Decomposing Value Added Growth over Sectors into Explanatory Factors

W. Erwin Diewert and Kevin J. Fox,[1] December 29, 2016
Discussion Paper 16-07,
Vancouver School of Economics,
The University of British Columbia,
Vancouver, Canada, V6T 1L4.

Abstract

A decomposition of nominal value added growth over multiple sectors of an economy into explanatory factors is presented. The explanatory factors over a single sector are changes in the efficiency of the sector, growth of primary inputs, changes in sectoral output and input prices, technical progress and returns to scale. In order to implement the decomposition for a sector, an estimate of the sector's cost constrained value added function for the two periods under consideration is required, which is taken to be the free disposal hull of past observations. The problems associated with aggregating over sectors are also considered. The methodology is illustrated using U.S. data for two sectors over the years 1960-2014.

Keywords

Measurement of output, input and productivity, value added functions, revenue functions, variable profit functions, duality theory, economic price and quantity indexes, technical progress, total factor productivity, revenue efficiency, aggregation over sectors.

JEL Classification Numbers

C43, D24, D61, E23, H44, O47.

[1] W. Erwin Diewert: Vancouver School of Economics, University of British Columbia, Vancouver B.C., Canada, V6T 1Z1 and the School of Economics, UNSW Australia, Sydney 2052, (email: erwin.diewert@ubc.ca) and Kevin J. Fox: CAER & School of Economics, UNSW Australia, Sydney 2052 (email: K.Fox@unsw.edu.au). The authors thank Knox Lovell, Marcel Timmer and seminar participants at the US Bureau of Economic Analysis for helpful comments. The first author gratefully acknowledges the financial support of the SSHRC of Canada, and both authors gratefully acknowledge the financial support of the Australian Research Council (DP150100830).

1. Introduction

Understanding sources of economic growth has long been of interest to academics and policy makers. A better understanding of the determinants of value added growth can provide insights into the potential for policies to address inefficiencies and a deeper understanding of the drivers of productivity, a topic of heightened recent interest given the slowdown in productivity growth across many developed countries; see e.g. Gordon (2016), Mokyr, Vickers and Ziebarth (2015), Byrne, Fernald and Reinsdorf (2016), and Syverson (2016).

While there has been much attention to growth at the aggregate economy level, there has perhaps been less at the sectoral level, and especially in determining the relationship between sectoral growth determinants and aggregate growth determinants. To address this, we derive exact and approximate decompositions of nominal value added growth over multiple sectors of an economy into explanatory factors, and illustrate these using data for the US Corporate Nonfinancial and the Noncorporate Nonfinancial sectors, 1960 to 2014.

We take the explanatory factors of value added growth in a sector to be as follows:

- efficiency changes,
- changes in output prices,
- changes in primary inputs,
- changes in input prices,
- technical progress, and
- returns to scale.

We start by decomposing value added growth in a single production sector into these components, before considering the relationship with aggregate national value added growth. In order to implement our decomposition, an estimate of the sector's best practice technology for the two periods under consideration is required. This could be obtained using econometric techniques or nonparametric frontier modelling, such as Data Envelopment Analysis (DEA) type techniques; see e.g. Charnes and Cooper (1985) and Färe, Grosskopf and Lovell (1985). We do not make any of the convexity assumptions that are typical in this literature, and instead use the Free Disposal Hull (FDH) approach of Tulkens (1986)(1993) and his co-authors; see also Diewert and Fox (2014)(2016a).

We show that if sectoral production is assumed to be efficient in both periods, then knowledge of the best practice technology is not required. Rather, the sectoral decomposition of value added growth can be obtained using the index number techniques pioneered by Diewert and Morrison (1986) and Kohli (1990), drawing on exactness results from index number theory on the relationship between the translog functional form and the Törnqvist index. However, during recessions, it seems unlikely that production units are operating on their production frontiers (fixed capital stock components cannot be readily reduced in the light of reduced output demands) and thus it is important for a growth accounting methodology to allow for technical and allocative inefficiency. Our methodological approach does this. It has the advantages that it does not involve any econometric estimation, and involves only observable data on input and output prices and quantities for the sector. Thus it is simple enough to be implemented by statistical agencies.

Another positive feature of our approach is that it rules out technical regress, which is a problematic concept for a broad range of economic models; see e.g. Aiyar, Dalgaard and Moav (2008) and Diewert and Fox (2016b). A consequence of ruling out technical regress is that when there is a recession, for example, the loss of efficiency is gross loss of efficiency less any technical progress that occurs during recession years. Hence, in this case estimates of efficiency losses may be offset by technical progress, and what is measured as efficiency change is the net effect.

After explaining our methodological approach for a single production sector, we consider the problems associated with aggregating over sectors. Two alternative approaches for relating the sectoral growth components to national value added growth are considered. The first approach is a weighted average sectoral approach, which yields an approximate decomposition of aggregate value added growth into weighted sectoral explanatory factors. The second approach starts at the aggregate level and derives the corresponding sectoral contributions to aggregate growth.

The rest of the paper is organized as follows. The core methodology is explained in the following section, where we introduce the *cost constrained value added function* which is used throughout. In section 3, the method for decomposing value added growth into our six components for each sector is derived. Section 4 describes our nonparametric approach to obtaining empirical estimates for the best practice cost constrained value added functions, which allows us to decompose TFP growth for a sector into explanatory components. The following two sections deal with aggregating over sectors. Section 5 introduces the first approach to obtaining a decomposition of national value added growth through the aggregation of the sectoral explanatory factors. This can be described as a "bottom up" approach, as it starts at the sector level and aggregates to the national level. Here it is also noted that under the (efficiency) assumptions of Diewert and Morrison (1986) and Kohli (1990), it is possible to get a decomposition that can be implemented by using only index numbers. An approximate decomposition into national explanatory factors is also provided. Section 6, following the example of Balk (2016b), starts with a decomposition of national value added growth and relates this to sectoral explanatory factors. This approach relies on hypothetical sector shares. It is shown that if we instead use observed shares then we get an approximate decomposition into sector effects that is exactly equal to the approximate decomposition of section 5. Using our data on two major sectors of the U.S. economy, sections 7 through 10 provide empirical applications of the approaches of sections 3 through 6, respectively, with the results shedding light on sources of value added and productivity growth for the U.S. over a 55 year period. Section 11 concludes.

2. The Cost Constrained Value Added Function for a Sector

Suppose that a sector produces M net outputs,[2] $y \equiv [y_1,...,y_M]$, using N primary inputs $x \equiv [x_1,...,x_N] \geq 0_N$, while facing the strictly positive vector of net output prices $p \equiv [p_1,...,p_M] >> 0_M$ and the strictly positive vector of input prices $w \equiv [w_1,...,w_N] >> 0_N$. The value of primary inputs used by the sector during period t is then $w \cdot x \equiv \Sigma_{n=1}^{N} w_n x_n$. Denote the *period t production possibilities set* for the sector by S^t.[3] Define the sector's *period t cost constrained value added function,* $R^t(p,w,x)$ as follows:[4]

(1) $R^t(p,w,x) \equiv \max_{y,z} \{p \cdot y : (y,z) \in S^t ; w \cdot z \leq w \cdot x\}$.

If (y^*,z^*) solves the constrained maximization problem defined by (1), then sectoral value added $p \cdot y$ is maximized subject to the constraints that (y,z) is a feasible production vector and primary input expenditure $w \cdot z$ is equal to or less than "observed" primary input expenditure $w \cdot x$. Thus if the sector faces the prices $p^t >> 0_M$ and $w^t >> 0_N$ during period t and (y^t, x^t) is the sector's observed production vector, then production will be *value added efficient* if the observed value added, $p^t \cdot y^t$, is equal to the optimal value added, $R^t(p^t,w^t,x^t)$. However, production may not be efficient and so the following inequality will hold:

(2) $p^t \cdot y^t \leq R^t(p^t,w^t,x^t)$.

Following the example of Balk (1998; 82), we define the *value added* or *net revenue efficiency* of the sector during period t, e^t, as follows:

(3) $e^t \equiv p^t \cdot y^t / R^t(p^t,w^t,x^t) \leq 1$

where the inequality in (3) follows from (2). Thus if $e^t = 1$, then production is allocatively efficient in period t and if $e^t < 1$, then production for the sector during period t is allocatively inefficient. Note that the above definition of value added efficiency is a net revenue counterpart to Farrell's (1957; 255) cost based measure of *overall efficiency* in the DEA context, which combined his measures of technical and (cost) allocative

[2] Let $(y,x) \in S^t$ where $y = [y_1,...,y_M]$ and $x \equiv [x_1,...,x_N] \geq 0_N$. If $y_m > 0$, then the sector produces the mth net output during period t while if $y_m < 0$, then the sector uses the mth net output as an intermediate input.

[3] We assume that S^t satisfies the following regularity conditions: (i) S^t is a closed set; (ii) for every $x \geq 0_N$, $(0_M,x) \in S^t$; (iii) if $(y,x) \in S^t$ and $y^* \leq y$, then $(y^*,x) \in S^t$ (free disposability of net outputs); (iv) if $(y,x) \in S^t$ and $x^* \geq x$, then $(y,x^*) \in S^t$ (free disposability of primary inputs); (v) if $x \geq 0_N$ and $(y,x) \in S^t$, then $y \leq b(x)$ where the upper bounding vector b can depend on x (bounded primary inputs implies bounded from above net outputs). When applying our methodology, we will need somewhat stronger conditions that will imply that that the cost constrained value added function is *positive* when evaluated at observed data points.

[4] Note that $R^t(p,w,x)$ is well defined even if there are increasing returns to scale in production; i.e., the constraint $w \cdot z \leq w \cdot x$ leads to a finite value for $R^t(p,w,x)$. The cost constrained value added function is analogous to Diewert's (1983; 1086) *balance of trade restricted value added function*, Diewert and Morrison's (1986; 669) *domestic sales function*, Fisher and Shell's (1998; 48) *cost restricted sales function* and Balk's (2003; 34) *indirect revenue function*. Fisher and Shell and Balk defined their functions as $IR^t(p,w,c) \equiv \max_{y,z} \{p \cdot y : w \cdot z \leq c ; (y,z) \in S^t\}$ where $c > 0$ is a scalar cost constraint. It can be seen that our cost constrained value added function replaces c in the above definition by $w \cdot x$.

efficiency. DEA or *Data Envelopment Analysis* is the term used by Charnes and Cooper (1985) and their co-workers to denote an area of analysis which is called the nonparametric approach to production theory[5] or the measurement of the efficiency of production[6] by economists.

The cost constrained value added function has some interesting mathematical properties. For fixed w and x, $R^t(p,w,x)$ is a convex and linearly homogeneous function of p.[7] For fixed p and w, $R^t(p,w,x)$ is nondecreasing in x. If S^t is a convex set, then $R^t(p,w,x)$ is also concave in x. For fixed p and x, $R^t(p,w,x)$ is homogeneous of degree 0 in w.

It is possible to get more insight into the properties of R^t if we introduce the sector's *period t value added function* $\Pi^t(p,x)$. Thus for $p >> 0_M$ and $x \geq 0_N$, define $\Pi^t(p,x)$ as follows:[8]

$$(4)\ \Pi^t(p,x) \equiv \max_y \{p \cdot y : (y,x) \in S^t\}.$$

Using definitions (1) and (4), it can be seen that the cost constrained value added function $R^t(p,w,x)$ has the following representation:

$$(5)\ R^t(p,w,x) \equiv \max_{y,z} \{p \cdot y : (y,z) \in S^t;\ w \cdot z \leq w \cdot x;\}$$
$$= \max_z \{\Pi^t(p,z) : w \cdot z \leq w \cdot x;\ z \geq 0_N\}.$$

Holding p constant, we can define the period t *"utility" function* $f^t(z) \equiv \Pi^t(p,z)$ and the second maximization problem in (5) becomes the following "utility" maximization problem:

$$(6)\ \max_z \{f^t(z) : w \cdot z \leq w \cdot x;\ z \geq 0_N\}$$

where $w \cdot x$ is the consumer's "income". For u in the range of $\Pi^t(p,z)$ over the set of nonnegative z vectors and for $w >> 0_N$, we can define the *cost function* $C^t(u,w)$ that corresponds to $f^t(z)$ as follows:[9]

---

[5] See Hanoch and Rothschild (1972), Diewert and Parkan (1983), Varian (1984) and Diewert and Mendoza (2007).

[6] See Farrell (1957), Afriat (1972), Färe and Lovell (1978), Färe, Grosskopf and Lovell (1985), Coelli, Rao and Battese (1997) and Balk (1998) (2003).

[7] A version of Hotelling's Lemma also holds for $R^t(p,w,x)$. Suppose y^*,x^* is a solution to the constrained maximization problem that defines $R^t(p^*,w^*,x^*)$ and $\nabla_p R^t(p^*,w^*,x^*)$ exists. Then $y^* = \nabla_p R^t(p^*,w^*,x^*)$. See Diewert and Morrison (1986; 670) for the analogous properties for their sales function.

[8] This function is known as the GDP function or the national product function in the international trade literature; see Kohli (1978) (1990), Woodland (1982) and Feenstra (2004; 76). It is known as the gross, restricted or variable profit function in the duality literature; see Gorman (1968), McFadden (1978) and Diewert (1973) (1974). Sato (1976) called it a value added function. It was introduced into the economics literature by Samuelson (1953). We use the cost constrained value added function as our basic building block in this paper rather than the conceptually simpler GDP function because the cost constrained value added function allows us to deal with technologies which exhibit global increasing returns to scale.

[9] Of course, $f^t(z)$ should be denoted as $f^t(z,p)$ and $C^t(u,w)$ should be denoted as $C^t(u,w,p)$.

(7) $C^t(u,w) \equiv \min_z \{w \cdot z : f^t(z) \geq u;\ z \geq 0_N\} = \min_z \{w \cdot z : \Pi^t(p,z) \geq u;\ z \geq 0_N\}$.

If $\Pi^t(p,z)$ increases as all components of z increase, then $C^t(u,w)$ will be increasing in u and we can solve the following maximization problem for a unique u^*:

(8) $\max_u \{u: C^t(u,w) \leq w \cdot x\}$.

Using the solution to (8), we will have the following solution for the maximization problem that defines $R^t(p,w,x)$:

(9) $R^t(p,w,x) = u^*$

with $C^t(u^*,w) = w \cdot x$.

The above formulae simplify considerably if S^t is a cone, so that production is subject to constant returns to scale. If S^t is a cone, then $\Pi^t(p,z)$ is linearly homogeneous in z and hence, so is $f^t(z) \equiv \Pi^t(p,z)$. Define the unit cost function c^t that corresponds to f^t as follows:[10]

(10) $c^t(w,p) \equiv \min_z \{w \cdot z : \Pi^t(p,z) \geq 1;\ z \geq 0_N\}$.

The total cost function, $C^t(u,w) = C^t(u,w,p)$ is now equal to $uc^t(w,p)$ and the solution to (8) is the following u^*:

(11) $u^* = R^t(p,w,x) \equiv w \cdot x/c^t(w,p)$.

3. Decomposing Value Added Growth for a Sector into Explanatory Factors

In this section, we will assume that we can observe the outputs and inputs used by the sector or production unit for two consecutive periods, say period t−1 and t. Thus the observed net output and input vectors for the two periods are y^{t-1}, y^t, $x^{t-1} >> 0_N$ and $x^t >> 0_N$. The observed output and input price vectors are the strictly positive vectors p^{t-1}, p^t, w^{t-1} and w^t. We also assume that $p^i \cdot y^j > 0$ and $w^i \cdot x^j > 0$ for i = t−1,t and j = t−1,t. Our task in this section is to decompose (one plus) the growth in observed nominal value added over the two periods, $p^t \cdot y^t/p^{t-1} \cdot y^{t-1}$, into explanatory growth factors.

One of the explanatory factors will be the *growth in the value added efficiency* of the sector or production unit. In the previous section, we defined the period t value added efficiency as $e^t \equiv p^t \cdot y^t/R^t(p^t,w^t,x^t)$. Define the corresponding period t−1 efficiency as $e^{t-1} \equiv p^{t-1} \cdot y^{t-1}/R^{t-1}(p^{t-1},w^{t-1},x^{t-1})$. Given the above definitions of revenue efficiency in periods t−1 and t, we can define an index of the *change in value added efficiency* ε^t for the sector over the two periods ε^t as follows:

[10] $c^t(w,p)$ will be linearly homogeneous and concave in w for fixed p and it will be homogeneous of degree minus one in p for fixed w. If $\Pi^t(p,z)$ is increasing in p_m, then $c^t(w,p)$ will be decreasing in p_m.

(12) $\varepsilon^t \equiv e^t/e^{t-1} = [p^t \cdot y^t/R^t(p^t,w^t,x^t)]/[p^{t-1} \cdot y^{t-1}/R^{t-1}(p^{t-1},w^{t-1},x^{t-1})]$.

Thus if $\varepsilon^t > 1$, then value added efficiency has *improved* going from period t−1 to t whereas it has *fallen* if $\varepsilon^t < 1$.

Notice that the cost constrained value added function for the production unit in period t, $R^t(p,w,x)$, depends on four sets of variables:

- The time period t and this index t serves to indicate that the period t technology set S^t is used to define the period t value added function;
- The vector of output prices p that the production unit faces;
- The vector of primary input prices w that the production unit faces and
- The vector of primary inputs x which is available for use by the production unit during period t.

At this point, we will follow the methodology that is used in the economic approach to index number theory that originated with Konüs (1939) and Allen (1949) and we will use the value added function to define various *families of indexes* that vary only *one* of the four sets of variables, t, p, w and x, between the two periods under consideration and hold constant the other sets of variables.[11]

Our first family of factors that explain sectoral value added growth is a family of *output price indexes*, $\alpha(p^{t-1},p^t,x,t)$:

(13) $\alpha(p^{t-1},p^t,w,x,s) \equiv R^s(p^t,w,x)/R^s(p^{t-1},w,x)$.

Thus the output price index $\alpha(p^{t-1},p^t,w,x,s)$ defined by (13) is equal to the (hypothetical) cost constrained value added $R^s(p^t,w,x)$ generated by the best practice technology of period s while facing the period t output prices p^t and the reference primary input prices w and using the reference primary input vector x, divided by the cost constrained value added $R^s(p^{t-1},w,x)$ generated by the best practice technology of period s while facing the period t−1 output prices p^{t-1} and the reference primary input prices w and using the same reference primary input vector x. Thus for each choice of technology (i.e., s could equal t−1 or t) and for each choice of reference vectors of input prices w and quantities x, we obtain a possibly different output price index.

Following the example of Konüs (1939) in his analysis of the true cost of living index, it is natural to single out two special cases of the family of input price indexes defined by (13): one choice where we use the period t−1 technology and set the reference input

[11] The theory which follows is largely adapted from Diewert (1980a; 455-461) (1983, 1054-1076) (2014), Diewert and Morrison (1986), Kohli (1990), Fox and Kohli (1998) and the IMF, ILO, OECD, UN and the World Bank (2004; 455-456). This approach to the output quantity and input price indexes is an adaptation of the earlier work on theoretical price and quantity indexes by Konüs (1939), Allen (1949), Fisher and Shell (1972) (1998), Samuelson and Swamy (1974), Archibald (1977) and Balk (1998).

prices and quantities equal to the period t−1 input prices and quantities w^{t-1} and x^{t-1} (which gives rise to a *Laspeyres type output price index*) and another choice where we use the period t technology and set the reference input prices and quantities equal to the period t prices and quantities w^t and x^t (which gives rise to a *Paasche type output price index*). We define these special cases α_L^t and α_P^t as follows:

$$(14)\ \alpha_L^t \equiv \alpha(p^{t-1},p^t,w^{t-1},x^{t-1},t-1) \equiv R^{t-1}(p^t,w^{t-1},x^{t-1})/R^{t-1}(p^{t-1},w^{t-1},x^{t-1})\ ;$$
$$(15)\ \alpha_P^t \equiv \alpha(p^{t-1},p^t,w^t,x^t,t) \qquad \equiv R^t(p^t,w^t,x^t)/R^t(p^{t-1},w^t,x^t).$$

Since both output price indexes, α_L^t and α_P^t, are equally representative, a single estimate of output price change should be set equal to a symmetric average of these two estimates. We choose the geometric mean as our preferred symmetric average and thus our preferred overall measure of output price growth is the following overall *output price index*, α^t:[12]

$$(16)\ \alpha^t \equiv [\alpha_L^t\, \alpha_P^t]^{1/2}\ .$$

Our second family of factors that explain value added growth is a family of *input quantity indexes*, $\beta(x^{t-1},x^t,w)$:

$$(17)\ \beta(x^{t-1},x^t,w) \equiv w{\cdot}x^t/w{\cdot}x^{t-1}.$$

The input quantity index $\beta(x^{t-1},x^t,w)$ defined by (17) is equal to a ratio of simple linear aggregates of the observed input vectors for periods t−1 and t, x^{t-1} and x^t, where we use the vector of strictly positive input prices $w >> 0_N$ as weights. We note that this family of input quantity index does not use the cost constrained value added function. An alternative definition for a family of input quantity indexes that uses the cost restricted value added function for period s and reference vectors p and w is $\beta^*(x^{t-1},x^t,p,w,s) \equiv R^s(p^s,w^s,x^t)/R^s(p^s,w^s,x^{t-1})$.[13] If the period s technology set is a cone, then using (11), it can be seen that $\beta^*(x^{t-1},x^t,p,w,s) = w{\cdot}x^t/w{\cdot}x^{t-1} = \beta(x^{t-1},x^t,w)$. In the general case where the period s technology is not a cone, the input growth measure $\beta^*(x^{t-1},x^t,p,w,s)$ will also incorporate the effects of nonconstant returns to scale. In this general case, it seems preferable to isolate the effects of nonconstant returns to scale and the use of the simple input quantity indexes defined by (17) will allow us to do this as will be seen below.

It is natural to single out two special cases of the family of input quantity indexes defined by (17): one choice where we use the period t−1 input prices w^t which gives rise to the *Laspeyres input quantity index* β_L^t and another choice where we set the reference input

[12] Choosing the geometric mean leads to a measure of output price inflation that satisfies the time reversal test; i.e., the resulting index has the property that it is equal to the reciprocal of the corresponding index that measures price change going backwards in time rather than forward in time; see Diewert (1997) and Diewert and Fox (2017) on this point.

[13] The counterpart to this family of input quantity indexes was defined by Sato (1976; 438) and Diewert (1980; 456) using value added functions (i.e., the functions $\Pi^s(p,x)$) with the assumption that there was no technical progress between the two periods being compared.

prices equal to w^t (which gives rise to the *Paasche input quantity index* $\beta_P{}^t$. Thus define these special cases $\beta_L{}^t$ and $\beta_P{}^t$ as follows:

(18) $\beta_L{}^t \equiv w^{t-1}\cdot x^t/w^{t-1}\cdot x^{t-1}$;
(19) $\beta_P{}^t \equiv w^t\cdot x^t/w^t\cdot x^{t-1}$.

Since both input quantity indexes, $\beta_L{}^t$ and $\beta_P{}^t$, are equally representative, a single estimate of input quantity change should be set equal to a symmetric average of these two estimates. We choose the geometric mean as our preferred symmetric average and thus our preferred overall measure of input quantity growth is the following overall *input quantity index*, β^t:[14]

(20) $\beta^t \equiv [\beta_L{}^t\,\beta_P{}^t]^{1/2}$.

Our next family of indexes will measure the effects on cost constrained value added of a change in input prices going from period t−1 to t. We consider a family of measures of the relative change in cost constrained value added of the form $R^s(p,w^t,x)/R^s(p,w^{t-1},x)$. Since $R^s(p,w,x)$ is homogeneous of degree 0 in the components of w, it can be seen that we cannot interpret $R^s(p,w^t,x)/R^s(p,w^{t-1},x)$ as an input price index. If there is only one primary input, $R^s(p,w^t,x)/R^s(p,w^{t-1},x)$ is identically equal to unity and this measure of input price change will be independent of changes in the price of the single input. It is best to interpret $R^s(p,w^t,x)/R^s(p,w^{t-1},x)$ as measuring the effects on cost constrained value added of a change in the relative proportions of primary inputs used in production or in the *mix* of inputs used in production that is induced by a change in relative input prices when there is more than one primary input. Thus define the family of *input mix indexes* $\gamma(w^{t-1},w^t,p,x,s)$ as follows:[15]

(21) $\gamma(w^{t-1},w^t,p,x,s) \equiv R^s(p,w^t,x)/R^s(p,w^{t-1},x)$.

As usual, we will consider two special cases of the above family of input mix indexes, a Laspeyres case and a Paasche case. However, the Laspeyres case $\gamma_L{}^t$ will use the period t cost constrained value added function and the period t−1 reference vectors p^{t-1} and x^{t-1} while the Paasche case $\gamma_P{}^t$ will use the use the period t−1 cost constrained value added function and the period t reference vectors p^t and x^t:

(22) $\gamma_{LPP}{}^t \equiv \gamma(w^{t-1},w^t,p^{t-1},x^t,t) \quad \equiv R^t(p^{t-1},w^t,x^t)/R^t(p^{t-1},w^{t-1},x^t)$;

---

[14] This index is Fisher's (1922) ideal input quantity index.

[15] It would be more accurate to say that $\gamma(w^{t-1},w^t,p,x,s)$ represents the hypothetical proportional change in cost constrained value added for the period s reference technology due to the effects of a change in the input price vector from w^{t-1} to w^t when facing the reference net output prices p and the reference vector of inputs x. Thus we shorten this description to say that γ is an "input mix index". If there is only one primary input, then since $R^s(p,w,x)$ is homogeneous of degree 0 in w, it can be seen that $\gamma(w^{t-1},w^t,p,x,s) \equiv R^s(p,w^t,x)/R^s(p,w^{t-1},x) = [(w_1{}^t)^0\ R^s(p,1,x)]/[(w_1{}^{t-1})^0\ R^s(p,1,x)] = 1$; i.e., if there is only one primary input, then the input mix index is identically equal to 1. For alternative mix definitions, see Balk (2001) and Diewert (2014; 62).

(23) $\gamma_{PLL}{}^t \equiv \gamma(w^{t-1},w^t,p^t,x^{t-1},t-1) \equiv R^{t-1}(p^t,w^t,x^{t-1})/R^{t-1}(p^t,w^{t-1},x^{t-1})$.

The reason for these rather odd looking choices for reference vectors will be justified below in more detail but, basically, we make these choices in order to have value added growth decompositions into explanatory factors that are exact without making restrictive assumptions on the technology sets.

As usual, the above two indexes are equally representative and so it is natural to take an average of these two measures. We choose the geometric mean as our preferred symmetric average and thus our preferred overall measure of input mix change is the following overall *input mix index*, γ^t:

(24) $\gamma^t \equiv [\gamma_{LPP}{}^t\gamma_{PLL}{}^t]^{1/2}$.

We turn now to the effects on cost constrained value added due to the effects of technical progress; i.e., as time marches on, new techniques are developed that allow increased outputs using the same inputs or that allow the same outputs to be produced by fewer inputs. Thus we use the cost constrained value added function in order to define a family of *technical progress indexes* going from period t−1 to t, $\tau(p,w,x)$, for reference vectors of output and input prices, p and w, and a reference vector of input quantities x as follows:[16]

(25) $\tau(t-1,t,p,w,x) \equiv R^t(p,w,x)/R^{t-1}(p,w,x)$.

Technical progress measures are usually defined in terms of upward shifts in production functions or outward shifts of production possibilities sets due to the discovery of new techniques or managerial innovations over time. If there is positive technical progress going from period t−1 to t, then $R^t(p,w,x)$ will be greater than $R^{t-1}(p,w,x)$ and hence $\tau(p,w,x)$ will be greater than one and this measure of technical progress is equal to the proportional increase in value added that results from the expansion of the underlying best practice technology sets due to the passage of time. For each choice of reference vectors of output and input prices, p and w, and reference vector of input quantities x, we obtain a possibly different measure of technical progress.

Again, we will consider two special cases of the above family of technical progress indexes, a Laspeyres case and a Paasche case. However, the Laspeyres case $\tau_L{}^t$ will use the period t input vector x^t as the reference input vector and the period t−1 reference output and input price vectors p^{t-1} and w^{t-1} while the Paasche case $\tau_P{}^t$ will use the use the period t−1 input vector x^{t-1} as the reference input and the period t reference output and input price vectors p^t and w^t:

[16] This family of technical progress measures was defined by Diewert and Morrison (1986; 662) using the value added function $\Pi^t(p,x)$. A special case of the family was defined earlier by Diewert (1983; 1063). Balk (1998; 99) also used this definition and Balk (1998; 58), following the example of Salter (1960), also used the joint cost function to define a similar family of technical progress indexes.

(26) $\tau_L{}^t \equiv \tau(t-1,t,p^{t-1},w^{t-1},x^t) \equiv R^t(p^{t-1},w^{t-1},x^t)/R^{t-1}(p^{t-1},w^{t-1},x^t)$.
(27) $\tau_P{}^t \equiv \tau(t-1,t,p^t,w^t,x^{t-1}) \equiv R^t(p^t,w^t,x^{t-1})/R^{t-1}(p^t,w^t,x^{t-1})$.

Using (11), recall that if the reference technologies in periods t and t−1 are cones, then we have $R^t(p,w,x) = w\cdot x/c^t(w,p)$ and $R^{t-1}(p,w,x) = w\cdot x/c^{t-1}(w,p)$. Thus in the case where the reference technology is subject to constant returns to scale, $\tau_L{}^t \equiv \tau(t-1,t,p^{t-1},w^{t-1},x^t)$ turns out to be independent of x^t and $\tau_P{}^t \equiv \tau(t-1,t,p^t,w^t,x^{t-1})$ turns out to be independent of x^{t-1}. These "mixed" indexes of technical progress are then true Laspeyres and Paasche type indexes.

We have one more family of indexes to define and that is a family of returns to scale measures. Our measures are analogous to the global measures of returns to scale that were introduced by Diewert (2014; 62) using cost functions. Here we will use the cost restricted value added function in place of the cost function. Our returns to scale measure will be a measure of output growth divided by input growth from period t−1 to t but the technology is held constant when we compute the output growth measure. Our measure of input growth will be $w\cdot x^t/w\cdot x^{t-1}$ where w is a positive vector of reference input prices. Now pick positive reference price vector p that will value our M net outputs. If we hold the technology constant at period t−1 levels, our measure of output growth will be $R^{t-1}(p,w,x^t)/R^{t-1}(p,w,x^{t-1})$. If we hold the technology constant at period t levels, our measure of output growth will be $R^t(p,w,x^t)/R^t(p,w,x^{t-1})$. Thus for the reference technology set indexed by s (equal to t−1 or t) and reference price vectors p and w, define the family of *returns to scale measures* $\delta(x^{t-1},x^t,p,w,s)$ as follows:

(28) $\delta(x^{t-1},x^t,p,w,s) \equiv [R^s(p,w,x^t)/R^s(p,w,x^{t-1})]/[w\cdot x^t/w\cdot x^{t-1}]$.

We define the Laspeyres and Paasche special cases of (28):

(29) $\delta_L{}^t \equiv \delta(x^{t-1},x^t,p^{t-1},w^{t-1},t-1) \equiv [R^{t-1}(p^{t-1},w^{t-1},x^t)/R^{t-1}(p^{t-1},w^{t-1},x^{t-1})]/[w^{t-1}\cdot x^t/w^{t-1}\cdot x^{t-1}]$;
(30) $\delta_P{}^t \equiv \delta(x^{t-1},x^t,p^t,w^t,t) \equiv [R^t(p^t,w^t,x^t)/R^t(p^t,w^t,x^{t-1})]/[w^t\cdot x^t/w^t\cdot x^{t-1}]$.

In the case where the period t−1 reference production possibilities set S^{t-1} is a cone so that production is subject to constant returns to scale, then using (11), it can be seen that $\delta_L{}^t$ is equal to 1 and if S^t is a cone, then $\delta_P{}^t$ defined by (30) is also equal to 1.

Our preferred measure of returns to scale to be used in empirical applications is the geometric mean of the above special cases:

(31) $\delta^t \equiv [\delta_L{}^t\,\delta_P{}^t]^{1/2}$.

We are now in a position to decompose (one plus) the growth in value added for the production unit going from period t−1 to t as the product of six explanatory growth factors:

- The change in cost constrained value added efficiency over the two periods; i.e., $\varepsilon^t \equiv e^t/e^{t-1}$ defined by (12) above;
- Growth (or changes) in output prices; i.e., a factor of the form $\alpha(p^{t-1},p^t,w,x,s)$ defined above by (13);
- Growth (or changes) in input quantities; i.e., a factor of the form $\beta(x^{t-1},x^t,w)$ defined by (17);
- Growth (or changes) in input prices; i.e., an input mix index of the form $\gamma(w^{t-1},w^t,p,x,s)$ defined by (21);
- Changes due to technical progress; i.e., a factor of the form $\tau(t-1,t,p,w,x)$ defined by (25) and
- A returns to scale measure $\delta(x^{t-1},x^t,p,w,s)$ of the type defined by (28).

Straightforward algebra using the above definitions shows that we have the following exact decompositions of the observed value added ratio going from period t−1 to t into explanatory factors of the above type:[17]

(32) $p^t{\cdot}y^t/p^{t-1}{\cdot}y^{t-1} = \varepsilon^t\,\alpha_P{}^t\,\beta_L{}^t\,\gamma_{LPP}{}^t\,\delta_L{}^t\,\tau_L{}^t$;
(33) $p^t{\cdot}y^t/p^{t-1}{\cdot}y^{t-1} = \varepsilon^t\,\alpha_L{}^t\,\beta_P{}^t\,\gamma_{PLL}{}^t\,\delta_P{}^t\,\tau_P{}^t$.

Now multiply the above decompositions together and take the geometric mean of both sides of the resulting equation. Using the above definitions, it can be seen that we obtain the following *exact decomposition of value added growth into the product of six explanatory growth factors*:[18]

(34) $p^t{\cdot}y^t/p^{t-1}{\cdot}y^{t-1} = \varepsilon^t\,\alpha^t\,\beta^t\,\gamma^t\,\delta^t\,\tau^t$.

If the reference technology exhibits constant returns to scale in periods t−1 and t, then $\delta_L{}^t = \delta_P{}^t = \delta^t = 1$ and the returns to scale factors drop out of the decompositions on the right hand sides of (32)-(34).

Total Factor Productivity growth for the production unit under consideration going from period t−1 to t can be defined (following Jorgenson and Griliches (1967)) as an index of output growth divided by an index of input growth. An appropriate index of output growth is the value added ratio divided by the value added price index α^t. An appropriate

[17] Diewert (2011) obtained decompositions of cost growth analogous to (32) and (33) under the assumption that the production unit was cost efficient in each period.
[18] Balk (2003; 9-10) introduced the term "profitability" to describe the period t ratio of revenue to cost π^t but he considered this concept earlier; see Balk (1998; 66) for historical references. Diewert and Nakamura (2003; 129) described the same concept by the term "margin". If we divide both sides of (34) through by (one plus) the rate of cost growth, $w^t{\cdot}x^t/w^{t-1}{\cdot}x^{t-1}$, we obtain an expression for (one plus) the rate of growth of profitability, π^t/π^{t-1}, which will equal $\varepsilon^t\,\alpha^t\,\gamma^t\,\delta^t\,\tau^t/\beta^{t**}$ where β^{t**} is the Fisher ideal input price index that matches up with the Fisher ideal input quantity index $\beta^{t;}$ i.e., $\beta^t\beta^{t**} = w^t{\cdot}x^t/w^{t-1}{\cdot}x^{t-1}$. This decomposition of profitability growth can be compared to the alternative profitability growth decompositions obtained by Balk (2003; 22), Diewert and Nakamura (2003), O'Donnell (2009; 11) (2010; 531) and Diewert (2014; 63). Our present decomposition of profitability is closest to that derived by Diewert. The problem with Diewert's decomposition is that his measure of returns to scale combined returns to scale with mix effects.

index of input growth is β^t. Thus define the *period t TFP growth rate*, $TFPG^t$, for the production unit as follows:[19]

$$(35)\ TFPG^t \equiv \{[p^t\cdot y^t/p^{t-1}\cdot y^{t-1}]/\alpha^t\}/\beta^t = \varepsilon^t\,\gamma^t\,\delta^t\,\tau^t$$

where the last equality in (35) follows from (34). Thus in general, period t TFP growth is equal to the product of period t value added efficiency ε^t, a period t input mix index γ^t (which typically will be small in magnitude), period t technical progress τ^t and period t returns to scale for the best practice technology δ^t. If the reference best practice technologies are subject to constant returns to scale, then the returns to scale term is identically equal to 1 and drops out of the decomposition given by (35).

We follow the example of Kohli (1990) and obtain a levels decomposition for the observed level of nominal value added in period t, $p^t\cdot y^t$, relative to its observed value in period 1, $p^1\cdot y^1$. We assume that we have price and quantity data for the primary inputs used and net outputs produced by the production unit (p^t,w^t,y^t,x^t) for periods t = 1, 2,...,T. We also assume that we have estimates for the cost constrained value added functions, $R^t(p,w,x)$, that correspond to the best practice technology sets S^t for t = 1,2,...,T. Thus for t = 2,3,...,T, we can calculate the period to period growth factors ε^t, α^t, β^t, γ^t, τ^t and δ^t. Define the cumulated explanatory variables as follows:

$$(36)\ E^1 \equiv 1;\ A^1 \equiv 1;\ B^1 \equiv 1;\ C^1 \equiv 1;\ D^1 \equiv 1;\ T^1 \equiv 1.$$

For t = 2,3,...,T, define the above variables recursively as follows:

$$(37)\ E^t \equiv \varepsilon^t E^{t-1};\ A^t \equiv \alpha^t A^{t-1};\ B^t \equiv \beta^t B^{t-1};\ C^t \equiv \gamma^t C^{t-1};\ D^t \equiv \delta^t D^{t-1};\ T^t \equiv \tau^t T^{t-1}.$$

Using the above definitions and (34), it can be seen that we have the following *levels decomposition* for the level of period t observed value added relative to its period 1 level:

$$(38)\ p^t\cdot y^t/p^1\cdot y^1 = A^t\,B^t\,C^t\,D^t\,E^t\,T^t\,; \qquad t = 2,...,T.$$

Define the period t level of Total Factor Productivity, TFP^t, as follows:

$$(39)\ TFP^1 \equiv 1;\ \text{for } t = 2,...,T, \text{ define } TFP^t \equiv (TFPG^t)(TFP^{t-1})$$

where $TFPG^t$ is defined by (35) for t = 2,...,T. Using (35)-(39), it can be seen that we have the following *levels decomposition for TFP* using the cumulated explanatory factors defined by (36) and (37):

[19] There are similar decompositions for TFP growth using just quantity data and Malmquist gross output and input indexes; see Diewert and Fox (2014) (2017). For additional decompositions of TFP growth using both price and quantity data, see Balk (1998), (2001), (2003), Caves, Christensen and Diewert (1982), Diewert and Morrison (1986), Kohli (1990) and Diewert and Fox (2008) (2010). However, we believe that our present decomposition is the most comprehensive decomposition of TFP growth into explanatory factors that makes use of observable price and quantity data for both outputs and inputs.

(40) $TFP^t = [p^t \cdot y^t / p^1 \cdot y^1]/[A^t\, B^t] = C^t\, D^t\, E^t\, T^t$; $t = 2,...,T$.

In the following section, we explain a practical method for obtaining estimates for the cost constrained value added function for a sector.

4. A Nonparametric Approximation to the Cost Constrained Value Added Function

We assume that (y^t,x^t) is the production unit's observed net output and primary input vector respectively, where $x^t > 0_N$ and the observed vector of net output and primary input prices is (p^t,w^t), with $p^t >> 0_M$ and $w^t >> 0_N$ for $t = 1,2,...,T$.[20] We assume that the production unit's period t production possibilities set S^t is the conical free disposal hull of the period t actual production vector and past production vectors that are in our sample of time series observations for the unit.[21] Using this assumption, for strictly positive price vectors p and w and nonnegative input quantity vector x, we define the *period t cost constrained value added function* $R^t(p,w,x)$ for the production unit as follows:

$$
\begin{aligned}
(41)\ R^t(p,w,x) &\equiv \max_{\lambda_1,...,\lambda_t} \{p\cdot(\Sigma_{s=1}^{t}\, y^s\lambda_s) \, ; \, w\cdot(\Sigma_{s=1}^{t}\, x^s\lambda_s) \le w\cdot x \, ; \, \lambda_1 \ge 0 \, ,..., \, \lambda_t \ge 0\} \\
&= \max_s \{p\cdot y^s\, w\cdot x/w\cdot x^s : s = 1,2,...,t\} \\
&= w\cdot x \max_s \{p\cdot y^s/w\cdot x^s : s = 1,2,...,t\}.
\end{aligned}
$$

The second line in (41) follows from the first line since there is only one constraint in the linear programming (LP) problem defined by the first line. So the solution to the LP problem is a simple maximum of the t + 1 numbers 0 and $p\cdot y^s\, w\cdot x/w\cdot x^s$ for $s = 1,...,t$. Our assumption that all inner products of the form $p\cdot y^s$ and $w\cdot x^s$ are positive rules out the all $\lambda_s = 0$ solution to (41). The last expression in (41) shows that when we assume constant returns to scale for our nonparametric representation for S^t, the resulting $R^t(p,w,x)$ is linear and nondecreasing in x, is convex and linearly homogeneous in p and is homogeneous of degree 0 in w.

If t numbers, $\mu_1,...,\mu_t$ are all positive, then it can be seen that:

$$(42)\ \max_s \{\mu_s : s = 1,...,t\} = 1/\min_s \{1/\mu_s : s = 1,...,t\}.$$

Using (41) and (42), it can be seen that we can rewrite $R^t(p,w,x)$ as follows:

$$(43)\ R^t(p,w,x) = w\cdot x \max_s \{p\cdot y^s/w\cdot x^s : s = 1,2,...,t\}$$

---

[20] We also assume that $p^s\cdot y^t > 0$ for $s = 1,...,T$ and $t = 1,...,T$. This will ensure that all of our explanatory factors are strictly positive.

[21] Diewert (1980b; 264) suggested that the convex, conical, free disposal hull of past and current production vectors be used as an approximation to the period t technology set S^t when measuring TFP growth. Tulkens (1993; 201-206), Tulkens and Eeckaut (1995a) (1995b) and Diewert and Fox (2014)(2017) dropped the convexity and constant returns to scale assumptions and used free disposal hulls of past and current production vectors to represent the period t technology sets. In the present paper, we also drop the convexity assumption but maintain the free disposal and constant returns to scale assumptions. We also follow Diewert and Parkan (1983; 153-157) and Balk (2003; 37) in introducing price data into the computations.

$= w{\cdot}x/\min_s \{w{\cdot}x^s/p{\cdot}y^s : s = 1,2,...,t\}$
$= w{\cdot}x/c^t(w,p)$

where we define the *period t nonparametric unit cost function* $c^t(w,p)$ as follows:

(44) $c^t(w,p) \equiv \min_s \{w{\cdot}x^s/p{\cdot}y^s : s = 1,2,...,t\}$.

Thus we have an explicit functional form for the unit cost function $c^t(w,p)$ that was defined earlier by (10) above. It can be seen that $c^t(w,p)$ defined by (44) is a linear nondecreasing function of w (and hence is linearly homogeneous and concave in w which is a necessary property for unit cost functions) and is convex and homogeneous of degree minus one in p.

Now we are in a position to apply the decompositions of value added growth (34), of TFP growth (35) and for the level of TFP (40), using the specific functional form for a sector's cost constrained value added function defined by (43). However, with the assumption of constant returns to scale in production, the returns to scale growth factor δ^t is identically equal to one and so this factor vanishes from the decompositions of value added and TFP growth defined by (34) and (35) above. The levels return to scale growth factor D^t in (40) is also identically equal to one and hence vanishes from the decomposition (40).

In the following two sections we return to the more general model that was described in Section 3, but we assume that we have constructed cost constrained value added functions for the K sectors. We will study two alternative approaches to the problems associated with aggregating over sectors. For brevity, we refer to the aggregate of the K sectors as the national economy.

5. National Value Added Growth Decompositions: The Sectoral Weighted Average Approach

In this first approach to deriving a decomposition of national value added growth into explanatory components, we simply use weighted averages of the sectoral decompositions.

Suppose we have $K \geq 2$ sectors and we apply the sectoral value added decomposition methodology that was explained in section 3 to each sector. Denote the net output vector, primary input vector and the corresponding price vectors for sector k and period t by y^{kt}, x^{kt}, p^{kt}, w^{kt} for k = 1,...,K and t = 1,...,T.[22] Denote the period t nominal value added for sector k by $v^{kt} \equiv p^{kt}{\cdot}y^{kt}$ for k = 1,...,K and t = 1,...,T. Define the sector k decomposition factors $\alpha^{kt}, \beta^{kt}, \gamma^{kt}, \delta^{kt}, \varepsilon^{kt}, \tau^{kt}$ and A^{kt}, B^{kt}, C^{kt}, D^{kt}, E^{kt}, T^t using the same definitions as were used in section 3 above where sector k uses the constrained value added functions $R^{kt}(p^{kt},w^{kt},x^{kt})$ to construct the sector k explanatory factors $\alpha^{kt}, \beta^{kt}, \gamma^{kt}, \delta^{kt}, \varepsilon^{kt}, \tau^{kt}$ for k =

---

[22] The number of net outputs and primary inputs do not have to be the same across sectors but we do assume that all inner products of the form $p^{ks}.y^{kt}$ and $w^{ks}{\cdot}x^{kt}$ are positive for all k = 1,...,K, s = 1,...,T and t = 1,...,T.

1,...,K and t = 2,...,T. Then the sectoral counterparts to the decompositions (34), (35) and (40) are as follows for k = 1,...,K and t = 2,...,T:

(45) $v^{kt}/v^{k,t-1} = \alpha^{kt}\,\beta^{kt}\,\gamma^{kt}\,\delta^{kt}\,\varepsilon^{kt}\,\tau^{kt}$;
(46) $TFPG^{kt} \equiv [v^{kt}/v^{k,t-1}]/\alpha^{kt}\,\beta^{kt} = \gamma^{kt}\,\delta^{kt}\,\varepsilon^{kt}\,\tau^{kt}$;
(47) $TFP^{kt} = [v^{kt}/v^{k,1}]/A^{kt}\,B^{kt} = C^{kt}\,D^{kt}\,E^{kt}\,T^{kt}$.

Define the *period t national value added* v^t as the sum of the period t sectoral value added for each sector:

(48) $v^t \equiv \Sigma_{k=1}^{K}\, v^{kt}$; t = 1,...,T.

Define the *period t share of national value added for sector k*, s^{kt}, as follows:

(49) $s^{kt} \equiv v^{kt}/v^t$; k = 1,...,K; t = 1,...,T.

Using definitions (48) and (49), it is easy to see that we have the following *exact decomposition* of period t national value added growth into sectoral explanatory components:

(50) $v^t/v^{t-1} = \Sigma_{k=1}^{K}\, v^{kt}/\Sigma_{k=1}^{K}\, v^{k,t-1}$; t = 2,...,T
$= \Sigma_{k=1}^{K}\, s^{k,t-1}\,(v^{kt}/v^{k,t-1})$
$= \Sigma_{k=1}^{K}\, s^{k,t-1}\alpha^{kt}\,\beta^{kt}\,\gamma^{kt}\,\delta^{kt}\,\varepsilon^{kt}\,\tau^{kt}$

where the last equality follows using the sectoral decompositions (45).

Note that the national value added growth decomposition defined by (50) uses the period t−1 sectoral value added shares. It is possible to obtain an *alternative exact decomposition* of period t national value added growth using the period t sectoral value added shares:

(51) $v^t/v^{t-1} = \Sigma_{k=1}^{K}\, v^{kt}/\Sigma_{k=1}^{K}\, v^{k,t-1}$; t = 2,...,T
$= 1/[\Sigma_{k=1}^{K}\, v^{k,t-1}/\Sigma_{k=1}^{K}\, v^{kt}]$
$= 1/[\Sigma_{k=1}^{K}\, s^{kt}\,(v^{k,t-1}/v^{kt})]$
$= [\Sigma_{k=1}^{K}\, s^{kt}\,(v^{kt}/v^{k,t-1})^{-1}]^{-1}$
$= [\Sigma_{k=1}^{K}\, s^{kt}\,(\alpha^{kt}\,\beta^{kt}\,\gamma^{kt}\,\delta^{kt}\,\varepsilon^{kt}\,\tau^{kt})^{-1}]^{-1}$

where the last equality again follows using the sectoral decompositions (45).

Comparing (50) and (51), we see that (50) uses a period t−1 share weighted arithmetic average of the sectoral decompositions whereas (51) uses a period t share weighted harmonic average of the sectoral decompositions. Both decompositions are exact. Obviously, the decomposition defined by (50) is much simpler to use in practical applications.

The above decompositions are useful if we want a decomposition of aggregate value added growth into sectoral contributions but they do not lead to simple decompositions into *national* explanatory factors for value added efficiency, output price effects, input quantity effects and so on.[23] In order to accomplish this task, we resort to the use of approximations to the exact decompositions defined by (50) and (51). Before introducing these approximations, we first define period t national weighted averages of the sectoral explanatory factors. The period t logarithms of these *national explanatory factors* are defined as follows for t = 2,...,T:

(52) $\ln \alpha^{t\bullet} \equiv \Sigma_{k=1}^{K} (1/2)(s^{kt} + s^{k,t-1})\ln \alpha^{kt}$;
(53) $\ln \beta^{t\bullet} \equiv \Sigma_{k=1}^{K} (1/2)(s^{kt} + s^{k,t-1})\ln \beta^{kt}$;
(54) $\ln \gamma^{t\bullet} \equiv \Sigma_{k=1}^{K} (1/2)(s^{kt} + s^{k,t-1})\ln \gamma^{kt}$;
(55) $\ln \delta^{t\bullet} \equiv \Sigma_{k=1}^{K} (1/2)(s^{kt} + s^{k,t-1})\ln \delta^{kt}$;
(56) $\ln \varepsilon^{t\bullet} \equiv \Sigma_{k=1}^{K} (1/2)(s^{kt} + s^{k,t-1})\ln \varepsilon^{kt}$;
(57) $\ln \tau^{t\bullet} \equiv \Sigma_{k=1}^{K} (1/2)(s^{kt} + s^{k,t-1})\ln \tau^{kt}$.

Thus $\alpha^{t\bullet}$ is a weighted geometric average of the period t sectoral value added price indexes α^{kt} where the weight for sector k in period t is $(1/2)(s^{kt} + s^{k,t-1})$, the arithmetic average of sector k's shares of national value added in periods t−1 and t.[24] Similarly, $\beta^{t\bullet}$ is a weighted geometric average of the period t sectoral primary input quantity indexes β^{kt} where the weight for sector k in period t is also $(1/2)(s^{kt} + s^{k,t-1})$.[25] The interpretation of the other national explanatory factors is similar.

Recall the exact decompositions given by (50) and (51). Approximate the weighted arithmetic mean on the right hand side of (50) by the corresponding weighted geometric mean. Approximate the weighted harmonic mean on the right hand side of (51) by the corresponding weighted geometric mean. Now take the equally weighted geometric mean of the resulting two approximations and we obtain the following approximation to the logarithm of the national value ratio going from period t−1 to period t for t = 2,...,T:[26]

$$(58)\ \ln v^t/v^{t-1} \approx \Sigma_{k=1}^{K} (1/2)(s^{kt} + s^{k,t-1})\ln(v^{kt}/v^{k,t-1})$$
$$= \Sigma_{k=1}^{K} (1/2)(s^{kt} + s^{k,t-1})\ln(\alpha^{kt}\beta^{kt}\gamma^{kt}\delta^{kt}\varepsilon^{kt}\tau^{kt}) \qquad \text{using (45)}$$

[23] It is possible to adapt the technique used by Diewert (2015; 373) who decomposed a weighted sum of industry multiplicative effects into a sum of national explanatory effects by assigning interaction terms into industry effects.
[24] Thus $\alpha^{t\bullet}$ is a two stage price index: in the first stage, we aggregate over prices of outputs and intermediate inputs in sector k and in the second stage, we aggregate these sectoral price indexes over sectors. Note that the second stage of aggregation uses the Törnqvist price index formula; see Diewert (1976) who noted the connection of this index to the economic approach to index number theory.
[25] Thus $\beta^{t\bullet}$ is a two stage quantity index: in the first stage, we aggregate over quantities of primary inputs in sector k and in the second stage, we aggregate these sectoral quantity indexes over sectors. Note that the second stage of aggregation uses the Törnqvist quantity index formula.
[26] The approximation defined by (58) will generally be quite accurate; the geometric mean approximation to the right hand side of (50) will be equal to or less than the truth while the geometric mean approximation to the right hand side of (51) will be equal to or greater than the truth due to Schlömilch's inequality; see Hardy, Littlewood and Polya (1934; 26). Thus the biases will tend to cancel out. This type of approximation argument was used by Diewert (2016).

$$= \ln \alpha^{t\bullet} + \ln \beta^{t\bullet} + \ln \gamma^{t\bullet} + \ln \delta^{t\bullet} + \ln \varepsilon^{t\bullet} + \ln \tau^{t\bullet}$$

where the last equality follows using definitions (52)-(57). Exponentiating both sides of (58) leads to the following *approximate decomposition of period t national value added growth into explanatory factors* for t = 2,...,T:

$$(59)\ v^t/v^{t-1} \approx \alpha^{t\bullet}\, \beta^{t\bullet}\, \gamma^{t\bullet}\, \delta^{t\bullet}\, \varepsilon^{t\bullet}\, \tau^{t\bullet}.$$

A measure of *period t national real value added growth* is $[v^t/v^{t-1}]/\alpha^{t\bullet}$, which is national nominal value added growth v^t/v^{t-1} divided by the national value added price index $\alpha^{t\bullet}$. A measure of *period t national primary input growth* is $\beta^{t\bullet}$. *Period t national Total Factor Productivity Growth*, $TFPG^t$, can then be defined as national real value added growth divided by national primary input growth:

$$(60)\ TFPG^t \equiv [v^t/v^{t-1}]/\alpha^{t\bullet}\, \beta^{t\bullet} \approx \gamma^{t\bullet}\, \delta^{t\bullet}\, \varepsilon^{t\bullet}\, \tau^{t\bullet}\ ; \qquad t = 2,...,T$$

where the approximate equality in (60) follows from the approximate equality (59). Thus (60) provides an approximate decomposition of national (one plus) TFP growth into the product of various national explanatory growth factors (mix effects, returns to scale effects, cost constrained value added efficiency effects and technical progress effects).

It is of some interest to determine what happens to the value added decomposition defined by (59) if we make stronger assumptions on the sectoral technology sets. Assume that the technology of each sector can be represented by a translog value added function with the restrictions on technical progress that are described in Diewert and Morrison (1986) and Kohli (1990) (DMK). These papers also assumed constant returns to scale and competitive profit maximizing behavior. Under the assumptions used by these authors, the sectoral value added growth decompositions defined by (45) simplify to the following decompositions:

$$(61)\ v^{kt}/v^{k,t-1} = \alpha^{kt}\, \beta^{kt}\, \tau^{kt}\ ; \qquad k = 1,...,K;\ t = 2,...,T$$

where α^{kt} turns out to be the period t Törnqvist value added output price index for sector k and β^{kt} is the period t Törnqvist primary input quantity index for sector k. The period t, sector k technical progress index τ^{kt} can be calculated residually as $[v^{kt}/v^{k,t-1}]/\alpha^{kt}\, \beta^{kt}$ so under the DMK assumptions, it is not necessary to have estimates for the sectoral best practice cost constrained value added functions: all of the explanatory factors on the right hand side of (61) can be calculated using index numbers. Under the DMK assumptions, it turns out that $\gamma^{kt} = \delta^{kt} = \varepsilon^{kt} = 1$; i.e., the mix effects, returns to scale effects and inefficiency factors are identically equal to unity and hence are not present on the right hand side of equations (61). Definitions (52)-(57) can still be used to form the national explanatory factors $\alpha^{t\bullet}$, $\beta^{t\bullet}$, $\gamma^{t\bullet}$, $\delta^{t\bullet}$, $\varepsilon^{t\bullet}$, and $\tau^{t\bullet}$ but of course, under the DMK assumptions, $\gamma^{t\bullet} = \delta^{t\bullet} = \varepsilon^{t\bullet} = 1$ and the approximate national value added decomposition for period t defined by (59) becomes:

$$(62)\ v^t/v^{t-1} \approx \alpha^{t\bullet}\, \beta^{t\bullet}\, \tau^{t\bullet}\ ; \qquad t = 2,...,T.$$

The approximate national TFP growth decomposition that corresponds to (62) is the following one:

(63) $TFPG^t \equiv [v^t/v^{t-1}]/\alpha^{t\bullet}\,\beta^{t\bullet} \approx \tau^{t\bullet}$; $t = 2,\ldots,T.$

The decomposition of national value added growth defined by (62) is of some interest for the following reasons:

- It can be implemented using index numbers; i.e., it is not necessary to have estimates for the sectoral best practice functions which describe the sectoral technologies.
- The national value added deflator $\alpha^{t\bullet}$ turns out to be a two stage Törnqvist price index, where the first stage aggregates over outputs and intermediate inputs in a sector and the second stage aggregates over the sectoral net output price indexes.[27]
- The national primary input quantity index $\beta^{t\bullet}$ turns out to be a two stage Törnqvist quantity index where the first stage aggregates over primary inputs in a sector and the second stage aggregates over the sectoral primary input quantity indexes.[28]
- The decompositions of nominal value added growth defined by (59) and (62) indicate that our approach is a *bottom up* rather than a *top down approach*; i.e., our approach starts at the sector level and aggregates up to the national level, rather than starting at the level of national aggregate inputs and outputs.[29]

Although the national TFP growth decompositions defined by (60) in the general case (and by (62) when the DMK assumptions are made) are intuitively plausible, they are not exact decompositions.[30] Moreover, it is not completely clear that the "correct" definition for national TFP growth for period t is nominal value added growth v^t/v^{t-1} deflated by the product of the aggregate value added price index $\alpha^{t\bullet}$ times the aggregate primary input quantity index $\beta^{t\bullet}$. In the following section, we will provide an alternative exact decomposition of national value added growth into explanatory components that are based on national definitions of explanatory factors.

[27] Diewert (1978) showed that a single stage Törnqvist price index will generally provide a close approximation to its two stage counterpart.
[28] Diewert (1978) also showed that a single stage Törnqvist quantity index will generally provide a close approximation to its two stage counterpart. Diewert (1980; 495-498) discussed two stage aggregation versus single stage aggregation when aggregating over sectors.
[29] For discussions on these two approaches to aggregate productivity measurement, see Gu (2012), Diewert (2012), Schreyer (2012) and Balk (2015) (2016a) (2016b).
[30] However, note that the decompositions of aggregate value added growth given by (50) and (51) are exact. Moreover, the sectoral explanatory factors in these decompositions are theoretically well defined.

6. National Value Added Growth Decompositions: The National Cost Constrained Value Added Function Approach

In our second approach to providing decompositions of national value added growth and national Total Factor Productivity growth, we will utilize a general approach recommended by Balk (2016b).[31] Specifically, we will adapt the same definitions that we used to explain sectoral value added growth to explain national value added growth.

Recall the definitions for the period t, sector k value added price vectors p^{kt}, net output vectors y^{kt}, input quantity vectors x^{kt} and input price vectors w^{kt} that were introduced in the previous section. We use these sectoral prices and quantities in order to define their *national counterparts*, where the national output price and net quantity vectors for period t are defined as $p^t \equiv [p^{1t},p^{2t},...,p^{Kt}]$ and $y^t \equiv [y^{1t},y^{2t},...,y^{Kt}]$ respectively and the national input price and quantity vectors are defined as $w^t \equiv [w^{1t},w^{2t},...,w^{Kt}]$ and $x^t \equiv [x^{1t},x^{2t},...,x^{Kt}]$ respectively.[32] *Period t national value added* v^t is again defined by (48) but it can also be defined as the inner product of the national p^r and y^t vectors:

$$(64)\ v^t \equiv \Sigma_{k=1}^K v^t = \Sigma_{k=1}^K p^t \cdot y^t \equiv p^t \cdot y^t.$$

As in the previous section, the *sector k cost constrained value added function for period t* $R^{kt}(p^k,w^k,x^k)$ is defined as follows for a suitable sector k reference net output price vector p^k, primary input price vector w^k and primary input quantity vector x^k:

$$(65)\ R^{kt}(p^k,w^k,x^k) \equiv \max_{y^k,z^k} \{p^k \cdot y^k : (y^k,z^k) \in S^{kt} ;\ w^k \cdot z^k \leq w^k \cdot x^k\}; \quad k = 1,...,K;\ t = 1,...,T$$

where S^{kt} is the technology set for sector k in period t.

Let $p \equiv [p^1,...,p^K]$ and $w \equiv [w^1,...,w^K]$ be strictly positive national net output and primary input reference price vectors and let $x \equiv [x^1,...,x^K]$ be a strictly positive national primary input reference vector. Define the period t *national cost constrained value added function* as follows:

$$(66)\ R^t(p,w,x) \equiv \max_{y^1,...,y^K,z^1,...,z^K} \{\Sigma_{k=1}^K p^k \cdot y^k : (y^k,z^k) \in S^{kt};\ w^k \cdot z^k \leq w^k \cdot x^k ;\ k = 1,...,K\}$$
$$= \Sigma_{k=1}^K R^{kt}(p^k,w^k,x^k)$$

where the second line above follows using definitions (65). Thus the period t national cost constrained value added function is equal to the sum of the period t sectoral value added functions.[33] The national cost constrained value added function defined by (66) is

[31] "The point of departure of this paper is that aggregate productivity should be interpreted as productivity of the aggregate." Bert M. Balk (2016b; 2).

[32] Note that the p^t, y^t, w^t and x^t now denote national price and quantity vectors, whereas in Sections 2 and 3 above, these vectors denoted sectoral price and quantity vectors.

[33] It should be noted that our definition of the national cost constrained value added function is not an exact counterpart to the corresponding sectoral definition since we have a separate cost constraint for each sector in definition (66) instead of the single aggregate cost constraint which is $\Sigma_{k=1}^K w^k \cdot z^k \leq \Sigma_{k=1}^K w^k \cdot x^k$. Thus our

the key function that we use to define a decomposition of national value added growth into explanatory components.[34]

Recall the single sector analysis which was presented in section 3. It turns out that all of the analysis in that section can be applied to the present national context: simply reinterpret equations (12)-(40) using the national cost constrained value added function defined by (66) in place of the sectoral cost constrained value added function which appeared in section 3. In particular, we obtain the national value added growth decomposition defined by (34) which is $p^t\cdot y^t/p^{t-1}\cdot y^{t-1} = \varepsilon^t \alpha^t \beta^t \gamma^t \delta^t \tau^t$.[35]

Our next task is to describe the national growth factors, the α^t, β^t, γ^t, δ^t, ε^t and τ^t that appear in our reinterpreted national decomposition (34), in terms of their sectoral counterparts, the α^{kt}, β^{kt}, γ^{kt}, δ^{kt}, ε^{kt} and τ^{kt} that were defined in section 5. In order to do this, we will need to define the *sector k share of national best practice value added in period t*, σ^{kt}, as follows:[36]

$$(67)\ \sigma^{kt} \equiv R^{kt}(p^{kt},w^{kt},x^{kt})/R^t(p^t,w^t,x^t)\ ; \qquad k=1,\ldots,K;\ t=1,\ldots,T.$$

The *period t national cost constrained value added efficiency factor* e^t is defined as the ratio of actual value added v^t to optimal cost constrained value added for t = 1,...,T:

$$\begin{aligned}(68)\ e^t &\equiv v^t/R^t(p^t,w^t,x^t)\\ &= [\Sigma_{k=1}^K v^{kt}]/[\Sigma_{i=1}^K R^{it}(p^{it},w^{it},x^{it})] && \text{using (64) and (65)}\\ &= \Sigma_{k=1}^K [v^{kt}/R^{kt}(p^{kt},w^{kt},x^{kt})][R^{kt}(p^{kt},w^{kt},x^{kt})]/[\Sigma_{i=1}^K R^{it}(p^{it},w^{it},x^{it})] && \text{using (67)}\\ &= \Sigma_{k=1}^K \sigma^{kt} e^{kt}\end{aligned}$$

where the last line follows using definitions (67) and $e^{kt} \equiv v^{kt}/R^{kt}(p^{kt},w^{kt},x^{kt})$. The national value added efficiency for period t, e^t, is then equal to a share weighted average of the sectoral efficiencies e^{kt} for period t. Thus the national value added efficiency growth factor ε^t is equal to the following expression:

concept of cost restricted national value added efficiency is a more limited one compared to a concept which would allow for input reallocation effects between sectors. The problem with the broader efficiency concept is that many primary inputs are fixed and cannot be readily moved across sectors.

[34] We note that national GDP functions have been used to construct national price and quantity indexes; e.g., see Chapter 18 of the IMF, ILO, OECD, UN and World Bank (2004; 464). However, the idea of using an aggregate value added function to aggregate over sectors was noted by Bliss (1975; 146) and many others; see Diewert (1980a; 464-465) for additional references to the literature.

[35] Because our definition of the national cost constrained value added function has K sectoral cost constraints instead of a single national cost constraint, there is a problem with our national returns to scale definition; i.e., the δ^t in our reinterpreted decomposition (34). The problem is that if we assume that the sectoral production possibility sets are cones, it is no longer the case that our national global measure of returns to scale δ^t is necessarily equal to one. We need some additional assumptions to ensure that the assumption that the sectors are subject to constant returns to scale implies that our measure of national returns to scale equals unity. We will discuss these additional assumptions below.

[36] Of course, knowledge of the sectoral best practice revenue functions is required in order to calculate these hypothetical shares.

(69) $\varepsilon^t \equiv e^t/e^{t-1} = [\Sigma_{k=1}^K \sigma^{kt} e^{kt}]/[\Sigma_{k=1}^K \sigma^{k,t-1} e^{k,t-1}]$; $t = 2,...,T.$

We now turn to the national output price indexes α_L^t and α_P^t defined by the national counterparts to (14) and (15). These counterparts are the following national indexes:

(70) $\alpha_L^t \equiv R^{t-1}(p^t,w^{t-1},x^{t-1})/R^{t-1}(p^{t-1},w^{t-1},x^{t-1})$ $t = 2,...,T$
$= \Sigma_{k=1}^K R^{k,t-1}(p^{kt},w^{k,t-1},x^{k,t-1})/\Sigma_{k=1}^K R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{k,t-1})$
$= \Sigma_{k=1}^K \sigma^{k,t-1} R^{k,t-1}(p^{kt},w^{k,t-1},x^{k,t-1})/R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{k,t-1})$ using (67)
$= \Sigma_{k=1}^K \sigma^{k,t-1} \alpha_L^{kt}$

where the last equality follows using the definition of the sector k Laspeyres type price index for period t, $\alpha_L^{kt} \equiv R^{k,t-1}(p^{k,t},w^{k,t-1},x^{k,t-1})/R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{k,t-1})$ for $k = 1,...,K$. Thus the national Laspeyres type output price index for period t, α_L^t, is equal to a share weighted average of the period t sectoral Laspeyres type output price indexes. We can establish a similar decomposition for the national Paasche type output price index for period t, α_P^t:

(71) $\alpha_P^t \equiv R^t(p^t,w^t,x^t)/R^t(p^{t-1},w^t,x^t)$ $t = 2,...,T$
$= 1/\{\Sigma_{k=1}^K R^{kt}(p^{k,t-1},w^{kt},x^{kt})/\Sigma_{k=1}^K R^{kt}(p^{kt},w^{kt},x^{kt})\}$
$= 1/\{\Sigma_{k=1}^K \sigma^{kt} [R^{kt}(p^{kt},w^{kt},x^{kt})/R^{k,t}(p^{k,t-1},w^{kt},x^{kt})]^{-1}\}$ using (67)
$= \{\Sigma_{k=1}^K \sigma^{kt} [\alpha_P^{kt}]^{-1}\}^{-1}$

where the last equality follows using the definition of the sector k Paasche type price index for period t, $\alpha_P^{kt} \equiv R^{kt}(p^{kt},w^{kt},x^{kt})/R^{kt}(p^{k,t-1},w^{kt},x^{kt})$ for $k = 1,...,K$. Thus the national Paasche type output price index for period t, α_P^t, is equal to a share weighted harmonic average of the period t sectoral Paasche type output price indexes.

As in section 3, our preferred overall national price index for period t is defined by the geometric mean of the above two indexes; i.e., $\alpha^t \equiv [\alpha_L^t \, \alpha_P^t]^{1/2}$.

We now turn to the national input quantity indexes $\beta_L^t \equiv w^{t-1}\cdot x^t/w^{t-1}\cdot x^{t-1}$ and $\beta_P^t \equiv w^t\cdot x^t/w^t\cdot x^{t-1}$ defined by the national counterparts to (18) and (19). These counterparts are just the ordinary Laspeyres and Paasche input quantity indexes so that definitions (18) and (19) also serve to define the national indexes but of course, the price and quantity vectors that enter into these definitions are now national vectors instead of sectoral vectors. Our preferred national primary input quantity index for period t is the Fisher index $\beta^t \equiv [\beta_L^t \, \beta_P^t]^{1/2}$.

We now consider national input mix indexes; i.e., indexes that measure the effects of changes in the reference input prices going from period t−1 to period t. The period t and sector k *input mix indexes* are defined as follows:[37]

(72) $\gamma_{LPP}^{kt} \equiv R^{kt}(p^{k,t-1},w^{kt},x^{kt})/R^{kt}(p^{k,t-1},w^{k,t-1},x^{kt})$ $k = 1,...,K ; t = 2,...,T$

[37] Recall definitions (22) and (23) in section 3.

(73) $\gamma_{PLL}{}^{kt} \equiv R^{k,t-1}(p^{kt},w^{kt},x^{k,t-1})/R^{k,t-1}(p^{kt},w^{k,t-1},x^{k,t-1})$.

The *national period t input mix index* which is a counterpart to the sectoral input mix indexes defined by (72) is defined as follows:

$$(74)\ \gamma_{LPP}{}^{t} \equiv R^{t}(p^{t-1},w^{t},x^{t})/R^{t}(p^{t-1},w^{t-1},x^{t}) \qquad t = 2,...,T$$
$$= \Sigma_{k=1}{}^{K} R^{kt}(p^{k,t-1},w^{kt},x^{kt})/\Sigma_{i=1}{}^{K} R^{it}(p^{i,t-1},w^{i,t-1},x^{it})$$
$$= \Sigma_{k=1}{}^{K} \sigma_{LPP}{}^{kt} R^{kt}(p^{k,t-1},w^{kt},x^{kt})/R^{kt}(p^{k,t-1},w^{k,t-1},x^{kt})$$
$$= \Sigma_{k=1}{}^{K} \sigma_{LPP}{}^{kt} \gamma_{LPP}{}^{kt}$$

where the period t sector shares $\sigma_{LPP}{}^{kt}$ are defined as follows:

$$(75)\ \sigma_{LPP}{}^{kt} \equiv R^{kt}(p^{k,t-1},w^{k,t-1},x^{kt})/\Sigma_{i=1}{}^{K} R^{it}(p^{i,t-1},w^{i,t-1},x^{it})\ ; \qquad k = 1,...,K\ ;\ t = 2,...,T.$$

Thus the national period t input mix index $\gamma_{LPP}{}^{t}$ is a share weighted arithmetic average of the period t sectoral input mix indexes $\gamma_{LPP}{}^{kt}$.

The *national period t input mix index* which is a counterpart to the sectoral indexes defined by (73) is defined as follows:

$$(76)\ \gamma_{PLL}{}^{t} \equiv R^{t-1}(p^{t},w^{t},x^{t-1})/R^{t-1}(p^{t},w^{t-1},{}^{,t-1}) \qquad t = 2,...,T$$
$$= \Sigma_{k=1}{}^{K} R^{k,t-1}(p^{kt},w^{kt},x^{k,t-1})/\Sigma_{i=1}{}^{K} R^{i,t-1}(p^{it},w^{i,t-1},x^{i,t-1})$$
$$= [\Sigma_{i=1}{}^{K} R^{i,t-1}(p^{it},w^{i,t-1},x^{i,t-1})/\Sigma_{k=1}{}^{K} R^{k,t-1}(p^{kt},w^{kt},x^{k,t-1})]^{-1}$$
$$= [\Sigma_{k=1}{}^{K} \sigma_{PLL}{}^{kt} \{R^{k,t-1}(p^{kt},w^{kt},x^{k,t-1})/R^{k,t-1}(p^{kt},w^{k,t-1},x^{k,t-1})\}^{-1}]^{-1}$$
$$= [\Sigma_{k=1}{}^{K} \sigma_{PLL}{}^{kt} \{\gamma_{LPP}{}^{kt}\}^{-1}]^{-1}$$

where the period t sector k shares $\sigma_{PLL}{}^{kt}$ are defined as follows:

$$(77)\ \sigma_{PLL}{}^{kt} \equiv R^{k,t-1}(p^{kt},w^{kt},x^{k,t-1})/\Sigma_{i=1}{}^{K} R^{i,t-1}(p^{it},w^{it},x^{i,t-1})\ ; \qquad k = 1,...,K\ ;\ t = 2,...,T.$$

The final equation in (76) tells us that the national period t input mix index $\gamma_{PLL}{}^{t}$ is a share weighted harmonic mean of the period t sectoral input mix indexes $\gamma_{PLL}{}^{kt}$.

As in section 3, our preferred overall national input mix index for period t is defined by the geometric mean of the above two indexes; i.e., $\gamma^{t} \equiv [\gamma_{LPP}{}^{t}\ \gamma_{PLL}{}^{t}]^{1/2}$.

The period t Laspeyres and Paasche type national measures of technical progress can be defined using definitions (26) and (27) in section 3 with the understanding that the price and quantity vectors are interpreted as national vectors rather than sectoral vectors. The period t, sector k Laspeyres and Paasche type indexes of technical progress are defined as follows for k = 1,...,K:

$$(78)\ \tau_{L}{}^{kt} \equiv R^{kt}(p^{k,t-1},w^{k,t-1},x^{kt})/R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{kt})$$
$$(79)\ \tau_{P}{}^{kt} \equiv R^{kt}(p^{kt},w^{kt},x^{k,t-1})/R^{k,t-1}(p^{kt},w^{kt},x^{k,t-1}).$$

The national counterpart to the sectoral indexes defined by (78) is defined as follows:

$$(80)\ \tau_L{}^t \equiv R^t(p^{t-1},w^{t-1},x^t)/R^{t-1}(p^{t-1},w^{t-1},x^t) \qquad t = 2,...,T$$
$$= \Sigma_{k=1}{}^K R^{kt}(p^{k,t-1},w^{k,t-1},x^{kt})/\Sigma_{i=1}{}^K R^{i,t-1}(p^{i,t-1},w^{i,t-1},x^{it})$$
$$= \Sigma_{k=1}{}^K \sigma_L{}^{kt} R^{kt}(p^{k,t-1},w^{k,t-1},x^{kt})/R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{kt})$$
$$= \Sigma_{k=1}{}^K \sigma_L{}^{kt} \tau_L{}^{kt}$$

where the period t sector k shares of (hypothetical) national value added $\sigma_L{}^{kt}$ are defined as follows:[38]

$$(81)\ \sigma_L{}^{kt} \equiv R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{kt})/\Sigma_{i=1}{}^K R^{i,t-1}(p^{i,t-1},w^{i,t-1},x^{it}) ; \qquad k = 1,...,K ; t = 2,...,T.$$

Thus the national period t Laspeyres type technical progress index $\tau_L{}^t$ is a share weighted arithmetic average of the period t Laspeyres type sectoral technical progress indexes $\tau_L{}^{kt}$.

The national counterpart to the Paasche type sectoral indexes defined by (79) is defined as follows:

$$(82)\ \tau_P{}^t \equiv R^t(p^t,w^t,x^{t-1})/R^{t-1}(p^t,w^t,x^{t-1}) \qquad t = 2,...,T$$
$$= [\Sigma_{k=1}{}^K R^{k,t-1}(p^{kt},w^{kt},x^{k,t-1})/\Sigma_{i=1}{}^K R^{it}(p^{it},w^{it},x^{i,t-1})]^{-1}$$
$$= [\Sigma_{k=1}{}^K \sigma_P{}^{kt} \{R^{kt}(p^{kt},w^{kt},x^{k,t-1})/R^{k,t-1}(p^{kt},w^{kt},x^{k,t-1})\}^{-1}]^{-1}$$
$$= [\Sigma_{k=1}{}^K \sigma_P{}^{kt} \{\tau_P{}^{kt}\}^{-1}]^{-1}$$

where the period t sector k shares of (hypothetical) national value added $\sigma_P{}^{kt}$ are defined as follows:[39]

$$(83)\ \sigma_P{}^{kt} \equiv R^{kt}(p^{kt},w^{kt},x^{k,t-1})/\Sigma_{i=1}{}^K R^{it}(p^{it},w^{it},x^{i,t-1}) ; \qquad k = 1,...,K ; t = 2,...,T.$$

Thus the national period t Paasche type technical progress index $\tau_P{}^t$ is a share weighted harmonic average of the period t Paasche type sectoral technical progress indexes $\tau_P{}^{kt}$.

As in section 3, our preferred overall national technical progress index for period t is defined by the geometric mean of the above two indexes; i.e., $\tau^t \equiv [\tau_L{}^t\ \tau_P{}^t]^{1/2}$.

We conclude by introducing national measures of returns to scale. We first define the *Laspeyres and Paasche sector k, period t measures of returns to scale* that are the sectoral counterparts to definitions (29) and (30) in section 3 for k = 1,...,K and t = 2,...,T:

[38] If all of the sectoral technology sets are cones, then using (9), it can be seen that $R^{kt}(p,w,x)$ is equal to $w{\cdot}x/c^{kt}(w,p)$ and hence $R^{kt}(p,w,x)/R^{k,t-1}(p,w,x) = c^{k,t-1}(w,p)/c^{kt}(w,p)$ which is independent of x. Hence in the case where all of the sectoral technology sets are subject to constant returns to scale, the shares $\sigma_L{}^{kt}$ defined by (81) turn out to be equal to the Laspeyres shares $\sigma^{k,t-1} \equiv R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{k,t-1})/R^{t-1}(p^{t-1},w^{t-1},x^{t-1})$ defined by (67).

[39] If all of the sectoral technology sets are cones, then the shares $\sigma_P{}^{kt}$ defined by (83) turn out to be equal to the Paasche shares $\sigma^{kt} \equiv R^{kt}(p^{kt},w^{kt},x^{kt})/R^t(p^t,w^t,x^t)$ defined by (67).

(84) $\delta_L^{kt} \equiv [R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{kt})/R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{k,t-1})]/[w^{k,t-1}\cdot x^{kt}/w^{k,t-1}\cdot x^{k,t-1}]$;
(85) $\delta_P^{kt} \equiv [R^{kt}(p^{kt},w^{kt},x^{kt})/R^{kt}(p^{kt},w^{kt},x^{k,t-1})]/[w^{kt}\cdot x^{kt}/w^{kt}\cdot x^{k,t-1}]$.

In order to provide a more intuitive interpretation for our national measure of returns to scale, it is useful to decompose the above measures of sectoral returns to scale into two components: one that reflects the rate of output growth due to input growth, holding constant technology and prices, and another measure that reflects input growth. The numerators in (84) and (85) are the desired measures of constant technology output growth while the denominators in (84) and (85) are reasonable measures of input growth. We then define the *sector k, period t Laspeyres and Paasche measures of constant technology output growth*, λ_L^{kt} and λ_P^{kt}, and the *Laspeyres and Paasche measures of input growth*, β_L^{kt} and β_P^{kt}, as follows for k = 1,...,K and t = 2,...,T:

(86) $\lambda_L^{kt} \equiv R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{kt})/R^{k,t-1}(p^{k,t-1},w^{k,t-1},x^{k,t-1})$;
(87) $\lambda_P^{kt} \equiv R^{kt}(p^{kt},w^{kt},x^{kt})/R^{kt}(p^{kt},w^{kt},x^{k,t-1})$;
(88) $\beta_L^{kt} \equiv w^{k,t-1}\cdot x^{kt}/w^{k,t-1}\cdot x^{k,t-1}$;
(89) $\beta_P^{kt} \equiv w^{kt}\cdot x^{kt}/w^{kt}\cdot x^{k,t-1}$.

Thus $\delta_L^{kt} = \lambda_L^{kt}/\beta_L^{kt}$ and $\delta_P^{kt} = \lambda_P^{kt}/\beta_P^{kt}$. Note also that β_L^{kt} and β_P^{kt} are just the period t, sector k ordinary Laspeyres and Paasche input quantity indexes. We also define the period t, *sector k (observed) primary input cost share of total period t primary input cost*, $\boldsymbol{S}^{kt}$, as follows:

(90) $\boldsymbol{S}^{kt} \equiv w^{kt}\cdot x^{kt}/\, w^t\cdot x^t$; k = 1,...,K; t = 1,...,T.

The *national period t Laspeyres measure of returns to scale*, δ_L^t, is defined as the following national counterpart to definition (29) in section 3:

(91) $\delta_L^t \equiv [R^{t-1}(p^{t-1},w^{t-1},x^t)/R^{t-1}(p^{t-1},w^{t-1},x^{t-1})]/[w^{t-1}\cdot x^t/w^{t-1}\cdot x^{t-1}]$ t = 2,...,T
$= [\Sigma_{k=1}^K R^{kt}(p^{k,t-1},w^{k,t-1},x^{kt})/\Sigma_{i=1}^K R^{i,t-1}(p^{i,t-1},w^{i,t-1},x^{i,t-1})]/[\Sigma_{k=1}^K w^{i,t-1}\cdot x^{it}/\Sigma_{k=1}^K w^{i,t-1}\cdot x^{i,t-1}]$
$= [\Sigma_{k=1}^K \sigma^{k,t-1} \lambda_L^{kt}]/[\Sigma_{k=1}^K \boldsymbol{S}^{k,t-1} \beta_L^{kt}]$ using (67), (86), (88) and (90)
$= [\Sigma_{k=1}^K \sigma^{k,t-1} \beta_L^{kt} (\lambda_L^{kt}/\beta_L^{kt})]/[\Sigma_{k=1}^K \boldsymbol{S}^{k,t-1} \beta_L^{kt}]$
$= [\Sigma_{k=1}^K \sigma^{k,t-1} \beta_L^{kt} \delta_L^{kt}]/[\Sigma_{k=1}^K \boldsymbol{S}^{k,t-1} \beta_L^{kt}]$

where the last equation follows using $\delta_L^{kt} = \lambda_L^{kt}/\beta_L^{kt}$. Thus in general, δ_L^t is not equal to a weighted average of the sectoral measures of returns to scale. However, the third equation in (91) shows that δ_L^t is equal to a weighted arithmetic average of the sectoral constant technology constant price output growth rates λ_L^{kt} divided by a weighted average of the sectoral input growth rates β_L^{kt}. So the Laspeyres type definition of national returns to scale does capture the spirit of the sectoral definition of returns to scale as an output growth index using a constant technology divided by an input growth index.

The *national period t Paasche measure of returns to scale*, δ_P^t, is defined as the following national counterpart to definition (30) in section 3:

$$(92)\ \delta_P{}^t \equiv [R^t(p^t,w^t,x^t)/R^t(p^t,w^t,x^{t-1})]/[w^t\cdot x^t/w^t\cdot x^{t-1}] \qquad t = 2,...,T$$
$$= [\Sigma_{k=1}{}^K R^{kt}(p^{kt},w^{kt},x^{kt})/\Sigma_{i=1}{}^K R^{it}(p^{it},w^{it},x^{i,t-1})]/[\Sigma_{k=1}{}^K w^{it}\cdot x^{it}/\Sigma_{k=1}{}^K w^{it}\cdot x^{i,t-1}]$$
$$= [\Sigma_{k=1}{}^K \sigma^{kt}(\lambda_P{}^{kt})^{-1}]^{-1}/[\Sigma_{k=1}{}^K \boldsymbol{S}^{kt}(\beta_P{}^{kt})^{-1}]^{-1} \qquad \text{using (67), (87), (89) and (90)}$$
$$= [\Sigma_{k=1}{}^K \sigma^{kt}(\beta_P{}^{kt})^{-1}(\delta_P{}^{kt})^{-1}]^{-1}/[\Sigma_{k=1}{}^K \boldsymbol{S}^{kt}(\beta_P{}^{kt})^{-1}]^{-1}$$

where the last equation follows using $\delta_P{}^{kt} = \lambda_P{}^{kt}/\beta_P{}^{kt}$. Thus in general, $\delta_P{}^t$ is not equal to a weighted harmonic average of the Paasche type sectoral measures of returns to scale, $(\delta_P{}^{kt})$. However, the third equation in (92) shows that $\delta_P{}^t$ is equal to a weighted harmonic mean of the Paasche type sectoral constant technology constant price output growth rates $\lambda_P{}^{kt}$ divided by a weighted harmonic mean of the Paasche sectoral input growth rates $\beta_P{}^{kt}$. So, as was the case with the Laspeyres type index of returns to scale, the Paasche type definition of national returns to scale does capture the spirit of the sectoral definition of returns to scale as an output growth index using a constant technology divided by an input growth index. As usual, our preferred measure of national returns to scale is $\delta^t \equiv [\delta_L{}^t\,\delta_P{}^t]^{1/2}$.

The exact aggregate returns to scale decompositions into sectoral effects defined by (91) and (92) show that it is not necessarily the case that $\delta_L{}^t$ and $\delta_P{}^t$ will equal one if all of the sectoral technologies are subject to constant returns to scale, in which case all of the sectoral returns to scale measures are equal to unity. Recall that if each S^{kt} is a cone, then we have shown that $\delta_L{}^{kt} = \delta_P{}^{kt} = 1$ for k = 1,...,K and t = 2,...,T. Assuming that each S^{kt} is a cone, then examining equations (91), it can be seen that if $\sigma^{k,t-1} = \boldsymbol{S}^{k,t-1}$ for all k = 1,...,K and t = 2,...,T, then $\delta_L{}^{kt}$ will equal unity. Similarly, examining equations (92), it can be seen if $\sigma^{kt} = \boldsymbol{S}^{kt}$ for all k = 1,...,K and t = 2,...,T, then $\delta_P{}^{kt}$ will equal unity. Sufficient conditions that will ensure that the theoretical sector k value added shares of national value added σ^{kt} defined by (67) equal the corresponding observed sector k share of national primary input value S^{kt} defined by (90) are the following conditions:

$$(93)\ R^{kt}(p^{kt},w^{kt},x^{kt}) = w^{kt}\cdot x^{kt}\ ; \qquad k = 1,...,K;\ t = 1,...,T.$$

Equations (93) imply that maximal cost constrained net revenue for sector k in period t, $R^{kt}(p^{kt},w^{kt},x^{kt})$, is equal to observed sector k expenditures on primary inputs for period t, $w^{kt}\cdot x^{kt}$, for all k and t. Equations (93) are the usual sectoral adding up conditions that are implied by (efficient) competitive price taking behavior when there are constant returns to scale in production in each sector.

This completes our discussion of the decomposition of each of our aggregate explanatory factors into sectoral effects. As noted at the beginning of this section, we can simply apply the three exact decompositions of value added that we obtained in section 3 (namely (32), (33) and (34)) to the present national context and we obtain the following three exact decompositions of national value added growth into explanatory components for t = 2,...,T:

$$(94)\ v^t/v^{t-1} = \alpha_P{}^t\,\beta_L{}^t\,\gamma_{LPP}{}^t\,\delta_L{}^t\,\varepsilon^t\,\tau_L{}^t\ ;$$

(95) $v^t/v^{t-1} = \alpha_L{}^t \beta_P{}^t \gamma_{PLL}{}^t \delta_P{}^t \varepsilon^t \tau_P{}^t$;
(96) $v^t/v^{t-1} = \alpha^t \beta^t \gamma^t \delta^t \varepsilon^t \tau^t$.

We conclude this section by deriving a useful approximation to the exact decomposition defined by (96).

Recall the exact decompositions for $\alpha_L{}^t$ and $\alpha_P{}^t$ defined by (70) and (71). Approximate the theoretical sector k, period t−1 shares $\sigma^{k,t-1}$ in (70) by the observed sector k, period t−1 value added shares $s^{k,t-1}$ defined by (49) and approximate the theoretical sector k, period t shares σ^{kt} in (71) by the observed sector k, period t value added shares s^{kt}. Then approximate the resulting weighted arithmetic mean $\Sigma_{k=1}{}^K s^{k,t-1} \alpha_L{}^{kt}$ by the corresponding weighted geometric mean and approximate the resulting weighted harmonic mean $\{\Sigma_{k=1}{}^K s^{kt} [\alpha_P{}^{kt}]^{-1}\}^{-1}$ by the corresponding weighted geometric mean. Thus $\alpha^t \equiv [\alpha_L{}^t \alpha_P{}^t]^{1/2}$ can be approximated by $\alpha^{t\bullet}$ defined by (52) in the previous section.

We note that $\beta_L{}^t \equiv \Sigma_{k=1}{}^K \boldsymbol{S}^{k,t-1} \beta_L{}^t$. Approximate the sector k input shares $\boldsymbol{S}^{kt}$ by the corresponding output shares $s^{k,t-1}$ and then approximate the weighted arithmetic mean, $\Sigma_{k=1}{}^K s^{k,t-1} \beta_L{}^t$, by the corresponding weighted geometric mean. Similarly, $\beta_P{}^t \equiv [\Sigma_{k=1}{}^K \boldsymbol{S}^{kt} (\beta_P{}^t)^{-1}]^{-1}$. Approximate the sector k input shares $\boldsymbol{S}^{kt}$ by the corresponding sectoral value added output shares s^{kt} and then approximate the weighted harmonic mean, $[\Sigma_{k=1}{}^K s^{kt} (\beta_P{}^t)^{-1}]^{-1}$, by the corresponding weighted geometric mean. Thus $\beta^t \equiv [\beta_L{}^t \beta_P{}^t]^{1/2}$ can be approximated by $\beta^{t\bullet}$ defined by (53) in the previous section.

Recall the exact decompositions for $\gamma_{LPP}{}^t$ and $\gamma_{PLL}{}^t$ defined by (74) and (76). Approximate the theoretical sector k, period t shares $\sigma_{LPP}{}^{kt}$ in (74) by the observed sector k, period t−1 value added shares $s^{k,t-1}$ defined by (49) and approximate the theoretical sector k, period t shares $\sigma_{PLL}{}^{kt}$ in (76) by the observed sector k, period t value added shares s^{kt}. Then approximate the resulting weighted arithmetic mean $\Sigma_{k=1}{}^K s^{k,t-1} \gamma_{LPP}{}^{kt}$ by the corresponding weighted geometric mean and approximate the resulting weighted harmonic mean $\{\Sigma_{k=1}{}^K s^{kt} [\gamma_{LPP}{}^{kt}]^{-1}\}^{-1}$ by the corresponding weighted geometric mean. Thus $\gamma^t \equiv [\gamma_{LPP}{}^t \gamma_{PLL}{}^t]^{1/2}$ can be approximated by $\gamma^{t\bullet}$ defined by (54) in the previous section.

We will now approximate the third lines of equations (91) and (92) which provide exact decompositions for the Laspeyres and Paasche measures of national returns to scale effects. We have $\delta_L{}^t = \Sigma_{k=1}{}^K \sigma^{k,t-1} \lambda_L{}^{kt}/\Sigma_{k=1}{}^K \boldsymbol{S}^{k,t-1} \beta_L{}^{kt}$. Approximate the theoretical value added output shares for sector k and period t−1 in the numerator, $\sigma^{k,t-1}$, by the corresponding observed output shares, $s^{k,t-1}$, and then approximate the resulting weighted arithmetic mean, $\Sigma_{k=1}{}^K s^{k,t-1} \lambda_L{}^{kt}$, by the corresponding geometric mean. Approximate the observed value added input shares for sector k and period t−1 in the denominator, $\boldsymbol{S}^{k,t-1}$, by the corresponding observed output shares, $s^{k,t-1}$, and then approximate the resulting weighted arithmetic mean, $\Sigma_{k=1}{}^K s^{k,t-1} \beta_L{}^{kt}$, by the corresponding weighted geometric mean. Thus an approximation to the logarithm of $\delta_L{}^t$ is $\ln \delta_L{}^t \approx \Sigma_{k=1}{}^K s^{k,t-1} \ln \delta^{kt}$ where $\delta^{kt} \equiv \lambda^{kt}/\beta^{kt}$. Approximating the shares σ^{kt} and $\boldsymbol{S}^{kt}$ which appear in (92) by s^{kt} and applying similar geometric approximations to the harmonic means in the third line of (92), we can show that an approximation to the logarithm of δ_P^t is $\ln \delta_P^t \approx \Sigma_{k=1}^K s^{kt} \ln \delta^{kt}$. Thus $\delta^{t\bullet}$ defined by (55) in the previous section is an approximation to $\delta^t \equiv [\delta_L^t \delta_P^t]^{1/2}$.

Equation (69) above provided an exact expression for ε^t, the rate of growth of national value added efficiency: $\varepsilon^t = [\Sigma_{k=1}^K \sigma^{kt} e^{kt}]/[\Sigma_{k=1}^K \sigma^{k,t-1} e^{k,t-1}]$. Approximate the theoretical value added output shares σ^{kt} and $\sigma^{k,t-1}$ in the numerator and denominator of ε^t by the arithmetic average of their observed counterparts, $(1/2)(s^{kt} + s^{k,t-1})$.[40] We then have the following approximation for $\varepsilon^{t:}$

(97) $\varepsilon^t \approx [\Sigma_{k=1}^K (1/2)(s^{kt} + s^{k,t-1})e^{kt}]/[\Sigma_{k=1}^K (1/2)(s^{kt} + s^{k,t-1})e^{k,t-1}]$; $\qquad t = 2,...,T.$

Finally, approximate the weighted arithmetic means in the numerator and denominator of the right hand side of (97) by the corresponding weighted geometric means. The logarithm of ε^t is then approximately equal to the following expression:

(98) $\ln \varepsilon^t \approx \Sigma_{k=1}^K (1/2)(s^{kt} + s^{k,t-1})\ln e^{kt} - \Sigma_{k=1}^K (1/2)(s^{kt} + s^{k,t-1})\ln e^{k,t-1}$ $\qquad t = 2,...,T$
$= \Sigma_{k=1}^K (1/2)(s^{kt} + s^{k,t-1})\ln (e^{kt}/e^{k,t-1})$ $\qquad$ since $\varepsilon^{kt} \equiv e^{kt}/e^{k,t-1}$
$= \Sigma_{k=1}^K (1/2)(s^{kt} + s^{k,t-1})\ln \varepsilon^{kt}$
$= \ln \varepsilon^{t\bullet}$

where $\varepsilon^{t\bullet}$ was defined in the previous section by (56).

We obtained exact decompositions for τ_L^t and τ_P^t; see (80) and (82) above. Approximate the theoretical sector k, period t shares σ_L^{kt} in (80) by the observed sector k, period t−1 value added shares $s^{k,t-1}$ defined by (49) and approximate the theoretical sector k, period t shares σ_P^{kt} in (82) by the observed sector k, period t value added shares s^{kt}. Then approximate the resulting weighted arithmetic mean $\Sigma_{k=1}^K s^{k,t-1} \tau_L^t$ by the corresponding weighted geometric mean and approximate the resulting weighted harmonic mean $\{\Sigma_{k=1}^K s^{kt} [\tau_P^t]^{-1}\}^{-1}$ by the corresponding weighted geometric mean. Thus $\tau^t \equiv [\tau_L^t \tau_P^t]^{1/2}$ can be approximated by $\tau^{t\bullet}$ defined by (57) in the previous section.

We have provided approximations to all six national explanatory factors that explain the period t growth of national value added, so that we have the following approximation to the exact decomposition defined by (96):

(99) $v^t/v^{t-1} = \alpha^t \beta^t \gamma^t \delta^t \varepsilon^t \tau^t \approx \alpha^{t\bullet} \beta^{t\bullet} \gamma^{t\bullet} \delta^{t\bullet} \varepsilon^{t\bullet} \tau^{t\bullet}$; $\qquad t = 2,...,T$

where the last set of explanatory factors on the right hand side of (99) are defined by (52)-(57) in the previous section. Thus the approximate decomposition (99) that we derived in the present section coincides with the approximate decomposition (59) that we derived in the previous section.

We turn now to an empirical application of our methodology.

[40] This approximation is of lower quality than our previous approximations.

7. TFP Growth for the U.S. Corporate Nonfinancial Sector, 1960-2014

The US Bureau of Economic Analysis (BEA), in conjunction with the Bureau of Labor Statistics (BLS) and the Board of Governors of the Federal Reserve, have developed a new set of production accounts (the Integrated Macroeconomic Accounts or IMA) for two major private sectors of the US economy: the Corporate Nonfinancial Sector and the Noncorporate Nonfinancial Sector. The Balance Sheet Accounts in the IMA cover the years 1960-2014 but do not provide a decomposition of output, input and asset values into price and quantity components. Diewert and Fox (2016a) provided such a decomposition and we will use their data in this study.

In this section, we will use their output and input data for the U.S. Corporate Nonfinancial Sector (which we denote as Sector 1) for the 55 years 1960-2014. The year t output y^{1t} is real value added[41] and the corresponding year t value added deflator is denoted as p^{1t}. The ten inputs used by this sector are labour and the services of nine types of asset.[42] The output and input data are listed in Appendix A of Diewert and Fox (2016b). The year t input vector for this sector is $x^{1t} \equiv [x_1^{1t},x_2^{1t},...,x_{10}^{1t}]$ where x_1^{1t} is year t labour input measured in billions of 1960 dollars and $x_2^{1t},...,x_{10}^{1t}$ are capital service inputs measured in billions of 1960 capital stock dollars. The corresponding year t input price vector for Sector 1 is $w^{1t} \equiv [w_1^{1t},w_2^{1t},...,w_{10}^{1t}]$ for t = 1960,...,2014.

Our year t technology set for Sector 1, S^{1t}, is defined as the free disposal cone spanned by the observed output and input vectors for the sector up to and including the year t observation. For convenience, we label the years 1960-2014 as years 1-55 in definitions (100)-(104) below. Thus S^{1t} is defined as follows:

$$(100)\quad S^{1t} \equiv \{(y,x): y \leq \Sigma_{s=1}^{t} y^{1s} \lambda_s;\ x \geq \Sigma_{s=1}^{t} x^{1s} \lambda_s;\ \lambda_1 \geq 0, ..., \lambda_s \geq 0\}; \qquad t = 1,...,55.$$

We adapt definition (41) of section 4 to the present situation and define the *Sector 1 year t cost constrained value added function* $R^{1t}(p,w,x)$ for $p > 0$, $w >> 0_{10}$ and $x >> 0_{10}$ as follows:

$$\begin{aligned}(101)\quad R^{1t}(p,w,x) &\equiv \max_{y,z} \{py: (y,z) \in S^{1t};\ w \cdot z \leq w \cdot x\} \qquad t = 1,...,55\\ &= \max_{\lambda_1,...,\lambda_t} \{p(\Sigma_{s=1}^{t} y^{1s}\lambda_s)\ ;\ w \cdot (\Sigma_{s=1}^{t} x^{1s}\lambda_s) \leq w \cdot x\ ;\ \lambda_1 \geq 0 ,..., \lambda_t \geq 0\}\\ &= \max_s \{py^{1s}\, w \cdot x / w \cdot x^{1s} : s = 1,2,...,t\}\\ &= w \cdot x \max_s \{py^{1s}/w \cdot x^{1s} : s = 1,2,...,t\}.\end{aligned}$$

[41] There is only a single value added output for this sector. The published data for this sector did not allow Diewert and Fox (2016a) to decompose real value added into gross output and intermediate input components.

[42] The nine types of asset used in this sector and the corresponding input numbers are as follows: 2 = Equipment; 3 = Intellectual property products; 4 = Nonresidential structures; 5 = Residential structures; 6 = Residential land; 7 = Farm land; 8 = Commercial land; 9 = Beginning of year inventory stocks and 10 = Beginning of the year real holdings of currency and deposits. The prices are user costs that use predicted asset inflation rates rather than ex post inflation rates but balancing rates of return were used that make the value of input in each year equal to the corresponding value of output.

Using the cost constrained value added functions defined by (101), we can readily calculate the Sector 1 counterparts to the year t generic value added growth decompositions (32)-(33) that we derived in section 3 above. Using our present notation for the Sector 1 prices and quantities, these decompositions can be written as follows for t = 2,...,55:[43]

(102) $v^{1t}/v^{1,t-1} = p^{1t}y^{1t}/p^{1,t-1}y^{1,t-1} = \varepsilon^{1t}\,\alpha_P{}^{1t}\,\beta_L{}^{1t}\,\gamma_{LPP}{}^{1t}\,\tau_L{}^{1t}$;
(103) $v^{1t}/v^{1,t-1} = \varepsilon^{1t}\,\alpha_L{}^{1t}\,\beta_P{}^{1t}\,\gamma_{PLL}{}^{1t}\,\tau_P{}^{1t}$;
(104) $v^{1t}/v^{1,t-1} = \varepsilon^{1t}\,\alpha^{1t}\,\beta^{1t}\,\gamma^{1t}\,\tau^{1t}$.

As in section 3, we define year t Total Factor Productivity Growth for Sector 1 as value added growth divided by output price growth α^{1t} times input quantity growth β^{1t}:

(105) $TFPG^{1t} \equiv [v^{1t}/v^{1.t-1}]/[\alpha^{1t}\beta^{1t}] = \varepsilon^{1t}\,\gamma^{1t}\,\tau^{1t}$; t = 1961,...,2014.

Since we have only a single value added output, $\alpha^{1t} \equiv p^{1t}/p^{1,t-1}$ can be interpreted as a Fisher output price index and $[v^{1t}/v^{1.t-1}]/\alpha^{1t}$ can be interpreted as a Fisher output quantity index going from year t−1 to year t. β^{1t} is the Fisher input quantity index going from year t−1 to year t. Thus $TFPG^{1t}$ is equal to a conventional Fisher productivity growth index in this one output case.

The (one plus) growth factors for our Sector 1 that appear in the decomposition given by (105) are listed in Table 1 below. In addition, we list the cost constrained value added efficiency levels e^{1t} that are the Sector 1 counterparts to the e^t defined by (3).

---

[43] Since our Sector 1 technology sets are cones, our returns to scale explanatory factors are all equal to unity; i.e., $\delta_L{}^{1t} = \delta_P{}^{1t} = \delta^{1t} = 1$ for t =2,...,55. Thus these explanatory factors do not appear in the decompositions (102)-(104). Since there is only one output for Sector 1, we have $\alpha_L{}^{1t} = \alpha_P{}^{1t} = \alpha^{1t} = p^{1t}/p^{1,t-1}$.

Table 1: U.S. Corporate Nonfinancial Value Added Growth $v^{1t}/v^{1.t-1}$, Output Price Growth α^{1t}, Input Quantity Growth β^{1t}, TFP Growth $TFPG^{1t}$, Value Added Efficiency Growth ε^{1t}, Input Mix Growth Factors γ^{1t} and Technical Progress Growth Factors τ^{1t} and Value Added Efficiency Factors e^{1t}

Year t	$v^{1t}/v^{1,t-1}$	α^{1t}	β^{1t}	$TFPG^{1t}$	ε^{1t}	γ^{1t}	τ^{1t}	e^{1t}
1961	1.02696	1.00305	1.00469	1.01906	1.00000	1.00000	1.01906	1.00000
1962	1.09170	1.00647	1.03460	1.04841	1.00000	1.00000	1.04842	1.00000
1963	1.06692	1.00495	1.02435	1.03642	1.00000	1.00000	1.03641	1.00000
1964	1.08005	1.00921	1.02752	1.04153	1.00000	1.00000	1.04154	1.00000
1965	1.10313	1.01758	1.04519	1.03721	1.00000	1.00000	1.03722	1.00000
1966	1.10528	1.02920	1.05110	1.02171	1.00000	1.00000	1.02172	1.00000
1967	1.05161	1.02232	1.02739	1.00123	1.00000	1.00000	1.00123	1.00000
1968	1.09790	1.03102	1.03470	1.02914	1.00000	1.00000	1.02914	1.00000
1969	1.08381	1.04212	1.04058	0.99944	0.99950	0.99994	1.00000	0.99950
1970	1.02816	1.03715	0.99899	0.99233	0.99337	0.99894	1.00000	0.99287
1971	1.07692	1.03612	1.00738	1.03176	1.00718	1.00006	1.02436	1.00000
1972	1.11407	1.03557	1.04114	1.03330	1.00000	1.00000	1.03330	1.00000
1973	1.12293	1.05864	1.04779	1.01236	1.00000	1.00000	1.01235	1.00000
1974	1.08158	1.09825	1.01129	0.97383	0.97416	0.99966	1.00000	0.97416
1975	1.08256	1.09815	0.98241	1.00345	1.00312	1.00035	1.00000	0.97720
1976	1.13447	1.04863	1.03550	1.04476	1.02333	1.00061	1.02032	1.00000
1977	1.13447	1.05665	1.04253	1.02985	1.00000	1.00000	1.02985	1.00000
1978	1.14104	1.07144	1.05167	1.01263	1.00000	1.00000	1.01262	1.00000
1979	1.11671	1.08202	1.03863	0.99368	0.99367	1.00001	1.00000	0.99367
1980	1.08280	1.09350	1.00423	0.98604	0.98644	0.99960	1.00000	0.98019
1981	1.13011	1.08602	1.01929	1.02090	1.02021	1.00019	1.00048	1.00000
1982	1.03636	1.05950	0.98579	0.99226	0.99289	0.99937	1.00000	0.99289
1983	1.06824	1.01840	1.01903	1.02936	1.00716	1.00019	1.02185	1.00000
1984	1.12268	1.03086	1.04890	1.03830	1.00000	1.00000	1.03830	1.00000
1985	1.06489	1.01773	1.02518	1.02064	1.00000	1.00000	1.02064	1.00000
1986	1.04040	1.01396	1.01493	1.01098	1.00000	1.00000	1.01099	1.00000
1987	1.07299	1.01895	1.02831	1.02405	1.00000	1.00000	1.02404	1.00000
1988	1.08863	1.02562	1.02632	1.03421	1.00000	1.00000	1.03422	1.00000
1989	1.04995	1.03037	1.02506	0.99409	0.99408	1.00002	1.00000	0.99408
1990	1.04513	1.03022	1.00733	1.00709	1.00596	1.00003	1.00108	1.00000
1991	1.01671	1.02198	0.98427	1.01074	1.00000	1.00000	1.01073	1.00000
1992	1.04361	1.01273	1.00954	1.02075	1.00000	1.00000	1.02077	1.00000
1993	1.04598	1.02083	1.02097	1.00359	1.00000	1.00000	1.00360	1.00000
1994	1.07768	1.01521	1.03256	1.02806	1.00000	1.00000	1.02807	1.00000
1995	1.06258	1.01365	1.03237	1.01541	1.00000	1.00000	1.01541	1.00000
1996	1.06562	1.00663	1.02243	1.03537	1.00000	1.00000	1.03538	1.00000
1997	1.07519	1.00793	1.03774	1.02794	1.00000	1.00000	1.02795	1.00000
1998	1.05955	1.00264	1.02397	1.03203	1.00000	1.00000	1.03204	1.00000
1999	1.06140	1.00662	1.03224	1.02149	1.00000	1.00000	1.02148	1.00000
2000	1.06697	1.01154	1.02730	1.02676	1.00000	1.00000	1.02677	1.00000
2001	0.99271	1.01425	0.98433	0.99434	0.99529	0.99905	1.00000	0.99529
2002	1.00792	0.99938	0.98456	1.02437	1.00473	0.99999	1.01955	1.00000
2003	1.03213	1.01016	0.99016	1.03190	1.00000	1.00000	1.03189	1.00000
2004	1.06661	1.02069	1.00978	1.03488	1.00000	1.00000	1.03488	1.00000
2005	1.06847	1.03438	1.01215	1.02056	1.00000	1.00000	1.02057	1.00000
2006	1.07032	1.03063	1.01851	1.01964	1.00000	1.00000	1.01964	1.00000
2007	1.03033	1.02012	1.01042	0.99959	0.99969	0.99991	1.00000	0.99969
2008	1.00803	1.02121	0.99622	0.99084	0.99113	0.99970	1.00000	0.99082
2009	0.94412	1.01625	0.95309	0.97475	0.97620	0.99851	1.00000	0.96724
2010	1.05625	1.00079	1.00150	1.05384	1.03387	1.00112	1.01818	1.00000
2011	1.04785	1.02221	1.02168	1.00332	1.00000	1.00000	1.00332	1.00000
2012	1.05777	1.01674	1.02335	1.01661	1.00000	1.00000	1.01660	1.00000
2013	1.03693	1.00644	1.02162	1.00849	1.00000	1.00000	1.00849	1.00000
2014	1.04003	1.00803	1.02686	1.00476	1.00000	1.00000	1.00476	1.00000
Mean	1.06620	1.02880	1.01910	1.01700	1.00000	0.99995	1.01700	0.99736

It can be verified that the TFP growth decomposition defined by (105) holds; i.e., for each year t, nonparametric TFP growth $TFPG^t$ equals the product of value added efficiency growth ε^{1t} times the year t input mix growth factor γ^{1t} times the year t technical progress measure τ^{1t}. It can be seen that the input mix factors are all very close to one. It can also be seen when value added efficiency in year t, e^{1t}, is less than one, then the year t technical progress measure τ^{1t} always equals one so that there is no technical progress in years where the value added efficiency is less than one. Our nonparametric measure of technical progress τ^{1t} is always equal to or greater than one; i.e., our measure never indicates technological regress. Another important empirical regularity emerges from Table 1: since the input mix growth factors γ^{1t} are always very close to one, then when the year t value added efficiency growth factor ε^t is equal to one, our nonparametric measure of TFP growth, $TFPG^t$, is virtually equal to our year t measure of technical progress τ^{1t}. Finally, the last row of Table 1 lists the arithmetic averages of the various growth factors. It can be seen that the arithmetic average rate of TFP growth (and of technical progress) for Sector 1 is 1.70% per year which is a very high average rate of TFP growth over 55 years.

To conclude this section, apply the definitions (37)-(40) to our present Sector 1 in order to obtain the following levels decomposition for Total Factor Productivity in year t relative to the year 1960, TFP^{1t}:

(106) $TFP^{1t} = [v^{1t}/v^{1,1960}]/[A^{1t}\,B^{1t}] = C^{1t}\,E^{1t}\,T^{1t}$;   t = 1960,...,2014.

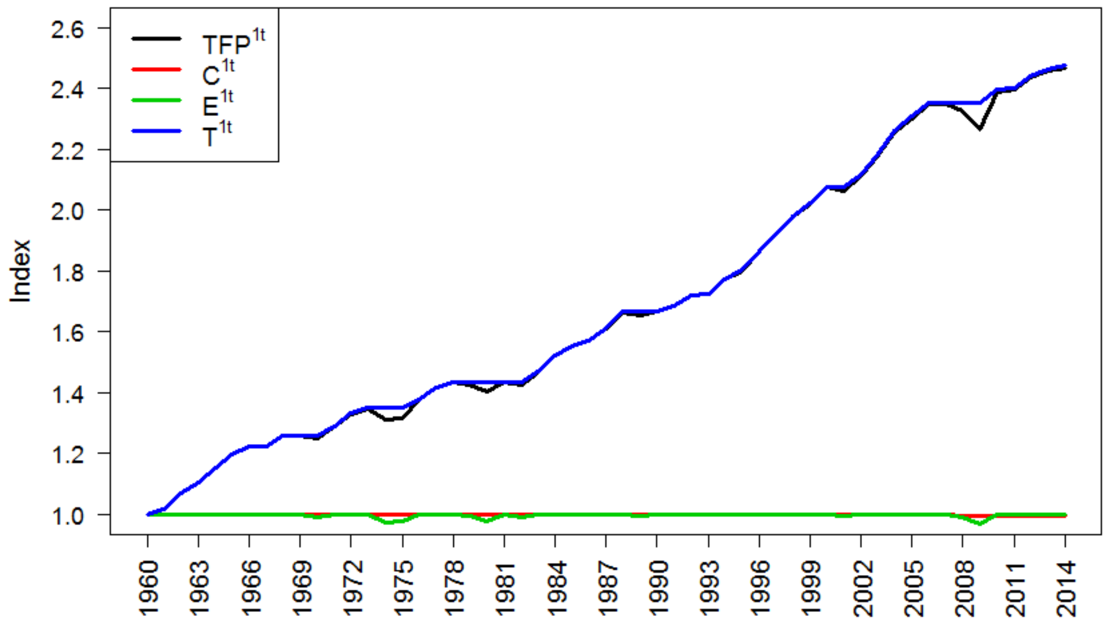
Table 2 lists the various levels that appear in (106). We note that the returns to scale level for Sector 1 in year t relative to 1960, D^{1t}, is identically equal to one and so it does not appear in the decomposition defined by (106).

Table 2: U.S. Corporate Nonfinancial Value Added Year t Levels $v^{1t}/v^{1.1960}$, Output Price Levels A^{1t}, Input Quantity Levels B^{1t}, TFP Levels TFP^{1t}, Input Mix Levels C^{1t}, Value Added Efficiency Levels E^{1t} and Technical Progress Levels T^{1t} where all Levels are Relative to 1960

Year t	$v^{1t}/v^{1,1960}$	A^{1t}	B^{1t}	TFP^{1t}	C^{1t}	E^{1t}	T^{1t}
1960	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1961	1.02696	1.00305	1.00469	1.01906	1.00000	1.00000	1.01906
1962	1.12113	1.00954	1.03945	1.06839	1.00000	1.00000	1.06840
1963	1.19615	1.01454	1.06476	1.10730	1.00000	1.00000	1.10730
1964	1.29190	1.02388	1.09407	1.15329	1.00000	1.00000	1.15330
1965	1.42514	1.04188	1.14350	1.19620	1.00000	1.00000	1.19622
1966	1.57518	1.07230	1.20194	1.22217	1.00000	1.00000	1.22220
1967	1.65648	1.09623	1.23486	1.22368	1.00000	1.00000	1.22370
1968	1.81864	1.13024	1.27771	1.25934	1.00000	1.00000	1.25937
1969	1.97106	1.17785	1.32956	1.25864	0.99994	0.99950	1.25937
1970	2.02656	1.22161	1.32822	1.24899	0.99888	0.99287	1.25937
1971	2.18245	1.26574	1.33802	1.28866	0.99894	1.00000	1.29004
1972	2.43141	1.31076	1.39306	1.33157	0.99894	1.00000	1.33300
1973	2.73032	1.38762	1.45963	1.34803	0.99894	1.00000	1.34947
1974	2.95307	1.52395	1.47611	1.31275	0.99860	0.97416	1.34947
1975	3.19687	1.67352	1.45016	1.31728	0.99894	0.97720	1.34947
1976	3.62675	1.75491	1.50164	1.37625	0.99955	1.00000	1.37689
1977	4.11443	1.85432	1.56550	1.41733	0.99955	1.00000	1.41799
1978	4.69473	1.98679	1.64640	1.43524	0.99955	1.00000	1.43590
1979	5.24266	2.14974	1.70999	1.42617	0.99956	0.99367	1.43590
1980	5.67676	2.35075	1.71722	1.40627	0.99916	0.98019	1.43590
1981	6.41535	2.55297	1.75035	1.43566	0.99935	1.00000	1.43659
1982	6.64861	2.70487	1.72548	1.42454	0.99872	0.99289	1.43659
1983	7.10233	2.75464	1.75831	1.46636	0.99891	1.00000	1.46798
1984	7.97366	2.83966	1.84429	1.52252	0.99891	1.00000	1.52421
1985	8.49111	2.89001	1.89073	1.55395	0.99891	1.00000	1.55567
1986	8.83418	2.93035	1.91896	1.57102	0.99891	1.00000	1.57276
1987	9.47898	2.98587	1.97328	1.60880	0.99891	1.00000	1.61057
1988	10.31908	3.06238	2.02521	1.66384	0.99891	1.00000	1.66569
1989	10.83452	3.15539	2.07596	1.65401	0.99893	0.99408	1.66569
1990	11.32345	3.25074	2.09118	1.66573	0.99896	1.00000	1.66748
1991	11.51263	3.32219	2.05828	1.68363	0.99896	1.00000	1.68537
1992	12.01468	3.36449	2.07791	1.71857	0.99896	1.00000	1.72038
1993	12.56706	3.43456	2.12149	1.72473	0.99896	1.00000	1.72656
1994	13.54323	3.48679	2.19055	1.77314	0.99896	1.00000	1.77502
1995	14.39079	3.53437	2.26146	1.80046	0.99896	1.00000	1.80238
1996	15.33507	3.55779	2.31219	1.86415	0.99896	1.00000	1.86615
1997	16.48815	3.58599	2.39946	1.91624	0.99896	1.00000	1.91830
1998	17.47003	3.59545	2.45697	1.97761	0.99896	1.00000	1.97976
1999	18.54264	3.61926	2.53617	2.02010	0.99896	1.00000	2.02229
2000	19.78448	3.66103	2.60542	2.07417	0.99896	1.00000	2.07642
2001	19.64021	3.71320	2.56459	2.06243	0.99802	0.99529	2.07642
2002	19.79580	3.71089	2.52498	2.11269	0.99801	1.00000	2.11702
2003	20.43174	3.74861	2.50013	2.18008	0.99801	1.00000	2.18453
2004	21.79279	3.82615	2.52457	2.25612	0.99801	1.00000	2.26072
2005	23.28502	3.95768	2.55525	2.30252	0.99801	1.00000	2.30721
2006	24.92242	4.07891	2.60254	2.34773	0.99801	1.00000	2.35252
2007	25.67843	4.16098	2.62966	2.34678	0.99791	0.99969	2.35252
2008	25.88458	4.24922	2.61973	2.32528	0.99762	0.99082	2.35252
2009	24.43813	4.31825	2.49684	2.26658	0.99613	0.96724	2.35252
2010	25.81284	4.32164	2.50058	2.38861	0.99725	1.00000	2.39529
2011	27.04786	4.41763	2.55480	2.39655	0.99725	1.00000	2.40323
2012	28.61031	4.49159	2.61446	2.43635	0.99725	1.00000	2.44312
2013	29.66699	4.52053	2.67099	2.45703	0.99725	1.00000	2.46386
2014	30.85463	4.55684	2.74273	2.46873	0.99725	1.00000	2.47559

Note that the final level of TFP in 2014, 2.46873, is slightly less than the level of technology in 2014, which was 2.47559. This small difference is explained by the fact that the cumulative input mix level, 0.99725, is slightly less than 1 in 2014. We plot TFP^{1t}, C^{1t}, E^{1t} and T^{1t} in Figure 1.

Figure 1: Sector 1 Level of TFP, Input Mix, Value Added Efficiency and Technology



It can be seen that there was a substantial decline in value added efficiency over the years 2006-2009 and in fact, TFP has grown at a slower than average rate since 2006. The level of TFP also fell in the 1974, 1979, 1982, 1989 and 2001 recessions when efficiency growth dipped below one. However, on the whole, TFP growth in the U.S. Corporate Nonfinancial Sector has been satisfactory.

We turn now to an analysis of the productivity performance of the U.S. Noncorporate Nonfinancial Sector.

8. TFP Growth for the U.S. Noncorporate Nonfinancial Sector, 1960-2014

In this section, we use the Diewert and Fox (2016b) output and input data for the U.S. Noncorporate Nonfinancial Sector (which we denote as Sector 2) for the 55 years 1960-2014. The year t output y^{2t} is real value added for this sector and the corresponding year t value added deflator is denoted as p^{2t}. The 15 inputs used by this sector are labour and the services of 14 types of asset.[44] The output and input data are listed in Appendix A of

[44] The 14 types of asset used in this sector and the corresponding input numbers are as follows: 2 = Equipment held by sole proprietors; 3 = Equipment held by partners; 4 = Equipment held by cooperatives; 5 = Intellectual property products held by sole proprietors; 6 = Intellectual property products held by partners; 7 = Nonresidential structures held by sole proprietors; 8 = Nonresidential structures held by partners; 9 = Nonresidential structures held by cooperatives; ; 10 = Residential structures held by the noncorporate nonfinancial sector; 11 = Residential land held by the noncorporate nonfinancial sector; 12 =

Diewert and Fox (2016b). The year t input vector for this sector is $x^{2t} \equiv [x_1^{2t},x_2^{2t},...,x_{15}^{2t}]$ where x_1^{2t} is year t labour input measured in billions of 1960 dollars and $x_2^{2t},...,x_{15}^{2t}$ are capital service inputs measured in billions of 1960 capital stock dollars. The corresponding year t input price vector for sector 2 is $w^{2t} \equiv [w_1^{2t},w_2^{2t},...,w_{15}^{2t}]$ for t = 1960,...,2014.

Our year t technology set for Sector 2, S^{2t}, is defined as the free disposal cone spanned by the observed output and input vectors for sector 2 up to and including the year t observation. Again for convenience, we label the years 1960-2014 as years 1-55 in equations (107)-(108) below. Thus S^{2t} is defined as follows:

$$(107)\ S^{2t} \equiv \{(y,x): y \le \Sigma_{s=1}^{t} y^{2s} \lambda_s;\ x \ge \Sigma_{s=1}^{t} x^{2s} \lambda_s;\ \lambda_1 \ge 0, ..., \lambda_s \ge 0\}; \qquad t = 1,...,55.$$

We adapt definition (41) in section 4 to the present situation and define the *Sector 2 year t cost constrained value added function* $R^{2t}(p,w,x)$ for $p > 0$, $w >> 0_{15}$ and $x >> 0_{15}$ as follows:

$$\begin{aligned}(108)\ R^{2t}(p,w,x) &\equiv \max_{y,z} \{py: (y,z)\in S^{2t};\ w\cdot z \le w\cdot x\} \qquad t = 1,...,55\\ &= \max_{\lambda_1,...,\lambda_t} \{p(\Sigma_{s=1}^{t} y^{2s}\lambda_s)\ ;\ w\cdot(\Sigma_{s=1}^{t} x^{2s}\lambda_s) \le w\cdot x\ ;\ \lambda_1 \ge 0\ ,...,\ \lambda_t \ge 0\}\\ &= \max_s \{py^{2s}\ w\cdot x/w\cdot x^{2s} : s = 1,2,...,t\}\\ &= w\cdot x \max_s \{py^{2s}/w\cdot x^{2s} : s = 1,2,...,t\}.\end{aligned}$$

Using the cost constrained value added functions defined by (108), we can readily calculate the Sector 2 counterparts to the year t generic value added growth decompositions (32)-(33) that we derived in section 3 above. Using our present notation for the Sector 2 prices and quantities, these decompositions can be written in the same manner as (102)-(104) except that the superscript 2 replaces the superscript 1 in these decompositions of Sector 2 value added. As in section 3, we define year t Total Factor Productivity Growth for Sector 2 as value added growth divided by output price growth α^{2t} times input quantity growth β^{2t}, which leads to the following year t decomposition of TFP growth for Sector 2:[45]

$$(109)\ TFPG^{2t} \equiv [v^{2t}/v^{2.t-1}]/[\alpha^{2t}\beta^{2t}] = \varepsilon^{2t}\ \gamma^{2t}\ \tau^{2t}\ ; \qquad t = 1961,...,2014.$$

Since we have only a single value added output, $\alpha^{2t} \equiv p^{2t}/p^{2,t-1}$ can be interpreted as a Fisher output price index and $[v^{2t}/v^{2.t-1}]/\alpha^{2t}$ can be interpreted as a Fisher output quantity index going from year t−1 to year t. β^{2t} is the Fisher input quantity index going from year t−1 to year t for Sector 2. Thus $TFPG^{2t}$ is equal to a conventional Fisher productivity growth index in this one output case.

Farm land held by the noncorporate nonfinancial sector; 13 = Commercial land held by noncorporate nonfinancial sector; 14 = Beginning of the year inventories held by the noncorporate nonfinancial sector and 15 = Beginning of the year real holdings of currency and deposits by noncorporate nonfinancial sector..

[45] The returns to scale measures δ^{2t} for Sector 2 are all equal to one and thus these growth factors do not appear in (109).

Table 3: U.S. Noncorporate Nonfinancial Value Added Growth $v^{2t}/v^{2,t-1}$, Output Price Growth α^{2t}, Input Quantity Growth β^{2t}, TFP Growth $TFPG^{2t}$, Value Added Efficiency Growth Factors ε^{2t}, Input Mix Growth Factors γ^{2t} and Technical Progress Growth Factors τ^{2t} and Value Added Efficiency Factors e^{2t}

Year t	$v^{2t}/v^{2,t-1}$	α^{2t}	β^{2t}	$TFPG^{2t}$	ε^{2t}	γ^{2t}	τ^{2t}	e^{2t}
1961	1.02611	1.01608	0.98336	1.02696	1.00000	1.00000	1.02695	1.00000
1962	1.03628	1.01432	0.99047	1.03148	1.00000	1.00000	1.03149	1.00000
1963	1.02537	1.01138	0.98980	1.02428	1.00000	1.00000	1.02426	1.00000
1964	1.05036	1.01681	1.00129	1.03166	1.00000	1.00000	1.03169	1.00000
1965	1.05693	1.01991	0.99748	1.03891	1.00000	1.00000	1.03889	1.00000
1966	1.06541	1.03070	1.00046	1.03319	1.00000	1.00000	1.03320	1.00000
1967	1.02600	1.03158	0.99474	0.99984	1.00000	0.99976	1.00003	1.00000
1968	1.05414	1.04532	0.99242	1.01614	1.00000	1.00000	1.01619	1.00000
1969	1.05337	1.04377	1.00590	1.00327	1.00000	1.00000	1.00330	1.00000
1970	1.03555	1.03923	0.99661	0.99986	1.00000	0.99959	1.00023	1.00000
1971	1.05931	1.04708	0.99649	1.01525	1.00000	1.00000	1.01529	1.00000
1972	1.10456	1.04810	1.01373	1.03960	1.00000	1.00000	1.03965	1.00000
1973	1.16895	1.04374	1.03526	1.08181	1.00000	1.00000	1.08176	1.00000
1974	1.05229	1.09421	1.02312	0.93996	0.94041	0.99956	1.00000	0.94041
1975	1.07140	1.10309	0.99317	0.97796	0.97815	0.99985	1.00000	0.91986
1976	1.09366	1.08706	1.00684	0.99924	0.99895	1.00030	1.00000	0.91889
1977	1.09102	1.08397	1.01453	0.99208	0.99188	1.00024	1.00000	0.91143
1978	1.13298	1.07146	1.02691	1.02970	1.02912	1.00063	1.00000	0.93797
1979	1.11623	1.11798	1.02918	0.97012	0.97030	0.99984	1.00000	0.91011
1980	1.05125	1.06681	1.01524	0.97062	0.97138	0.99923	1.00000	0.88406
1981	1.08929	1.09691	1.00518	0.98793	0.98828	0.99960	1.00000	0.87370
1982	1.04364	1.06451	1.00908	0.97157	0.97353	0.99798	1.00000	0.85057
1983	1.05757	1.07541	1.01796	0.96606	0.96782	0.99826	1.00000	0.82319
1984	1.15730	1.01282	1.02387	1.11602	1.11439	1.00148	1.00000	0.91736
1985	1.07921	1.04855	1.01081	1.01823	1.01809	1.00017	1.00000	0.93396
1986	1.05984	1.01768	1.01298	1.02808	1.03079	0.99734	1.00000	0.96272
1987	1.04901	1.04904	1.01400	0.98616	0.98582	1.00038	1.00000	0.94906
1988	1.08921	1.04848	1.01531	1.02319	1.02319	1.00004	1.00000	0.97107
1989	1.06461	1.05355	1.02123	0.98950	0.99138	0.99813	1.00000	0.96269
1990	1.04318	1.04174	1.01110	0.99039	0.99286	0.99751	1.00000	0.95582
1991	1.00925	1.04153	1.00458	0.96459	0.96850	0.99596	1.00000	0.92571
1992	1.06726	1.01569	0.98668	1.06495	1.06571	0.99930	1.00000	0.98654
1993	1.03876	1.02356	1.02364	0.99141	0.99190	0.99947	1.00000	0.97855
1994	1.05278	1.01154	1.01235	1.02807	1.02192	0.99940	1.00665	1.00000
1995	1.04291	1.04735	1.00941	0.98649	0.98648	0.99999	1.00000	0.98648
1996	1.07813	1.05142	1.00882	1.01645	1.01370	0.99997	1.00272	1.00000
1997	1.06300	1.03341	1.01934	1.00912	1.00000	1.00000	1.00916	1.00000
1998	1.08134	1.02197	1.00929	1.04836	1.00000	1.00000	1.04834	1.00000
1999	1.06748	1.01844	1.00857	1.03925	1.00000	1.00000	1.03922	1.00000
2000	1.08263	1.05231	1.01848	1.01015	1.00000	1.00000	1.01012	1.00000
2001	1.15232	1.04426	1.08093	1.02086	1.00000	1.00000	1.02087	1.00000
2002	1.04271	1.00449	1.01851	1.01918	1.00000	1.00000	1.01918	1.00000
2003	1.05478	1.00840	1.02963	1.01589	1.00000	1.00000	1.01587	1.00000
2004	1.08508	1.01994	1.03304	1.02984	1.00000	1.00000	1.02985	1.00000
2005	1.06903	1.02104	1.03269	1.01386	1.00000	1.00000	1.01387	1.00000
2006	1.09790	1.01809	1.03878	1.03814	1.00000	1.00000	1.03815	1.00000
2007	1.02757	1.02451	1.02994	0.97382	0.97366	1.00015	1.00000	0.97366
2008	1.05018	1.00395	1.00167	1.04431	1.02705	0.99990	1.01689	1.00000
2009	0.93797	0.98063	0.98224	0.97379	0.97479	0.99898	1.00000	0.97479
2010	1.03208	1.03383	0.99483	1.00349	1.00318	1.00033	1.00000	0.97788
2011	1.08243	1.01786	1.00189	1.06143	1.02262	1.00079	1.03713	1.00000
2012	1.05758	1.02064	1.01541	1.02047	1.00000	1.00000	1.02049	1.00000
2013	1.03553	1.02191	1.01072	1.00258	1.00000	1.00000	1.00261	1.00000
2014	1.04452	1.02358	1.01703	1.00338	1.00000	1.00000	1.00340	1.00000
Mean	1.06400	1.03890	1.01180	1.01260	0.99971	1.00030	1.01250	0.97086

The growth factors for our Sector 2 that appear in the decomposition given by (109) are listed in Table 3. In addition, we list the cost constrained value added efficiency levels e^{2t} that are the Sector 2 counterparts to the e^t defined by (3).

It can be verified that the TFP growth decomposition defined by (109) holds; i.e., for each year t, nonparametric TFP growth in Sector 2, $TFPG^{2t}$, equals the product of value added efficiency growth ε^{2t} times the year t input mix growth factor γ^{2t} times the year t technical progress measure τ^{2t}. The arithmetic average rate of TFP growth for Sector 2 was 1.26% per year, which is well below the 1.70% per year rate of TFP growth for Sector 1, but is still quite good. As was the case with Sector 1, the Sector 2 input mix growth factors are all close to one and hence are not a significant determinant of TFP growth for the Noncorporate Nonfinancial Sector of the U.S. economy. Again, we see that when the year t efficiency factor e^{2t} is below one, then the year t rate of technological change τ^{2t} is equal to one. Moreover, the rate of technological change τ^{2t} is always greater than or equal to one. What is very surprising is the very large number of years where value added efficiency e^{2t} is below unity, indicating that Sector 2 is operating well within the production frontier during those years.[46] The mean level of the value added efficiency factors is equal to 0.97086. Compare this very low average level of efficiency with the corresponding average level of efficiency for Sector 1, which was 0.99736.[47] Nevertheless, we see that the average rate of TFP growth for Sector 2 was 1.26% per year which is very close to the average rate of technical progress for Sector 2, which was 1.25% per year.

To conclude this section, apply definitions (37)-(40) to our present Sector 2 in order to obtain the following levels decomposition for Total Factor Productivity in year t relative to the year 1960, TFP^{1t}:

(110) $TFP^{2t} = [v^{2t}/v^{2,1960}]/[A^{2t}\,B^{2t}] = C^{2t}\,E^{2t}\,T^{2t}$; t = 1960,...,2014.

Table 4 lists the various levels that appear in (110). Again, we note that the returns to scale level for Sector 2 in year t relative to 1960, D^{2t}, is identically equal to one and so it does not appear in the decomposition defined by (110).

---

[46] Recall that as we our approach rules out technical regress, loss of efficiency is gross loss of efficiency less any technical progress that occurs during recession years. Hence estimates of efficiency loss are a bit too low in magnitude, and our estimates of technical progress are biased downward.

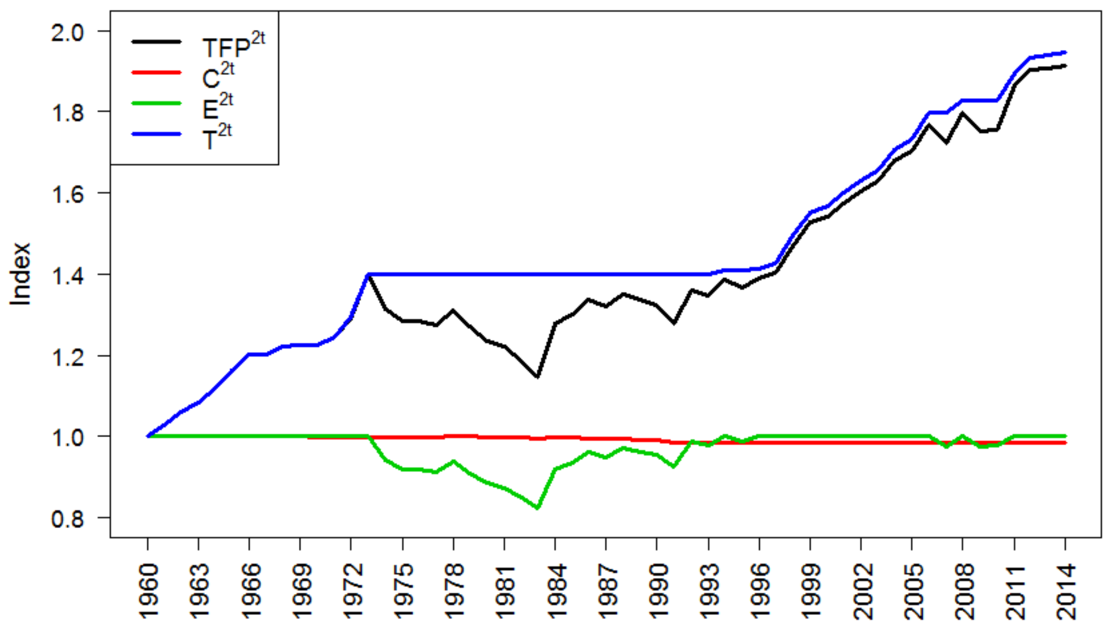
[47] The efficiency level e^{2t} was below unity for the years 1974-1993, 1995, 2007, 2009 and 2010, which is a total of 24 years. A possible explanation for the long stretch of inefficient years 1974-1993 is that Sector 2 uses a high proportion of structures and land to produce its outputs and there may have been a boom in these investments prior to 1974. Once the recession of 1974 occurred, these relatively fixed inputs could not be contracted in line with the outputs produced by this sector, leading to the long string of inefficient years. An alternative explanation is that there are measurement errors in our data for Sector 2.

Table 4: U.S. Noncorporate Nonfinancial Value Added Year t Levels $v^{2t}/v^{2.1960}$, Output Price Levels A^{2t}, Input Quantity Levels B^{2t}, TFP Levels TFP^{2t}, Input Mix Levels C^{2t}, Value Added Efficiency Levels E^{2t} and Technical Progress Levels T^{2t} where all Levels are Relative to 1960

Year t	$v^{2t}/v^{2,1960}$	A^{2t}	B^{2t}	TFP^{2t}	C^{2t}	E^{2t}	T^{2t}
1960	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
1961	1.02611	1.01608	0.98336	1.02696	1.00000	1.00000	1.02695
1962	1.06334	1.03063	0.97399	1.05928	1.00000	1.00000	1.05930
1963	1.09031	1.04236	0.96406	1.08500	1.00000	1.00000	1.08499
1964	1.14522	1.05988	0.96531	1.11935	1.00000	1.00000	1.11938
1965	1.21041	1.08098	0.96288	1.16291	1.00000	1.00000	1.16291
1966	1.28958	1.11417	0.96332	1.20151	1.00000	1.00000	1.20152
1967	1.32310	1.14936	0.95825	1.20132	0.99976	1.00000	1.20156
1968	1.39474	1.20145	0.95099	1.22070	0.99976	1.00000	1.22102
1969	1.46917	1.25404	0.95660	1.22470	0.99976	1.00000	1.22505
1970	1.52140	1.30323	0.95336	1.22452	0.99935	1.00000	1.22532
1971	1.61164	1.36458	0.95001	1.24320	0.99935	1.00000	1.24406
1972	1.78014	1.43021	0.96305	1.29243	0.99935	1.00000	1.29339
1973	2.08089	1.49277	0.99701	1.39816	0.99935	1.00000	1.39913
1974	2.18970	1.63340	1.02006	1.31422	0.99891	0.94041	1.39913
1975	2.34606	1.80178	1.01309	1.28526	0.99877	0.91986	1.39913
1976	2.56580	1.95864	1.02002	1.28428	0.99906	0.91889	1.39913
1977	2.79932	2.12311	1.03484	1.27411	0.99930	0.91143	1.39913
1978	3.17157	2.27483	1.06269	1.31196	0.99993	0.93797	1.39913
1979	3.54018	2.54321	1.09370	1.27276	0.99977	0.91011	1.39913
1980	3.72163	2.71313	1.11037	1.23536	0.99900	0.88406	1.39913
1981	4.05393	2.97606	1.11613	1.22045	0.99861	0.87370	1.39913
1982	4.23086	3.16805	1.12627	1.18576	0.99659	0.85057	1.39913
1983	4.47442	3.40696	1.14649	1.14551	0.99485	0.82319	1.39913
1984	5.17824	3.45063	1.17385	1.27841	0.99632	0.91736	1.39913
1985	5.58841	3.61816	1.18655	1.30171	0.99649	0.93396	1.39913
1986	5.92280	3.68213	1.20195	1.33826	0.99385	0.96272	1.39913
1987	6.21307	3.86270	1.21878	1.31974	0.99422	0.94906	1.39913
1988	6.76733	4.04995	1.23744	1.35034	0.99426	0.97107	1.39913
1989	7.20459	4.26681	1.26371	1.33616	0.99240	0.96269	1.39913
1990	7.51567	4.44489	1.27773	1.32333	0.98993	0.95582	1.39913
1991	7.58523	4.62950	1.28359	1.27647	0.98593	0.92571	1.39913
1992	8.09540	4.70216	1.26649	1.35937	0.98524	0.98654	1.39913
1993	8.40916	4.81293	1.29644	1.34770	0.98472	0.97855	1.39913
1994	8.85297	4.86845	1.31245	1.38553	0.98413	1.00000	1.40844
1995	9.23288	5.09895	1.32480	1.36681	0.98412	0.98648	1.40844
1996	9.95424	5.36112	1.33647	1.38929	0.98410	1.00000	1.41227
1997	10.58131	5.54023	1.36232	1.40195	0.98410	1.00000	1.42521
1998	11.44201	5.66193	1.37498	1.46974	0.98410	1.00000	1.49411
1999	12.21415	5.76635	1.38676	1.52743	0.98410	1.00000	1.55271
2000	13.22343	6.06797	1.41238	1.54294	0.98410	1.00000	1.56842
2001	15.23758	6.33652	1.52669	1.57512	0.98410	1.00000	1.60115
2002	15.88833	6.36495	1.55495	1.60534	0.98410	1.00000	1.63186
2003	16.75867	6.41842	1.60103	1.63085	0.98410	1.00000	1.65776
2004	18.18448	6.54641	1.65393	1.67950	0.98410	1.00000	1.70725
2005	19.43982	6.68416	1.70799	1.70279	0.98410	1.00000	1.73094
2006	21.34295	6.80506	1.77422	1.76773	0.98410	1.00000	1.79697
2007	21.93133	6.97187	1.82734	1.72145	0.98425	0.97366	1.79697
2008	23.03184	6.99939	1.83040	1.79772	0.98415	1.00000	1.82732
2009	21.60307	6.86380	1.79789	1.75060	0.98315	0.97479	1.82732
2010	22.29602	7.09602	1.78859	1.75672	0.98347	0.97788	1.82732
2011	24.13380	7.22277	1.79196	1.86463	0.98424	1.00000	1.89516
2012	25.52348	7.37188	1.81957	1.90280	0.98424	1.00000	1.93399
2013	26.43038	7.53339	1.83908	1.90771	0.98424	1.00000	1.93904
2014	27.60715	7.71100	1.87039	1.91416	0.98424	1.00000	1.94563

Note that the final level of TFP for Sector 2 in 2014, 1.91416, is somewhat less than the level of technology in 2014, which was 1.94563. This small difference is explained by the fact that the cumulative input mix level, 0.98424, is 1.5% less than 1 in 2014.[48] Note also that the final level of TFP in Sector 2, 1.91416, is much lower than the final level of TFP for Sector 1, which was 2.46873. We plot TFP^{2t}, C^{2t}, E^{2t} and T^{2t} in Figure 2.

Figure 2: Sector 2 Level of TFP, Input Mix, Value Added Efficiency and Technology



It can be seen that the loss of value added efficiency in Sector 2 was massive over the 20 years 1974-1993 and this loss of efficiency dragged down the level of Sector 2 TFP over these years. However, TFP growth resumed in 1994 and was excellent until 2006 when TFP growth again stalled with the exception of two good years of growth in 2011 and 2012.

It can be seen that our nonparametric methodology provides a useful supplement to traditional index number methods for calculating TFP growth. It illustrates the adverse influence of recessions when output falls but inputs cannot be adjusted optimally due to the fixity of many capital stock (and labour) components of aggregate input. Under these circumstances, production takes place in the interior of the production possibilities set and for Sector 2, the resulting waste of resources was substantial.[49]

In the following two sections, we will apply the aggregation over sectors methodology that we developed in sections 5 and 6.

---

[48] For most observations, γ^{2t} is only slightly less than one. But over time, the product of these γ^{2t} cumulate to 0.984 which is significantly below one.

[49] We note that our empirical results in this section and the previous one, which use the cost restricted value added function, are very similar to our previous results for these sectors in Diewert and Fox (2016b), which used a cost function approach. However, our previous approach relied on the fact that we had only a single output in each sector. Our present approach is preferred if there are many sectoral outputs.

9. Empirical Results Using the Sectoral Weighted Average Approach

In this section, we implement the approximate national value added decomposition approach that was explained in section 5.

Let $v^t \equiv v^{1t} + v^{2t}$ denote "national" value added and define the sector k share of national value added by $s^{kt} \equiv v^{kt}/v^t$ for k = 1,2. The Sector 1 year t shares of value added are listed in Table 5 below.[50] The year t sector k explanatory growth factors, α^{kt}, β^{kt}, γ^{kt}, ε^{kt} and τ^{kt}, are listed in Tables 1 and 3 above. Now apply definitions (52)-(57) in order to define the *year t national growth factors*, $\alpha^{t\bullet}$, $\beta^{t\bullet}$, $\gamma^{t\bullet}$, $\varepsilon^{t\bullet}$ and $\tau^{t\bullet}$. These national growth factors are listed in Table 5. Define the national year t TFP growth factor as $TFPG^t \equiv [v^t/v^{t-1}]/\alpha^{t\bullet}\,\beta^{t\bullet}$. The approximate TFP growth decomposition defined by (60) becomes:

$$(111)\ TFPG^{t\bullet} \equiv [v^t/v^{t-1}]/\alpha^{t\bullet}\,\beta^{t\bullet} \approx \gamma^{t\bullet}\,\varepsilon^{t\bullet}\,\tau^{t\bullet}\,; \qquad t = 2,...,T.$$

The national returns to scale growth factor, $\delta^{t\bullet}$, which appeared in (60) is not present in (111) because the sectoral growth factors, δ^{1t} and δ^{2t}, are all equal to unity due to the fact that our year t production possibilities sets, S^{1t} and S^{2t}, are cones. The growth decomposition components that appear in (111) are listed in Table 5.

The arithmetic means of the growth rates over the 54 years 1961-2014 are listed in the last row of Table 5. It can be seen that the average rate of national TFP growth over these years was 1.60 percent per year, which is equal to the average rate of technical progress. Technical progress growth was not present for 8 years: 1974, 1975, 1979, 1980, 1982, 1989, 2007 and 2009. For those years, the rate of growth of value added efficiency was below unity and this translated into negative rates of TFP growth for those years. The national approximate input mix growth factors, the $\gamma^{t\bullet}$, are all very close to unity. The approximate equality in (111) was very close to being an equality: the absolute value of the difference between $TFPG^{t\bullet}$ and $\gamma^{t\bullet}\,\varepsilon^{t\bullet}\,\tau^{t\bullet}$ was always less than 0.00003 and the mean difference was -0.0000034. Thus for our particular data set, the approximate decomposition of national TFP growth seems to be quite satisfactory.

To conclude this section, apply definitions (37)-(40) to national value added in order to obtain the following levels decomposition for approximate national Total Factor Productivity in year t relative to the year 1960, $TFP^{t\bullet}$:

$$(112)\ TFP^{t\bullet} = [v^t/v^{1960}]/[A^{t\bullet}\,B^{t\bullet}] \approx C^{t\bullet}\,E^{t\bullet}\,T^{t\bullet}\,; \qquad t = 1960,...,2014.$$

Table 6 lists the various levels that appear in (112). We note that the returns to scale national factor in year t relative to 1960, $D^{t\bullet}$, is identically equal to one and so it does not appear in the decomposition defined by (112).

[50] Of course, $s^{2t} \equiv 1 - s^{1t}$ for each year t.

Table 5: U.S. National Nonfinancial Value Added Growth v^t/v^{t-1}, Output Price Growth $\alpha^{t\bullet}$, Input Quantity Growth $\beta^{t\bullet}$, TFP Growth $TFPG^{t\bullet}$, Input Mix Growth Factors $\gamma^{t\bullet}$, Value Added Efficiency Growth Factors $\varepsilon^{t\bullet}$ and Technical Progress Growth Factors $\tau^{t\bullet}$ and Sector 1 Shares of National Value Added s^{1t}

Year t	v^t/v^{t-1}	$\alpha^{t\bullet}$	$\beta^{t\bullet}$	$TFPG^{t\bullet}$	$\gamma^{t\bullet}$	$\varepsilon^{t\bullet}$	$\tau^{t\bullet}$	s^{1t}
1960	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.70438
1961	1.02671	1.00688	0.99834	1.02139	1.00000	1.00000	1.02139	0.70455
1962	1.07533	1.00874	1.02160	1.04347	1.00000	1.00000	1.04348	0.71528
1963	1.05509	1.00675	1.01453	1.03300	1.00000	1.00000	1.03298	0.72330
1964	1.07183	1.01129	1.02027	1.03882	1.00000	1.00000	1.03883	0.72884
1965	1.09060	1.01820	1.03223	1.03766	1.00000	1.00000	1.03767	0.73721
1966	1.09480	1.02959	1.03773	1.02468	1.00000	1.00000	1.02468	0.74427
1967	1.04506	1.02466	1.01902	1.00088	0.99994	1.00000	1.00093	0.74893
1968	1.08691	1.03454	1.02408	1.02592	1.00000	1.00000	1.02592	0.75650
1969	1.07640	1.04252	1.03212	1.00037	0.99995	0.99962	1.00079	0.76171
1970	1.02992	1.03765	0.99842	0.99412	0.99910	0.99495	1.00005	0.76041
1971	1.07270	1.03872	1.00478	1.02781	1.00005	1.00547	1.02219	0.76340
1972	1.11182	1.03851	1.03461	1.03478	1.00000	1.00000	1.03479	0.76495
1973	1.13375	1.05506	1.04479	1.02851	1.00000	1.00000	1.02850	0.75765
1974	1.07448	1.09728	1.01411	0.96559	0.99964	0.96596	1.00000	0.76265
1975	1.07991	1.09932	0.98494	0.99737	1.00023	0.99716	1.00000	0.76452
1976	1.12486	1.05743	1.02877	1.03402	1.00054	1.01762	1.01556	0.77105
1977	1.12452	1.06275	1.03615	1.02121	1.00005	0.99816	1.02304	0.77788
1978	1.13925	1.07144	1.04613	1.01639	1.00014	1.00638	1.00981	0.77910
1979	1.11660	1.08986	1.03654	0.98843	0.99997	0.98846	1.00000	0.77917
1980	1.07583	1.08762	1.00662	0.98266	0.99952	0.98313	1.00000	0.78422
1981	1.12130	1.08833	1.01627	1.01380	1.00006	1.01333	1.00038	0.79038
1982	1.03789	1.06055	0.99064	0.98787	0.99908	0.98879	1.00000	0.78922
1983	1.06599	1.03011	1.01881	1.01573	0.99978	0.99877	1.01722	0.79088
1984	1.12992	1.02702	1.04355	1.05428	1.00031	1.02319	1.03007	0.78581
1985	1.06796	1.02429	1.02207	1.02012	1.00004	1.00387	1.01616	0.78356
1986	1.04461	1.01477	1.01450	1.01469	0.99942	1.00663	1.00858	0.78040
1987	1.06772	1.02543	1.02518	1.01568	1.00008	0.99690	1.01876	0.78425
1988	1.08876	1.03051	1.02393	1.03182	1.00001	1.00496	1.02674	0.78416
1989	1.05311	1.03536	1.02423	0.99309	0.99961	0.99349	1.00000	0.78180
1990	1.04470	1.03272	1.00815	1.00343	0.99948	1.00309	1.00084	0.78212
1991	1.01508	1.02620	0.98865	1.00053	0.99912	0.99307	1.00839	0.78337
1992	1.04873	1.01338	1.00450	1.03025	0.99985	1.01401	1.01619	0.77955
1993	1.04439	1.02143	1.02156	1.00090	0.99988	0.99821	1.00281	0.78074
1994	1.07222	1.01441	1.02813	1.02806	0.99987	1.00472	1.02338	0.78471
1995	1.05835	1.02076	1.02742	1.00915	1.00000	0.99710	1.01210	0.78785
1996	1.06827	1.01601	1.01951	1.03131	0.99999	1.00290	1.02833	0.78589
1997	1.07258	1.01331	1.03379	1.02390	1.00000	1.00000	1.02392	0.78781
1998	1.06417	1.00674	1.02081	1.03549	1.00000	1.00000	1.03551	0.78438
1999	1.06271	1.00916	1.02708	1.02530	1.00000	1.00000	1.02529	0.78342
2000	1.07036	1.02028	1.02537	1.02312	1.00000	1.00000	1.02312	0.78093
2001	1.02768	1.02115	1.00598	1.00042	0.99927	0.99638	1.00481	0.75436
2002	1.01647	1.00065	0.99290	1.02307	0.99999	1.00355	1.01946	0.74802
2003	1.03784	1.00971	1.00004	1.02781	1.00000	1.00000	1.02780	0.74391
2004	1.07134	1.02050	1.01572	1.03357	1.00000	1.00000	1.03358	0.74062
2005	1.06862	1.03090	1.01744	1.01881	1.00000	1.00000	1.01883	0.74052
2006	1.07748	1.02733	1.02378	1.02445	1.00000	1.00000	1.02446	0.73560
2007	1.02960	1.02128	1.01554	0.99272	0.99997	0.99275	1.00000	0.73612
2008	1.01915	1.01656	0.99768	1.00489	0.99975	1.00063	1.00450	0.72809
2009	0.94245	1.00646	0.96091	0.97449	0.99864	0.97582	1.00000	0.72938
2010	1.04971	1.00955	0.99971	1.04008	1.00091	1.02554	1.01327	0.73393
2011	1.05705	1.02104	1.01631	1.01865	1.00021	1.00604	1.01231	0.72754
2012	1.05772	1.01780	1.02118	1.01766	1.00000	1.00000	1.01766	0.72758
2013	1.03655	1.01063	1.01864	1.00688	1.00000	1.00000	1.00689	0.72784
2014	1.04125	1.01225	1.02417	1.00438	1.00000	1.00000	1.00439	0.72699
Mean	1.06550	1.03100	1.01720	1.01600	0.99990	1.00000	1.01600	0.76069

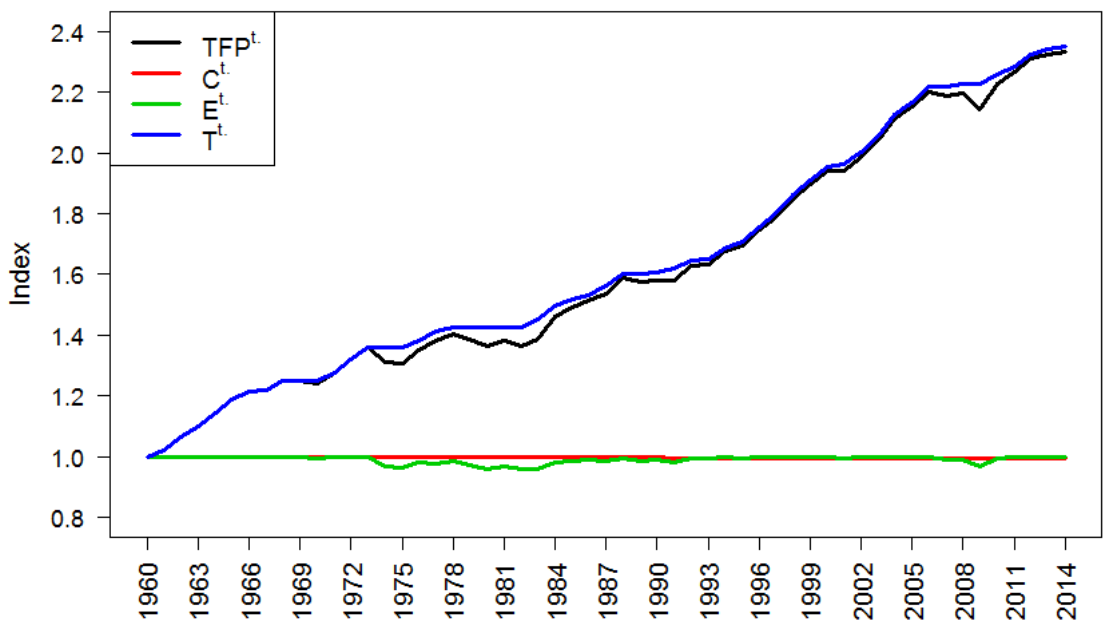
Table 6: U.S. National Value Added Year t Levels v^t/v^{1960}, Approximate Output Price Levels $A^{t\bullet}$, Input Quantity Levels $B^{t\bullet}$, TFP Levels $TFP^{t\bullet}$, Input Mix Levels $C^{t\bullet}$, Value Added Efficiency Levels $E^{t\bullet}$ and Technical Progress Levels $T^{t\bullet}$ where all Levels are Relative to 1960

Year t	v^t/v^{1960}	$A^{t\bullet}$	$B^{t\bullet}$	$TFP^{t\bullet}$	$C^{t\bullet}$	$E^{t\bullet}$	$T^{t\bullet}$	$Check^t$
1960	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.00000
1961	1.02671	1.00688	0.99834	1.02139	1.00000	1.00000	1.02139	0.00000
1962	1.10405	1.01568	1.01990	1.06579	1.00000	1.00000	1.06580	0.00001
1963	1.16487	1.02254	1.03472	1.10096	1.00000	1.00000	1.10095	-0.00001
1964	1.24855	1.03408	1.05569	1.14370	1.00000	1.00000	1.14370	0.00000
1965	1.36167	1.05290	1.08972	1.18677	1.00000	1.00000	1.18678	0.00001
1966	1.49076	1.08406	1.13084	1.21606	1.00000	1.00000	1.21608	0.00002
1967	1.55793	1.11079	1.15234	1.21713	0.99994	1.00000	1.21720	0.00000
1968	1.69334	1.14915	1.18009	1.24868	0.99994	1.00000	1.24875	0.00000
1969	1.82270	1.19801	1.21799	1.24914	0.99989	0.99962	1.24975	0.00000
1970	1.87724	1.24311	1.21607	1.24180	0.99899	0.99457	1.24982	-0.00002
1971	2.01372	1.29125	1.22188	1.27633	0.99903	1.00001	1.27755	0.00000
1972	2.23889	1.34097	1.26417	1.32071	0.99903	1.00001	1.32200	0.00003
1973	2.53834	1.41481	1.32078	1.35837	0.99903	1.00001	1.35968	0.00001
1974	2.72740	1.55244	1.33943	1.31164	0.99867	0.96596	1.35968	0.00003
1975	2.94535	1.70663	1.31926	1.30818	0.99890	0.96322	1.35968	0.00006
1976	3.31310	1.80464	1.35722	1.35268	0.99944	0.98019	1.38085	0.00006
1977	3.72566	1.91788	1.40628	1.38137	0.99949	0.97839	1.41266	0.00007
1978	4.24446	2.05491	1.47116	1.40401	0.99963	0.98463	1.42652	0.00007
1979	4.73938	2.23956	1.52491	1.38776	0.99961	0.97327	1.42652	0.00008
1980	5.09878	2.43579	1.53501	1.36369	0.99913	0.95685	1.42652	0.00008
1981	5.71727	2.65093	1.55998	1.38251	0.99919	0.96961	1.42706	0.00006
1982	5.93387	2.81145	1.54538	1.36575	0.99827	0.95874	1.42706	0.00006
1983	6.32545	2.89611	1.57445	1.38723	0.99805	0.95756	1.45164	0.00009
1984	7.14725	2.97435	1.64302	1.46253	0.99837	0.97976	1.49530	0.00011
1985	7.63296	3.04659	1.67927	1.49196	0.99840	0.98355	1.51946	0.00012
1986	7.97345	3.09159	1.70363	1.51387	0.99782	0.99008	1.53250	0.00012
1987	8.51344	3.17020	1.74653	1.53760	0.99790	0.98700	1.56125	0.00013
1988	9.26906	3.26692	1.78833	1.58654	0.99791	0.99190	1.60299	0.00015
1989	9.76137	3.38243	1.83165	1.57557	0.99752	0.98544	1.60299	0.00018
1990	10.19775	3.49310	1.84658	1.58097	0.99701	0.98849	1.60435	0.00016
1991	10.35158	3.58461	1.82562	1.58181	0.99613	0.98164	1.61781	0.00014
1992	10.85605	3.63256	1.83383	1.62967	0.99598	0.99539	1.64401	0.00017
1993	11.33793	3.71040	1.87336	1.63114	0.99586	0.99361	1.64862	0.00017
1994	12.15676	3.76387	1.92607	1.67691	0.99573	0.99830	1.68716	0.00019
1995	12.86605	3.84201	1.97888	1.69226	0.99573	0.99540	1.70757	0.00020
1996	13.74446	3.90353	2.01750	1.74525	0.99572	0.99829	1.75595	0.00021
1997	14.74204	3.95548	2.08567	1.78695	0.99572	0.99829	1.79794	0.00025
1998	15.68809	3.98216	2.12908	1.85038	0.99572	0.99829	1.86178	0.00027
1999	16.67190	4.01864	2.18673	1.89719	0.99572	0.99829	1.90886	0.00026
2000	17.84497	4.10016	2.24221	1.94106	0.99572	0.99829	1.95299	0.00026
2001	18.33883	4.18686	2.25561	1.94187	0.99500	0.99468	1.96239	0.00032
2002	18.64079	4.18957	2.23960	1.98667	0.99499	0.99821	2.00057	0.00032
2003	19.34611	4.23027	2.23969	2.04192	0.99499	0.99821	2.05618	0.00030
2004	20.72626	4.31697	2.27491	2.11047	0.99499	0.99821	2.12523	0.00033
2005	22.14840	4.45038	2.31458	2.15018	0.99499	0.99821	2.16524	0.00037
2006	23.86438	4.57201	2.36962	2.20275	0.99499	0.99821	2.21820	0.00039
2007	24.57077	4.66929	2.40644	2.18672	0.99496	0.99097	2.21820	0.00038
2008	25.04136	4.74660	2.40085	2.19740	0.99472	0.99160	2.22817	0.00037
2009	23.60017	4.77727	2.30701	2.14134	0.99336	0.96762	2.22817	0.00036
2010	24.77332	4.82290	2.30633	2.22718	0.99426	0.99233	2.25774	0.00040
2011	26.18666	4.92436	2.34395	2.26873	0.99447	0.99833	2.28554	0.00038
2012	27.69810	5.01202	2.39360	2.30880	0.99447	0.99833	2.32590	0.00038
2013	28.71043	5.06529	2.43821	2.32468	0.99447	0.99833	2.34192	0.00040
2014	29.89479	5.12731	2.49715	2.33486	0.99447	0.99833	2.35219	0.00042

From Table 6, it can be seen that the final level of approximate national TFP relative to 1960 was 2.33486 which is somewhat below the final level of technology, which was 2.35219. This difference is explained by the below unity final level of the input mix explanatory factor, $C^{2014\bullet} = 0.99447$, and the below unity final level of the value added efficiency factor, $E^{2014\bullet} = 0.99833$.[51] The final column in Table 6 is $Check^t$ defined as the difference between $TFP^{t\bullet}$ and $C^{t\bullet} E^{t\bullet} T^{t\bullet}$. It can be seen that this difference is very small and thus the approximate decomposition of the level of national TFP given by (112) is almost exact. We plot $TFP^{t\bullet}$, $C^{t\bullet}$, $E^{t\bullet}$ and $T^{t\bullet}$ in Figure 3.

Since Sector 1 is almost three times as big as Sector 2, it can be seen that the overall national results are closer to the Sector 1 results. In particular, the huge value added inefficiency results that showed up in Sector 2 are no longer so huge in the national results. However, inefficiency effects which are a result of recessions still show up as significant determinants of TFP at the national level.

Figure 3: Approximate National Level of TFP, Input Mix, Value Added Efficiency and Technology



10. Empirical Results Using the National Cost Constrained Value Added Approach

In this section, we implement the exact national value added decomposition approach that was explained in section 6.

Defined the sector k shares of national best practice value added in year t by $\sigma^{kt} \equiv R^{kt}(p^{kt},w^{kt},x^{kt})/R^t(p^t,w^t,x^t)$ for k = 1,2 and t = 1960,...,2014 where $R^{kt}(p^{kt},w^{kt},x^{kt})$ is defined by (65) and $R^t(p^t,w^t,x^t)$ is defined by (66). These shares for Sector 1 are listed in Table 7.

[51] Note that 0.99447×0.99833×2.35219 = 2.33528. If we subtract this number from the 2014 level of TFP, 2.33486, we get 0.00042 which is equal to the value of Check for 2014.

As was noted in the previous section, the year t sector k explanatory growth factors, α^{kt}, β^{kt}, γ^{kt}, ε^{kt} and τ^{kt}, are listed in Tables 1 and 3 above. Now apply definitions (69)-(92) in order to define the *year t national growth factors*, α^t, β^t, γ^t, δ^t, ε^t and τ^t. The year t exact decomposition of national value added growth is given by (96), which we repeat here as equations (113):

(113) $v^t/v^{t-1} = \alpha^t \beta^t \gamma^t \delta^t \varepsilon^t \tau^t$; t = 1961,...,2014.

Define the *national year t TFP growth factor* as $TFPG^t \equiv [v^t/v^{t-1}]/\alpha^t\beta^t$. This definition and equations (113) imply that we have the following national exact TFP growth decomposition into explanatory factors:

(114) $TFPG^t \equiv [v^t/v^{t-1}]/\alpha^t \beta^t = \gamma^t \delta^t \varepsilon^t \tau^t$; t = 1961,...,2014.

The growth decomposition components that appear in (114) are listed in Table 7.

The arithmetic means of the growth rates over the 54 years 1961-2014 are listed in the last row of Table 7.[52] It can be seen that the average rate of national TFP growth over these years was 1.59 percent per year, just below the average rate of technical progress, which was 1.60 percent per year. The results in Table 7 are very similar to the results listed in Table 5 above. Again, technical progress growth was not present for 8 years: 1974, 1975, 1979, 1980, 1982, 1989, 2007 and 2009. For those years, the rate of growth of value added efficiency was below unity and this translated into negative rates of TFP growth for those years. The national exact input mix and returns to scale growth factors, the γ^t and δ^t, are all very close to unity. The exact equality in (114) was not quite exact due to rounding errors: the absolute value of the difference between $TFPG^t$ and $\gamma^t \delta^t \varepsilon^t \tau^t$ was always less than 0.00003 and the mean difference was -0.0000027.

To conclude this section, apply definitions (37)-(40) to national value added in order to obtain the following exact levels decomposition for national Total Factor Productivity in year t relative to the year 1960, $TFP^{t\bullet}$:

(115) $TFP^t = [v^t/v^{1960}]/[A^t B^t] = C^t D^t E^t T^t$; t = 1960,...,2014.

Table 8 lists the various levels that appear in (115) along with the variable $Check^t$, which is defined as TFP^t minus $C^t D^t E^t T^t$. It can be seen that $Check^t$ is quite small although the rounding errors build up as time goes on.

---

[52] The mean for σ^{1t} is for all 55 observations.

Table 7: U.S. National Nonfinancial Value Added Growth v^t/v^{t-1}, Output Price Growth α^t, Input Quantity Growth β^t, TFP Growth $TFPG^t$, Input Mix Growth Factors γ^t, Returns to Scale Growth Factors δ^t, Value Added Efficiency Growth Factors ε^t, Technical Progress Growth Factors τ^t, Sector 1 Best Practice Share of National Value Added σ^{1t} and National Levels of Efficiency e^t

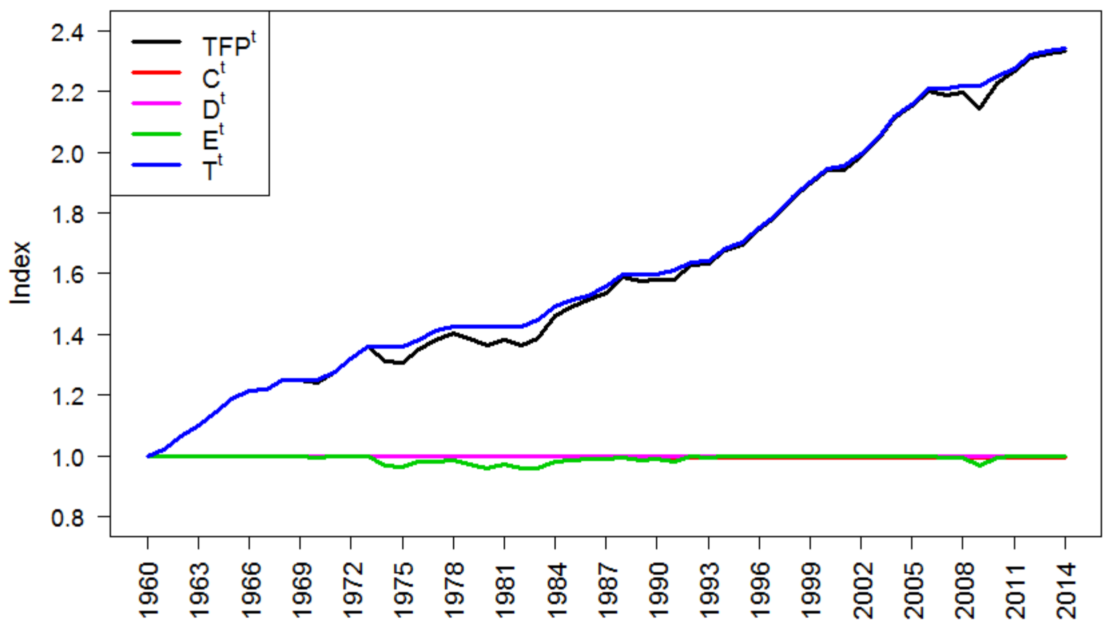
Year t	v^t/v^{t-1}	α^t	β^t	$TFPG^t$	γ^t	δ^t	ε^t	τ^t	σ^{1t}	e^t
1960	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.70438	1.00000
1961	1.02671	1.00688	0.99834	1.02139	1.00000	1.00000	1.00000	1.02139	0.70455	1.00000
1962	1.07533	1.00874	1.02160	1.04347	1.00000	1.00000	1.00000	1.04348	0.71528	1.00000
1963	1.05509	1.00675	1.01453	1.03300	1.00000	1.00000	1.00000	1.03299	0.72330	1.00000
1964	1.07183	1.01129	1.02027	1.03882	1.00000	1.00000	1.00000	1.03883	0.72884	1.00000
1965	1.09060	1.01820	1.03223	1.03766	1.00000	1.00000	1.00000	1.03766	0.73721	1.00000
1966	1.09480	1.02959	1.03773	1.02468	1.00000	1.00000	1.00000	1.02468	0.74427	1.00000
1967	1.04506	1.02466	1.01901	1.00088	0.99994	1.00000	1.00000	1.00093	0.74893	1.00000
1968	1.08691	1.03454	1.02408	1.02592	1.00000	1.00000	1.00000	1.02592	0.75650	1.00000
1969	1.07640	1.04252	1.03212	1.00037	0.99995	1.00000	0.99962	1.00079	0.76180	0.99962
1970	1.02992	1.03765	0.99842	0.99413	0.99910	1.00000	0.99495	1.00005	0.76171	0.99457
1971	1.07270	1.03871	1.00478	1.02781	1.00004	1.00001	1.00546	1.02220	0.76340	1.00000
1972	1.11182	1.03851	1.03460	1.03478	1.00000	1.00000	1.00000	1.03479	0.76495	1.00000
1973	1.13375	1.05506	1.04479	1.02851	1.00000	1.00000	1.00000	1.02851	0.75765	1.00000
1974	1.07448	1.09727	1.01412	0.96560	0.99964	1.00004	0.96593	1.00000	0.75621	0.96593
1975	1.07991	1.09936	0.98495	0.99732	1.00023	1.00010	0.99703	1.00000	0.75346	0.96306
1976	1.12486	1.05793	1.02877	1.03352	1.00053	0.99963	1.01778	1.01530	0.75578	0.98019
1977	1.12452	1.06318	1.03615	1.02080	1.00006	0.99957	0.99865	1.02256	0.76144	0.97887
1978	1.13925	1.07144	1.04614	1.01639	1.00015	0.99967	1.00688	1.00964	0.76788	0.98560
1979	1.11660	1.09034	1.03654	0.98799	0.99997	0.99988	0.98815	1.00000	0.76369	0.97392
1980	1.07583	1.08717	1.00662	0.98306	0.99951	1.00018	0.98336	1.00000	0.76624	0.95772
1981	1.12130	1.08855	1.01627	1.01359	1.00005	0.99971	1.01344	1.00037	0.76713	0.97059
1982	1.03789	1.06068	0.99065	0.98775	0.99904	1.00058	0.98813	1.00000	0.76233	0.95906
1983	1.06599	1.03182	1.01881	1.01404	0.99972	0.99997	0.99786	1.01655	0.75688	0.95701
1984	1.12992	1.02657	1.04355	1.05474	1.00035	0.99941	1.02514	1.02913	0.77094	0.98107
1985	1.06796	1.02470	1.02207	1.01971	1.00004	0.99981	1.00393	1.01588	0.77175	0.98493
1986	1.04461	1.01480	1.01451	1.01465	0.99940	0.99998	1.00674	1.00848	0.77382	0.99157
1987	1.06772	1.02566	1.02518	1.01545	1.00008	0.99989	0.99696	1.01857	0.77527	0.98855
1988	1.08876	1.03067	1.02393	1.03167	1.00001	0.99992	1.00512	1.02650	0.77915	0.99361
1989	1.05311	1.03548	1.02422	0.99298	0.99960	0.99998	0.99340	1.00000	0.77628	0.98706
1990	1.04470	1.03280	1.00815	1.00335	0.99947	1.00002	1.00301	1.00084	0.77432	0.99003
1991	1.01508	1.02640	0.98865	1.00033	0.99908	1.00022	0.99281	1.00827	0.76999	0.98291
1992	1.04873	1.01340	1.00450	1.03023	0.99984	0.99982	1.01433	1.01603	0.77721	0.99700
1993	1.04439	1.02144	1.02156	1.00089	0.99988	1.00001	0.99821	1.00280	0.77700	0.99522
1994	1.07222	1.01440	1.02813	1.02807	0.99987	0.99996	1.00481	1.02333	0.78471	1.00000
1995	1.05835	1.02080	1.02742	1.00911	1.00000	0.99997	0.99710	1.01208	0.78557	0.99710
1996	1.06827	1.01607	1.01951	1.03126	0.99999	0.99998	1.00291	1.02829	0.78589	1.00000
1997	1.07258	1.01331	1.03379	1.02389	1.00000	1.00000	1.00000	1.02391	0.78781	1.00000
1998	1.06417	1.00674	1.02081	1.03550	1.00000	1.00000	1.00000	1.03550	0.78438	1.00000
1999	1.06271	1.00916	1.02707	1.02530	1.00000	1.00000	1.00000	1.02529	0.78342	1.00000
2000	1.07036	1.02028	1.02537	1.02312	1.00000	1.00000	1.00000	1.02312	0.78093	1.00000
2001	1.02768	1.02113	1.00596	1.00045	0.99928	0.99996	0.99644	1.00481	0.75524	0.99644
2002	1.01647	1.00065	0.99290	1.02307	0.99999	0.99999	1.00357	1.01946	0.74802	1.00000
2003	1.03784	1.00971	1.00004	1.02782	1.00000	1.00000	1.00000	1.02780	0.74391	1.00000
2004	1.07134	1.02050	1.01572	1.03357	1.00000	1.00000	1.00000	1.03358	0.74062	1.00000
2005	1.06862	1.03090	1.01744	1.01882	1.00000	1.00000	1.00000	1.01882	0.74052	1.00000
2006	1.07748	1.02733	1.02378	1.02445	1.00000	1.00000	1.00000	1.02445	0.73560	1.00000
2007	1.02960	1.02129	1.01555	0.99271	0.99998	1.00005	0.99269	1.00000	0.73097	0.99269
2008	1.01915	1.01653	0.99768	1.00492	0.99976	1.00001	1.00062	1.00452	0.72991	0.99330
2009	0.94245	1.00652	0.96091	0.97443	0.99863	0.99995	0.97581	1.00000	0.73091	0.96927
2010	1.04971	1.00960	0.99971	1.04004	1.00090	0.99999	1.02553	1.01325	0.72954	0.99402
2011	1.05705	1.02103	1.01631	1.01867	1.00021	0.99996	1.00602	1.01239	0.72754	1.00000
2012	1.05772	1.01780	1.02118	1.01766	1.00000	1.00000	1.00000	1.01766	0.72758	1.00000
2013	1.03655	1.01063	1.01864	1.00688	1.00000	1.00000	1.00000	1.00689	0.72784	1.00000
2014	1.04125	1.01225	1.02417	1.00438	1.00000	1.00000	1.00000	1.00438	0.72699	1.00000
Mean	1.06550	1.03110	1.01720	1.01590	0.99989	0.99997	1.00000	1.0160	0.75486	0.99129

Table 8: U.S. National Value Added Year t Levels v^t/v^{1960}, Output Price Levels A^t, Input Quantity Levels B^t, TFP Levels TFP^t, Input Mix Levels C^t, Returns to Scale Levels D^t, Value Added Efficiency Levels E^t and Technical Progress Levels T^t where all Levels are Relative to 1960

Year t	v^t/v^{1960}	A^t	B^t	TFP^t	C^t	D^t	E^t	T^t	$Check^t$
1960	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	0.00000
1961	1.02671	1.00688	0.99834	1.02139	1.00000	1.00000	1.00000	1.02139	0.00000
1962	1.10405	1.01568	1.01990	1.06578	1.00000	1.00000	1.00000	1.06580	0.00001
1963	1.16487	1.02254	1.03472	1.10096	1.00000	1.00000	1.00000	1.10096	0.00000
1964	1.24855	1.03408	1.05570	1.14370	1.00000	1.00000	1.00000	1.14371	0.00001
1965	1.36167	1.05290	1.08973	1.18676	1.00000	1.00000	1.00000	1.18679	0.00002
1966	1.49076	1.08406	1.13085	1.21605	1.00000	1.00000	1.00000	1.21608	0.00003
1967	1.55793	1.11079	1.15235	1.21712	0.99994	1.00000	1.00000	1.21721	0.00001
1968	1.69334	1.14915	1.18010	1.24867	0.99994	1.00000	1.00000	1.24876	0.00001
1969	1.82270	1.19801	1.21800	1.24913	0.99989	1.00000	0.99962	1.24975	0.00001
1970	1.87724	1.24311	1.21607	1.24179	0.99899	1.00000	0.99457	1.24982	-0.00002
1971	2.01372	1.29124	1.22188	1.27633	0.99903	1.00001	1.00000	1.27756	0.00000
1972	2.23889	1.34096	1.26416	1.32073	0.99903	1.00001	1.00000	1.32201	0.00002
1973	2.53834	1.41480	1.32078	1.35839	0.99903	1.00001	1.00000	1.35970	0.00001
1974	2.72740	1.55241	1.33942	1.31166	0.99867	1.00005	0.96593	1.35970	0.00003
1975	2.94535	1.70666	1.31926	1.30815	0.99889	1.00014	0.96306	1.35970	0.00007
1976	3.31310	1.80553	1.35722	1.35201	0.99942	0.99977	0.98019	1.38050	0.00006
1977	3.72566	1.91961	1.40628	1.38013	0.99948	0.99934	0.97887	1.41165	0.00007
1978	4.24446	2.05676	1.47116	1.40274	0.99963	0.99901	0.98560	1.42526	0.00008
1979	4.73938	2.24256	1.52491	1.38590	0.99960	0.99889	0.97392	1.42526	0.00010
1980	5.09878	2.43804	1.53502	1.36243	0.99911	0.99907	0.95772	1.42526	0.00009
1981	5.71727	2.65393	1.55999	1.38095	0.99917	0.99879	0.97059	1.42578	0.00007
1982	5.93387	2.81496	1.54540	1.36403	0.99821	0.99937	0.95906	1.42578	0.00008
1983	6.32545	2.90454	1.57446	1.38319	0.99793	0.99934	0.95701	1.44939	0.00011
1984	7.14725	2.98172	1.64304	1.45890	0.99828	0.99875	0.98107	1.49161	0.00012
1985	7.63296	3.05536	1.67930	1.48766	0.99832	0.99856	0.98493	1.51530	0.00014
1986	7.97345	3.10059	1.70366	1.50945	0.99772	0.99854	0.99157	1.52814	0.00014
1987	8.51344	3.18015	1.74655	1.53277	0.99780	0.99843	0.98855	1.55653	0.00015
1988	9.26906	3.27768	1.78834	1.58131	0.99781	0.99836	0.99361	1.59777	0.00017
1989	9.76137	3.39397	1.83166	1.57021	0.99741	0.99834	0.98706	1.59777	0.00018
1990	10.19775	3.50528	1.84660	1.57547	0.99688	0.99836	0.99003	1.59911	0.00017
1991	10.35158	3.59783	1.82564	1.57599	0.99596	0.99858	0.98291	1.61234	0.00015
1992	10.85605	3.64603	1.83385	1.62363	0.99580	0.99840	0.99700	1.63818	0.00018
1993	11.33793	3.72420	1.87338	1.62508	0.99569	0.99841	0.99522	1.64276	0.00018
1994	12.15676	3.77784	1.92609	1.67070	0.99556	0.99837	1.00000	1.68110	0.00020
1995	12.86605	3.85642	1.97890	1.68592	0.99555	0.99835	0.99710	1.70140	0.00021
1996	13.74446	3.91837	2.01751	1.73863	0.99555	0.99833	1.00000	1.74954	0.00022
1997	14.74204	3.97052	2.08569	1.78017	0.99555	0.99833	1.00000	1.79138	0.00025
1998	15.68809	3.99729	2.12909	1.84336	0.99555	0.99833	1.00000	1.85497	0.00027
1999	16.67190	4.03392	2.18673	1.89001	0.99555	0.99833	1.00000	1.90189	0.00025
2000	17.84497	4.11575	2.24221	1.93371	0.99555	0.99833	1.00000	1.94585	0.00025
2001	18.33883	4.20271	2.25557	1.93458	0.99483	0.99829	0.99644	1.95521	0.00029
2002	18.64079	4.20543	2.23956	1.97921	0.99482	0.99828	1.00000	1.99326	0.00031
2003	19.34611	4.24627	2.23964	2.03427	0.99482	0.99828	1.00000	2.04867	0.00028
2004	20.72626	4.33331	2.27485	2.10256	0.99482	0.99828	1.00000	2.11746	0.00030
2005	22.14840	4.46722	2.31451	2.14213	0.99482	0.99828	1.00000	2.15732	0.00032
2006	23.86438	4.58931	2.36955	2.19450	0.99482	0.99828	1.00000	2.21007	0.00034
2007	24.57077	4.68701	2.40639	2.17850	0.99480	0.99832	0.99269	2.21007	0.00035
2008	25.04136	4.76448	2.40080	2.18921	0.99456	0.99833	0.99330	2.22007	0.00033
2009	23.60017	4.79555	2.30695	2.13323	0.99320	0.99828	0.96927	2.22007	0.00032
2010	24.77332	4.84158	2.30628	2.21864	0.99409	0.99827	0.99402	2.24948	0.00034
2011	26.18666	4.94338	2.34389	2.26005	0.99430	0.99823	1.00000	2.27735	0.00032
2012	27.69810	5.03138	2.39354	2.29997	0.99430	0.99823	1.00000	2.31756	0.00032
2013	28.71043	5.08486	2.43816	2.31579	0.99430	0.99823	1.00000	2.33352	0.00033
2014	29.89479	5.14712	2.49708	2.32594	0.99430	0.99823	1.00000	2.34375	0.00034

From Table 8, it can be seen that the final level of national TFP relative to 1960 was 2.32594 which is somewhat below the final level of technology, which was 2.34375. This difference is explained by the below unity final level of the input mix explanatory factor, C^{2014} = 0.99430, and the below unity final level of the returns to scale cumulated factor, D^{2014} = 0.99823. Note that the difference variable $Check^t$ is small; the 2014 value of $Check^t$ is the largest at 0.00034. Thus our theoretically exact decomposition of the level of national TFP given by (115) is almost numerically exact. We plot TFP^t, C^t, D^t, E^t and T^t in Figure 4.

Figure 4: National Level of TFP, Input Mix, Returns to Scale, Value Added Efficiency and Technology



It can be seen that the input mix and returns to scale explanatory factors are not important in explaining U.S. Nonfinancial Private Sector TFP growth over the period 1960-2014. The most important explanatory factor is the level of technical progress but during recession years, the level of value added efficiency plays an important role. Also noteworthy is the very high rate of TFP growth for the Nonfinancial Sector over this long period: the geometric average rate of TFP growth was 1.575% per year using the exact decomposition explained in this section and it was 1.583% per year using the approximate decomposition that was explained in the previous section.[53]

[53] The variable B^t that is listed in Table 8 is just the chained Fisher input quantity index that treats each input in each Sector as a separate commodity when forming the index. In a similar fashion, one can form a chained Fisher quantity index of the outputs in the two Sectors and then form a Fisher TFP index as the ratio of the Fisher output index to the Fisher input index. We calculated this Fisher TFP index using our data and found that it was very close to our approximate TFP index that was calculated in the previous section. The geometric average rate of TFP growth for the Fisher productivity index was also 1.583% per year, the same rate as the approximate TFP index achieved over our sample period.

11. Conclusion

We have derived decompositions of nominal value added growth (and TFP growth) for a single sector into explanatory factors. We also two alternative approaches to relating the sectoral decompositions to a national growth decomposition: a weighted average sectoral approach and a national value added function approach.

Starting with Denison (1962), various authors have presented decompositions of either aggregate labour productivity growth or TFP growth into sectoral explanatory factors by manipulating the index number formulae that are used to define the relevant aggregate.[54] The approach taken here relied instead on the economic approach to index number theory that started with Konüs (1939).

Rather than using the consumer's expenditure function in order to define various economic indexes, we used the sectoral and national cost constrained value added functions, $R^t(p,w,x)$, as the basic building blocks in our approach. These functions depend on four sets of variables: t (indicating which technology set S^{kt} or S^t is in scope), the output price vector p, the primary input price vector w and the primary input quantity vector x. Ratios of the cost constrained value added functions were used to define various explanatory "economic" indexes where three of the four sets of variables are held constant in the numerator and denominator and the remaining variable changes from a period $t-1$ level in the denominator to a period t level in the numerator.

With the goal of decomposing value added growth into a product of economic indexes, we operationalized our approach by assuming that an adequate approximation to a period t technology set can be obtained by taking the conical free disposal hull of past quantity observations for the sector under consideration. We found that our approach generated numerical estimates of TFP growth that are fairly close to standard index number estimates of TFP growth.

A main advantage of our new approach is that our new nonparametric measure of technical progress never indicates technical regress. During recessions, value added efficiency drops below unity and depresses TFP growth. For our U.S. data set, TFP growth is well explained as the product of value added efficiency growth times the rate of technical progress. For the U.S. Noncorporate Nonfinancial Sector, we found that the cost of recessions was particularly high.

Implementation of the decompositions can provide key insights into the drivers of economic growth at a detailed sectoral level. Hence, we believe that they will provide new insights into the sources of economic growth. Our decompositions may also indicate data mismeasurement problems that can then be addressed by statistical agencies.

---

[54] See for example, Tang and Wang (2004), Dumagan (2012), Balk (2014) (2015) (2016a) (2016b) and Diewert (2015) (2016).

References

Afriat, S.N. (1972), “Efficiency Estimation of Production Function”, *International Economic Review* 13, 568-598.

Aiyar, S., C-J. Dalgaard and O. Moav (2008), “Technological Progress and Regress in Pre-industrial Times”, *Journal of Economic Growth* 13, 125-144.

Allen, R.D.G. (1949), “The Economic Theory of Index Numbers”, *Economica* 16, 197-203.

Archibald, R.B. (1977), “On the Theory of Industrial Price Measurement: Output Price Indexes”, *Annals of Economic and Social Measurement* 6, 57-62.

Balk, B.M. (1998), *Industrial Price, Quantity and Productivity Indices*, Boston: Kluwer Academic Publishers.

Balk, B.M. (2001), “Scale Efficiency and Productivity Change”, *Journal of Productivity Analysis* 15, 159-183.

Balk, B.M. (2003), “The Residual: On Monitoring and Benchmarking Firms, Industries and Economies with respect to Productivity”, *Journal of Productivity Analysis* 20, 5-47.

Balk, B.M. (2014), “Dissecting Aggregate Labour and Output Productivity Change”, *Journal of Productivity Analysis* 42, 35-43.

Balk, B.M. (2015), “Measuring and Relating Aggregate and Subaggregate Total Factor Productivity Change Without Neoclassical Assumptions”, *Statistica Neerlandica* 69, 21-48.

Balk, B.M. (2016a), “The Dynamics of Productivity Change: A Review of the Bottom-up Approach”, pp. 15-49 in *Productivity and Efficiency Analysis*, W.H. Greene, L. Khalaf, R.C. Sickles, M. Veall and M.-C. Voia (eds.), New York: Springer Cham Heidelberg.

Balk, B.M. (2016b), “Aggregate Productivity and the Productivity of the Aggregate: Connecting the Bottom-Up and Top-Down Approaches”, paper presented at the North American Productivity Workshop IX, June 15-18, Quebec City.

Bliss, C.J. (1975), *Capital Theory and the Distribution of Income*, Amsterdam: North-Holland.

Byrne, D., J. Fernald and M. Reinsdorf (2016), “Does the United States Have a Productivity Slowdown or a Measurement Problem?” in J. Eberly and J. Stock (eds.), *Brookings Papers on Economic Activity: Spring 2016*, Washington, D.C.: Brookings Institute.

Caves, D.W., L.R. Christensen and W.E. Diewert (1982), “The Economic Theory of Index Numbers and the Measurement of Input, Output and Productivity”, *Econometrica* 50, 1393-1414.

Charnes, A. and W.W. Cooper (1985), “Preface to Topics in Data Envelopment Analysis”, *Annals of Operations Research* 2, 59-94.

Coelli, T., D.S. Prasada Rao and G. Battese (1997), *An Introduction to Efficiency and Productivity Analysis*, Boston: Kluwer Academic Publishers.

Denison, E.F. (1962), *The Sources of Economic Growth in the United States and the Alternatives Before Us*, New York: Committee for Economic Development.

Diewert, W.E. (1973), “Functional Forms for Profit and Transformation Functions”, *Journal of Economic Theory* 6, 284-316.

Diewert, W.E., (1974), “Applications of Duality Theory,” pp. 106-171 in M.D. Intriligator and D.A. Kendrick (ed.), *Frontiers of Quantitative Economics*, Vol. II, Amsterdam: North-Holland.

Diewert, W.E. (1976), “Exact and Superlative Index Numbers”, *Journal of Econometrics* 4, 114-145.

Diewert, W.E. (1978), “Superlative Index Numbers and Consistency in Aggregation”, *Econometrica* 46, 883-900.

Diewert, W.E. (1980a), “Aggregation Problems in the Measurement of Capital”, pp. 433-528 in *The Measurement of Capital*, D. Usher (ed.), Chicago: The University of Chicago Press.

Diewert, W.E. (1980b), “Capital and the Theory of Productivity Measurement”, *American Economic Review* 70, 260-267.

Diewert, W.E. (1983), “The Theory of the Output Price Index and the Measurement of Real Output Change”, pp. 1049-1113 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

Diewert, W.E. (1997), “Commentary on Mathew D. Shapiro and David W. Wilcox: Alternative Strategies for Aggregating Prices in the CPI”, *The Federal Reserve Bank of St. Louis Review*, Vol. 79:3, 127-137.

Diewert, W.E. (2011), “Measuring Productivity in the Public Sector: Some Conceptual Problems”, *Journal of Productivity Analysis* 36; 177-191.

Diewert, W.E. (2012), "Rejoinder to Gu on 'Estimating Capital Input for Measuring Business Sector Multifactor Productivity Growth in Canada' ", *International Productivity Monitor*, No. 24, Fall, 59-68.

Diewert, W.E. (2014), "Decompositions of Profitability Change using Cost Functions, *Journal of Econometrics* 183, 58-66.

Diewert, W.E. (2015), "Decompositions of Productivity Growth into Sectoral Effects", *Journal of Productivity Analysis* 43, 367-387.

Diewert, W.E. (2016), "Decompositions of Productivity Growth into Sectoral Effects: Some Puzzles Explained", pp. 1-14 in *Productivity and Efficiency Analysis*, W.H. Greene, L. Khalaf, R.C. Sickles, M. Veall and M.-C. Voia (eds.), New York: Springer Cham Heidelberg.

Diewert, W.E. and K.J. Fox (2008), "On the Estimation of Returns to Scale, Technical Progress and Monopolistic Markups", *Journal of Econometrics* 145, 174-193.

Diewert, W.E. and K.J. Fox (2010), "Malmquist and Törnqvist Productivity Indexes: Returns to Scale and Technical Progress with Imperfect Competition", *Journal of Economics* 101:1, 73-95.

Diewert, W.E. and K.J. Fox (2014), "Reference Technology Sets, Free Disposal Hulls and Productivity Decompositions", *Economics Letters* 122, 238–242.

Diewert, W.E. and K.J. Fox (2017), "Decomposing Productivity Indexes into Explanatory Factors", *European Journal of Operational Research* 256, 275–291.

Diewert, W.E. and K.J. Fox (2016a), "Alternative User Costs, Rates of Return and TFP Growth Rates for the US Nonfinancial Corporate and Noncorporate Business Sectors: 1960-2014", Discussion Paper 16-03, Vancouver School of Economics, The University of British Columbia, Vancouver, Canada, V6T 1L4.

Diewert, W.E. and K.J. Fox (2016b), "A Decomposition of U.S. Business Sector TFP Growth into Technical Progress and Cost Efficiency Components", Discussion Paper 16-04, Vancouver School of Economics, The University of British Columbia, Vancouver, Canada, V6T 1L4.

Diewert, W.E. and N.F. Mendoza (2007), "The Le Chatelier Principle in Data Envelopment Analysis", pp. 63-82 in *Aggregation, Efficiency, and Measurement*, Rolf Färe, Shawna Grosskopf and Daniel Primont (eds.), New York.

Diewert, W.E. and C.J. Morrison (1986), "Adjusting Output and Productivity Indexes for Changes in the Terms of Trade", *The Economic Journal* 96, 659-679.

Diewert, W.E. and A.O. Nakamura (2003), "Index Number Concepts, Measures and Decompositions of Productivity Growth", *Journal of Productivity Analysis* 19, 127-159.

Diewert, W.E. and C. Parkan (1983), "Linear Programming Tests of Regularity Conditions for Production Functions," pp. 131-158 in *Quantitative Studies on Production and Prices,* W. Eichhorn, R. Henn, K. Neumann and R.W. Shephard (eds.), Vienna: Physica Verlag.

Dumagan, J.C. (2013), "A Generalized Exactly Additive Decomposition of Aggregate Labor Productivity Growth", Review of Income and Wealth 59, 157–168.

Färe, R. and C.A.K. Lovell (1978), "Measuring the Technical Efficiency of Production", *Journal of Economic Theory* 19, 150-162.

Färe, R., S. Grosskopf and C.A.K. Lovell (1985), *The Measurement of Efficiency of Production*, Boston: Kluwer-Nijhoff.

Farrell, M.J. (1957), "The Measurement of Production Efficiency", *Journal of the Royal Statistical Society*, Series A, 120, 253-278.

Feenstra, R.C. (2004), *Advanced International Trade: Theory and Evidence,* Princeton N.J.: Princeton University Press.

Fisher, I. (1922), *The Making of Index Numbers*, Boston: Houghton-Mifflin.

Fisher, F.M. and K. Shell (1972), "The Pure Theory of the National Output Deflator", pp. 49-113 in *The Economic Theory of Price Indexes*, New York: Academic Press.

Fisher, F.M. and K. Shell (1998), *Economic Analysis of Production Price Indexes*, New York: *Cambridge* University Press.

Fox, K.J. and U. Kohli (1998), "GDP Growth, Terms of Trade Effects and Total Factor Productivity", *Journal of International Trade and Economic Development* 7, 87-110.

Gordon, R. (2016), *The Rise and Fall of American Growth: The U.S. Standard of Living since the Civil War*, New Jersey: Princeton University Press.

Gorman, W.M. (1968), "Measuring the Quantities of Fixed Factors", pp. 141-172 in *Value, Capital and Growth: Papers in Honour of Sir John Hicks,* J.N Wolfe (ed.), Chicago: Aldine Press.

Gu, W. (2012), "Estimating Capital Input for Measuring Business Sector Multifactor Productivity Growth in Canada: Response to Diewert and Yu," *International Productivity Monitor*, No. 24, Fall, 47-58.

Hanoch, G. and M. Rothschild (1972), "Testing the Assumptions of Production Theory: A Nonparametric Approach", *Journal of Political Economy* 80, 256-275.

Hardy, G.H., J.E. Littlewood and G. Polya, (1934), *Inequalities*, Cambridge, England: Cambridge University Press.

IMF, ILO, OECD, UN and the World Bank (2004), *Producer Price Index Manual: Theory and Practice*, Washington: The International Monetary Fund.

Jorgenson, D.W. and Z. Griliches (1967). "The Explanation of Productivity Change", *Review of Economic Studies* 34, 249–283.

Kohli, U. (1978), "A Gross National Product Function and the Derived Demand for Imports and Supply of Exports", *Canadian Journal of Economics* 11, 167-182.

Kohli, U. (1990), "Growth Accounting in the Open Economy: Parametric and Nonparametric Estimates", *Journal of Economic and Social Measurement* 16, 125-136.

Konüs, A.A. (1939), "The Problem of the True Index of the Cost of Living", *Econometrica* 7, 10-29.

McFadden, D. (1978), "Cost, Revenue and Profit Functions", pp. 3-109 in *Production Economics: A Dual Approach to Theory and Applications*, Volume 1, M. Fuss and D. McFadden (eds.), Amsterdam: North-Holland.

Mokyr, J., C. Vickers and N.L. Ziebarth (2015), "The History of Technological Anxiety and the Future of Economic Growth: Is This Time Different?" *Journal of Economic Perspectives* 29(3), 31–50.

O'Donnell, C.J. (2009), "An Aggregate Quantity-Price Framework for Measuring and Decomposing Productivity and Profitability Change", Working Paper WP07/2008, Centre for Efficiency and Productivity Analysis, School of Economics, University of Queensland, St. Lucia, Queensland 4072.

O'Donnell, C.J. (2010), "Measuring and Decomposing Agricultural Productivity and Profitability Change", *Australian Journal of Agricultural and Resource Economics* 54:4, 527-560.

Salter, W. E. G. (1960), *Productivity and Technical Change*, Cambridge U.K.: Cambridge University Press.

Samuelson, P.A. (1953), "Prices of Factors and Goods in General Equilibrium', *Review of Economic Studies* 21, 1-20.

Samuelson, P.A. and S. Swamy (1974), "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis", *American Economic Review* 64, 566-593.

Sato, K. (1976), "The Meaning and Measurement of the Real Value Added Index", *Review of Economics and Statistics* 58, 434-442.

Schreyer, P. (2012), "Comment on 'Estimating Capital Input for Measuring Business Sector Multifactor Productivity Growth in Canada' ", *International Productivity Monitor*, No. 24, Fall, 73-75.

Syverson, C. (2016), "Challenges to Mismeasurement Explanations for the U.S. Productivity Slowdown," NBER Working Paper 21974.

Tang, J. and W. Wang (2004), "Sources of Aggregate Labour Productivity Growth in Canada and the United States", *Canadian Journal of Economics* 37, 421-444.

Tulkens, H. (1993), "On FDH Efficiency Analysis: Some Methodological Issues and Application to Retail Banking, Courts, and Urban Transit", *Journal of Productivity Analysis* 4, 183–210.

Tulkens, H. and P.V. Eeckaut (1995a), "Non-Frontier Measures of Efficiency, Progress and Regress for Time Series Data", *International Journal of Production Economics* 39, 83-97.

Tulkens, H. and P.V. Eeckaut (1995b), "Nonparametric Efficiency, Progress and Regress Measures for Panel Data: Methodological Aspects", *European Journal of Operational Research* 80, 474-499.

Varian, H.R. (1984), "The Nonparametric Approach to Production Analysis", *Econometrica* 52, 579-597.

Woodland, A.D. (1982), *International Trade and Resource Allocation*, Amsterdam: North-Holland.