ln(p_1^t/p_1^1)+(½)(s_{T3}^1+s_{T3}^t)\ln(p_2^t/p_2^1)]$$

where $s_{T1}^t \equiv V_Y^t/V_O^t$, $s_{T2}^t \equiv -p_1^t z_1^t/V_O^t$ and $s_{T3}^t \equiv -p_2^t z_2^t/V_O^t$ for t = 1,...,5. Note that $s_{T1}^t > 0$, $s_{T2}^t < 0$, $s_{T3}^t < 0$ but these value added "shares" sum up to one; i.e., $s_{T1}^t + s_{T2}^t + s_{T3}^t = 1$ for t = 1,...,5. The value added price indexes defined by (12)-(15) are listed in Table 7 below. The fixed base price index formulae defined by (12)-(15) can be modified to provide the corresponding chain link price indexes and these links can be chained together to defined the corresponding Laspeyres, Paasche, Fisher and Törnqvist value added chained indexes P_{LO}^{t*}, P_{PO}^{t*}, P_{FO}^{t*} and P_{TO}^{t*}. These chained indexes are also listed in Table 7.

Table 7: Fixed Base and Chained Laspeyres, Paasche, Fisher and Törnqvist Value Added Output Price Indexes

Period t	P_{LO}^t	P_{PO}^t	P_{FO}^t	P_{TO}^t	P_{LO}^{t*}	P_{PO}^{t*}	P_{FO}^{t*}	P_{TO}^{t*}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.99200	0.99089	0.99142	0.99144	0.99200	0.99089	0.99142	0.99144
3	0.98400	0.97519	0.97909	0.97963	0.98288	0.97844	0.98052	0.98065
4	0.95800	0.93500	0.94445	0.94662	0.95096	0.94314	0.94681	0.94705
5	0.93600	0.89347	0.90920	0.91494	0.92045	0.90957	0.91465	0.91500

As usual, the Laspeyres type price indexes are higher (after period 1) than the corresponding Paasche type indexes and the spread between the fixed base Laspeyres and

[34] See the IMF/Eurostat/ILO/OECD/UNECE/The World Bank (2004).
[35] Justifications for this procedure that are based on the economic approach to index number theory can be found in Diewert (1976), Diewert and Morrison (1986) and Kohli (1990) who justified the use of Törnqvist indexes using the economic approach to the measurement of productivity. Diewert (1992a) (2012) justified the use of Fisher indexes using the economic approach. If the public sector production unit sells some outputs at market prices, then these sales should become a part of the unit's intermediate input subaggregate except that the positive signs associated with the quantities of such outputs should be replaced by negative signs.

Paasche price indexes is larger than the spread between their chained counterparts. As usual, the superlative indexes are generally close to each other. Note also that the value added price indexes listed in Table 7 end up below unity in period 5 whereas the gross output price indexes listed in Table 5 end up above unity in period 5 (except for the fixed base Paasche index).

The period t implicit real value added output quantities or volumes that correspond to the four fixed base value added price indexes defined by (12)-(15) are defined by (16) and the corresponding chained indexes are defined by (17) for t = 1,...,5:

(16) $Q_{LO}{}^{t} \equiv V_{O}{}^{t}/P_{LO}{}^{t}$; $Q_{PO}{}^{t} \equiv V_{O}{}^{t}/P_{PO}{}^{t}$; $Q_{PO}{}^{t} \equiv V_{O}{}^{t}/P_{PO}{}^{t}$; $Q_{PO}{}^{t} \equiv V_{O}{}^{t}/P_{PO}{}^{t}$;
(17) $Q_{LO}{}^{t*} \equiv V_{O}{}^{t}/P_{LO}{}^{t*}$; $Q_{PO}{}^{t*} \equiv V_{O}{}^{t}/P_{PO}{}^{t*}$; $Q_{PO}{}^{t*} \equiv V_{O}{}^{t}/P_{PO}{}^{t*}$; $Q_{PO}{}^{t*} \equiv V_{O}{}^{t}/P_{PO}{}^{t*}$.

The above implicit quantity indexes of real value added are listed in Table 8 below. The substantial difference between the fixed base implicit Laspeyres and Paasche indexes of real value added in period 5 is noteworthy.

Table 8: Fixed Base and Chained Implicit Laspeyres, Paasche, Fisher and Törnqvist Indexes of Real Value Added

Period t	$Q_{LO}{}^{t}$	$Q_{PO}{}^{t}$	$Q_{FO}{}^{t}$	$Q_{TO}{}^{t}$	$Q_{LO}{}^{t*}$	$Q_{PO}{}^{t*}$	$Q_{FO}{}^{t*}$	$Q_{TO}{}^{t*}$
1	100.000	100.000	100.000	100.000	100.000	100.000	100.000	100.000
2	100.887	101.000	100.946	100.944	100.887	101.000	100.946	100.944
3	107.033	108.000	107.570	107.510	107.154	107.640	107.412	107.398
4	111.263	114.000	112.859	112.601	112.086	113.016	112.578	112.550
5	115.502	121.000	118.907	118.161	117.453	118.859	118.199	118.153

The primary input quantity indexes listed in Table 3 above along with the real value added output indexes listed in Table 8 can be used to form TFP indexes by dividing period t real value added by the corresponding measure of real input. Thus the *period t Value Added Total Factor Productivity level* using the fixed base Laspeyres price index formula is defined as $TFP_{LO}{}^{t} \equiv Q_{LO}{}^{t}/Q_{LX}{}^{t}$ for each t. Using the other seven methods for aggregating inputs similarly leads to the following alternative measures of real value added TFP for period t: $TFP_{PO}{}^{t} \equiv Q_{PO}{}^{t}/Q_{PX}{}^{t}$; $TFP_{FO}{}^{t} \equiv Q_{FO}{}^{t}/Q_{Fx}{}^{t}$; $TFP_{TO}{}^{t} \equiv Q_{TO}{}^{t}/Q_{TX}{}^{t}$; $TFP_{LO}{}^{t*} \equiv Q_{LO}{}^{t*}/Q_{LX}{}^{t}$; $TFP_{PO}{}^{t*} \equiv Q_{PO}{}^{t*}/Q_{PX}{}^{t}$; $TFP_{FO}{}^{t*} \equiv Q_{FO}{}^{t*}/Q_{Fx}{}^{t}$; $TFP_{TO}{}^{t*} \equiv Q_{TO}{}^{t*}/Q_{TX}{}^{t}$. These alternative measures of value added productivity are listed in Table 9.[36]

Table 9: Fixed Base and Chained Laspeyres, Paasche, Fisher and Törnqvist Indexes of Total Factor Productivity Based on Real Value Added

Period t	$TFP_{LO}{}^{t}$	$TFP_{PO}{}^{t}$	$TFP_{FO}{}^{t}$	$TFP_{TO}{}^{t}$	$TFP_{LO}{}^{t*}$	$TFP_{PO}{}^{t*}$	$TFP_{FO}{}^{t*}$	$TFP_{TO}{}^{t*}$
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	1.00000	1.00000	1.00003	1.00001	1.00000	1.00000	1.00003	1.00001

[36] Note that the TFP levels listed in Table 9 can also be generated by dividing the input price indexes listed in Table 2 by the corresponding real value added output prices listed in Table 7. This result was first established by Jorgenson and Griliches (1967; 252) and is a consequence of the fact that for our example, the value of inputs is always exactly equal to the value of outputs for each period.

3	**1.00000**	**1.00000**	**1.00051**	**1.00000**	**1.00000**	**1.00000**	**1.00014**	**1.00002**
4	**1.00000**	**1.00000**	**1.00210**	**0.99999**	**1.00000**	**1.00000**	**1.00025**	**1.00002**
5	**1.00000**	**1.00000**	**1.00582**	**0.99997**	**1.00000**	**1.00000**	**1.00038**	**1.00002**

Since there are no independent measures of output for our production unit, it should be the case that all TFP levels for our public sector production unit should equal unity. Looking at Table 9, it can be seen that this result holds whenever Laspeyres or Paasche indexes are used. For the Fisher and Törnqvist methods of aggregation, it can be seen that this result does not hold but it does hold to a high degree of approximation.[37]

This has been a rather lengthy discussion on how to construct price and quantity estimates for a public sector production unit when information on the price and quantity of the unit's outputs is not available. In the end, we have seen that it does not really matter very much whether we measure the productivity of the unit on the basis of its gross output or on its value added output: in the first case, the level of TFP is always equal to unity and in the second case, it is either equal to unity if Laspeyres or Paasche aggregation is used or almost equal to unity if superlative index aggregation is used. Thus it would seem that the choice of index number formula is not very relevant for public sector production units that do not have independent output measures. While this is true as far as the calculation of TFP is concerned, it is definitely relevant when measuring the size of the real value added (and gross output) of public sector production units. From Table 6 above, it can be seen that the fixed base Laspeyres and Paasche estimates of real gross output in period 5 were 209.1 versus 221.0 while from Table 8 above, the fixed base Laspeyres and Paasche estimates of real value added in period 5 were 115.5 versus 121.0. These are substantial differences: the choice of index number formula does matter.

5. Cost Based Methods for Valuing Public Sector Outputs when Information on the Quantities of Nonmarket Outputs is Available

In this section, we assume that there is quantity information on the nonmarket outputs produced by a public sector production unit but there are no corresponding market prices to value the nonmarket outputs. The main public sectors at the national or state level where this situation arises are the health, education and social services sectors.[38] At the municipal or local level, the same situation arises for provision of some services such as waste disposal, water and sewage services.[39] The provision of road and highway services

[37] The reason why TFP = 1 when Laspeyres or Paasche indexes are used as the method of aggregation over commodities is due to the fact that these two formulae are *consistent in aggregation*; i.e., if a Laspeyres aggregate is formed by using the Laspeyres formula to aggregate two or more subaggregates and then the resulting subaggregate price and quantities are aggregated in a second stage using the Laspeyres formula again, then the resulting two stage Laspeyres indexes of price and quantity are exactly equal to the corresponding Laspeyres indexes of price and quantity that are constructed using a single stage of aggregation. On the other hand, the two superlative indexes are not exactly consistent in aggregation but they are approximately consistent in aggregation; see Diewert (1978) for an explanation of these results.

[38] We will discuss in more detail the problems associated with measuring health and education outputs in section 8 below. The Atkinson (2005) Report on measuring government outputs in the context of the national accounts has a much more detailed discussion of the associated measurement problems.

[39] Data Envelopment Analysis or benchmarking production units for relative performance was initiated by Farrell (1957) and Charnes, Cooper and Rhodes (1978) and these methods have been widely applied to

arises at both levels of government. The *System of National Accounts 1993* recommends valuing publicly provided services at their unit costs of production.[40] In particular, Chapter 16 in *SNA 1993* notes that if we have quantity information on the numbers of various different types of outputs produced by a public sector production unit, then Laspeyres or Paasche indexes can be calculated using sales as values for market services and unit costs times quantities produced as values for nonmarket services.[41] We will indicate below exactly how this can be done.

We will consider the following two cases:

- Case 1: The production unit produces only one nonmarket output.
- Case 2: The production unit produces many nonmarket outputs.

Case 1 is easy to deal with if we make use of the algebra that was developed in the previous section. Thus define the period t price and quantity vectors for primary inputs by w^t and x^t and the period t price and quantity vectors for intermediate inputs by p^t and z^t as before. However, now we have information on the *quantity of output produced* by the production unit during period t, say q^t for t = 1,...,5. In order to apply the algebra that was developed in the previous section, we need to change the units of measurement for the single output so that output in period 1 is equal to input cost in period 1. Thus define the *normalized output* for the production unit for period t, Q^t, for t = 1,...,5 as follows:

$$(18)\ Q^t \equiv [p^1 \cdot z^1 + w^1 \cdot x^1][q^t/q^1].$$

Thus when t = 1, the quantity of output produced Q^1 is equal to total input cost in period 1. The corresponding period t *cost based output price* P^t is defined as period t total cost divided by period t normalized output; i.e., define P^t for t = 1,...,5 as follows:[42]

$$(19)\ P^t \equiv (p^t \cdot z^t + w^t \cdot x^t)/Q^t.$$

The Q^t and P^t defined by (19) and (19) are now independent estimates for the quantity and price of gross output produced by the public sector production unit during period t. Thus period t *fixed base Laspeyres and Paasche type gross output Total Factor*

---

public sector production units. We will not discuss these methods in this chapter since they generally do not generate reasonable output prices in the case where there are many outputs.

[40] Scitovsky (1967) suggested this method for imputing prices to outputs produced by the public health sector. Hill (1975; 19-20) noted that unit costs should equal selling prices for competitive market producers and advocated the general use of unit costs to value outputs for nonmarket producers. Hill carried over his 1975 advice into the *System of National Accounts 1993* where he was a principle contributor. Schreyer (2012a) formally developed the price equals unit cost methodology to value nonmarket outputs in much more detail.

[41] See paragraphs 16.133 and 16.134 of Eurostat, IMF, OECD, UN and the World Bank (1993). If the public sector production unit produces some outputs that are sold at market prices, then these outputs can be reclassified as negative intermediate inputs for our purposes; i.e., these outputs would appear as negative components in the z^t vectors while the corresponding prices would appear as positive components in the p^t vectors.

[42] It can be seen that $P^1 = 1$.

Productivity for the public sector production unit can be defined as $TFP_{GOL}{}^{t} \equiv Q^{t}/Q_{LY}{}^{t}$ and $TFP_{GOP}{}^{t} \equiv Q^{t}/Q_{PY}{}^{t}$ respectively. The remaining 6 input indexes listed in Table 6 can be used to define analogous gross output TFP indexes.

Value added TFP indexes can also be defined by adapting the algebra used in the previous section. The new fixed base Laspeyres and Paasche value added price indexes that are counterparts to definitions (12) and (13) in the previous section are now defined as follows:

(20) $P_{LO}{}^{t} \equiv [P^{t}Q^{1} - p^{t}\cdot z^{1}]/[P^{1}Q^{1} - p^{1}\cdot z^{1}]$;
(21) $P_{PO}{}^{t} \equiv [P^{t}Q^{t} - p^{t}\cdot z^{t}]/[P^{1}Q^{t} - p^{1}\cdot z^{t}]$.

Thus P^{t} defined by (19) has replaced $P_{LY}{}^{t}$ in (12) and P^{t} defined by (19) has replaced $P_{PY}{}^{t}$ in (13) and Q^{t} defined by (18) has replaced $Q_{LY}{}^{t}$ in (12) and $Q_{PY}{}^{t}$ in (13) in equations (20) and (21). The remaining 6 value added price indexes that are counterparts to the remaining 6 value added price indexes that are listed in Table 7 can be defined in an analogous manner. Denote these remaining 6 indexes for period t by $P_{FO}{}^{t}$, $P_{TO}{}^{t}$, $P_{LO}{}^{t*}$, $P_{PO}{}^{t*}$, $P_{FO}{}^{t*}$ and $P_{TO}{}^{t*}$. The value of value added in period t, $V_{O}{}^{t}$, is still equal to the value of primary input in period t, $w^{t}\cdot x^{t}$. Now period t real value added can be defined in 8 different ways by deflating $V_{O}{}^{t}$ by our new 8 alternative real value added price indexes. Thus we obtain counterparts to the 8 real value added quantity indexes that appeared in Table 8; i.e., we have $Q_{LO}{}^{t} \equiv V_{O}{}^{t}/P_{LO}{}^{t}$; $Q_{PO}{}^{t} \equiv V_{O}{}^{t}/P_{PO}{}^{t}$; ..., $Q_{TO}{}^{t*} \equiv V_{O}{}^{t}/P_{TO}{}^{t*}$. Finally, 8 alternative measures of TFP that are counterparts to the Table 9 measures of TFP can be obtained by dividing the new real value added output indexes by the corresponding primary input quantity indexes; i.e., we have $TFP_{LO}{}^{t} \equiv Q_{LO}{}^{t}/Q_{LX}{}^{t}$; $TFP_{PO}{}^{t} \equiv Q_{PO}{}^{t}/Q_{PX}{}^{t}$; ..., $TFP_{TO}{}^{t*} \equiv Q_{TO}{}^{t*}/Q_{TX}{}^{t*}$. Now that we have independent measures of the quantity of gross nonmarket outputs produced by the public sector production unit, it will no longer be the case that TFP (either on a gross output or a value added output concept) will be identically (or approximately) equal to unity in all periods. The choice of index number formula is now important for the measurement of TFP as well as for the measurement of sectoral real output.

We will now consider the second case where the public sector production unit produces many outputs. Obviously, if data on the price and quantity for each input that is used to produce each output can be found, then the production activities of the public sector unit can be decomposed into separate production functions and the index number treatment that was explained above for a single output can be applied to each separate production activity. Typically, it will be difficult to allocate the fixed inputs used by the public sector establishment to the separate activities that produce each output.[43] However, it may be

[43] If it is possible to estimate a period t joint cost function for the public sector production unit, say $C^{t}(q,p,w)$, and this cost function is differentiable with respect to the components of the output vector q when evaluated at the period t data so that the vector of first order partial derivatives $\nabla_{q}C^{t}(q^{t},p^{t},w^{t}) \equiv P^{t}$ exists, then this vector of marginal costs can serve as an appropriate vector of cost based output prices. In addition, if the technology is subject to constant returns to scale, then the value of period t output, $P^{t}\cdot q^{t}$, will be equal to period t total cost, $C^{t}(q^{t},p^{t},w^{t}) = p^{t}\cdot z^{t} + w^{t}\cdot x^{t}$. This cost function based methodology for the

possible to obtain estimates of the *fraction* of total establishment costs in a period that can be imputed to each production activity. Thus define the overall period t price and quantity vectors used by the production unit for primary inputs by w^t and x^t and the period t price and quantity vectors for intermediate inputs by p^t and z^t as before. Suppose that the unit produces K outputs and information on the quantity of each output produced by the production unit during period t is available, say q_k^t for k = 1,...,K and t = 1,...,5. Suppose in addition, that $f_k^t > 0$ is the fraction of period t total cost that can be attributed to the production of output k during period t. Approximate cost based period t output prices for the K outputs can now be defined as follows for all t and k:

(22) $P_k^t \equiv f_k^t(p^t \cdot z^t + w^t \cdot x^t)/q_k^t$.

Thus we will have period t output price and quantity vectors, $P^t \equiv [P_1^t,...,P_K^t]$ and $q^t \equiv [q_1^t,...,q_K^t]$, for the production unit and normal index number theory can be used to form output aggregates[44] which in turn can be matched up with the corresponding input aggregates to form TFP estimates.

The problem with this method for the valuation of nonmarket outputs is that it will generally be difficult to determine the appropriate cost fractions f_k^t.[45] In other words, typically it will be possible to measure establishment outputs and total cost in each period, but it will be difficult to decompose the total cost into cost components that can be allocated to each individual output so that the vector of unit costs can be calculated.

6. The Use of Quality Adjusted Output Weights

In this section, we discuss the use of purchaser or recipient weights to aggregate the nonmarket outputs of a public sector production unit. The basic idea is easy to explain. Consider a public sector establishment that is producing K outputs over T periods, say q_k^t for k = 1,...,K and t = 1,...,T. Let $q^t \equiv [q_1^t,...,q_K^t]$ be the period t vector of nonmarket outputs. We assume that these output quantities can be observed. Define the period t price and quantity vectors of intermediate and primary inputs by the usual p^r, w^t for prices and z^t and x^t for quantities. The aggregation of inputs proceeds as was explained in Section 4 above. The problem is how exactly can we construct a period t aggregate output price, say P^t and the corresponding aggregate gross output quantity, say Q^t?

A possible solution to the problem is to use a vector of user based relative valuations for the K outputs.[46] Let $\omega_k > 0$ represent the *relative value to users or recipients* of output k for k = 1,...,K and let $\omega \equiv [\omega_1,...,\omega_K]$ be the vector of weights. These weights can be

---

measurement of the productivity growth of a public sector production unit is developed in some detail in Diewert (2011) (2012) and in the Appendix to this chapter.

[44] The resulting gross output aggregates should be normalized so that aggregate real output equals aggregate real input in the base period.

[45] Suppose that the producer's total cost function for period t, $C^t(q_1^t,...,q_K^t,p^t,w^t) \equiv C^t$, has been estimated and it is differentiable with respect to the outputs, q_k. If in addition, production is subject to constant returns to scale, then the fractions f_k^t should be defined as $f_k^t \equiv [q_k^t \partial C^t(q_1^t,...,q_K^t,p^t,w^t)/\partial q_k]/C^t$ for k = 1,...,K.

[46] Our suggested methodology is simply an elaboration of a methodology suggested by Schreyer (2010; 21).

regarded as *quality adjustment factors*: the higher the weight, the more recipients or users of the nonmarket outputs of the establishment value the particular output.[47] The period t output aggregate can be defined as the weighted sum of the individual period to output quantities using the vector ω as weights and the corresponding period t nonmarket aggregate price P^{t*} can be defined as period t total cost divided by Q^{t*}; i.e., we have the following definitions for t = 1,...,T:[48]

(23) $Q^{t*} \equiv \omega \cdot q^t$;
(24) $P^{t*} \equiv [p^t \cdot z^t + w^t \cdot x^t]/Q^{t*}$.

Now define the aggregate normalized period t output price and quantity by $P^t \equiv P^{t*}/P^{1*}$ and $Q^t \equiv P^{1*}Q^{t*}$ and we can apply the algebra that developed for Case 1 in Section 5 above to the normalized output price and quantity that we have just defined.[49]

It should be noted that this method can in principle deal with the introduction of new nonmarket goods and services: all that is required is a valuation weight for a new commodity relative to the weights for continuing commodities.[50]

The advantage of this welfare weights method for valuing the nonmarket outputs of a public sector production unit over the section 5 method is that the present method does not require estimates for the unit cost of production for each nonmarket output. Of course, the problem with the present method for valuing nonmarket outputs is that it will be difficult to determine the appropriate vector of output weights ω.[51] If experts cannot agree on the appropriate weights, this puts statistical agencies in a difficult position since their estimates of output and input should be *objective* and *reproducible*. In the following

[47] Sections 7 and 8 below suggest some methods for determining these weights but there is no generally applicable method for choosing these welfare oriented weights. In the concluding section of this chapter, we indicate that it is very difficult to determine these valuation weights in a rigorous fashion.
[48] Suppose the welfare weights change over time. Thus let ω^1 and ω^2 be the weights for periods 1 and 2. Define $Q^{1*} \equiv 1$ and $Q^{2*} \equiv [\omega^1 \cdot q^2/\omega^1 \cdot q^1]^{1/2}[\omega^2 \cdot q^2/\omega^2 \cdot q^1]^{1/2}$. Now use (24) to define P^{1*} and P^{2*} and define the normalized prices and quantities by $P^t \equiv P^{t*}/P^{1*}$ and $Q^t \equiv P^{1*}Q^{t*}$ for t = 1,2. The resulting P^t and Q^t are Fisher type aggregate prices and quantities for periods 1 and 2. Thus the methodology can be generalized to deal with changing welfare weights.
[49] Note that we are forcing the aggregate price times quantity in each period to equal period t total cost; i.e., we have $P^tQ^t = P^{t*}Q^{t*} = p^t \cdot z^t + w^t \cdot x^t$ for each t. This follows the convention applied to the valuation of nonmarket production that is recommended by Schreyer (2010; 76): "Throughout this handbook, it is understood that the value of output of institutional units in the health care industry is measured by the observed money value of output in the case of market producers and by the sum of costs in the case of non-market producers. This follows national accounts conventions." It is not necessary to perform the normalizations of output prices and quantities to make the value of nonmarket output equal to the total net cost of producing the nonmarket outputs in period t, but if this is not done, then the nonmarket production unit will make a profit or loss which is more or less arbitrary. Thus the Schreyer-Hill normalizing convention leads to a value of aggregate output which will exactly exhaust period t cost.
[50] If nonmarket commodity 1 is not present in period 1 but is present in subsequent periods, then this implies that $q_1^1 = 0$ and for subsequent periods $t > 1$, $q_1^t > 0$. For a more explicit treatment of the new commodity problem in the nonmarket context, see the following section or Diewert (2012; 220-221).
[51] See Atkinson (2005; 88-90) and Schreyer (2012) for nice discussions on the valuation of nonmarket outputs and the differences between marginal cost and final demander valuations.

two sections, we will consider examples of how the use of output valuation weights could work in practice.

7. Quality Change, Unit Values and Linking Bias

Technical progress occurs in the public sector, just as it occurs in the private sector. When a new product appears in the private sector during a time period, statistical agencies that construct price indexes face a problem: there is no price in the previous period that can be matched to the new product. Hence, historically, statistical agencies have ignored the existence of the new product during the period where it first appears but in the second (or later) period of its existence, the new product can be treated in the normal way where a Laspeyres, Paasche or other index going from one period to the next is constructed because price and quantity information on the new product is now available for two consecutive periods. Thus the new product is *linked in* to an existing index that excluded the new product. The problem with this procedure is that it can lead to biased price and quantity indexes.[52] We will illustrate the problem by analyzing an artificial example that is due to Schreyer (2010; 21).

Schreyer supposes that there is a clinic that offers treatments for eye surgery. In period 1, only a traditional treatment exists. In period 2, a laser surgery alternative is introduced that is equivalent to the traditional treatment but has a lower unit cost. His data cover three periods. Let q_1^t and q_2^t denote the number of traditional and laser treatments done in period t. Schreyer assumes that the period t total costs for performing the traditional and laser surgeries are C_1^t and C_2^t respectively. Thus period t unit costs for the two types of treatment, c_1^t and c_2^t, are defined as $c_k^t \equiv C_k^t/q_k^t$ for k = 1,2 and t = 1,2,3. These data are listed in Table 10 below. Following the cost based methodology explained in Section 5 above, the normalized unit costs for each sector can be used as prices to value the outputs of each sector. Thus the *normalized cost based output prices* for the first sector are defined as $P_1^t \equiv c_1^t/c_1^1$ for t =1,2,3 and the (normalized) cost based output prices for the second sector are defined as $P_2^t \equiv c_2^t/c_2^2$ for t = 2,3. The *normalized quantities* for each sector k in period t, Q_k^t, are defined as the sectoral total costs C_k^t divided by the corresponding normalized output price, P_k^t, so we have $Q_1^t \equiv C_1^t/P_1^t = q_1^tc_1^t$ for t = 1,2,3 and $Q_2^t \equiv C_2^t/P_2^t = q_2^tc_2^t$ for t = 2,3. These normalized output prices and quantities are also listed in Table 10.

Since we are interested in productivity measurement in this survey, we will augment Schreyer's data by adding some detail on the decomposition of treatment costs into price and quantity components. For simplicity, we suppose that there is only one input that is used in each treatment and the price of this input in period t for treatment k, w_k^t, is always equal to unity so we have $w_k^t = 1$ for k = 1,2 and t = 1,2,3. Thus the quantity of input used

[52] This problem was pointed out by Griliches (1979; 97): "What happens to price indices will depend on whether they allow for the "quality" improvements embedded in the new item or not. By and large they do not make such quality adjustments. Instead, the new product is "linked in" at its introductory (or subsequent) price with the price indices left unchanged." Gordon (1981; 130-133) and Diewert (1996; 31) (1998; 51-54) also recognized this source of bias and suggested methods for measuring its magnitude.

in treatment k for period k, x_k^t, is equal to the corresponding total cost C_k^t divided by w_k^t. Thus for k = 1,2 and t = 1,2,3, we have:

(25) $x_k^t \equiv C_k^t/w_k^t = C_k^t$.

The input prices w_k^t and the corresponding input quantities $x_k^t = C_k^t$ are also listed in Table 10.

Table 10: Data for Schreyer's Laser Surgery Example

t	q_1^t	q_2^t	w_1^t	w_2^t	C_1^t (x_1^t)	C_2^t (x_2^t)	c_1^t	c_2^t	P_1^t	P_2^t	Q_1^t	Q_2^t
1	**50**	**0**	**1.0**	**1.0**	**5000**	**0**	**100**	**__**	**1.0**	**__**	**5000**	**0**
2	**40**	**10**	**1.0**	**1.0**	**4000**	**900**	**100**	**90**	**1.0**	**1.0**	**4000**	**900**
3	**5**	**45**	**1.0**	**1.0**	**500**	**4050**	**100**	**90**	**1.0**	**1.0**	**500**	**4050**

Period t *total cost*, C^t, is defined as the sum of the two treatment total costs; i.e., we have $C^t \equiv C_1^t + C_2^t$ for t = 1,2,3. Since the sectoral input prices w_k^t are always equal to unity, it can be seen that any reasonable index number estimate for an aggregate input price will be equal to unity as well. Thus let the period t aggregate input price be defined as $W^t \equiv w_1^t = w_2^t = 1$ for t = 1,2,3. Define period t aggregate input X^t as total cost divided C^t divided by the period t input price index W^t so we have $X^t \equiv C^t/W^t = x_1^t + x_2^t$ for t = 1,2.3. C^t, W^t and X^t are listed in Table 11.

From Table 10, it can be seen that the normalized cost based output prices for both treatments are equal to one for all periods except that the output price for laser treatments in period 1, P_2^1, is missing. How should an aggregate output price be constructed, given that there is a missing price for the second commodity in period 1? Following standard statistical agency practice in past years, it is natural to use the price movements in the first commodity as the deflator for the value of both outputs in period 2. Letting P^t be the aggregate output price index, this methodology leads to the following definitions for P^t for t = 1,2: $P^1 \equiv P_1^1 = 1$; $P^2 \equiv P_1^2 = 1$. The output price index that takes us from period 2 to period 3 could be the Laspeyres, Paasche, Fisher or Törnqvist index but since both output prices remain constant over these two periods, all of these bilateral price indexes will remain constant as well. Thus the aggregate output price index will remain constant over periods 2 and 3. Putting this all together, we will have $P^t = 1$ for t = 1,2,3. The corresponding period t aggregate quantity, Q^t, is defined as aggregate cost C^t divided by P^t and thus $Q^t \equiv C^t/P^t = C^t$ for t = 1,2,3. Total Factor Productivity, TFP^t, is defined as aggregate output Q^t divided by aggregate input X^t; i.e., $TFP^t \equiv Q^t/X^t = 1$ for t = 1,2,3 since both Q^t and X^t turn out to equal aggregate period cost, C^t, for all t. Thus using the usual linking methodology to deal with new products, we end up showing that the introduction of a new more productive technology has led to no measured productivity gains. C^t, W^t, X^t, P^t, Q^t and TFP^t are all listed in Table 11.

Table 11: Alternative Measures of Aggregate Output, Input and TFP for the Schreyer Example

t	C^t	W^t	X^t	P^t	Q^t	TFP^t	Q_u^{t*}	P_u^{t*}	Q_u^t	P_u^t	TFP_u^t
1	5000	1.0	5000	1.0	5000	1.000	50	100	5000	1.00	1.000
2	4900	1.0	4900	1.0	4900	1.000	50	98	5000	0.98	1.020
3	4550	1.0	4550	1.0	4550	1.000	50	91	5000	0.91	1.099

Since each treatment gives equivalent results to recipients of the treatment, an alternative measure of aggregate clinic output can be obtained by simply adding up the number of treatments. Thus define a *utility based measure of output* in period t, Q_u^{t*}, by performing this addition and define the corresponding *unit value price*, P_u^{t*}, by deflating total cost C^t by Q_u^{t*}. Thus for t = 1,2,3, we have:

(26) $Q_u^{t*} \equiv q_1^t + q_2^t$; $P_u^{t*} \equiv C^t/Q_u^{t*}$.

In keeping with our convention that the aggregate output and input prices should equal unity in the base period, we normalize the price series P_u^{t*} by dividing each price by the price in the base period P_u^{1*} and the quantity series Q_u^{t*} is normalized by multiplying each quantity by P_u^{1*}. Denote the resulting *normalized aggregate output price and quantity series* for period t by P_u^t and Q_u^t. For t = 1,2,3, we have the following definitions:

(27) $P_u^t \equiv P_u^{t*}/P_u^{1*}$; $Q_u^t \equiv Q_u^{t*} P_u^{1*} = Q_u^{t*} [C^1/Q_u^{1*}]$.

The utility based output measure, Q_u^t, can now be used to define a utility based measure of Total Factor Productivity that is equal to Q_u^t divided by our measure of aggregate input X^t; i.e., define $TFP_u^t \equiv Q_u^t/X^t$ for t = 1,2,3. The series P_u^{t*}, Q_u^{t*}, P_u^t, Q_u^t and TFP_u^t are listed in Table 11. Comparing TFP_u^t with our earlier measure TFP^t, it can be seen that our new measure shows that productivity increased substantially during periods 2 and 3 as compared to our old measure which showed no increase in productivity.[53]

An example of how linking bias can lead to estimates of output that have a downward bias occurred when Griliches and Cockburn (1994) discussed how statistical agencies treated the introduction of generic drugs into the marketplace. A generic drug has the same molecular composition as the corresponding brand name drug and so instead of treating the generic and brand name drug as separate products and linking in the generic to the price index for drugs when the generic drug first appears on the marketplace, it may be preferable to treat the products as being equivalent, which will lead to a higher aggregate output of drugs as in the above example.[54]

Recall that the utility oriented approach to output measurement initially measured period t aggregate output as $Q_u^{t*} = q_1^t + q_2^t$ and our final output measure Q_u^t was proportional to

[53] This analysis is essentially due to Schreyer (2010; 21). Diewert (2012; 220-221) presented a similar analysis of the linking problem that came to the same conclusion.

[54] Linking bias occurs not only with respect to the introduction of new products but also when new lower cost outlets come into existence. The resulting bias is called *outlet substitution bias* and it can occur in the public sector as well as in the private sector; see Reinsdorf (1993) and Diewert (1996; 31) (1998; 50-51) for references to the literature.

Q_u^{t*}. Thus it can be seen that Schreyer's model is a special case of the more general quality adjustment model that was defined by equation (23) in the previous section. This equation is $Q^{t*} \equiv \omega \cdot q^t$ which in turn is equal to $\omega_1 q_1^t + \omega_2 q_2^t$ when there are only two outputs. Thus if we set $\omega_1 = \omega_2 = 1$, the general quality adjustment model in the previous section reduces to the method used by Schreyer. For Schreyer's example, it was easy to determine the weights ω_1 and ω_2. In most real life examples, it will be more difficult to determine the appropriate weights.

There is an alternative method for dealing with new goods or services that is due to Hicks (1940; 114). In the period before the new commodity appears, we could imagine (or estimate)[55] an imputed price for the new product that would just cause potential purchasers to demand zero units of it. Now match up this imputed price with the corresponding quantity (which is 0) in the period that precedes the introduction of the new good and apply normal index number theory. For Schreyer's example, since the two commodities are close to being identical, an imputed price for the laser treatment that is slightly higher than the actual period 1 price for the traditional treatment would be appropriate. From Table 10, we see that the price for the traditional treatment in period 1 is $c_1^1 = 100$. Thus following the Hicks' methodology, set the imputed price for the laser treatment, c_2^1, equal to 100 as well. Now go to Table 10 and use the output quantity data that is in the q_1^t and q_2^t columns and the corresponding price data that is in the unit cost columns c_1^t and c_2^t but replace the missing price for c_1^1 by 100. Apply normal index number theory to this new set of price and quantity data. Calculate the fixed base Laspeyres, Paasche, Fisher and Törnqvist price indexes for each period t, P_L^t, P_P^t, P_F^t, P_T^t, and the corresponding quantity indexes, Q_L^t, Q_P^t, Q_F^t, Q_T^t, which are defined as total period t cost C^t divided by the corresponding price index for period t. These fixed base output price and quantity indexes are listed in Table 12.

Table 12: Fixed Base Laspeyres, Paasche, Fisher and Törnqvist Price and Quantity Indexes for the Schreyer Data with Hicksian Imputation

Period t	P_L^t	P_P^t	P_F^t	P_T^t	Q_L^t	Q_P^t	Q_F^t	Q_T^t
1	1.00000	1.00000	1.00000	1.00000	5000.000	5000.000	5000.000	5000.000
2	1.00000	0.98000	0.98995	0.99037	4900.000	5000.000	4949.747	4947.642
3	1.00000	0.91000	0.95394	0.95419	4550.000	5000.000	4769.696	4768.436

As usual, the superlative price indexes, P_F^t and P_T^t, are fairly close to each other and hence so are the companion superlative quantity indexes, Q_F^t and Q_T^t. The use of these superlative quantity indexes as the clinic's output measure would lead to TFP indexes that end up around 1.048 in period 3. The use of the fixed base Laspeyres formula to aggregate output prices leads to a constant price index (i.e., $P_L^t = 1$ for all periods t) and the corresponding output quantity index (listed as Q_L^t in Table 12) is actually a fixed base Paasche quantity index and it takes on exactly the same values as Q^t in Table 11. Thus for

[55] Hausman (1997) estimated these reservation prices econometrically for new breakfast cereals. For a diagram explaining the Hicks methodology, see Diewert (1996; 32). For additional approaches to the estimation of reservation prices, see Lovell and Zieschang (1994), Feenstra (1994) and Diewert (1998; 51-53).

this example, the use of the fixed base Laspeyres formula for aggregating output prices leads to the same (downward biased) TFP indexes that occurred when we used the linking methodology that was explained in the beginning of this section. What is striking is that the use of the fixed base Paasche formula to aggregate output prices leads to the output price index listed as P_P^t in Table 12 (the corresponding output quantity index is listed as Q_L^t), and this price index is exactly equal to the utility weighted output price index P_u^t that is also listed in Table 11. Thus the productivity index that results from the use of P_P^t is exactly equal to the utility weights Total Factor Productivity index, TFP_u^t, that is listed in Table 11. The reason why this equality occurs is due to the fact that the output quantity index Q_P^t that corresponds to P_P^t turns out to be equal to the fixed weight index $Q_P^t \equiv c_1^1 q_1^t + c_2^1 q_2^t = 100q_1^t + 100q_2^t = Q_u^t$.

The above analysis shows that there are at least two methods that can be used to mitigate possible bias due to the introduction of new products into the public sector: the use of utility weighting of outputs or the use of Hicksian imputed prices to value the new product in the period before its introduction. The first method requires estimates for relative welfare or utility weights for the new product relative to existing products. The second method requires estimates for shadow prices that value the new product relative to existing products (these shadow prices are essentially proportional to utility weights) in the period prior to the introduction of the new product. At first glance, it appears that the second method is preferable, since the use of Hicksian shadow prices is consistent with normal consumer theory. Using a superlative index number formula to aggregate either prices or quantities, the resulting volume or quantity indexes can be consistent with utility maximizing behavior on the part of users of the products, where the functional form for the underlying utility function is reasonably flexible. The use of the first method essentially assumes that purchasers of the products have linear subutility functions that remain constant from period to period. However, the apparent superiority of the second method rests on the assumption of utility maximizing behavior on the part of purchasers and on the existence of market prices for the commodities under consideration. In the case of goods and services supplied by the public sector, the usual justifications for the economic approach to index number theory do not apply and so it is not clear that the second method for mitigating new good bias is superior to the first method.

A final method that might be used to quality adjust the outputs of public sector production units is the use of hedonic regression methods.[56] A hedonic regression model regresses the price of a product on the price determining characteristics of the product. Once a hedonic regression has been determined, the relative value of a new product with certain characteristics can be determined relative to existing products using a hedonic regression model. Thus if a new public sector output with a mix of characteristics that is similar to products with similar characteristics that sell in the marketplace, then the value of the new public sector output relative to existing marketplace outputs that are similar

[56] For reviews and references to the literature on hedonic regression techniques, see Triplett (1983) (2004), de Haan (2010), de Haan and Diewert (2011), de Haan and Krsinich (2014) and Aizcorbe (2014). Diewert (2011; 180) and Schreyer (2012a; 260-266) present cost function and utility function based hedonic regression models respectively to adjust for quality change in the public sector. Schreyer's model is probably the more appropriate model.

could be determined using a hedonic regression that involves only marketed products. Thus a hedonic regression model may provide a scientific method for determining the valuation weights ω_k that made their appearance in the previous section. The problem with this suggestion is that the characteristics of the public sector outputs may be quite different from characteristics of "similar" products that appear in the market sector; i.e., there may be no such similar products. However, in some situations, such as the valuation of subsidized housing, the hedonic regression methodology may well work in a satisfactory manner.

8. Specific Measurement Issues

In this section, a few of the measurement issues that arise in measuring outputs in specific subsectors of the public sector will be discussed. The focus will be on possible methods for choosing the utility oriented valuation weights ω_k that made their appearance in section 6 above. For a comprehensive discussion on how to measure outputs for the entire public sector, see Hill (1975) and Atkinson (2005). For a detailed discussion on how to measure education and health outputs, see Schreyer (2010) (2012a).

8.1. The Education Sector

Hill (1975; 48) recommended that output in the public education sector should be measured by *pupil hours of instruction* with possible quality adjustment for the number of students in the classes of instruction under consideration.[57] Hill (1975; 46) did not favour any quality adjustment for class failure rates or for class performance on test scores. He argued that the output of private driving classes is measured by fees collected and hence is proportional to the number of students taking the driving course of instruction and he observed that there is no quality adjustment of the outputs of driving schools for subsequent failures when students take their driving tests. Hence, by analogy, there should be no adjustment for failures when students fail their classes. This argument is not convincing since it may be more reasonable to quality adjust the private sector driving school output for student failure of the subsequent driving tests.

Atkinson (2005; 128) noted that the UK uses the number of *full time equivalent students* as the output measure in the national accounts.[58] Atkinson (2005; 130) basically endorsed this method for measuring school outputs but noted that there should be a switch from registered pupil numbers to actual school attendance numbers and for pupils aged 16 and over, some account of school examination success should be taken into account.

Schreyer (2010; 37) recommended (as a first step) that education output for primary and secondary education services be measured by *pupil hours*, differentiated by the level of

[57] Eurostat (2001) also recommended this measure be used by countries belonging to the European Union.

[58] "The fte pupils in the four types of maintained school (nursery, primary, secondary and special schools) are added together using cost-weighting by type of school, based on total UK expenditure for that type of school. The cost weights have not been updated since 2000." Atkinson (2005; 128). He goes on to note that there was a small quality adjustment based on exam success.

education and possibly other characteristics. However later, Schreyer (2010; 42) indicated how pupil hours could be quality adjusted by average test results for the class under consideration:

"The target measure for the quality-adjusted volume change of education services is the change in the number of pupil hours (H) multiplied by the quality of teaching. The indicator for the quality of teaching is average scores (S) divided by the change of pupil hours per pupil (H/N). Division by H/N is necessary because pupil attainments are influenced by possible changes in the number of lessons and this influence should be eliminated to arrive at quality of one pupil hour."

Schreyer (2010; 42) went on to show that the change in the volume of educational services going from period t−1 to t for the class under consideration was equal to the following expression:

(28) Change in volume $= [H^t/H^{t-1}][S^t/S^{t-1}]/\{[H^t/N^t]/[H^{t-1}/N^{t-1}]\} = N^tS^t/N^{t-1}S^{t-1}$.

Thus the final Schreyer measure of output in period t for the class under consideration is proportional to N^tS^t, the number of students in the class, N^t, times the average class score, S^t, for a test that appropriately measures what has been learned in the class. Thus if there are K classes in scope for the educational output index where the number of pupils in class k in period t is N_k^t and the average test score for students in the class is S_k^t, then period t output is proportional to $Q^t = \Sigma_{k=1}^K S_k^t N_k^t$. This output measure fits in with the general class of quality adjusted output measures that were discussed in section 6 where Q^t was proportional to $\omega\cdot q^t \equiv \Sigma_{k=1}^K \omega_k q_k^t$ or more generally, proportional to $\omega^t\cdot q^t \equiv \Sigma_{k=1}^K \omega_k^t q_k^t$, where the vector of quality adjustment factors ω^t can change as t changes. Thus the average test score S_k^t plays the role of a period t quality adjustment factor ω_k^t and the number of students in class k during period t, N_k^t, plays the role of an output measure q_k^t that is not quality adjusted.

The Schreyer volume measure is a reasonable one but it suffers from two defects:

- The tests administered in each period may not adequately reflect the actual average increase in knowledge that students have acquired going from period t−1 to t;
- The above mentioned average test results do not take into account the capabilities of the class being tested.

The first problem is always a potential problem with using standardized tests to measure the increased capabilities of a class and so caution must be used in allowing test results to be the dominant factor in measuring educational outputs.[59]

[59] When measuring the output for a public school class that consists of young students, one could argue that the school is providing a day care component as well as augmenting student skills and knowledge. The day care component of school output would be proportional to student hours spent at school during the time period under consideration. With no change in school hours, student hours would be proportional to school attendance days. Determining non-arbitrary relative weights for the day care and knowledge acquisition components of school output would be problematic.

The second problem could potentially be addressed by testing students at the beginning of each class term and at the end of the class term. Suppose there are K classes in scope during a number of periods indexed by t. Suppose that the number of students in class k during period t is N_k^t for k = 1,...,K. Suppose further that students are tested at the beginning of period t and at the end of period t on the materials to be covered in the class and that the beginning and end of period test scores for student n in class k during year t are S_{kn}^{tb} and S_{kn}^{te} for k = 1,...,K and n = 1,...,$N_k^t \equiv N(k,t)$. Define the *beginning and end of year average test scores,* $S_{k\bullet}^{tb}$ and $S_{k\bullet}^{te}$, for class k in year t as follows:

$$(29)\ S_{k\bullet}^{tb} \equiv \Sigma_{n=1}^{N(k,t)}\, S_{kn}^{tb}/N_k^t\ ;\ S_{k\bullet}^{te} \equiv \Sigma_{n=1}^{N(k,t)}\, S_{kn}^{te}/N_k^t.$$

If we attempt to measure educational output by the increase in a student's knowledge and capabilities due to classroom teaching, then the quality adjusted output of class k in year t, Q_k^{t*}, could be measured as being proportional to the class sum of end of period t test scores less the corresponding sum of beginning of period t test scores; i.e., we have for k = 1,...,K:

$$\begin{aligned}(30)\ Q_k^{t*} &\equiv \Sigma_{n=1}^{N(k,t)}\, S_{kn}^{te} - \Sigma_{n=1}^{N(k,t)}\, S_{kn}^{tb} \\ &= S_{k\bullet}^{te} N_k^t - S_{k\bullet}^{tb} N_k^t \\ &= [S_{k\bullet}^{te} - S_{k\bullet}^{tb}]N_k^t.\end{aligned}$$

Now set $\omega_k^t \equiv S_{k\bullet}^{te} - S_{k\bullet}^{tb}$ and $q_k^t \equiv N_k^t$ and define (preliminary)[60] total period t output as $Q^{t*} \equiv \Sigma_{k=1}^{K}\, \omega_k^t q_k^t$. Thus we see that the number of students in class k can play the role of an unadjusted measure of class k output and the average difference in the class test scores between the end and beginning of the year can play the role of a quality adjustment factor.

It may be very difficult to design tests given at the beginning and end of a class that will accurately measure the effect of the teaching on increasing student knowledge and capabilities. Moreover, because the quality adjustment factor is a difference, the resulting output volume measures could turn out to be quite volatile.[61]

Obviously, the issues surrounding the quality adjustment of educational output measures are far from settled.

8.2. The Health Sector

Hill (1975) has an extensive (and thoughtful) discussion of the problems associated with the measurement of health services in both the private and public sectors. Hill (1975; 33) noted that there are two general approaches to the measurement of health service outputs:

---

[60] There is a final adjustment that makes the value of output in period t equal to total input cost as was explained in section 6.

[61] This potential volatility could be reduced by making the final quality adjustment factor equal to an average of the Schreyer quality adjustment factor that appears in (28) and the one that appears in (30) where Schreyer's S^t could be replaced by our $S_{k\bullet}^{te}$.

"As already explained, the appropriate measure of the output of the health industry or branch in the context of economic accounting is the treatment actually provided to consumers or patients. An alternative view is to regard medical treatment as only a means to an end, namely achieving an improvement in health, and to seek to measure the output in these terms."

Thus Hill endorsed output measures that are based on the amount of medical treatment provided to patients (such as patient days in hospitals or number of visits to general practitioners), irrespective of the outcome of such treatments. But if a particular public sector medical treatment accomplishes nothing, does it make sense to treat the expenditures associated with the treatment as a positive output?

Thus the issues are similar to the measurement of outputs in the public education sector. Patient hours (in hospitals) or patient days replace pupil hours or pupil days as unadjusted measures of output in the hospital sector compared to measures of output in the public education sector and number of patients treated replace number of students in classes as alternative unadjusted measures of output in the health sector as compared to the education sector. These measures do not reflect any improvements (or failures) in capabilities of patients treated or of pupils taught.

The quality adjustment methodology that was suggested at the end of section 8.1 could be modified to apply to the public health sector. Instead of a test score at the beginning and end of each class, the health sector counterpart would be some measure of the capabilities of a patient before and after the medical treatment. Thus suppose the medical treatment is a hip joint replacement. Before the operation, medical experts would have to devise a "test" score that rated the capabilities of the patient to perform a range of tasks with the impaired leg on a scale of say 1 to 10, with 10 being completely "normal" and 1 being essentially immobile. After the operation with suitable recovery time, the patient would be graded again for mobility using the same scale. However, mobility is not the only issue: before and after the operation, the patient could have various degrees of pain associated with the condition. Again, the degree of pain before and after the operation could be measured on a scale of 1 to 10 but now we would have to face the issue of how to weight the two scales in an overall scale. Assuming that these "test" design issues could be adequately addressed by medical experts, the algebra surrounding equations (29) and (30) could be adapted to the medical context.[62] Obviously, there would be many difficulties associated with implementing this outcome based methodology. Thus in the face of all the difficulties associated with implementing an outcomes methodology for health services, it may be necessary to implement Hill's preferred methodology and just measure the various services that the health sector provides without attempting to measure outcomes.

Hill (1975; 36) provided the following description of the type of hospital services that he would measure:

---

[62] This outcome oriented methodology is not a new idea. Consider the following quotation from Atkinson (2005; 118): "One approach would be to seek to use weights based on the value of health gain from each treatment rather than on its cost."

"For example, suppose an individual enters a hospital for treatment. In general, such treatment can be decomposed into a number of different elements. The following services may be itemized.

(1) The provision of food, accommodation and hotel type services.
(2) Nursing care.
(3) Medical examinations including diagnostic services such as: (i) laboratory tests; (ii) X-ray examinations; (iii) other forms of examinations such as cardiographs, etc.
(4) The provision of drugs and other similar remedial treatment.
(5) Various specialist services such as: (i) surgery; (ii) radiotherapy; (iii) physiotherapy, etc.

The above breakdown is only intended to be illustrative."

Using the above breakdown of hospital associated services, there are still some problems associated with measuring the outputs of each of the above five components. The number of patient days would probably suffice for measuring outputs for category (1); the number of nurse days could suffice for (2); the number of "standard" examinations would work for (3); the number of drugs administered for (4) and either the number of "standard" interventions for category (5) or the number of hours of each type of intervention that was administered by the hospital. Of course, working out the allocation of total hospital costs to each of the 5 types of activity would be difficult.

If instead of following Hill's service oriented methodology, we followed an outcome oriented methodology, then we would no longer measure all of the particular services listed in categories (2)-(5) above: the focus would be on the number of individual patient treatments for various ailments and the outcomes of the hospitalization process. However, we should still measure the "hotel" services that the hospital provides as separate outputs that should be added to the treatment outcome outputs. These food and accommodation services are substituting for the food and accommodation services that are no longer being consumed at the residences of hospital patients. The provision of hotel services for nursing homes and assisted living arrangements are very important components of the outputs of these subsectors of the public sector.

It can be seen that the measurement of public health sector outputs is an extremely difficult problem. In practice, national statistical agencies use only very rough measures of output for their public health sectors. Hill (1975; 42-43) listed how various countries measured the output of their health sectors as of 1975. For example, in the Netherlands, the output of the hospital sector was measured by the number of patient days while the output of other health services was proportional to an index of employment in the non-hospital health sector. For the UK, the output of the hospital sector was also proportional to an index of hospital employment, the output of general practitioners was measured by the number of general practitioners and the output of most other kinds of health services (including dental services) was measured by the number of treatments.

Atkinson (2005; 103-124) provided an extensive review of health output measurement issues in the UK. Atkinson (2005; 106) noted that prior to 2004, hospital outputs were primarily measured by patient days while other medical outputs were primarily measured by the number of consultations for specific treatments. After 2004, the number of treatments that were recognized as separate categories was greatly increased and cost weights for each category were constructed. This constitutes a big improvement over the UK output measures that were in place in 1975 as described by Hill in the above paragraph.

Schreyer (2010; 72-106) has an extensive treatment of the measurement issues surrounding health care, including references to the literature as well as a detailed description of methods used to measure health outputs in a large number of countries. Schreyer adopted a *treatment based definition* for the outputs of the health sector as his target output concept.[63] In practical terms, using a treatment approach to measuring health sector outputs means that the hospital output measure for the treatment of a narrowly defined medical condition would not use patient days as the output measure but would simply use the number of patients treated for the condition (assuming that average treatment outcomes remain constant from period to period). For an application of the treatment outcome approach to the problem of making international comparisons of health sector real output, see Koechlin, Konijn, Lorenzoni and Schreyer (2015).[64]

8.3. The Infrastructure, Distribution and Public Transportation Sectors

The public sector in every country provides a vast network of *roads and highways* that typically can be used free of charge to transport passengers and goods from place to place. From a utility or demander perspective, the output generated by a given stretch of homogeneous road or highway over a period of time should at least include the passenger kilometers travelled over the road as well as the ton kilometers of freight that is shipped over the road during the time period. However, even if a household or firm does not use the road in a given period, they may still value the *option* or *possibility* of using the road and thus the road network itself could also be regarded as a valued output from the demander perspective. From the cost or supplier perspective, building the road initially is definitely a cost determining output. If the public sector also maintains the road, then measures of the utilization of the road (such as passenger miles and ton miles generated by users of the road over the time period) will also be cost determining outputs. Note that the network weights and the utilization weights will generally be quite different from the user and supplier perspectives. Since we are taking the cost perspective to the valuation of public sector outputs, the supplier cost weights will be based on the initial cost of building the road and the resulting user costs[65] will form the network component of total

[63] "A complete treatment refers to the pathway that an individual takes through heterogeneous institutions in the health industry in order to receive full and final treatment for a disease or condition. ... Our target definition of health care services includes medical services to prevent a disease." Schreyer (2010; 73). "The target definition of health care volume output proposed earlier is the number of complete treatments with specified bundles of characteristics so as to capture quality change and new products." Schreyer (2010; 76). Schreyer goes on to explain why constructing measures of complete treatments is very difficult and hence why one might have to settle for measures of processes that are components of a complete treatment.

[64] For an application of the treatment approach in the time series context, see Gu and Morin (2014).

[65] Note that the land acquisition costs or opportunity costs for the road can be high. If interest is not allowed as a component of user cost, then the corresponding user cost of the land component of the road will be (mistakenly) set equal to zero and the contribution of public sector roads to the country's GDP will be greatly undervalued. It is difficult to determine what an appropriate depreciation rate for the road bed should be but it will probably be a very small number. The depreciation rate for the asphalt or concrete surface of the road can be relatively large and it will be related to the utilization of the road.

road cost in period t and the utilization cost component will be based on the sum of labour and material costs plus the maintenance capital equipment user costs that pertain to period t.

Lawrence and Diewert (2006; 215) noted the similarity in measuring the output of a road system with measuring the output of an *electricity distributor*:

"The distributor has the responsibility of providing the 'road' and keeping it in good condition but it has little, if any, control over the amount of 'traffic' that goes down the road. Consequently they argue it is inappropriate to measure the output of the distributor by a volume of sales or 'traffic' type measure. Rather the distributor's output should be measured by the availability of the infrastructure it has provided and the condition in which it has maintained it – essentially a supply side measure."

Lawrence and Diewert (2006; 215) go on to suggest that a comprehensive output measure for a regulated electricity distributor should consist of three components − throughput, network line capacity and the number of customers.[66] Moreover, they followed the national accounts treatment of public sector production and valued each of the three output components by their imputed cost of production.

Lawrence and Diewert (2006; 215) also suggested that the same methodology that treats throughput and the underlying network as separate outputs could be applied to passenger traffic on government owned or regulated *railways and transit systems*, to *pipelines*, to *telecommunication providers* and to *natural gas distributors*. The point is that many distribution production units have the opportunity to behave in a monopolistic manner and so they are regulated by the government. The regulators typically force the distribution unit to provide services to all potential customers in an area at regulated prices. Hence these prices are not necessarily the prices that would be generated by unregulated markets. This fact has implications for the economic approach to the measurement of TFP that relies on exact index numbers which in turn relies on the assumption of competitive behavior in both output and input markets.[67] While it is reasonable to assume that a regulated firm behaves competitively on input markets, it is not reasonable to assume that regulated firms behave competitively on output markets. Thus a different index number methodology that relies on the estimation of cost functions or on the estimation of unit costs to value outputs (as was explained in section 5 above) is needed to measure productivity, not only for public sector production units, but also for regulated production units in the distribution, telecommunications and transportation sectors.[68] Thus the unit cost based methodology to the measurement of public sector

[66] Each customer requires a separate (costly) connection plus each customer has a separate cost of billing.
[67] The exact index number approach to measuring productivity growth can be extended to encompass some limited forms of monopolistic behavior; see Diewert and Fox (2008; 179) (2010; 75). These papers also show that in more general models of monopolistic behavior, exact index number techniques can be used to obtain very simple estimating equations where the contributions of increasing returns to scale and technical progress to productivity growth can be separately identified.
[68] See Lawrence and Diewert (2006; 231-233) for an example of how the cost function approach to measuring TFP can be applied in the regulated firm context. For more general expositions of how cost

production and productivity that was pioneered by Scitovsky (1967), Hill (1975) and Schreyer (2010) has a wider application to the regulated part of the private sector.

National post offices that offer nationwide mail delivery at regulated prices are another example of a production unit that produces a network availability output as well as utilization or throughput outputs. The unit cost based methodology for output measurement could also be applied to this sector.

9. Conclusion

This chapter covered the three main classes of methods used by national income accountants to construct measures of real output for public sector establishments that produce nonmarket outputs. If it is difficult or impossible to construct unambiguous measures for the quantities of nonmarket outputs produced by a public sector production unit, then aggregate output is typically set equal to a measure of establishment aggregate input and the resulting TFP estimates will show no productivity gains. However, in section 4, we showed that measurement was not completely clear cut in this situation; i.e., we looked at the complications that arise if we want to measure the value added of the production unit versus its gross output and the consistency between the two measures. We also showed that the choice of index number formula will, in general, make a substantial difference to the resulting measures of gross output or value added growth of public sector production unit.

In section 5, we covered the second general method for constructing aggregate output measures for the nonmarket outputs of a public sector production unit. Using this methodology, the prices of nonmarket outputs are set equal to the unit costs of producing the outputs. Once these imputed prices have been determined, normal index number theory can be applied to construct estimates of aggregate nonmarket output and of Total Factor Productivity of the public sector unit. Of course the practical problem with this method is that it will typically be difficult to construct suitable measures of unit cost for the nonmarket outputs.

In section 6, we covered the final general method for constructing aggregate output measures for public sector production units. Using this approach, the nonmarket outputs of a public sector establishment are aggregated together by using a vector of weights that reflect the relative value of the nonmarket outputs to users or recipients of the nonmarket outputs. The corresponding aggregate nonmarket output price is determined by dividing total establishment cost (less the value of market outputs) by the welfare oriented nonmarket quantity index. The advantage of this method over the section 5 method is that it is not necessary to form estimates of the unit costs for the nonmarket outputs. This disadvantage of the section 6 method is that it will generally be difficult to determine the appropriate vector of nonmarket output weights.

---

functions can be utilized to measure TFP in the public and regulated sectors, see Diewert (2012) and Schreyer (2012a).

In section 7, we discussed the problem of quality adjustment of nonmarket outputs which boiled down to the problem of finding appropriate vectors of nonmarket output weights. We also discussed the problems associated with linking in new products.

In section 8, the problems associated with measuring nonmarket outputs in the education and health sectors were discussed and how the general measurement methods discussed in sections 5 and 6 could be applied to these sectors. We also noted that there were measurement problems for finding suitable prices for the outputs of regulated firms that are entirely analogous to the problems associated with finding reasonable prices for the nonmarket outputs of public sector units. Regulated firms are regulated because they have some sort of monopoly power. Typically, the regulatory authorities require the regulated firms to provide uniform levels of service over regions or locations at regulated prices. Thus the quantities produced by a regulated firm are typically not the quantities that a price taking competitive firm would provide at the prices that regulators set. Hence the same methods that are used to value the outputs of nonmarket producers should be used to measure aggregate output and TFP of regulated firms; i.e., the regulated prices should be replaced by marginal or average unit cost prices. Thus the methods discussed in this chapter have some applicability to the measurement of TFP in regulated industries.

In section 3, we provided an extensive discussion on how to measure the value of capital services in the public sector. These measurement problems deserve a lot more attention than they have received in the past. Statistical agency measures of the value of capital services in the government sector do not include the imputed interest cost of the fixed capital that is used in this sector, due to national income accounting conventions. This convention has led to a very large downward bias in both the nominal and real GDP of all countries, with the bias being bigger for rich countries that generally have larger public sectors than poorer countries. The only cost associated with capital inputs used in the public sector that is allowed in the international System of National Accounts is depreciation. Thus the user costs of government buildings are vastly understated. In addition to the understatement of the costs associated with the use of structures, there is a further understatement due to the complete neglect of land user costs for government owned land. Because land does not depreciate, the costs associated with the land that sits under public schools and hospitals are set equal to zero, which is the ultimate understatement! If a government owned office building were instead rented from the private sector, the explicit rent would be recognized in the SNA and this explicit rent would include the interest opportunity cost of capital that is tied up in the structure and the land plot that supports the structure.[69] The neglect of the land that roads and

[69] Many national income accountants recognize that the current treatment of government owned capital in the SNA is not consistent with general accounting principles: "The fact that exactly the same kind of service may be provided on both a market and on a non-market basis raises an important question for this report. It is proposed as a matter of principle that the basic methodology used to measure changes in the volume of real output should always be the same irrespective whether the service is provided on a market or on a non-market basis." Hill (1975; 19). Atkinson (2005; 49) explicitly recommended that the opportunity cost of capital be added to depreciation charges to account for the cost of capital: "We recommend that the appropriate measure of capital input for production and productivity analysis is the flow of *capital services* of an asset type. This involves adding to the capital consumption an interest charge, with an agreed interest rate, on the entire owned capital."

government owned railways sit on also will lead to a substantial downward bias in the GDP of the public sector.

Finally, we conclude this chapter with some observations on the difficulties associated with determining the "right" prices for valuing public sector outputs that are allocated to households at very low or zero prices. Suppose that a household has preferences over market goods and services and over nonmarket goods and services that are provided to it by the public sector. Denote the household consumption vectors of nonmarket and market commodities by y and z respectively and suppose that the households preferences can be represented by the utility function f(y,z). Suppose that the household faces the price vector w^t for market goods and services and has "income" I^t to spend on these commodities. The various levels of government allocate the vector y^t of public goods and services to the household in period t. We assume that the vector z^t solves the following period t utility maximization problem for the household:

(31) $\max_{z} \{f(y^t,z) : w^t \cdot z \leq I^t\} \equiv u^t = f(y^t,z^t)$.

Thus $z^t = d(I^t,p^t,y^t)$ where d is the *household's system of conditional market demand functions for market goods and services*; conditional on: (i) income spent on market commodities I^t, (ii) the price vector for market goods and services p^t and the household's allocation of public goods, y^t. The *conditional expenditure function* e that is generated by the utility function f is defined as follows:

(32) $e(u^t,w^t,y^t) \equiv \min_{z} \{w^t \cdot z \, ; \, f(y^t,z) \geq u^t\} = w^t \cdot z^t$.

Suppose that $e(u^t,w^t,y)$ is differentiable with respect to the components of y when $y = y^t$ and let

(33) $p^t \equiv \nabla_y e(u^t,w^t,y^t)$

denote this vector of partial derivatives. Define the *household's augmented period t income*, I^{t*}, as follows:

(34) $I^{t*} \equiv I^t + p^t \cdot y^t$.

Then under suitable regularity conditions, it can be shown that y^t,z^t is a solution to the following *augmented income utility maximization problem*:

(35) $\max_{y,z} \{f(y,z) : p^t \cdot y + w^t \cdot z \leq I^{t*}\} \equiv u^t = f(y^t,z^t)$.

It can be seen that p^t defined by (33) is an appropriate vector of shadow prices that value the components of the public sector quantity vector y^t for this household; i.e., the components of p^t reflect the value of the public goods vector y^t from a household welfare perspective.

How could we calculate this vector of shadow prices in practice? It would be necessary to estimate the household's system of market demand functions, $d(I^t,p^t,y^t)$, given a time series of data on I^t, p^t and y^t for the household. We could then use the estimated market demand functions and attempt to recover the underlying utility function, $f(y,z)$, up to a cardinalization and then the corresponding dual expenditure function e could recovered and finally the welfare oriented prices p^t could be calculated. Normal index number theory could be applied at this point.[70]

But there is a problem with the above methodology: we cannot fully recover the preferences of the household using this methodology![71] To show why this is the case, replace the original household utility function $f(y,z)$ by $F(y,z) \equiv f(y,z) + g(y)$ where $g(y)$ is a subutility function which is just defined over the public goods. Now assume that the household solves the following conditional utility maximization problem:

(36) $\max_z \{F(y^t,z) : w^t\cdot z \le I^t\}$.

It can be seen that the $z^t = d(I^t,p^t,y^t)$ which was the solution to the original utility maximization problem defined by (31) is also a solution to the new utility maximization problem defined by (36). This shows that a knowledge of the household's system of conditional market demand functions is not sufficient to fully reconstruct household preferences over market and nonmarket goods and services and hence it will be difficult to construct prices for nonmarket commodities from the welfare perspective.

Appendix: The Measurement of Productivity Growth using Joint Cost Functions

This Appendix takes a *cost based approach* to the measurement of the price and quantity of the outputs produced by a public sector production unit.[72] This approach values outputs at their marginal costs but we indicated in the main text that this is only a second best method for valuation from the viewpoint of welfare economics. However, it is the "right" approach to valuation if we want to measure the technical progress or productivity performance of the production unit.

We assume that we can observe the vector of *inputs used* by a public sector establishment in period t, $x^t \equiv [x_1{}^t,...,x_N{}^t]$, and the corresponding vector of *nonmarket outputs produced* during period t, $y^t \equiv [y_1{}^t,...,y_M{}^t]$, for t = 0,1.[73] We assume that the set of feasible inputs and outputs for period t is the production possibilities set S^t for t = 0,1. We

[70] Alternatively, once f has been determined up to a cardinalization, we could use the resulting time series for u^t as our household quantity index.
[71] This recovery impossibility theorem does not apply to business demanders of the outputs of a public sector production unit; it only applies to household demanders.
[72] Our approach is based on Diewert (2011) (2012).
[73] If the establishment produces some market outputs, then these market outputs can be treated as negative components of the input vectors, x^t.

will assume that there are *constant returns to scale in production*; i.e., we will assume that in each period t, the production possibilities set S^t satisfies the following property:[74]

(A1) $(x,y) \in S^t$ implies that $(\lambda x,\lambda y) \in S^t$ for all scalars $\lambda > 0$.

The establishment's *period t joint cost function*, $C^t(y,w)$, is defined as follows:

$$\text{(A2) } C^t(y,w) \equiv \min_x \{w \cdot x : (x,y) \in S^t\} \,; \qquad t = 0,1$$

where $w >> 0_N$ is a vector of strictly positive input prices that the production unit faces. The joint cost function will satisfy various regularity conditions, given regularity conditions on the underlying production possibilities sets S^t. In particular, with our present assumptions, it can be shown that $C^t(y,w)$ will be *linearly homogeneous* in the components of y and in the components of w; i.e., C^t will satisfy the following conditions:[75]

(A3) $C^t(\lambda y,w) = \lambda C^t(y,w)$ for all $\lambda > 0$;
(A4) $C^t(y,\lambda w) = \lambda C^t(y,w)$ for all $\lambda > 0$.

We suppose that the production unit's period t observed output and input vectors are y^t and x^t respectively and the vector of input prices that is faced in period t is w^t for t = 0,1. We also assume that in each period t, the observed input vector x^t is a solution to the period t cost minimization problem so that we have:

$$\text{(A5) } C^t(y^t,w^t) = w^t \cdot x^t \,; \qquad t = 0,1.$$

A final assumption is that the period t cost function $C^t(y,w)$ is differentiable with respect to the components of y,w at the observed period t data, y^t,w^t, for t = 0,1. Given that $C^t(y^t,w^t)$ is differentiable with respect to the components of the input price vector, by Shephard's (1953; 11) Lemma, the observed period t input vector x^t is equal to the vector of first order partial derivatives of $C^t(y^t,w^t)$ with respect to the components of w; i.e., we have:[76]

$$\text{(A6) } x^t = \nabla_w C^t(y^t,w^t) \,; \qquad t = 0,1.$$

Note that (A5) and (A6) imply that the following equations will hold:

---

[74] More formally, we assume that for all z in a nonempty closed convex set of feasible quality vectors, the set of (x,y) such that $(x,y) \in S^t$ is a nonempty, closed, convex cone for t = 0,1. The cone assumption is the assumption that implies constant returns to scale for constant quality outputs, which is assumption (A1). With these regularity conditions, it can be shown that $C^t(y,w)$ will be a convex, linearly homogeneous function in the components of y and a concave, linearly homogeneous function in w. The assumption of constant returns to scale is somewhat restrictive since in many cases, the government provides a service because the underlying technology that provides the service is subject to increasing returns to scale.
[75] Use the techniques outlined in Diewert (1974b; 134-136) to establish these results.
[76] Notation: $w^t \cdot x^t \equiv \sum_{n=1}^N w_n^t x_n^t$ is the inner product of the vectors w^t and x^t and $\nabla_w C^t(y^t,w^t) \equiv [\partial C^t(y^t,w^t)/\partial w_1,\ldots,\partial C^t(y^t,w^t)/\partial w_N]$ is the vector of first order partial derivatives of $C^t(y^t,w^t)$ with respect to the components of w.

(A7) $C^t(y^t,w^t) = w^t \cdot x^t = w^t \cdot \nabla_w C^t(y^t,w^t)$; $t = 0,1.$

It is useful to introduce some notation for the vectors of first order partial derivatives of C^t with respect to the components of the output vector y:

(A8) $p^t \equiv \nabla_y C^t(y^t,w^t)$; $t = 0,1;$

From (A8), it can be seen that p^t is the period t vector of *marginal cost prices* for the nonmarket outputs produced by the production unit in period t; i.e., p_m^t is the *incremental cost* of producing an additional unit of output m in period t. These incremental costs could be approximated by *allocated accounting costs*; i.e., p_m^t could be approximated by the period t share of total period t cost that could be attributed to the production of output m, say C_m^t, divided by the total period t production of output m, y_m^t.

Since $C^t(y,w)$ is linearly homogeneous in the components of y, Euler's Theorem on homogeneous functions implies the following relations:

$$\begin{aligned} \text{(A9) } C^t(y^t,w^t) &= y^t \cdot \nabla_y C^t(y^t,w^t) && t = 0,1 \\ &= p^t \cdot y^t && \text{using (A8)} \\ &= w^t \cdot x^t && \text{using (A7).} \end{aligned}$$

Thus if we use period t marginal costs p^t as prices for the period t nonmarket outputs y^t, then the resulting period t *imputed revenues*, $p^t \cdot y^t$, will be exactly equal to period t *costs*, $w^t \cdot x^t$.

We use the two joint cost functions, C^0 and C^1, in order to define a *family of cost based output quantity or volume indexes*, $\alpha(y^0,y^1,w,t)$, as follows:

(A10) $\alpha(y^0,y^1,w,t) \equiv C^t(y^1,w)/C^t(y^0,w).$

Note that this output quantity index depends not only on the two quantity vectors for periods 0 and 1, y^0 and y^1, but it also depends on a reference period t technology and a reference vector of input prices w. Thus the theoretical output quantity index $\alpha(y^0,y^1,w,t)$ defined by (A10) is equal to the (hypothetical) total cost $C^t(y^1,w)$ of producing the vector of observed period 1 outputs y^1, divided by the (hypothetical) total cost $C^t(y^0,w)$ of producing the vector of observed period 0 procedure outputs y^0, where in both cases, we use the technology of period t and assume that the establishment faces the same vector of reference input prices, w. Thus for each choice of technology (i.e., t could equal 0 or 1) and for each choice of a reference vector of input prices w, we obtain a (different) cost based output quantity index.

Following the example of Konüs (1939), it is natural to single out two special cases of the family of output quantity indexes defined by (A11): one choice where we use the period 0 technology and set the reference prices equal to the period 0 input prices w^0 and another choice where we use the period 1 technology and set the reference prices equal to the period 1 input prices w^1. These special cases are defined as α_0 and α_1 below:

$$\begin{aligned}
\text{(A11)}\quad \alpha_0 &\equiv C^0(y^1,w^0)/C^0(y^0,w^0) \\
&= C^0(y^1,w^0)/p^0\cdot y^0 && \text{using (A9)} \\
&\cong [C^0(y^0,w^0) + \nabla_y C^0(y^0,w^0)\cdot(y^1-y^0)]/p^0\cdot y^0 \\
& && \text{forming a first order Taylor series approximation to } C^0(y^1,w^0) \\
&= p^0\cdot y^1/p^0\cdot y^0 && \text{using (A8) and (A9)} \\
&\equiv Q_L
\end{aligned}$$

where Q_L is the *Laspeyres output quantity or volume index*.

We turn now to our second special case of the family of output quantity indexes defined by (A10).

$$\begin{aligned}
\text{(A12)}\quad \alpha_1 &\equiv C^1(y^1,w^1)/C^1(y^0,w^1) \\
&= p^1\cdot y^1/C^1(y^0,w^1) && \text{using (A9)} \\
&\cong p^1\cdot y^1/[C^1(y^1,w^1) + \nabla_y C^1(y^1,w^1)\cdot(y^0-y^1)] \\
& && \text{forming a first order Taylor series approximation to } C^1(y^0,w^1) \\
&= p^1\cdot y^1/p^1\cdot y^0 && \text{using (A8)-(A10)} \\
&\equiv Q_P
\end{aligned}$$

where Q_P is the *Paasche output quantity or volume index*.

Since the theoretical output quantity indexes, α_0 and α_1, are both equally representative, a single estimate of cost based output quantity growth should be set equal to a symmetric average of these two estimates. We will choose the geometric mean as our preferred symmetric average[77] and thus our preferred theoretical measure of cost based output quantity growth is the following *theoretical Fisher type output index*, α_F:

$$\begin{aligned}
\text{(A13)}\quad \alpha_F &\equiv [\alpha_0\alpha_1]^{1/2} \\
&\cong [Q_L Q_P]^{1/2} && \text{using (A11) and (A12)} \\
&\equiv Q_F
\end{aligned}$$

where Q_F is the *Fisher output quantity index*.

It should be noted that while Q_L and Q_P only approximate the corresponding theoretical indexes α_0 and α_1 to the first order, it is likely that Q_F approximates the theoretical index α_F to the accuracy of a second order approximation.[78]

[77] Diewert (1997) explained why the geometric mean is a good choice for the symmetric average: the resulting index satisfies the important time reversal test.

[78] See Diewert (2002).

We now turn our attention to theoretical measures of input price growth. We use the joint cost function C^t in order to define a *family of input price indexes*, $\beta(w^0,w^1,y,t)$, as follows:

$$\text{(A14)}\ \beta(w^0,w^1,y,t) \equiv C^t(y,w^1)/C^t(y,w^0).$$

Thus the theoretical input price index $\beta(w^0,w^1,y,t)$ defined by (A14) is equal to the (hypothetical) total cost $C^t(y,w^1)$ of producing the reference vector of outputs y facing the period 1 input prices w^1, divided by the (hypothetical) total cost of producing the reference vector of outputs y facing the period 0 input prices w^0, where in both cases, we use the technology of period t. Thus for each choice of technology (i.e., t could equal 0 or 1) and for each choice of a reference vector of outputs y, we obtain a (different) cost based input price index. Note that only the vector of input prices changes in the numerator and denominator of definition (A14).

Again following the example of Konüs (1939), it is natural to single out two special cases of the family of input price indexes defined by (A14): one choice where we use the period 0 technology and set the reference quantities equal to the period 0 quantities y^0 and another choice where we use the period 1 technology and set the reference quantities equal to the period 1 quantities y^1. These special cases are defined as β_0 and β_1 below:

$$\begin{aligned}
\text{(A15)}\ \beta_0 &\equiv C^0(y^0,w^1)/C^0(y^0,w^0) \\
&= C^0(y^0,w^1)/w^0\cdot x^0 && \text{using (A9)} \\
&\cong [C^0(y^0,w^0) + \nabla_w C^0(y^0,w^0)\cdot(w^1-w^0)]/w^0\cdot x^0 \\
& && \text{forming a first order Taylor series approximation to } C^0(y^0,w^1) \\
&= w^1\cdot x^0/w^0\cdot x^0 && \text{using (A6) and (A7)} \\
&\equiv P_L^*
\end{aligned}$$

where P_L^* is the ordinary Laspeyres input price index, $w^1\cdot x^0/w^0\cdot x^0$. Thus the theoretical cost function based input price index β_0 defined by the first line in (A15) is approximately equal to the Laspeyres input price index P_L^*. We turn now to our second special case of the family of input price indexes defined by (A14).

$$\begin{aligned}
\text{(A16)}\ \beta_1 &\equiv C^1(y^1,w^1)/C^1(y^1,w^0) \\
&= w^1\cdot x^1 /C^1(y^1,w^0) && \text{using (A10)} \\
&\cong w^1\cdot x^1/[C^1(y^1,w^1) + \nabla_w C^1(y^1,w^1)\cdot(w^0-w^1)] \\
& && \text{forming a first order Taylor series approximation to } C^1(y^1,w^0) \\
&= w^1\cdot x^1/w^0\cdot x^1 && \text{using (A6) and (A7)} \\
&\equiv P_P^*
\end{aligned}$$

where P_P^* is the ordinary Paasche input price index, $w^1\cdot x^1/w^0\cdot x^1$. Since both theoretical input price indexes, β_0 and β_1, are equally representative, a single estimate of input price change should be set equal to a symmetric average of these two estimates. We again choose the geometric mean as our preferred symmetric average and thus our preferred theoretical measure of input price growth is the following Fisher type theoretical input price index, β_F:

$$\begin{aligned} \text{(A17)}\ \beta_F &\equiv [\beta_0\beta_1]^{1/2} \\ &\cong [P_L^* P_P^*]^{1/2} && \text{using the approximations (A15) and (A16)} \\ &\equiv P_F^* \end{aligned}$$

where the Fisher (1922) index of input price change, P_F^*, is defined as the geometric mean of the Laspeyres and Paasche input price indexes. Given the fact that P_L^* is a first order approximation to β_0 and P_P^* is a first order approximation to β_1, it is obvious that P_F^* is at least a first order approximation to the theoretical input price index β_F. But in most cases, the approximation of P_F^* to β_F will be much better than a first order approximation since the usual upward bias in P_L^* will generally offset the usual downward bias in P_P^*.

We now define our last family of theoretical indexes. We again use the joint cost functions C^0 and C^1 in order to define a *family of reciprocal indexes of technical progress*, $\gamma(y,w)$, as follows:

$$\text{(A18)}\ \gamma(y,w) \equiv C^1(y,w)/C^0(y,w).$$

The family of theoretical reciprocal technical progress indexes (or reciprocal productivity indexes) $\gamma(y,w)$ defined by (A18) is equal to the (hypothetical) total cost $C^1(y,w)$ of producing the reference vector of outputs y when the public sector production unit faces the reference vector of input prices w using the period 1 technology, divided by the total cost $C^0(y,w)$ of producing the same reference vector of outputs y and facing the same reference vector of input prices w, where we now use the period 0 technology.[79] Thus $\gamma(y,w)$ is a measure of the *proportional reduction in costs* that occurs due to technical progress between periods 0 and 1 and it can be seen that this is an inverse measure of technical progress; i.e., there is positive technical progress between the two periods if $\gamma(y,w)$ is less than one. For each choice of a reference vector of output quantities y and reference vector of input prices w, we obtain a measure of exogenous cost reduction.

Instead of singling out the reference vectors y and w that appear in the definition of $\gamma(y,w)$ to be the period t quantity and price vectors (y^t,w^t) for t = 0,1, we will choose the *mixed reference vectors* (y^0,w^1) and (y^1,w^0) for our usual two special cases. The reason for these somewhat odd looking choices will be explained below.

We want to explain the growth in total costs going from period 0 to 1, $C^1(y^1,w^1)/C^0(y^0,w^0)$, as the product of 3 growth factors:

- Growth in outputs; i.e., a factor of the form $\alpha(y^0,y^1,w,t)$ defined above by (A10);

[79] Following Salter (1960), this is a cost function analogue to the value added function definitions of technical progress defined by Diewert (1983; 1063-1064), Diewert and Morrison (1986) and Kohli (1990).

- Growth in input prices; i.e., a factor of the form $\beta(w^0,w^1,y,t)$ defined by (A14) and
- Exogenous reduction in costs due to technical progress; i.e., a factor of the form $\gamma(y,w)$ defined by (A18).

Simple algebra shows that we have the following decompositions of the cost ratio $C^1(y^1,w^1)/C^0(y^0,w^0)$ into explanatory factors of the above type:[80]

$$\begin{aligned}
\text{(A19) } & C^1(y^1,w^1)/C^0(y^0,w^0) \\
& = [C^1(y^1,w^1)/C^1(y^0,w^1)][C^0(y^0,w^1)/C^0(y^0,w^0)][C^1(y^0,w^1)/C^0(y^0,w^1)] \\
& = \alpha_1\beta_0\gamma(y^0,w^1) \qquad \text{using definitions (A12), (A15) and (A18);}
\end{aligned}$$

$$\begin{aligned}
\text{(A20) } & C^1(y^1,w^1)/C^0(y^0,w^0) \\
& = [C^0(y^1,w^0)/C^0(y^0,w^0)][C^1(y^1,w^1)/C^1(y^1,w^0)][C^1(y^1,w^0)/C^0(y^1,w^0)] \\
& = \alpha_0\beta_1\gamma(y^1,w^0) \qquad \text{using definitions (A10), (A15) and (A22).}
\end{aligned}$$

The above decompositions show that the following two special cases of $\gamma(y,w)$ defined by (A18) are of particular interest:

$$\text{(A21) } \gamma(y^0,w^1) \equiv C^1(y^0,w^1)/C^0(y^0,w^1) \equiv \gamma_0 ;$$

$$\text{(A22) } \gamma(y^1,w^0) \equiv C^1(y^1,w^0)/C^0(y^1,w^0) \equiv \gamma_1 .$$

We will now work out potentially observable first order approximations to the two specific measures of reciprocal technical progress defined by (A21) and (A22). Using definition (A21) and taking first order approximations to $C^1(y^0,w^1)$ and $C^0(y^0,w^1)$, we have the following first order approximation to the *reciprocal productivity index* $\gamma(y^0,w^1)$:

$$\begin{aligned}
\text{(A23) } & \gamma(y^0,w^1) \\
& \cong [C^1(y^1,w^1) + \nabla_y C^1(y^1,w^1)\cdot(y^0-y^1)]/[C^0(y^0,w^0,z^0) + \nabla_w C^0(y^0,w^0)\cdot(w^1-w^0)] \\
& = [p^1\cdot y^1 + p^1\cdot(y^0-y^1)]/[w^0\cdot x^0 + x^0\cdot(w^1-w^0)] \qquad \text{using (A6)-(A9)} \\
& = p^1\cdot y^0/w^1\cdot x^0 \\
& = \{p^1\cdot y^0/p^1\cdot y^1\}/\{w^1\cdot x^0/w^1\cdot x^1\} \qquad \text{using } p^1\cdot y^1 = w^1\cdot x^1 \\
& = \{Q_P/Q_P^*\}^{-1}
\end{aligned}$$

where Q_P is the Paasche output index defined in (A12) and the *Paasche input quantity index* Q_P^* is defined as follows:

$$\text{(A24) } Q_P^* \equiv w^1\cdot x^1/w^1\cdot x^0.$$

Thus the cost function based theoretical index of reciprocal technical progress $\gamma(y^0,w^1)$ defined by (A21) above is approximately equal to the reciprocal of the Paasche output

[80] The decompositions of cost growth given by (A23) and (A25) are nonparametric analogues to the parametric revenue growth decompositions obtained by Diewert and Morrison (1986), Kohli (1990) and Fox and Kohli (1998) into explanatory factors.

quantity index Q_P divided by the Paasche input quantity index Q_P^*. Note however, that we require a knowledge of the period 1 vector of marginal costs, p^1, in order to form this empirical approximation to the theoretical technical progress index, $\gamma(y^0,w^1)$.

We can find a similar empirical approximation to the cost function based measure of technical progress $\gamma(y^1,w^0)$ defined by (A22):

$$\begin{aligned}
\text{(A25)}\ \gamma(y^1,w^0) &\cong [C^1(y^1,w^1) + \nabla_w C^1(y^1,w^1)\cdot(w^0-w^1)]/[C^0(y^0,w^0) + \nabla_y C^0(y^0,w^0)\cdot(y^1-y^0)] \\
&= [w^1\cdot x^1 + x^1\cdot(w^0-w^1)]/[p^0\cdot y^0 + p^0\cdot(y^1-y^0)] && \text{using (A6)-(A9)} \\
&= w^0\cdot x^1/p^0\cdot y^1 \\
&= \{w^0\cdot x^1/w^0\cdot x^0\}/\{p^0\cdot y^1/p^0\cdot y^0\} && \text{using } p^0\cdot y^0 = w^0\cdot x^0 \\
&= \{Q_L/Q_L^*\}^{-1}
\end{aligned}$$

where Q_L is the Laspeyres output index defined in (A11) and the *Laspeyres input quantity index* Q_L^* is defined as follows:

$$\text{(A26)}\ Q_L^* \equiv w^0\cdot x^1/w^0\cdot x^0.$$

Thus the cost function based theoretical index of reciprocal technical progress $\gamma(y^1,w^0)$ defined by (A22) above is approximately equal to the reciprocal of the Laspeyres output quantity index Q_L divided by the ordinary Laspeyres input quantity index Q_L^*. However, we do require a knowledge of the period 0 vector of marginal costs, p^0, in order to form this empirical approximation to the theoretical technical progress index, $\gamma(y^1,w^0)$.

Since the two cost decompositions for the rate of growth of cost, $C^1(y^1,w^1)/C^0(y^0,w^0)$, given by (A19) and (A20) are equally valid, we will take the geometric average of these two decompositions to obtain our preferred overall cost decomposition. This leads to the following theoretical decomposition of $w^1\cdot x^1/w^0\cdot x^0$ equal to $C^1(y^1,w^1)/C^0(y^0,w^0)$ into explanatory factors:

$$\text{(A27)}\ C^1(y^1,w^1)/C^0(y^0,w^0) = w^1\cdot x^1/w^0\cdot x^0 = \alpha_F\,\beta_F\,\gamma_F$$

where the theoretical Fisher type quality adjusted output quantity growth factor α_F is defined by (A13), the Fisher type theoretical input price growth factor β_F is defined by (A17) and the *Fisher type reciprocal measure of technical progress* γ_F is defined as the geometric mean of the two reciprocal productivity indexes defined by (A21) and (A22):

$$\text{(A28)}\ \gamma_F \equiv [\gamma(y^0,w^1)\gamma(y^1,w^0)]^{1/2} = [\gamma_0\,\gamma_1]^{1/2}.$$

The exact decomposition of (one plus) cost growth over the two periods under consideration given by (A27) is our preferred decomposition of cost growth into explanatory factors which can be implemented if the economic statistician has estimates for the period 0 and 1 cost functions at hand.

On the other hand, if full information on the cost functions for the two periods is not available but information on marginal costs is available in addition to basic price and quantity information, then the various theoretical indexes on the right hand side of (A27) can be replaced by their first order approximations. Thus using (A14), α_F can be approximated by the geometric mean of the quality adjusted Laspeyres and Paasche output volume indexes $[Q_{AL}Q_{AP}]^{1/2}$, using (A17), β_F can be approximated by the Fisher input price index P_F^* and using (A23) and (A25), γ_F can be approximated by the geometric mean of the reciprocal of the Paasche productivity index $\{Q_P/Q_P^*\}^{-1}$ and the Laspeyres productivity index $\{Q_L/Q_L^*\}^{-1}$. Substituting these first order approximations into (A27) leads to the following approximate decomposition of cost growth into explanatory factors:

$$\text{(A29)}\quad w^1\cdot x^1/w^0\cdot x^0 \cong [Q_LQ_P]^{1/2}\,[P_L^*P_P^*]^{1/2}\,[Q_P/Q_P^*]^{-1/2}\,[Q_L/Q_L^*]^{-1/2}$$
$$= [P_L^*P_P^*Q_L^*Q_P^*]^{1/2}$$
$$= P_F^*Q_F^*$$

where the Fisher (1922) *input price and quantity indexes* are defined as follows:

$$\text{(A30)}\quad P_F^* \equiv [P_L^*P_P^*]^{1/2} = [(w^1\cdot x^0/w^0\cdot x^0)(w^1\cdot x^1/w^0\cdot x^1)]^{1/2}\,;$$
$$\text{(A31)}\quad Q_F^* \equiv [Q_L^*Q_P^*]^{1/2} = [(w^0\cdot x^1/w^0\cdot x^0)(w^1\cdot x^1/w^1\cdot x^0)]^{1/2}.$$

However, as Fisher (1922) showed long ago, the product of the Fisher input price and quantity indexes, $P_F^*Q_F^*$, is exactly equal to the input value ratio, $w^1\cdot x^1/w^0\cdot x^0$. Thus equation (A29) holds as an *exact equality* rather than as an approximate equality. Thus our first order approximations for the explanatory factors α_F , β_F and γ_F in the exact decomposition (A28) also have the property that they are exact overall in the sense that the product of these approximate explanatory factors is exactly equal to the cost ratio $w^1\cdot x^1/w^0\cdot x^0$.

The above arguments are rather complex but can be summarized as follows:

- If cost functions for a government establishment have been estimated econometrically for two periods, then it is possible to decompose cost growth over the two periods under consideration into a product of explanatory factors, α_F β_F γ_F, where α_F is the cost function based measure of quality adjusted output growth defined by (A13), β_F is the cost function based measure of input growth defined by (A17) and γ_F is the cost function based measure of reciprocal productivity growth equal to the geometric mean of γ_0 and γ_1 defined by (A21) and (A22);
- If full cost function information is not available for the two periods but information on input and output quantities and input prices along with estimates of long run marginal costs for outputs in both periods is available, then it is possible to decompose cost growth over the two periods into the product of the Fisher output quantity index, Q_F, times the Fisher input price index, P_F^*, times the

reciprocal of the Fisher productivity index, $[Q_F/Q_F^*]^{-1}$ according to the first line in (A29).

The above material provides a justification for the use of the Fisher productivity index as a nonparametric approximation to the "true" productivity index. Moreover, this Appendix shows that public sector outputs should be valued at their marginal costs (and *not* at purchaser valuations) in order to measure the productivity growth of the production unit.

References

Aizcorbe, A.A. (2014), *A Practical Guide to Price Index and Hedonic Techniques*, Oxford: Oxford University Press.

Atkinson, Tony (2005), *Atkinson Review: Final Report; Measurement of Government Output and Productivity for the National Accounts*, New York: Palgrave Macmillan.

Babbage, C. (1835), *On the Economy of Machinery and Manufactures*, Fourth Edition, London: Charles Knight.

Balk, B.M. (1998), *Industrial Price, Quantity and Productivity Indices*, Boston: Kluwer Academic Publishers.

Balk, B.M. (2003), "The Residual: On Monitoring and Benchmarking Firms, Industries and Economies with respect to Productivity", *Journal of Productivity Analysis* 20, 5-47.

Cairns, R.D. (2013), "The Fundamental Problem of Accounting", *Canadian Journal of Economics* 46, 634-655.

Caves, D.W., L.R. Christensen and W.E. Diewert (1982a), "The Economic Theory of Index Numbers and the Measurement of Input, Output and Productivity", *Econometrica* 50, 1393-1414.

Caves, D.W., L.R. Christensen and W.E. Diewert (1982b), "Multilateral Comparisons of Output, Input and Productivity using Superlative Index Numbers", *Economic Journal* 96, 659-679.

Charnes, A., W.W. Cooper and E. Rhodes (1978), "Measuring the Efficiency of Decision Making Units", *European Journal of Operational Research* 2, 429–444.

Christensen, L.R. and D.W. Jorgenson (1969), "The Measurement of U.S. Real Capital Input, 1929-1967", *Review of Income and Wealth* 15, 293-320.

Church, A.H. (1901), "The Proper Distribution of Establishment Charges, Parts I, II, and III", *The Engineering Magazine* 21, 508-517; 725-734; 904-912.

de Haan, J. (2010), “Hedonic Price Indexes: A Comparison of Imputation, Time Dummy and ‘Re-pricing’ Methods’’, *Journal of Economics and Statistics* (*Jahrbücher fur Nationalökonomie und Statistik)* 230, 772-791.

de Haan, J. and W.E. Diewert (eds.) (2011), *Residential Property Price Indices Handbook*, Luxembourg: Eurostat.

de Haan, J. and F. Krsinich (2014), “Scanner Data and the Treatment of Quality Change in Nonrevisable Price Indexes,” *Journal of Business and Economic Statistics*, 32:3, 341-358.

de Haan, J. and H.A. van der Grient (2011), “Eliminating Chain drift in Price Indexes Based on Scanner Data”, *Journal of Econometrics* 161, 36-46.

Diewert, W.E. (1974a), “Intertemporal Consumer Theory and the Demand for Durables”, *Econometrica* 42, 497-516.

Diewert, W.E., (1974b), “Applications of Duality Theory,” pp. 106-171 in M.D. Intriligator and D.A. Kendrick (ed.), *Frontiers of Quantitative Economics*, Vol. II, Amsterdam: North-Holland.

Diewert, W.E. (1976), “Exact and Superlative Index Numbers”, *Journal of Econometrics* 4, 114-145.

Diewert, W.E. (1978), “Superlative Index Numbers and Consistency in Aggregation”, *Econometrica* 46, 883-900.

Diewert, W.E. (1980), “Aggregation Problems in the Measurement of Capital”, pp. 433-528 in *The Measurement of Capital*, D. Usher (ed.), Chicago: The University of Chicago Press.

Diewert, W.E. (1983), “The Theory of the Output Price Index and the Measurement of Real Output Change”, pp. 1049-1113 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

Diewert, W.E. (1992a), “Fisher Ideal Output, Input and Productivity Indexes Revisited”, *Journal of Productivity Analysis* 3, 211-248.

Diewert, W.E. (1992b), “The Measurement of Productivity*”, Bulletin of Economic Research* 44:3, 163-198.

Diewert, W.E. (1996), “Comments on CPI Biases”, *Business Economics* 32:2, 30-35.

Diewert, W.E. (1997), “Commentary on Mathew D. Shapiro and David W. Wilcox: Alternative Strategies for Aggregating Price in the CPI”, *The Federal Reserve Bank of St. Louis Review*, 79:3, 127-137.

Diewert, W.E. (1998), "Index Number Issues in the Consumer Price Index", *Journal of Economic Perspectives* 12:1, 47-58.

Diewert, W.E. (2002), "The Quadratic Approximation Lemma and Decompositions of Superlative Indexes", *Journal of Economic and Social Measurement* 28, 63-88.

Diewert, W.E. (2005), "Issues in the Measurement of Capital Services, Depreciation, Asset Price Changes and Interest Rates", pp. 479-542 in *Measuring Capital in the New Economy*, C. Corrado, J. Haltiwanger and D. Sichel (eds.), Chicago: University of Chicago Press.

Diewert, W.E. (2008), "OECD Workshop on Productivity Analysis and Measurement: Conclusions and Future Directions", pp. 11-36 in *Proceedings From the OECD Workshop on Productivity Measurement and Analysis*, Paris: OECD.

Diewert, W.E. (2009), "The Aggregation of Capital over Vintages in a Model of Embodied Technical Progress", *Journal of Productivity Analysis* 32, 1-19.

Diewert, W.E. (2010), "User Costs versus Waiting Services and Depreciation in a Model of Production", *Journal of Economics and Statistics* 230/6, 759-771.

Diewert, W.E. (2011), "Measuring Productivity in the Public Sector: Some Conceptual Problems", *Journal of Productivity Analysis* 36, 177-191.

Diewert, W.E. (2012), "The Measurement of Productivity in the Nonmarket Sector", *Journal of Productivity Analysis* 37, 217-229.

Diewert, W.E. (2014), "US TFP Growth and the Contribution of Changes in Export and Import Prices to Real Income Growth," *Journal of Productivity Analysis* 41, 19-39.

Diewert, W.E. (2015), "Reconciling Gross Output TFP Growth with Value Added TFP Growth", *International Productivity Monitor* 29:Fall, 17-24.

Diewert, W.E., J. de Haan and R. Hendriks (2015), "Hedonic Regressions and the Decomposition of a House Price Index into Land and Structure Components", *Econometric Reviews*, 34:1-2, 106-126.

Diewert, W.E. and K.J. Fox (2008), "On the Estimation of Returns to Scale, Technical Progress and Monopolistic Markups", *Journal of Econometrics* 145, 174-193.

Diewert, W.E. and K.J. Fox (2010), "Malmquist and Törnqvist Productivity Indexes: Returns to Scale and Technical Progress with Imperfect Competition", *Journal of Economics* 101, 73-95.

Diewert, W.E. and K.J. Fox (2016), “Alternative User Costs, Rates of Return and TFP Growth Rates for the US Nonfinancial Corporate and Noncorporate Business Sectors: 1960-2014”, Discussion Paper 16-03, Vancouver School of Economics, The University of British Columbia, Vancouver, Canada, V6T 1L4.

Diewert, W.E. and K.J. Fox (2016), “Sunk Costs and the Measurement of Commercial Property Depreciation”, *Canadian Journal of Economics* 49, 1340-1366.

Diewert, W.E., K.J. Fox and C. Shimizu (2016), “Commercial Property Price Indexes and the System of National Accounts”, *The Journal of Economic Surveys* 30, 913-943.

Diewert, W.E. and N. Huang (2011), “Capitalizing R&D Expenditures”, *Macroeconomic Dynamics*15:4, 537-564.

Diewert, W.E. and C.J. Morrison (1986), “Adjusting Output and Productivity Indexes for Changes in the Terms of Trade”, *The Economic Journal* 96, 659-679.

Diewert, W.E. and A.O. Nakamura (2003), “Index Number Concepts, Measures and Decompositions of Productivity Growth”, *Journal of Productivity Analysis* 19, 127-159.

Diewert, W.E., A.O. Nakamura and L. Nakamura (2009), “The Housing Bubble and a New Approach to Accounting for Housing in a CPI”, *Journal of Housing Economics* 18:3, 156-171.

Diewert, W.E. and C. Shimizu (2015a), “Residential Property Price Indices for Tokyo”, *Macroeconomic Dynamics* 19, 1659–1714.

Diewert, W.E. and C. Shimizu (2015b), “A Conceptual Framework for Commercial Property Price Indexes”, *Journal of Statistical Science and Application* 3:9-10, 131-152.

Diewert, W.E. and C. Shimizu (2016), “Alternative Approaches to Commercial Property Price Indexes for Tokyo”, *Review of Income and Wealth*, forthcoming.

Diewert, W.E. and C. Shimizu (2017), “Hedonic Regression Models for Tokyo Condominium Sales”, *Regional Science and Urban Economics*, forthcoming.

Diewert, W.E. and H. Wei (2017), “Getting Rental Prices Right for Computers”, *Review of Income and Wealth*, forthcoming.

Eurostat (2001), *Handbook on Price and Volume Measures in National Accounts*, Brussels: European Commission.

Eurostat, IMF, OECD, UN and the World Bank (1993), *System of National Accounts 1993*, New York: The United Nations.

Farrell, M.J. (1957), “The Measurement of Production Efficiency”, *Journal of the Royal Statistical Society*, Series A, 120, 253-278.

Feenstra, R.C. (1994), “New Product Varieties and the Measurement of International Prices”, *American Economic Review* 84, 157-177.

Fisher, I. (1922), *The Making of Index Numbers*, Boston: Houghton-Mifflin.

Fox, K.J. and U. Kohli (1998), “GDP Growth, Terms of Trade Effects and Total Factor Productivity”, *Journal of International Trade and Economic Development* 7, 87-110.

Griliches, Z. (1979), “Issues in Assessing the Contribution of Research and Development to Productivity Growth”, *Bell Journal of Economics* 10 (Spring), 92-116.

Griliches, Z. and I. Cockburn (1994), “Generics and New Goods in Pharmaceutical Price Indexes”, *American Economic Review* 84, 1213-1232.

Gordon, R. (1981), “The Consumer Price Index: Measuring Inflation and Causing It”, *The Public Interest* 63 (Spring), 112-134.

Gu, W. and S. Morin (2014), “Experimental Measures of Output and Productivity in the Canadian Hospital Sector, 2002 to 2010”, *Canadian Productivity Review*, Catalogue no. 15-206-X — No. 034, Ottawa: Statistics Canada.

Harper, M.J., E.R. Berndt and D.O. Wood (1989), “Rates of Return and Capital Aggregation Using Alternative Rental Prices”, pp. 331-372 in *Technology and Capital Formation*, D.W. Jorgenson and R. Landau (eds.), Cambridge MA: The MIT Press.

Hausman, J. (1997), “Valuation of New Goods Under Perfect and Imperfect Competition”, pp. 209-237 in *The Economics of New Goods*, T. Bresnahan and R. Gordon (eds.), NBER Studies in Income and Wealth, Volume 58, Chicago: The University of Chicago Press.

Hicks, J.R. (1940), “The Valuation of the Social Income”, *Economica* 7, 105-140.

Hill, P. (1975); *Price and Volume Measures for Non-Market Services*, Brussels: Statistical Office of the European Communities.

Hill, R.J. and T.P. Hill (2003), “Expectations, Capital Gains and Income”, *Economic Inquiry* 41, 607-619.

Inklaar, R. and W.E. Diewert (2016), “Measuring Industry Productivity and Cross-Country Convergence”, *Journal of Econometrics* 191, 426-433.

IMF/Eurostat/ILO/OECD/UNECE/The World Bank (2004), *Producer Price Index Manual: Theory and Practice*, Paul Armknecht (ed.), Washington DC: International Monetary Fund.

Ivancic, L., W.E. Diewert and K.J. Fox (2011), "Scanner Data, Time Aggregation and the Construction of Price Indexes", *Journal of Econometrics* 161, 24-35.

Jorgenson, D.W. (1989), "Capital as a Factor of Production", pp. 1-35 in *Technology and Capital Formation*, D.W. Jorgenson and R. Landau (eds.), Cambridge MA: The MIT Press.

Jorgenson, D.W. (1996), *Investment: Volume 2; Tax Policy and the Cost of Capital*, Cambridge, Massachusetts: The MIT Press.

Jorgenson, D.W. and Z. Griliches (1967), The Explanation of Productivity Change", *The Review of Economic Studies* 34, 249-283.

Jorgenson, D.W. and Z. Griliches (1972), "Issues in Growth Accounting: A Reply to Edward F. Denison", *Survey of Current Business* 52:4, Part II (May), 65-94.

Kohli, U. (1990), "Growth Accounting in the Open Economy: Parametric and Nonparametric Estimates", *Journal of Economic and Social Measurement* 16, 125-136.

Konüs, A.A. (1939), "The Problem of the True Index of the Cost of Living", *Econometrica* 7, 10-29.

Koechlin, F., P. Konijn, Lorenzoni, L. and P. Schreyer (2015), "Comparing Hospitals and Health Prices and Volumes Across Countries: A New Approach", *Social Indicators Research*; December, 1-22.

Lawrence, D. and W.E. Diewert (2006), "Regulating Electricity Networks: The ABC of Setting X in New Zealand," pp. 207-241 in T. Coelli and D. Lawrence (eds.), *Performance Measurement and Regulation of Network Utilities*, Cheltenham: Edward Elgar.

Lovell, C.A.K. and K. Zieschang (1994), "The Problem of New and Disappearing Commodities in the Construction of Price Indexes", pp. 353-369 in *Data Envelopment Analysis: Theory, Methodology and Application*, New York: Springer.

Peasnell, K.V. (1981), "On Capital Budgeting and Income Measurement", *Abacus* 17:1, 52-67.

Reinsdorf, M. (1993), "The Effect of Outlet Price Differentials in the U.S. Consumer Price Index", pp. 227-254 in *Price Measurement and Their Uses*, M. F. Foss, M.

E. Manser, and A. H. Young (eds.), NBER Studies in Income and Wealth 57, Chicago: University of Chicago Press.

Salter, W. E. G. (1960), *Productivity and Technical Change*, Cambridge U.K.: Cambridge University Press.

Schreyer, P. (2001), *Measuring Productivity: Measuring Aggregate and Industry Level Productivity Growth*, Paris: OECD.

Schreyer, P. (2009), *Measuring Capital - OECD Manual 2009*, Second Edition, Paris: OECD.

Schreyer, P. (2010), "Towards Measuring the Volume Output of Education and Health Services: A Handbook"; OECD Statistics Working Paper No 31, Paris: OECD.

Schreyer, P. (2012a), "Output, Outcome and Quality Adjustment in Measuring Health and Education Services"; *The Review of Income and Wealth* 58:2, 257-278.

Schreyer, P. (2012b), "Measuring Multifactor Productivity when Rates of Return are Endogenous", pp. 13-40 in *Price and Productivity Measurement: Volume 6: Index Number Theory*, W.E. Diewert, B.M. Balk, D. Fixler, K.J. Fox and A.O. Nakamura (eds.), Victoria: Trafford Publishing.

Scitovsky, A.A. (1967), "Changes in the Cost of Treatment of Selected Illnesses, 1951-65", *American Economic Review* 57, 1182-1195.

Shephard, R.W. (1953), *Cost and Production Functions*, Princeton N.J.: Princeton University Press.

Szulc, B.J. (Schultz) (1983), "Linking Price Index Numbers", pp/ 537-566 in W.E. Diewert and C. Montmarquette (eds.): *Price Level Measurement*, Ottawa: Statistics Canada.

Triplett, J. E. (1983), "Concepts of Quality in Input and Output Price Measures: A Resolution of the User Value and Resource Cost Debate", pp. 269-311 in *The U. S. National Income and Product Accounts: Selected Topics*, M. F. Foss (ed.), NBER Studies in Income and Wealth Volume 47, Chicago: The University of Chicago Press.

Triplett, J. E. (2004); *Handbook on Hedonic Indexes and Quality Adjustments in Price Indexes: Special Application to Information Technology Products*, STI Working Paper 2004/9, Paris: OECD.

Walras, L. (1954), *Elements of Pure Economics*, a translation by W. Jaffé of the Edition Définitive (1926) of the *Eléments d'économie pure*, first edition published in 1874, Homewood, Illinois: Richard D. Irwin.