

# INDEX NUMBER THEORY AND MEASUREMENT ECONOMICS

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## CHAPTER 2: FUNCTIONAL EQUATIONS

### 1. Introduction

*Functional equations* are very useful in many branches of pure and applied economics. For example, consider the problem of choosing an index number formula. A *price index*,  $P(p^0, p^1, q^0, q^1)$ , is a function of the price and quantity vectors,  $p^t \equiv [p_1^t, \dots, p_N^t]$  and  $q^t \equiv [q_1^t, \dots, q_N^t]$  respectively, pertaining to two periods,  $t = 0$  and  $1$  say, and  $P(p^0, p^1, q^0, q^1)$  is supposed to be a summary representation of the amount of price change that took place between periods 0 and 1 in the  $N$  commodities being aggregated. In the test approach to index number theory, we assume that the function  $P(p^0, p^1, q^0, q^1)$  satisfies various mathematical properties that seem “reasonable”. We place enough properties on the function  $P$  so that the functional form is completely determined. As another application of the theory of functional equations, consider the problem of taking the mean of  $N$  positive numbers,  $x \equiv [x_1, \dots, x_N]$ , by means of some function  $m(x)$ . If we place enough conditions on the function  $m$ , we can characterize the functional form for  $m$  by its desired properties. There are many other applications of functional equations to economics including developing measures of income inequality, social welfare, industrial concentration and tax progressivity. Eichhorn (1978) has yet other applications of functional equations to economics.

In this chapter, we will not develop these applications of functional equations to economics; rather, we will state and prove some of the simplest functional equations, due to Cauchy (1821) and Pexider (1903). These “simple” functional equations are often used to solve much more complicated functional equations. Our proofs are modifications of the proofs found in Aczél (1966) (1969) and Eichhorn (1978).<sup>1</sup>

The *four fundamental Cauchy functional equations* are:

- (1)  $f(x + y) = f(x) + f(y)$ ;
- (2)  $f(x + y) = f(x)f(y)$ ;
- (3)  $f(xy) = f(x) + f(y)$ ;  $x > 0 ; y > 0$ ;
- (4)  $f(xy) = f(x)f(y)$ ;  $x > 0 ; y > 0$ .

The domains of definition for  $x$  and  $y$  for equations (1) and (2) could also be the set of positive real numbers as is the case with equations (3) and (4) or (1) and (2) could hold for all  $x$  and  $y$  where  $x$  and  $y$  are arbitrary real numbers; i.e.,  $x \in \mathbb{R}^1$  and  $y \in \mathbb{R}^1$ . In general, the domain of definition for the functions in a functional equation must be carefully specified. For each of the equations (1) to (4), we look for the class of functions  $f$  that will satisfy the equation for all  $x$  and  $y$  in the domain of definition for  $f$ .

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<sup>1</sup> These references deal with solutions to the Cauchy and Pexider functional equations that are not necessarily continuous functions. We will cover only the case of continuous solutions since these are the relevant solutions for economic applications.

We shall see that the nonconstant continuous solutions<sup>2</sup> to the functional equations (1)-(4) are  $f(x) = cx$ ,  $f(x) = e^{cx}$ ,  $f(x) = \ln x$  and  $f(x) = x^c$  respectively, where  $c \neq 0$ .

The four fundamental Pexider functional equations are:

- (5)  $f(x + y) = g(x) + h(y)$ ;
- (6)  $f(x + y) = g(x)h(y)$ ;
- (7)  $f(xy) = g(x) + h(y)$ ;  $x > 0 ; y > 0$ ;
- (8)  $f(xy) = g(x)h(y)$ ;  $x > 0 ; y > 0$ .

Thus the Pexider functional equations have the same general structure as the Cauchy equations except instead of each equation having only a single unknown function, each Pexider equation has three unknown functions of one variable:  $f$ ,  $g$  and  $h$ .

**2. The First Fundamental Cauchy Functional Equation**

**Proposition 1:** Let  $f(x)$  be a continuous function of one variable defined for  $x > 0$ . Suppose  $f$  satisfies the following equation for all positive  $x$  and  $y$ :

(9)  $f(x + y) = f(x) + f(y)$  ;  $x > 0 ; y > 0$ .

Then  $f$  must be equal to the following function:

(10)  $f(x) \equiv cx$  ;  $x > 0$

where  $c$  is an arbitrary constant.

**Proof:** Let  $x_i > 0$  for  $i = 1, 2, \dots, n$ . Then making repeated use of (9), we have:

(11)  $f(\sum_{i=1}^n x_i) = f(x_1 + \{\sum_{i=2}^n x_i\})$   
 $= f(x_1) + f(\sum_{i=2}^n x_i)$  using (9)  
 $= f(x_1) + f(x_2) + f(\sum_{i=3}^n x_i)$  using (9) again  
 $= \dots$   
 $= \sum_{i=1}^n f(x_i)$ .

Now let each  $x_i = x > 0$  and (11) becomes:

(12)  $f(nx) = nf(x)$  ;  $n$  a positive integer and  $x > 0$ .

Let  $m$  and  $n$  be positive integers and let  $u > 0$  be an arbitrary positive number. Define  $x$  as:

(13)  $x \equiv [m/n]u$ .

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<sup>2</sup> Note that all four of the functional equations (1)-(4) have the zero function,  $f(x) \equiv 0$  for all  $x$  in the domain of definition, as solutions. This solution function is also continuous.

Using (13), we have:

$$(14) \quad nx = mu.$$

Now apply (12) to both sides of (14) and obtain the following equation:

$$(15) \quad nf(x) = mf(u).$$

Substitute (13) into (15) and we obtain the following equation:

$$(16) \quad f([m/n]u) = [m/n]f(u); \quad m \text{ and } n \text{ positive integers, } u > 0.$$

Define the constant  $c$  as

$$(17) \quad c \equiv f(1).$$

Now let  $r = m/n$  and let  $u = 1$  and substitute these values into (16) and we obtain the following equation:

$$(18) \quad \begin{aligned} f(r) &= rf(1) && \text{for } r = m/n \\ &= rc && \text{using (17).} \end{aligned}$$

The continuity of  $f$  and (18) imply that  $f(x)$  is defined by (10) where  $x$  is an arbitrary positive number. Q.E.D.

We now modify the above proof to cover the case where the functional equation (9) is to be satisfied for all  $x$  and  $y$  (not just positive  $x$  and  $y$ ).

**Proposition 2:** Let  $f(x)$  be a continuous function of one variable defined for  $x \in \mathbb{R}^1$ . Suppose  $f$  satisfies the following equation for all  $x$  and  $y$ :

$$(19) \quad f(x + y) = f(x) + f(y); \quad x \in \mathbb{R}^1; y \in \mathbb{R}^1.$$

Then  $f$  must be equal to the following function:

$$(20) \quad f(x) \equiv cx; \quad x \in \mathbb{R}^1$$

where  $c$  is an arbitrary constant.

**Proof:** From the proof of Proposition 1, we know  $f(x)$  satisfies (20) if  $x > 0$ . Using the continuity of  $f$ , we see that  $f$  will also satisfy (20) if  $x = 0$  and we will have:

$$(21) \quad f(0) = 0 = c \cdot 0.$$

Now let  $x > 0$  so that  $-x$  is an arbitrary negative number. Using (19) with  $y \equiv -x$ , we have

$$\begin{aligned}
 (22) \quad f(x) + f(-x) &= f(x - x) \\
 &= f(0) \\
 &= 0
 \end{aligned}
 \qquad \text{using (21).}$$

But (22) implies

$$\begin{aligned}
 (23) \quad f(-x) &= -f(x) \\
 &= -cx \\
 &= c(-x).
 \end{aligned}
 \qquad \text{using (10) since } x > 0$$

Thus  $f(x)$  satisfies (20) no matter whether  $x$  is positive, negative or zero. Q.E.D.

We turn now to the second Cauchy equation.

### 3. The Second Fundamental Cauchy Functional Equation

Let  $f(x)$  be a function of one variable defined for  $x > 0$ . Suppose  $f$  satisfies the following equation for all positive  $x$  and  $y$ :

$$(24) \quad f(x + y) = f(x)f(y); \qquad x > 0; y > 0.$$

Let  $x_i > 0$  for  $i = 1, 2, \dots, n$ . Then making repeated use of (24), we have:

$$\begin{aligned}
 (25) \quad f(\sum_{i=1}^n x_i) &= f(x_1 + \{\sum_{i=2}^n x_i\}) \\
 &= f(x_1)f(\sum_{i=2}^n x_i) && \text{using (24)} \\
 &= f(x_1)f(x_2)f(\sum_{i=3}^n x_i) && \text{using (24) again} \\
 &= \dots \\
 &= \prod_{i=1}^n f(x_i).
 \end{aligned}$$

Now let each  $x_i = x > 0$  and (25) becomes:

$$(26) \quad f(nx) = f(x)^n; \qquad n \text{ a positive integer and } x > 0.$$

We shall use (26) in the following Lemma.

**Lemma 1:** Suppose  $f(x)$  is a function defined for  $x > 0$  and satisfies (24). Then if there exists an  $x^* > 0$  such that

$$(27) \quad f(x^*) = 0,$$

then  $f$  must be identically equal to zero; i.e., we have

$$(28) \quad f(x) = 0; \qquad x > 0.$$

**Proof:** Let  $x > 0$ . Then using (24) with  $y = x^*$ , we have:

$$(29) \quad \begin{aligned} f(x + x^*) &= f(x)f(x^*) \\ &= 0 \end{aligned} \quad \text{using (27).}$$

Letting  $y \equiv x + x^*$ , we see that (29) implies

$$(30) \quad f(y) = 0; \quad y > x^*.$$

Now *suppose* there exists a  $u^*$  such that  $0 < u^* < x^*$  with

$$(31) \quad f(u^*) \neq 0.$$

Then since  $u^* > 0$ , there exists an integer  $n$  such that  $nu^* > x^*$  and hence, using (30),

$$(32) \quad f(nu^*) = 0.$$

But we can also apply equation (26) to  $f(nu^*)$ :

$$(33) \quad f(nu^*) = f(u^*)^n \neq 0 \quad \text{using our supposition (31).}$$

However, (33) contradicts (32) so our supposition (31) is false and the lemma follows. Q.E.D.

Looking at the functional equation (24), it is evident that

$$(34) \quad f(x) \equiv 0$$

for all  $x$  in the domain of definition of  $f$  is a solution to (24). Let us see if we can find other solutions to (24).

**Lemma 2:** Suppose  $f(x)$  is a function defined for  $x > 0$  and satisfies (24). Then

$$(35) \quad f(u) \geq 0 \quad \text{for all } u > 0.$$

**Proof:** Let  $u > 0$  and define  $x \equiv u/2$  and  $y \equiv u/2$  and apply (24):

$$(36) \quad \begin{aligned} f(u) &= f([u/2] + [u/2]) \\ &= f(u/2)^2 \\ &\geq 0. \end{aligned} \quad \begin{array}{l} \text{using (24)} \\ \\ \text{Q.E.D.} \end{array}$$

Lemmas 1 and 2 show that solutions  $f(x)$  to the functional equation (24) are either of the form  $f(x) \equiv 0$  for all  $x > 0$  or we must have

$$(37) \quad f(x) > 0 \quad \text{for all } x > 0.$$

Now we are ready to exhibit the complete solution to (24) when  $f$  is continuous.

**Proposition 3:** Let  $f(x)$  be a continuous function of one variable defined for  $x > 0$ . Suppose  $f$  satisfies the following equation for all positive  $x$  and  $y$ :

$$(38) \quad f(x + y) = f(x)f(y); \quad x > 0; y > 0.$$

Then the solution to (38) is *either*  $f(x) \equiv 0$  for all  $x > 0$  *or*  $f$  must be equal to the following function:

$$(39) \quad f(x) \equiv e^{cx}; \quad x > 0$$

where  $c$  is an arbitrary constant.

**Proof:** Lemmas 1 and 2 above tell us if we look for a nonzero solution, we must have  $f(x) > 0$  for all  $x > 0$ . Hence, we assume  $f$  satisfies (37) and thus we can take the natural logarithm of both sides of (38). Defining the function  $g(x) \equiv \ln f(x)$ , the transformed version of (38) becomes the following functional equation:

$$(40) \quad g(x + y) = g(x) + g(y); \quad x > 0; y > 0.$$

But this is the same functional equation as appeared in Proposition 1 above, with  $g$  replacing  $f$ . Upon noting that the continuity of  $f(x)$  implies the continuity of  $\ln f(x)$ , we see that  $g(x)$  in (40) is continuous and hence we can apply Proposition 1 to conclude  $g(x) = cx$  for some constant  $c$ . Since  $f(x) = e^{g(x)}$ , Proposition 3 follows. Q.E.D.

Now let us change the domain of definition in (24) from positive values for  $x$  and  $y$  to unrestricted values. Thus suppose  $f$  satisfies the following equation for all positive  $x$  and  $y$ :

$$(41) \quad f(x + y) = f(x)f(y); \quad x \in \mathbb{R}^1; y \in \mathbb{R}^1.$$

Lemmas 3 and 4 below are counterparts to Lemmas 1 and 2 above.

**Lemma 3:** Suppose  $f(x)$  is a function defined for  $x \in \mathbb{R}^1$  and satisfies (41). Then if there exists an  $x^* \in \mathbb{R}^1$  such that

$$(42) \quad f(x^*) = 0,$$

then  $f$  must be identically equal to zero; i.e., we have

$$(43) \quad f(x) = 0; \quad x \in \mathbb{R}^1.$$

**Proof:** Let  $x \in \mathbb{R}^1$  and define  $y$  by

$$(44) \quad y \equiv x - x^*.$$

Note that (44) implies  $x = x^* + y$ . Now apply (41) with  $x^*$  replacing  $x$ :

$$(45) \quad f(x) = f(x^* + y) \quad \text{using (44)}$$

$$\begin{array}{ll}
 = f(x^*)f(y) & \text{using (41)} \\
 = 0 & \text{using (42).}
 \end{array}
 \qquad \text{Q.E.D.}$$

**Lemma 4:** Suppose  $f(x)$  is a function defined for  $x \in \mathbb{R}^1$  and satisfies (41). Then

$$(46) \quad f(u) \geq 0 \qquad \text{for all } u \in \mathbb{R}^1.$$

**Proof:** Let  $u \in \mathbb{R}^1$  and define  $x \equiv u/2$  and  $y \equiv u/2$  and apply (41):

$$\begin{array}{ll}
 (47) \quad f(u) = f([u/2] + [u/2]) & \\
 \quad \quad = f(u/2)^2 & \text{using (41)} \\
 \quad \quad \geq 0. &
 \end{array}
 \qquad \text{Q.E.D.}$$

Lemmas 3 and 4 show that solutions  $f(x)$  to the functional equation (41) are either of the form  $f(x) \equiv 0$  for all  $x \in \mathbb{R}^1$  or we must have

$$(48) \quad f(x) > 0 \qquad \text{for all } x \in \mathbb{R}^1.$$

Now we are ready to exhibit the complete solution to (41) when  $f$  is continuous.

**Proposition 4:** Let  $f(x)$  be a continuous function of one variable defined for  $x \in \mathbb{R}^1$ . Suppose  $f$  satisfies the functional equation (41) for all  $x$  and  $y$ . Then the solutions  $f$  are given by *either*  $f(x) \equiv 0$  for all  $x \in \mathbb{R}^1$  *or*  $f$  must be equal to the following function:

$$(49) \quad f(x) \equiv e^{cx}; \qquad x \in \mathbb{R}^1$$

where  $c$  is an arbitrary constant.

**Proof:** The proof is the same as the proof of Proposition 3 above except that now we apply Proposition 2 instead of Proposition 1. Q.E.D.

The proofs in this section illustrate a general technique that is often used in solving functional equations: namely, try to transform the given functional equation into another equivalent functional equation for which you can find the solution!

#### 4. The Third Fundamental Cauchy Functional Equation

Let  $f(x)$  be a function of one variable defined for  $x > 0$ . Suppose  $f$  satisfies the following equation for all positive  $x$  and  $y$ :

$$(50) \quad f(xy) = f(x) + f(y); \qquad x > 0; y > 0.$$

**Proposition 5:** Let  $f(x)$  be a continuous function of one variable for  $x > 0$  and suppose  $f$  satisfies the functional equation (50). Then  $f$  must be equal to the following function:

$$(51) \quad f(x) = c \ln x; \qquad x > 0$$

where  $c$  is an arbitrary constant and  $\ln x$  denotes the natural logarithm of  $x$ .

**Proof:** Let  $x > 0$  and  $y > 0$ . Since  $x$  and  $y$  are positive, we can write  $x$  and  $y$  as follows:

$$(52) \quad x = e^{\ln x};$$

$$(53) \quad y = e^{\ln y}.$$

Substitute (52) and (53) into (50) and get the following functional equation:

$$(54) \quad f(e^{\ln x} e^{\ln y}) = f(e^{\ln x + \ln y}) = f(e^{\ln x}) + f(e^{\ln y}); \quad x > 0; y > 0.$$

Define the function of one variable  $g(u)$  for  $u \in \mathbb{R}^1$  in terms of  $f$  as follows:

$$(55) \quad g(u) \equiv f(e^u); \quad u \in \mathbb{R}^1.$$

Note that the continuity of  $f$  will imply the continuity of  $g$ . Using definition (55), we can rewrite (54) as follows:

$$(56) \quad g(\ln x + \ln y) = g(\ln x) + g(\ln y); \quad x > 0; y > 0.$$

Now define  $u \equiv \ln x$  and  $v \equiv \ln y$  and substitute these definitions into (56). Hence if  $f$  satisfies (50),  $g$  will satisfy the following functional equation:

$$(57) \quad g(u + v) = g(u) + g(v); \quad u \in \mathbb{R}^1; v \in \mathbb{R}^1.$$

But by Proposition 2, the set of solutions to (57) is:

$$(58) \quad g(u) = cu; \quad u \in \mathbb{R}^1; c \text{ is an arbitrary constant.}$$

Using (52), we have:

$$(59) \quad \begin{aligned} f(x) &= f(e^{\ln x}) \\ &= g(\ln x) && \text{using definition (55)} \\ &= c \ln x && \text{using (58)} \end{aligned}$$

which establishes (51). Q.E.D.

We have only one more Cauchy functional equation to solve.

## 5. The Fourth Fundamental Cauchy Functional Equation

Let  $f(x)$  be a function of one variable defined for  $x > 0$ . Suppose  $f$  satisfies the following equation for all positive  $x$  and  $y$ :

$$(60) \quad f(xy) = f(x)f(y); \quad x > 0; y > 0.$$

**Proposition 6:** Let  $f(x)$  be a continuous function of one variable for  $x > 0$  and suppose  $f$  satisfies the functional equation (60). Then *either*  $f(x) = 0$  for all  $x > 0$  *or*  $f$  must be equal to the following function:

$$(61) \quad f(x) = x^c; \quad x > 0; \quad c \text{ is an arbitrary constant.}$$

**Proof:** It is very easy to verify that  $f(x) = 0$  for all  $x$  is a solution to the functional equation (60). We now show that if  $f(x^*) = 0$  for any  $x^* > 0$ , then  $f(x)$  is identically equal to 0.

Suppose  $x^* > 0$  exists such that

$$(62) \quad f(x^*) = 0.$$

Let  $x > 0$ . Now define  $y$  by

$$(63) \quad y \equiv x/x^* > 0 \quad \text{since both } x \text{ and } x^* \text{ are positive.}$$

Note that (63) implies that  $x = x^*y$ . Thus we have for an arbitrary  $x > 0$ :

$$(64) \quad \begin{aligned} f(x) &= f(x^*y) \\ &= f(x^*)f(y) && \text{using (60)} \\ &= 0 && \text{using (62).} \end{aligned}$$

Thus if  $f(x^*) = 0$  for a single point  $x^*$ , then  $f(x)$  is identically equal to zero.

We now show that  $f(x)$  must be nonnegative. Let  $x > 0$  and define

$$(65) \quad y \equiv x^{1/2};$$

i.e., define  $y$  as the positive square root of  $x$ . Thus we have for an arbitrary  $x > 0$ :

$$(66) \quad \begin{aligned} f(x) &= f(yy) && \text{using definition (65)} \\ &= f(y)f(y) && \text{using (60)} \\ &\geq 0. \end{aligned}$$

Hence  $f(x)$  must be nonnegative over its domain of definition. This result and the earlier result (64) means that we can assume that:

$$(67) \quad f(x) > 0 \quad \text{for } x > 0$$

if we want to find a nonzero solution to the functional equation (60). Thus, we now assume that (67) holds and hence the natural logarithm of  $f(x)$ ,  $\ln f(x)$ , is well defined for each  $x > 0$ . Thus define the function of one variable,  $g(x)$ , as follows:

$$(68) \quad g(x) \equiv \ln f(x); \quad x > 0.$$

Since  $f$  is continuous, so is  $g$ . Now take logarithms of both sides of (60) and substitute definition (68) into the resulting equation in order to obtain the following equation:

$$(69) \quad g(xy) = g(x) + g(y); \quad x > 0; y > 0.$$

But (69) is the functional equation (50), with  $g$  replacing  $f$ . Hence by Proposition 5 above, the solution to (69) is:

$$(70) \quad g(x) \equiv c \ln x; \quad x > 0; c \text{ is an arbitrary constant.}$$

Exponentiating both sides of (68) shows that

$$(71) \quad \begin{aligned} f(x) &= e^{g(x)}; & x > 0 \\ &= e^{c \ln x} & \text{using (70)} \\ &= [e^{\ln x}]^c \\ &= x^c \end{aligned}$$

which is (61).

Q.E.D.

We turn now to Pexider's generalizations of Cauchy's functional equations.

## 6. The First Fundamental Pexider Functional Equation

Recall that the first Pexider functional equation was  $f(x + y) = g(x) + h(y)$ . We replace the functions  $f$ ,  $g$  and  $h$  by  $f^1$ ,  $f^2$  and  $f^3$  respectively and rewrite the equation as follows:

$$(72) \quad f^1(x + y) = f^2(x) + f^3(y); \quad x \in \mathbb{R}^1; y \in \mathbb{R}^1.$$

Note that we are assuming that the functions are defined for all real  $x$  and  $y$ , not just positive  $x$  and  $y$ . The following Proposition is a counterpart to Proposition 2 above.

**Proposition 7:** Let  $f^1(x)$ ,  $f^2(x)$  and  $f^3(x)$  be continuous functions of one variable defined for  $x \in \mathbb{R}^1$ . Suppose that these functions satisfy (72) above for all  $x$  and  $y$ . Then these functions must be equal to the following functions:

$$\begin{aligned} (73) \quad f^1(x) &\equiv cx + a + b; & x \in \mathbb{R}^1; \\ (74) \quad f^2(x) &\equiv cx + a; & x \in \mathbb{R}^1; \\ (75) \quad f^3(x) &\equiv cx + b; & x \in \mathbb{R}^1 \end{aligned}$$

where  $a$ ,  $b$  and  $c$  are arbitrary constants.

**Proof:** Let  $x = 0$  and equation (72) becomes:

$$(76) \quad f^1(y) = f^2(0) + f^3(y); \quad y \in \mathbb{R}^1.$$

Define the constant  $a$  as follows:

$$(77) \quad a \equiv f^2(0).$$

Now use (76) and (77) in order to define the function  $f^3$  in terms of the function  $f^1$ :

$$(78) \quad f^3(y) = f^1(y) - a ; \quad y \in \mathbb{R}^1.$$

Now let  $y = 0$  and equation (72) becomes:

$$(79) \quad f^1(x) = f^2(x) + f^3(0) ; \quad x \in \mathbb{R}^1.$$

Define the constant  $b$  as follows:

$$(80) \quad b \equiv f^3(0).$$

Now use (79) and (80) in order to define the function  $f^2$  in terms of the function  $f^1$ :

$$(81) \quad f^2(x) = f^1(x) - b ; \quad x \in \mathbb{R}^1.$$

Now substitute (78) and (81) into (72) and we find that  $f^1$  must satisfy the following functional equation:

$$(82) \quad f^1(x + y) = f^1(x) - b + f^1(y) - a ; \quad x \in \mathbb{R}^1 ; y \in \mathbb{R}^1$$

Define the function  $f$  in terms of  $f^1$  as follows:

$$(83) \quad f(x) \equiv f^1(x) - a - b ; \quad x \in \mathbb{R}^1.$$

Use equation (83) to solve for  $f^1$  in terms of  $f$ :

$$(84) \quad f^1(x) = f(x) + a + b.$$

Now substitute (84) into (82) and we obtain the following functional equation in  $f$ :

$$(85) \quad \begin{aligned} f(x + y) + a + b &= [f(x) + a + b] - b + [f(y) + a + b] - a ; & x \in \mathbb{R}^1 ; y \in \mathbb{R}^1 \text{ or} \\ f(x + y) &= f(x) + f(y) \end{aligned}$$

which is the first Cauchy equation. Hence, applying Proposition 2, we conclude that  $f(x)$  must equal  $cx$  for some constant  $c$ . With  $f(x)$  determined,  $f^1$  is determined using (84),  $f^2$  is determined using (81) and  $f^3$  is determined using (78). Q.E.D.

In the above Proposition, the domain of definition for  $f^1$ ,  $f^2$  and  $f^3$  was all of  $\mathbb{R}^1$ . We can obtain a counterpart to Proposition 7 where the functions are defined only over positive real numbers but we need to extend this domain of definition to include the origin as well. Thus we need the following limits to exist and be finite numbers:

$$(86) \lim_{x \rightarrow 0, x > 0} f^k(x) = a^k ; \quad k = 1, 2, 3$$

where each  $a^k$  is a finite real number. Now we can change the domains of definition for the functions in (72) to  $x \geq 0$  and  $y \geq 0$ ; i.e., we now consider the following functional equation:

$$(87) f^1(x + y) = f^2(x) + f^3(y) ; \quad x \geq 0 ; y \geq 0.$$

**Proposition 8:** Let  $f^1(x)$ ,  $f^2(x)$  and  $f^3(x)$  be continuous functions of one variable defined for nonnegative  $x$ . Suppose that these functions satisfy (87) above for all nonnegative  $x$  and  $y$ . Then these functions must be equal to the following functions:

$$(88) f^1(x) \equiv cx + a + b ; \quad x \geq 0;$$

$$(89) f^2(x) \equiv cx + a ; \quad x \geq 0 ;$$

$$(90) f^3(x) \equiv cx + b ; \quad x \geq 0.$$

where  $a$ ,  $b$  and  $c$  are arbitrary constants.

**Proof:** Repeat the proof of Proposition 7 above until the functional equation (85) is obtained. Now apply Proposition 1 to get the  $f$  solution  $f(x) \equiv cx$  for  $x > 0$ . Extend the domain of definition of this function to  $x = 0$  by continuity and the rest of the proof of Proposition 7 goes through. Q.E.D.

## 7. The Second Fundamental Pexider Functional Equation

Recall that the second Pexider functional equation was  $f(x + y) = g(x)h(y)$ . We replace the functions  $f$ ,  $g$  and  $h$  by  $f^1$ ,  $f^2$  and  $f^3$  respectively and rewrite the equation as follows:

$$(91) f^1(x + y) = f^2(x)f^3(y) ; \quad x \in \mathbb{R}^1 ; y \in \mathbb{R}^1.$$

It is obvious that

$$(92) f^1(x) \equiv 0 ; f^2(x) \equiv 0 \text{ and } f^3(x) \text{ is arbitrary;} \quad x \in \mathbb{R}^1$$

is a solution to (91). Similarly,

$$(93) f^1(x) \equiv 0 ; f^3(x) \equiv 0 \text{ and } f^2(x) \text{ is arbitrary;} \quad x \in \mathbb{R}^1$$

is also a solution to (91). We call these solutions to (91) the *trivial solutions*.

**Lemma 5:** Suppose

$$(94) f^2(0) = 0.$$

Then the only solution to (91) is a trivial solution.

**Proof:** Substitute  $x = 0$  into (91) and use (94) to obtain the following equation:

$$(95) \quad f^1(y) = 0f^3(y) = 0 \quad y \in \mathbb{R}^1.$$

Thus  $f^1$  must be identically equal to zero. Now substitute this fact back into (91) and we obtain the following equation:

$$(96) \quad 0 = f^2(x)f^3(y); \quad x \in \mathbb{R}^1; y \in \mathbb{R}^1.$$

If  $f^3(y^*) \neq 0$  for any  $y^* \in \mathbb{R}^1$ , then (96) implies that

$$(97) \quad f^2(x) = 0; \quad \text{for all } x \in \mathbb{R}^1.$$

Hence if  $f^3(y^*) \neq 0$  for any  $y^* \in \mathbb{R}^1$ , we obtain a trivial solution to (91). On the other hand, if  $f^3(y) = 0$  for all  $y \in \mathbb{R}^1$ , we again obtain a trivial solution to (91). Hence, under the supposition (94), we always obtain a trivial solution to (91). Q.E.D.

In a similar fashion, we can show if

$$(98) \quad f^3(0) = 0,$$

then again, we will always obtain a trivial solution to (91). Using the above results, we can now solve the second Pexider equation.

**Proposition 9:** Let  $f^1(x)$ ,  $f^2(x)$  and  $f^3(x)$  be continuous functions of one variable defined for  $x \in \mathbb{R}^1$ . Suppose that these functions satisfy (91) above for all  $x$  and  $y$ . Then *either* we have a trivial solution defined by (92) or (93) *or* these functions must be equal to the following functions:

$$\begin{aligned} (99) \quad f^1(x) &\equiv abe^{cx}; & x \in \mathbb{R}^1; \\ (100) \quad f^2(x) &\equiv ae^{cx}; & x \in \mathbb{R}^1; \\ (101) \quad f^3(x) &\equiv be^{cx}; & x \in \mathbb{R}^1 \end{aligned}$$

where  $a \neq 0$ ,  $b \neq 0$  and  $c \in \mathbb{R}^1$  are constants.

**Proof:** Substitute  $x = 0$  into (91) and we obtain the following equation:

$$(102) \quad f^1(y) = f^2(0)f^3(y); \quad y \in \mathbb{R}^1.$$

Define the constant  $a$  as:

$$(103) \quad a \equiv f^2(0).$$

Now if  $f^2(0) = 0$ , by Lemma 5 above, we obtain a trivial solution to (91). Hence, in what follows, we assume:

$$(104) \quad a \neq 0.$$

Using (102)-(104), we can define  $f^3$  in terms of  $f^1$  as follows:

$$(105) \quad f^3(x) = a^{-1}f^1(x); \quad x \in \mathbb{R}^1.$$

Now substitute  $y = 0$  into (91) and we obtain the following equation:

$$(106) \quad f^1(x) = f^2(x)f^3(0); \quad x \in \mathbb{R}^1.$$

Define the constant  $b$  as:

$$(107) \quad b \equiv f^3(0).$$

Now if  $f^3(0) = 0$ , by Lemma 5 above, we obtain a trivial solution to (91). Hence, in what follows, we assume:

$$(108) \quad b \neq 0.$$

Using (106)-(108), we can define  $f^2$  in terms of  $f^1$  as follows:

$$(109) \quad f^2(x) = b^{-1}f^1(x); \quad x \in \mathbb{R}^1.$$

Substitute (105) and (109) into (91), which eliminates  $f^2$  and  $f^3$  and we find that  $f^1$  must satisfy the following functional equation:

$$(110) \quad f^1(x+y) = a^{-1}b^{-1}f^1(x)f^1(y); \quad x \in \mathbb{R}^1; y \in \mathbb{R}^1.$$

Now define a new function  $f(x)$  in terms of  $f^1$  as follows:

$$(111) \quad f(x) \equiv a^{-1}b^{-1}f^1(x); \quad x \in \mathbb{R}^1.$$

Use (111) to solve for  $f^1$  in terms of  $f$ :

$$(112) \quad f^1(x) = abf(x).$$

Substitute this  $f^1$  solution into (110) and we obtain the following functional equation involving only  $f$ :

$$(113) \quad \begin{aligned} abf(x+y) &= a^{-1}b^{-1}[abf(x)][abf(y)]; & x \in \mathbb{R}^1; y \in \mathbb{R}^1 \text{ or} \\ f(x+y) &= f(x)f(y); & x \in \mathbb{R}^1; y \in \mathbb{R}^1. \end{aligned}$$

By Proposition 4 above, the only nontrivial solution to (113) is

$$(114) \quad f(x) \equiv e^{cx}; \quad x \in \mathbb{R}^1$$

where  $c$  is an arbitrary constant. Now substitute (114) back into (112), (109) and (105) and we obtain the nontrivial solutions to (91) defined by the functions (99)-(101). Q.E.D.

Note that if  $a = 0$  or  $b = 0$ , then the solution functions defined by (99)-(101) reduce to a trivial solution. Hence we could have simplified the statement of the Proposition by omitting the mention of trivial solutions in the first part of the Proposition and just defining the solution functions as (99)-(101) for arbitrary constants  $a, b$  and  $c$ .

In the above Proposition, the domain of definition for  $f^1$ ,  $f^2$  and  $f^3$  was all of  $\mathbb{R}^1$ . We can obtain a counterpart to Proposition 9 where the functions are defined only over positive real numbers but we need to extend this domain of definition to include the origin as well. Thus again we need the limits in (86) above to exist and be finite numbers. If this is the case, then we can change the domains of definition for the functions in (91) to  $x \geq 0$  and  $y \geq 0$ ; i.e., we now consider the following functional equation:

$$(115) \quad f^1(x+y) = f^2(x)f^3(y); \quad x \geq 0; y \geq 0.$$

**Proposition 10:** Let  $f^1(x)$ ,  $f^2(x)$  and  $f^3(x)$  be continuous functions of one variable defined for  $x$  nonnegative. Suppose that these functions satisfy (115) above for all  $x$  and  $y$ . Then these functions must be equal to the functions defined by (99)-(101) where  $a, b$  and  $c$  are arbitrary constants.

**Proof:** Repeat the proof of Proposition 9 above until the functional equation (113) is obtained. Now apply Proposition 3 to get the  $f$  solution  $f(x) \equiv e^{cx}$  for  $x > 0$ . Extend the domain of definition of this function to  $x = 0$  by continuity and the rest of the proof of Proposition 9 goes through. Q.E.D.

Propositions 8 and 10 are important in economic applications where the domains of definition of the relevant functions are usually restricted to positive or nonnegative  $x$ .

## 8. The Third Fundamental Pexider Functional Equation

Recall that the third Pexider functional equation was  $f(xy) = g(x) + h(y)$ . We replace the functions  $f, g$  and  $h$  by  $f^1, f^2$  and  $f^3$  respectively and rewrite the equation as follows:

$$(116) \quad f^1(xy) = f^2(x) + f^3(y); \quad x > 0; y > 0.$$

**Proposition 11:** Let  $f^1(x)$ ,  $f^2(x)$  and  $f^3(x)$  be continuous functions of one variable defined for positive  $x$ . Suppose that these functions satisfy (116) above for all  $x$  and  $y$ . Then these functions must be equal to the following functions:

$$(117) \quad f^1(x) \equiv c \ln x + a + b; \quad x > 0;$$

$$(118) \quad f^2(x) \equiv c \ln x + a; \quad x > 0;$$

$$(119) \quad f^3(x) \equiv c \ln x + b; \quad x > 0$$

where  $a, b$  and  $c$  are arbitrary constants.

**Proof:** Substitute  $x = 1$  into (116) and we obtain the following equation:

$$(120) f^1(y) = f^2(1) + f^3(y) ; \quad y > 0.$$

Define the constant a as:

$$(121) a \equiv f^2(1).$$

Using (120) and (121), we can define  $f^3$  in terms of  $f^1$  as follows:

$$(122) f^3(y) = f^1(y) - a ; \quad y > 0.$$

Now substitute  $y = 1$  into (116) and we obtain the following equation:

$$(123) f^1(x) = f^2(x) + f^3(1) ; \quad x > 0.$$

Define the constant b as:

$$(124) b \equiv f^3(1).$$

Using (123)-(124), we can define  $f^2$  in terms of  $f^1$  as follows:

$$(125) f^2(x) = f^1(x) - b ; \quad x > 0.$$

Now substitute (125) and (122) into (116) and we obtain the following equation that involves only  $f^1$ :

$$(126) f^1(xy) = f^1(x) - b + f^1(y) - a ; \quad x > 0 ; y > 0.$$

Use  $f^1$  in order to define a new function f as follows:

$$(127) f(x) \equiv f^1(x) - a - b ; \quad x > 0.$$

Now use (127) in order to write  $f^1$  in terms of f:

$$(128) f^1(x) = f(x) + a + b ; \quad x > 0.$$

Substitute (128) into (126) and obtain the following equation involving only f:

$$(129) f(xy) + a + b = [f(x) + a + b] - b + [f(y) + a + b] - a \quad x > 0 ; y > 0 \text{ or} \\ f(xy) = f(x) + f(y).$$

But (129) means f satisfies Cauchy's third functional equation and hence by Proposition 5, f must equal:

$$(130) f(x) \equiv \text{clnx} ; \quad x > 0$$

where  $c$  is an arbitrary constant. Substitute (130) into (128) and we find  $f^1$  is defined by (117). Then substitute (117) into (122) and (125) and we find that  $f^2$  and  $f^3$  must be defined by (118) and (119). Q.E.D.

## 9. The Fourth Fundamental Pexider Functional Equation

Recall that the fourth Pexider functional equation was  $f(xy) = g(x)h(y)$ . We replace the functions  $f$ ,  $g$  and  $h$  by  $f^1$ ,  $f^2$  and  $f^3$  respectively and rewrite the equation as follows:

$$(131) \quad f^1(xy) = f^2(x)f^3(y); \quad x > 0; y > 0.$$

It is obvious that

$$(132) \quad f^1(x) \equiv 0; f^2(x) \equiv 0 \text{ and } f^3(x) \text{ is arbitrary}; \quad x > 0$$

is a solution to (131). Similarly,

$$(133) \quad f^1(x) \equiv 0; f^3(x) \equiv 0 \text{ and } f^2(x) \text{ is arbitrary}; \quad x > 0$$

is also a solution to (131). We call these solutions to (131) the *trivial solutions*.

**Lemma 6:** Suppose

$$(134) \quad f^2(1) = 0.$$

Then the only solution to (131) is a trivial solution.

**Proof:** Substitute  $x = 1$  into (131) and use (134) to obtain the following equation:

$$(135) \quad f^1(y) = 0f^3(y) = 0 \quad y > 0.$$

Thus  $f^1$  must be identically equal to zero. Now substitute this fact back into (131) and we obtain the following equation:

$$(136) \quad 0 = f^2(x)f^3(y); \quad x > 0; y > 0.$$

If  $f^3(y^*) \neq 0$  for any  $y^* \in \mathbb{R}^1$ , then (136) implies that

$$(137) \quad f^2(x) = 0; \quad x > 0.$$

Hence if  $f^3(y^*) \neq 0$  for any  $y^* \in \mathbb{R}^1$ , we obtain a trivial solution to (131). On the other hand, if  $f^3(y) = 0$  for all  $y > 0$ , we again obtain a trivial solution to (131). Hence, under the supposition (134), we always obtain a trivial solution to (131). Q.E.D.

In a similar fashion, we can show if

$$(138) f^3(1) = 0,$$

then again, we will always obtain a trivial solution to (131). Using the above results, we can now solve the fourth Pexider equation.

**Proposition 12:** Let  $f^1(x)$ ,  $f^2(x)$  and  $f^3(x)$  be continuous functions of one variable defined for positive  $x$ . Suppose that these functions satisfy (131) above for all  $x$  and  $y$ . Then these functions must be equal to the following functions:

$$(139) f^1(x) \equiv abx^c; \quad x > 0;$$

$$(140) f^2(x) \equiv ax^c; \quad x > 0;$$

$$(141) f^3(x) \equiv bx^c; \quad x > 0$$

where  $a$ ,  $b$  and  $c$  are arbitrary constants.

**Proof:** Substitute  $x = 1$  into (131) and we obtain the following equation:

$$(142) f^1(y) = f^2(1)f^3(y); \quad y > 0.$$

Define the constant  $a$  as:

$$(143) a \equiv f^2(1).$$

If  $a = 0$ , then by Lemma 6, we obtain a trivial solution to (131). Hence, we will assume that  $a \neq 0$ . Using (142) and (143), we can define  $f^3$  in terms of  $f^1$  as follows:

$$(144) f^3(y) = a^{-1}f^1(y); \quad y > 0.$$

Now substitute  $y = 1$  into (131) and we obtain the following equation:

$$(145) f^1(x) = f^2(x)f^3(1); \quad x > 0.$$

Define the constant  $b$  as:

$$(146) b \equiv f^3(1).$$

If  $b = 0$ , then by Lemma 6, we obtain a trivial solution to (131). Hence, we will assume that  $b \neq 0$ . Using (145)-(146), we can define  $f^2$  in terms of  $f^1$  as follows:

$$(147) f^2(x) = b^{-1}f^1(x); \quad x > 0.$$

Now substitute (144) and (147) into (131) and we obtain the following equation that involves only  $f^1$ :

$$(148) f^1(xy) = a^{-1}b^{-1}f^1(x)f^1(y); \quad x > 0; y > 0.$$

Use  $f^1$  in order to define a new function  $f$  as follows:

$$(149) f(x) \equiv a^{-1}b^{-1}f^1(x); \quad x > 0.$$

Now use (149) in order to write  $f^1$  in terms of  $f$ :

$$(150) f^1(x) = abf(x); \quad x > 0.$$

Substitute (150) into (148) and obtain the following equation involving only  $f$ :

$$(151) \begin{aligned} abf(xy) &= a^{-1}b^{-1}[abf(x)][abf(y)] && x > 0; y > 0 \text{ or} \\ f(xy) &= f(x)f(y). \end{aligned}$$

But (151) means  $f$  satisfies Cauchy's fourth functional equation and hence by Proposition 6,  $f$  must equal:

$$(152) f(x) \equiv x^c; \quad x > 0$$

where  $c$  is an arbitrary constant. Substitute (152) into (150) and we find  $f^1$  is defined by (139) where  $a$  and  $b$  are not equal to zero. Then substitute (139) into (147) and (144) and we find that  $f^2$  and  $f^3$  must be defined by (140) and (141) with  $a \neq 0$  and  $b \neq 0$ . However, if we let  $a$  or  $b$  equal 0, then we obtain the trivial solutions to (131) so that all of the solutions to (131) are defined by (139)-(141) where  $a$ ,  $b$  and  $c$  are unrestricted constants. Q.E.D.

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