INDEX NUMBER THEORY AND MEASUREMENT ECONOMICS

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CHAPTER 1: EARLY APPROACHES TO INDEX NUMBER THEORY

1. Index Number Purpose and Overview.

"The answer to the question what is the *Mean* of a given set of magnitudes cannot in general be found, unless there is given also the object for the sake of which a mean value is required. There are as many kinds of average as there are purposes; and we may almost say in the matter of prices as many purposes as writers. Hence much vain controversy between persons who are literally at cross purposes." F.Y. Edgeworth (1888; 347).

The number of physically distinct goods and unique types of services that consumers can purchase is in the millions. On the business or production side of the economy, there are even more commodities that are actively traded. This is because firms not only produce commodities for final consumption, they also produce exports and intermediate commodities that are demanded by other producers. Firms collectively also use millions of imported goods and services, thousands of different types of labor services and hundreds of thousands of specific types of capital. If we further distinguish physical commodities by their geographic location or by the season or time of day that they are produced or consumed, then there are *billions* of commodities that are traded within each year of any advanced economy. Yet most macroeconomic models have only half a dozen quantity variables and many have only three: output, labor and capital. The models used in applied microeconomics generally have less than 20 or so quantity variables. The question that this book addresses is: *how exactly should the microeconomic information involving possibly millions of prices and quantities be aggregated into a smaller number of price and quantity variables?* This is the basic *index number problem.*[1]

Note that we have posed the index number problem in the context of microeconomic theory; i.e., given that we wish to implement some economic model based on producer or consumer theory, what is the "best" method for constructing a set of aggregates for the model? However, when constructing aggregate prices or quantities, other points of view (that do not rely on economics) are possible. We will also consider these alternative points of view in this book but the primary focus will be on economic approaches to index number theory. Thus in sections 2 to 7 below, we consider some of the early noneconomic approaches to index number theory. Another noneconomic approach is the *test* or *axiomatic* approach to index number theory. In this approach, we attempt to determine the functional form for the price and quantity aggregation functions by asking that these aggregation functions have various intuitively plausible *properties*. Given the importance of this approach, we devote chapter 3 below to it.[2]

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[1] For a comprehensive modern review of index number theory, see Balk (2008). For shorter more recent surveys that focus on the economic approach, see Kohli (2011) and Diewert (2009a) (2012).

[2] In many applications, the economic approach will not be applicable, perhaps because the underlying assumption of price taking optimizing behavior on the part of economic agents (consumers or producers) may not be satisfied. Under these conditions, we need an alternative aggregation principle. Two of the most promising alternative approaches are the stochastic approach considered in sections 3 and 4 below and the test approach considered in chapter 3.

In order to further define the index number problem, we introduce a bit of notation. We specify accounting periods, t = 0,1,…,T for which we have micro price and quantity data for N commodities pertaining to transactions by a definite consumer or producer (or a definite group of consumers or producers). Denote the price and quantity of commodity n in period t by p_n^t and q_n^t respectively for n = 1,2,…,N and t = 0,1,…,T. Before proceeding further, we need to discuss the exact meaning of the microeconomic prices and quantities if there are *multiple* transactions for say commodity n within period t. In this case, it is natural to interpret q_n^t as the *total* amount of commodity n transacted within period t. In order to conserve the value of transactions, it is necessary that p_n^t be defined as a *unit value*[3]; i.e., p_n^t must be equal to the value of transactions in commodity n for period t divided by the total quantity transacted, q_n^t. However, for now, we will follow Fisher[4] and Hicks[5] and assume that the accounting period has been chosen so that variations in commodity prices within a period are very small compared to their variations between periods. For t = 0,1,…,T, define *the value of transactions in period t* as:

(1) $V^t \equiv \sum_{n=1}^N p_n^t q_n^t \equiv p^t \cdot q^t$

where $p^t \equiv (p_1^t,\ldots, p_N^t)$ is the period t price vector, $q^t \equiv (q_1^t,\ldots, q_N^t)$ is the period t quantity vector and $p^t \cdot q^t$ denotes the inner product of these two vectors.

Using the above notation, we can now state the following *(levels) version of the index number problem using the axiomatic approach*: for t = 0,1,…,T, find scalar numbers P^t and Q^t such that

(2) $V^t = P^t Q^t$.

The number P^t is interpreted as an aggregate period t price level while the number Q^t is interpreted as an aggregate period t quantity level. The aggregate price level P^t is allowed to be a function of the period t price vector, p^t while the aggregate period t quantity level Q^t is allowed to be a function of the period t quantity vector, q^t; i.e., we have

(3) $P^t = c(p^t)$ and $Q^t = f(q^t)$; t = 0,1,…,T.

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[3] We will discuss in more detail what is the "right" concept of a price at the first stage of aggregation in section 6 below..

[4] "Throughout this book 'the price' of any commodity or 'the quantity' of it for any one year was assumed given. But what is such a price or quantity? Sometimes it is a single quotation for January 1 or July 1, but usually it is an average of several quotations scattered throughout the year. The question arises: On what principle should this average be constructed? The *practical* answer is *any* kind of average since, ordinarily, the variation during a year, so far, at least, as prices are concerned, are too little to make any perceptible difference in the result, whatever kind of average is used. Otherwise, there would be ground for subdividing the year into quarters or months until we reach a small enough period to be considered practically a point. The quantities sold will, of course, vary widely. What is needed is their sum for the year (which, of course, is the same thing as the simple arithmetic average of the per annum rates for the separate months or other subdivisions). In short, the simple arithmetic average, both of prices and of quantities, may be used. Or, if it is worth while to put any finer point on it, we may take the weighted arithmetic average for the prices, the weights being the quantities sold." Irving Fisher (1922; 318).

[5] "I shall define a week as that period of time during which variations in prices can be neglected. For theoretical purposes this means that prices will be supposed to change, not continuously, but at short intervals. The calendar length of the week is of course quite arbitrary; by taking it to be very short, our theoretical scheme can be fitted as closely as we like to that ceaseless oscillation which is a characteristic of prices in certain markets." J.R. Hicks (1946; 122).

The functions c and f are to be determined somehow. Note that we are requiring that the functional forms for the price aggregation function c and for the quantity aggregation function f be independent of time. This is a reasonable requirement since there is no reason (at this stage[6]) to change the method of aggregation as time changes.

Substituting (3) and (2) into (1) and dropping the superscripts t means that c and f must satisfy the following functional equation for all strictly positive price and quantity vectors:

(4) $c(p)f(q) = p \cdot q \equiv \sum_{n=1}^{N} p_n q_n$ for all $p >> 0_N$ and for all $q >> 0_N$.

We now could ask what properties should the price aggregation function c and the quantity aggregation function f have? We could assume that c and f satisfied various "reasonable" properties and hope that these properties would determine the functional form for c and f. However, it turns out that we only have to make the following very weak *positivity* assumptions on f and c:

(5) $c(p) > 0$ for all $p >> 0_N$; $f(q) > 0$ for all $q >> 0_N$.

Let 1_N denote an N dimensional vector of ones. Then (5) implies that when $p = 1_N$, $c(1_N)$ is a positive number, a say, and when $q = 1_N$, then $f(1_N)$ is also a positive number, b say; i.e., (5) implies that c and f satisfy:

(6) $c(1_N) = a > 0$; $f(1_N) = b > 0$.

Eichhorn (1978; 144) proved the following result:

Proposition 1: If the number of commodities $N > 1$, then there do not exist any functions c and f that satisfy (4) and (5).

Proof: Let $p = 1_N$ and substitute the first equation in (6) into (4) in order to obtain the following equation:

(7) $f(q) = 1_N \cdot q / a = \sum_{n=1}^{N} q_n / a$ for all $q \equiv (q_1,\ldots,q_N) >> 0_N$.

Now let $q = 1_N$ and substitute the second equation in (6) into (4) in order to obtain the following equation:

(8) $c(p) = 1_N \cdot p / b = \sum_{n=1}^{N} p_n / b$ for all $p \equiv (p_1,\ldots,p_N) >> 0_N$.

Now substitute (7) and (8) into the left hand side of (4) and we obtain the following equation:

(9) $\{ \sum_{n=1}^{N} p_n / b\}\{\sum_{n=1}^{N} q_n / a\} = \sum_{n=1}^{N} p_n q_n$ for all $p >> 0_N$ and for all $q >> 0_N$.

[6] At a later stage, we may want to introduce taste changes (see Caves, Christensen and Diewert (1982), Balk (1989) and Diewert (2009a) on this topic) or technological progress as factors which may cause our aggregator functions f and c to change over time.

If N is greater than one, it is obvious that equation (9) cannot be satisfied for all strictly positive p and q vectors. Q.E.D.

Thus this test approach to index number theory comes to an abrupt halt; it is fruitless to look for price and quantity level functions, $P^t = c(p^t)$ and $Q^t = f(q^t)$, that satisfy (2) or (4) and also satisfy the very reasonable positivity requirements, (5).[7]

Even though the above version of the test approach to index number theory fails, it turns out that the economic approach is somewhat more successful. This economic approach is due to Shephard (1953) (1970) and Samuelson and Swamy (1974) and we outline it below.

We now assume that f(q) is the utility function of a consumer and this consumer minimizes the cost of achieving the period t utility level $u^t \equiv f(q^t)$ for periods t = 0,1,…,T. Thus we assume that the observed period t consumption vector q^t solves the following period t cost minimization problem:

$$(10)\quad \min_q \{p^t\cdot q \ :\ f(q) \geq u^t = f(q^t)\ \} = p^t\cdot q^t\ ; \qquad t = 0,1,\ldots,T.$$

The period t price vector for the N commodities under consideration that the consumer faces is p^t. We place some regularity conditions on the utility function f. We assume that f is a continuous function defined over the nonnegative orthant, $\{q : q \geq 0_N\}$ and is positive, concave and (positively) linearly homogeneous over the strictly positive orthant, $\Omega \equiv \{q : q >> 0_N\}$.[8] The concavity property means that f satisfies the following inequalities:[9]

$$(11)\ f(\lambda q^1 + (1-\lambda)q^2) \geq \lambda f(q^1) + (1-\lambda)f(q^2) \ \text{ for all } 0 \leq \lambda \leq 1 \text{ and all } q^1 >> 0_N \text{ and } q^2 >> 0_N.$$

The linear homogeneity property means that f satisfies the following property:[10]

$$(12)\quad f(\lambda q) = \lambda f(q) \qquad \text{for all } \lambda > 0 \text{ and all } q >> 0_N.$$

Now use the utility function f to define the consumer's expenditure or cost function, C(u,p), as follows. For positive commodity prices $p >> 0_N$ and a positive utility level u, define the minimum cost of achieving the given utility level u as:

[7] The proof of Proposition 1 actually only requires the weaker positivity property (6), which is implied by (5).

[8] Note that $q \geq 0_N$ means that each component of the N dimensional vector q is nonnegative, $q >> 0_N$ means that each component of q is positive and $q > 0_N$ means that $q \geq 0_N$ but $q \neq 0_N$; i.e, q is nonnegative but at least one component is positive.

[9] Given that f is linearly homogeneous, the meaning of the concavity assumption is that each indifference curve of f has the curvature that is always assumed in elementary economics courses; i.e., for each utility level u > 0, the upper level set, $L(u) \equiv \{q : f(q) \geq u\}$ is a convex set. A set S is convex if and only if given any two points belonging to S, then the straight line segment joining those two points also belongs to S. In technical terms, S is convex if and only if $q^1\in S$, $q^2\in S$, $0 \leq \lambda \leq 1$ implies $\lambda q^1 + (1-\lambda)q^2\in S$.

[10] This assumption is fairly restrictive in the consumer context. It implies that all income elasticities of demand are unity, which is contradicted by empirical evidence.

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\begin{aligned}
(13)\ C(u,p) &\equiv \min_q \{p\cdot q : f(q) \geq u\} \\
&= \min_q \{p\cdot q : (1/u)f(q) \geq 1\} && \text{since } u > 0 \\
&= \min_q \{p\cdot q : f(q/u) \geq 1\} && \text{using the linear homogeneity property (12)} \\
&= u \min_q \{p\cdot q/u : f(q/u) \geq 1\} && \text{using } u > 0 \\
&= u \min_z \{p\cdot z : f(z) \geq 1\} && \text{letting } z = q/u \\
&= u\, C(1,p) && \text{using definition (13) with } u = 1 \\
(14)\qquad &= u\, c(p)
\end{aligned}
$$

where $c(p) \equiv C(1,p)$ is the *unit cost function* that is dual to f. It can be shown that the unit cost function c(p) satisfies the same regularity conditions that f satisfied; i.e., c(p) is positive, concave and (positively) linearly homogeneous over the strictly positive orthant in price space, $\Omega \equiv \{p : p >> 0_N\}$.[11] Substituting (14) into (10) leads to the following equations:

$$(15)\quad p^t\cdot q^t = c(p^t)f(q^t) \qquad \text{for } t = 0,1,\ldots,T.$$

Obviously, we can identify the period t unit cost, $c(p^t)$, as the period t price level P^t and the period t level of utility, $f(q^t)$, as the period t quantity level Q^t.[12]

Why does the economic approach to the determination of aggregate price and quantity levels work while the test approach fails? It is because the two approaches make *different assumptions*. In equations (15), the quantity vector, q^t, *cannot be specified independently of the price vector*, p^t; i.e., once p^t is specified, q^t is determined as the solution to the minimization problem in (10). In the test approach, p and q could vary *completely independently* in equation (4). This is the major difference between the economic and test approaches to index number theory.

Let us return to equations (15). Although the economic approach has successfully decomposed the period t expenditure $p^t\cdot q^t$ into a price component, $c(p^t)$, and a quantity component, $f(q^t)$, this is not a very useful decomposition for price statisticians working in statistical agencies. This is because the price statistician has no way of knowing exactly what the "correct" functional forms for c(p) or f(q) are. Thus although it is nice to know that the required theoretical decomposition exists, price statisticians need to know what the correct functional forms for c and f are in order to produce price and quantity indexes. From this "practical" point of view, both the test and economic approaches to the determination of aggregate price and quantity levels fail.[13]

Fortunately, this is not the end of the index number story. It turns out, that if we change the question that we are trying to answer slightly, then *both* the economic and test approaches yield

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[11] For additional material on duality theory and the properties of f and c, see Samuelson (1953), Shephard (1953) and Diewert (1974) (1993b; 107-123).

[12] There is also a producer theory interpretation of the above theory; i.e., let f be the producer's (constant returns to scale) production function, let p be a vector of input prices that the producer faces, let q be an input vector and let $u = f(q)$ be the maximum output that can be produced using the input vector q. $C(u,p) \equiv \min_q \{p\cdot q : f(q) \geq u\}$ is the producer's cost function in this case and $c(p^t)$ can be identified as the period t input price level while $f(q^t)$ is the period t aggregate input level.

[13] From the viewpoint of econometric methods however, the economic approach is not a failure: we could specify a flexible functional form for c or f and econometrically estimate the unknown parameters in these functions. This approach is not feasible for statistical agencies who have to aggregate hundreds of thousands of prices.

practical solutions to the index number problem. The change is that instead of trying to decompose the value of the aggregate into price and quantity components for a single period, we instead attempt to decompose a *value ratio* pertaining to two periods, say periods 0 and 1, into a *price change component* P times a *quantity change component* Q.[14] Thus we now look for two functions of 4N variables, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ such that:

(16) $p^1 \cdot q^1 / p^0 \cdot q^0 = P(p^0,p^1,q^0,q^1)\, Q(p^0,p^1,q^0,q^1)$.

If we take the test approach, then we want equation (16) to hold for all positive price and quantity vectors pertaining to the two periods under consideration, p^0,p^1,q^0,q^1. If we take the economic approach, then only the price vectors p^0 and p^1 are regarded as independent variables while the quantity vectors, q^0 and q^1, are regarded as dependent variables. In chapters 2 and 3 below, we will pursue the test approach and in chapters 4 to 6, we will take the economic approach. In all of these chapters, we will be taking a *bilateral approach to index number theory*; i.e., in making price and quantity comparisons between any two time periods, the relevant indexes use *only* price and quantity information that pertains to the two periods under consideration. It is also possible to take a *multilateral approach*; i.e., we look for functions, P^t and Q^t, that are functions of *all* of the price and quantity vectors, $p^0,p^1,\ldots,p^T,q^0,q^1,\ldots,q^T$. Thus we look for 2(T+1) functions, $P^t(p^0,p^1,\ldots,p^T,q^0,q^1,\ldots,q^T)$ and $Q^t(p^0,p^1,\ldots,p^T,q^0,q^1,\ldots,q^T)$, $t = 0,1,\ldots,T$, such that

(17) $p^t \cdot q^t = P^t(p^0,p^1,\ldots,p^T,q^0,q^1,\ldots,q^T)\, Q^t(p^0,p^1,\ldots,p^T,q^0,q^1,\ldots,q^T)$ for $t = 0,1,\ldots,T$.

Note the difference between (17) and equations (1)-(3) which can be rewritten as:

(18) $p^t \cdot q^t = c(p^t)f(q^t)$ for $t = 0,1,\ldots,T$.

The multilateral system of functions P^t and Q^t that might be solutions to (17) is much more general than the two functions c and f that might be solutions to (18).

We pursue the multilateral approach to index number theory in a subsequent chapter. However, this branch of index number theory is not nearly as well developed as the bilateral approaches and so the reader should not expect a complete treatment of all multilateral approaches.

The above introductory material should give the reader an idea of our main approaches to index number theory. To conclude this section, we briefly discuss two additional important topics; namely the *domain of definition problem* and the question of *index number purpose*.

The domain of definition problem is the problem of deciding what set of value transactions should be included in the value aggregates that are to be decomposed into price and quantity components. Over what time period should we aggregate transactions? Which set of economic agents should be included in the transactions? What set of commodities should be included in the transactions?

[14] In the economic approach, P is interpreted to be the ratio of unit cost functions, $c(p^1)/c(p^0)$, and Q is interpreted to be the utility ratio, $f(q^1)/f(q^0)$. Note that the linear homogeneity assumption on the utility function f effectively cardinalizes utility.

With respect to the time period, we will often be constrained by data availability. In general, the more rapid is general inflation, then the shorter we will want to make our time period. However, the expense of collecting data will increase as the time period becomes shorter and the "quality" of the data will also generally decline. With respect to producer commodities (business data), the shorter is the time period, then the difference between the period when say an input is used in production and when it is paid for can become significant. On the output side, the periods when an output is produced, when it is shipped, when it is sold and when it is finally paid for can differ if the length of the period is very short. In addition, daily, weekly or monthly seasonal fluctuations can cause substantial variability in the data when the time period is very short. With respect to consumer commodities (household data), the shorter is the time period, the more consumption will fluctuate due to seasonality in the pattern of household purchases. Thus in general, the shorter is the period, the more lumpy and erratic the quantity information will be.[15]

Which economic agents should have their transactions included in the domain of definition of the aggregate? The answer to this question depends on the purpose of the aggregation and we will discuss this below.

Which commodity transactions should be included in the aggregate? Again, the answer to this question depends on the purpose for which the aggregation is being done. If we are trying to measure the productivity performance of a firm, an industry or an entire economy, then it will be necessary to have price and quantity data on outputs produced by the production unit during the time periods under consideration and inputs used by the production unit during the same time periods, since productivity growth is generally defined as output growth divided by input growth. If we are trying to measure the consumer price inflation faced by a group of households (perhaps all households in a region, perhaps retired households or perhaps "poor" households), then we have to have information on all consumption commodity transactions by the group of households for the time periods under consideration. However, suppose some of these households undertake business activities at home. How are we to split the telephone bill between business use and personal consumption use? Suppose some of these households use their home computers for business and pleasure purposes? How are we to allocate the computer cost between consumption and production uses? How should we treat home renovation expenditures? At first sight, it seems to be an obvious consumption expenditure but suppose some households in our domain of definition group systematically buy old homes, renovate them, sell them and repeat the process. Then these renovation expenditures seem to be business expenditures rather than consumption expenditures. What we are saying here is that the problems of classification are not trivial!

There is another classification problem that should be mentioned here that occurs in both the consumer and producer contexts: namely, how should purchases of consumer and producer durables be treated? Should the entire purchase price of say a consumer durable good like a car or house be charged to the period of purchase? This is the *money outlays* or *acquisition cost approach* to the treatment of consumer durables. The problem with this approach is that the *services* of the purchased goods are not confined to period of purchase. By the definition of a

[15] If the time period is reduced to say one hour (which was actually necessary during the German hyperinflation of the 1920's), then obviously, most consumer purchases are zero with an occasional discrete jump in the purchase of one or two commodities. Aggregating over such a tiny time period will obviously not be too illuminating.

durable good (it lasts longer than one period), the purchase will yield a flow of services to the consumer for periods that follow the initial period of purchase. Thus it does not seem appropriate to charge the *entire* purchase price to the initial period of purchase. But how should the purchase price be distributed or allocated across periods? This is the *fundamental problem of accounting*. Accountants discuss this problem in the context of business accounting where a similar cost allocation problem occurs when a firm purchases a durable input. These accounting problems are discussed in the durables chapter for this course and in even more detail in Diewert (2005b) (2009b).

Suppose that we have a definite measurement goal in mind such as measuring consumer price change faced by a group of households. Statistical agencies produce Consumer Price Indexes (CPI's) to measure aggregate price change for the reference population covered by their indexes. However, they also produce subindexes for say food, clothing, shelter, transportation, recreation, services, etc. Similarly, statistical agencies often produce an aggregate producer price index and they decompose this index into various subindexes. Is there any rationale for the choice of these subindexes? In section 7 below, we will review the aggregation theorems of Hicks (1946; 312-313) and Leontief (1936; 54-57), which may cast some light on this question.[16]

What are the main purposes for index numbers? Obviously, the way we have framed the problem is that index numbers can be used to *summarize information* in an efficient way. In the economic approach, the 2N prices and quantities for period t, p^t and q^t, are reduced down to the aggregate period t price, $c(p^t)$ and the aggregate period t quantity, $f(q^t)$. Thus index number theory can be considered a part of *descriptive statistics*, which also tries to summarize information in an efficient manner. However, there are a number of specific uses of index numbers that arise in the context of various microeconomic and macroeconomic contexts:

- They are used to decompose value flows in the National Accounts into price and quantity components;
- they are used to compensate consumers for general changes in consumer prices (in the context of indexed pensions and for social welfare indexation purposes);[17]
- they are used by central bankers as measures of general price inflation in the economy;
- they are used for a wide variety of long term contract escalation purposes and
- they are used in labor negotiations.

Our focus in this book will be on the first purpose; i.e., the decomposition of value changes into price and quantity components. However, there will be some discussion of the second and third purposes listed above in later chapters.

We turn now to a discussion of the early approaches to index number theory. We consider a number of alternatives to the economic approach to the determination of the consumer price index. These alternatives are:

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[16] The best theoretical article on economic rationales for choosing industry and commodity classifications is Triplett (1990).
[17] Triplett (1983) has the best review paper in this area.

- the fixed basket approach;
- the test approach;
- the stochastic approach;
- the Divisia approach and
- the differences approach of Bennet and Montgomery.

These approaches are discussed in sections 2 to 6 below, with the exception of the test approach, which is discussed more fully in chapter 3.

2. The Fixed Basket Approach

"Of this, some idea may be formed from a table in the Appendix comprising a list of articles of general consumption, corn, butcher-meat, manufactures, tropical products, &c. and containing the probable amount of money expended on each by the public. This table is followed by explanatory remarks, of which the object is to show that contracts for a series of years ought to be made with a reference to the power of money in purchasing the necessaries and comforts of life; that after fixing a given sum, say 100*l*. as the amount of an annual salary, the payment in subsequent years should be not necessarily 100*l*., but either 95*l*., 100*l*., or 105*l*., according to the varying power of money in making purchases….
For the details of the table, and the calculations connected with it, we refer to the Appendix: at present we shall, for the sake of illustration, suppose it in operation, and bestow a few paragraphs on the effects that the adoption of such a measure would have on the interests of the country.
In what, it may be asked, would the benefits of it consist? In ascertaining on grounds that would admit of no doubt or dispute, the power in purchase of any given sum in one year, compared to its power of purchase in another. And what would be the practical application of this knowledge? The correction of a long list of anomalies in regard to rents, salaries, wages, &c., arising out of unforeseen fluctuations in our currency." Joseph Lowe (1823; 333-335).

It can be seen that Lowe had a very good grasp of the many uses that an index of consumer prices could be put to. His approach to measuring the price change between periods 0 and 1 was to specify an approximate representative commodity basket[18] quantity vector, $q \equiv (q_1,\ldots,q_N)$, and then calculate the level of prices in period 1 relative to period 0 as

$$(19)\quad P_{Lo}(p^0,p^1,q) \equiv p^1 \cdot q \,/\, p^0 \cdot q$$

where as usual p^0 and p^1 are the commodity price vectors that the consumer (or group[19] of consumers) face in periods 0 and 1 respectively. This first alternative approach to measuring aggregate consumer price change between periods 0 and 1 dates back to William Fleetwood, the Bishop of Ely, who advocated the above method in the book, *Chronicon Precosium* in 1707[20]. The fixed basket approach to measuring price change is intuitively very simple: we simply specify the commodity "list" q and calculate the price index as the ratio of the costs of buying this same list of goods in periods 1 and 0.

As time passed, economists and price statisticians demanded a bit more precision with respect to the specification of the basket vector q. There are two natural choices for the reference basket:

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[18] Lowe (1823; Appendix page 95) suggested that the commodity basket vector q should be updated every five years.
[19] Lowe (1823; 336) also advocated different indexes for different demographic groups of households.
[20] A good account of Fleetwood's contributions can be found in Ferger (1946). See also Diewert (1993a; 34-36) (1997; 128-129).

the period 0 commodity vector q^0 or the period 1 commodity vector q^1. These two choices lead to the Laspeyres (1871) price index P_L defined by (20) and the Paasche (1874) price index P_P defined by (21):[21]

(20) $P_L(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^0 / p^0 \cdot q^0$;

(21) $P_P(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1 / p^0 \cdot q^1$.

The above formulae can be rewritten in an alternative manner that is very useful for statistical agencies. Define the period t expenditure share on commodity n as follows:

(22) $s_n{}^t \equiv p_n{}^t q_n{}^t / p^t \cdot q^t$ for n = 1,…,N and t = 0,1.

Then the Laspeyres index (20) can be rewritten as follows:

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\begin{aligned}
(23)\quad P_L(p^0,p^1,q^0,q^1) &\equiv p^1 \cdot q^0 / p^0 \cdot q^0 \\
&= \textstyle\sum_{n=1}^N p_n{}^1 q_n{}^0 / p^0 \cdot q^0 \\
&= \textstyle\sum_{n=1}^N (p_n{}^1/p_n{}^0)\, p_n{}^0\, q_n{}^0 / p^0 \cdot q^0 \\
&= \textstyle\sum_{n=1}^N (p_n{}^1/p_n{}^0)\, s_n{}^0 \qquad \text{using definitions (22).}
\end{aligned}
$$

Thus the Laspeyres price index P_L can be written as a base period expenditure share weighted average of the N price ratios (or price relatives using index number terminology), $p_n{}^1/p_n{}^0$. The Laspeyres formula (until the very recent past) has been widely used as the intellectual base for country Consumer Price Indexes around the world. To implement it, the country statistical agency collects information on expenditure shares $s_n{}^0$ for the index domain of definition for the base period 0 and then collects information on *prices* alone on an ongoing basis. Thus the CPI can be produced on a timely basis without having to know current period quantity information.

The Paasche index can also be written in expenditure share and price ratio form as follows:

$$
\begin{aligned}
(24)\quad P_P(p^0,p^1,q^0,q^1) &\equiv p^1 \cdot q^1 / p^0 \cdot q^1 \\
&= 1/[\, p^0 \cdot q^1 / p^1 \cdot q^1] \\
&= 1/\, [\textstyle\sum_{n=1}^N p_n{}^0 q_n{}^1 / p^1 \cdot q^1] \\
&= 1/\, [\textstyle\sum_{n=1}^N (p_n{}^0/p_n{}^1)\, p_n{}^1\, q_n{}^1 / p^1 \cdot q^1] \\
&= 1/\, [\textstyle\sum_{n=1}^N (p_n{}^1/p_n{}^0)^{-1}\, s_n{}^1] \qquad \text{using definitions (22)} \\
&= [\textstyle\sum_{n=1}^N (p_n{}^1/p_n{}^0)^{-1}\, s_n{}^1]^{-1}.
\end{aligned}
$$

Thus the Paasche price index P_P can be written as a period 1 (or current period) expenditure share weighted *harmonic* average of the N price ratios.

The problem with these index number formulae is that they are equally plausible but in general, they will give different answers. This suggests that if we require a single estimate for the price

[21] Note that $P_L(p^0,p^1,q^0,q^1)$ does not actually depend on q^1 and $P_P(p^0,p^1,q^0,q^1)$ does not actually depend on q^0. However, it does no harm to include these vectors and the notation indicates that we are in the realm of bilateral index number theory.

change between the two periods, then we should take some sort of evenly weighted average of the two indexes as our final estimate of price change between periods 0 and 1. Examples of such symmetric averages[22] are the arithmetic mean, which leads to the Sidgwick (1883; 68) Bowley (1901; 227)[23] index, $(1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the Fisher[24] (1922) ideal index, P_F defined as

$$(25)\ P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)\ P_P(p^0,p^1,q^0,q^1)]^{1/2}\ .$$

At this point, the fixed basket approach to index number theory is transformed into the *test approach* to index number theory; i.e., in order to determine which of these fixed basket indexes or which averages of them might be "best", we need *criteria* or *tests* or *properties* that we would like our indexes to satisfy. We will pursue this topic in more detail in chapter 3 below but we will give the reader an introduction to this topic in the present section because some of these tests or properties are useful to evaluate other approaches to index number theory.

Let a and b be two positive numbers. Diewert (1993c; 361) defined a *symmetric mean* of a and b as a function m(a,b) that has the following properties:

(26) $m(a,a) = a$ for all $a > 0$ (mean property);
(27) $m(a,b) = m(b,a)$ for all $a > 0, b > 0$ (symmetry property);
(28) $m(a,b)$ is a continuous function for $a > 0, b > 0$ (continuity property);
(29) $m(a,b)$ is a strictly increasing function in each of its variables (increasingness property).

It can be shown that if m(a,b) satisfies the above properties, then it also satisfies the following property:[25]

(30) $\min\{a,b\} \leq m(a,b) \leq \max\{a,b\}$ (min-max property);

i.e., the mean of a and b, m(a,b), lies between the maximum and minimum of the numbers a and b. Since we have restricted the domain of definition of a and b to be positive numbers, it can be seen that an implication of (30) is that m also satisfies the following property:

(31) $m(a,b) > 0$ for all $a > 0, b > 0$ (positivity property).

If in addition, m satisfies the following property, then we say that m is a *homogeneous symmetric mean*:

(32) $m(\lambda a,\lambda b) = \lambda m(a,b)$ for all $\lambda > 0, a > 0, b > 0$.

What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical cost of living index? It is very desirable for a price index formula that depends on the price and

[22] For a discussion of the properties of symmetric averages, see Diewert (1993c).
[23] See Diewert (1993a; 36) and Balk (2008) for additional references to the early history of index number theory.
[24] Bowley (1899; 641) appears to have been the first to suggest the use of this index.
[25] To prove this, use the technique of proof used by Eichhorn and Voeller (1976; 10).

quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test*[26]. We say that the index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

(33) $P(p^1,p^0,q^1,q^0) = 1/ P(p^0,p^1,q^0,q^1)$;

i.e., if we interchange the period 0 and period 1 price and quantity data and evaluate the index, then this new index $P(p^1,p^0,q^1,q^0)$ is equal to the reciprocal of the original index $P(p^0,p^1,q^0,q^1)$.

Diewert (1997; 138) proved the following result:

Proposition 2: The Fisher Ideal price index defined by (25) above is the *only* index that is a homogeneous symmetric average of the Laspeyres and Paasche price indexes, P_L and P_P, and satisfies the time reversal test (33) above.

Proof: In order to prove this proposition, we only require the homogeneous mean function to satisfy the positivity and homogeneity properties, (31) and (32) above.

We define the mean price index P using the function m as follows:

(34) $P(p^0,p^1,q^0,q^1) \equiv m(P_L,P_P) = m(p^1\cdot q^0 / p^0\cdot q^0,\ p^1\cdot q^1 / p^0\cdot q^1)$

where we have used the definitions of P_L and P_P, (20) and (21) above. Since P is supposed to satisfy the time reversal test, we can substitute definition (34) into (33) in order to obtain the following equation:

(35) $m(p^0\cdot q^1/p^1\cdot q^1,\ p^0\cdot q^0/p^1\cdot q^0) = 1/ m(p^1\cdot q^0/p^0\cdot q^0,\ p^1\cdot q^1/p^0\cdot q^1)$.

Letting $a \equiv p^1\cdot q^0/p^0\cdot q^0$ and $b \equiv p^1\cdot q^1/p^0\cdot q^1$, we see that equation (35) can be rewritten as:

(36) $m(b^{-1},a^{-1}) = 1/ m(a,b)$.

Equation (36) can be rewritten as:

(37) $1 = m(a,b)\, m(b^{-1},a^{-1})$
$= a m(1,b/a)\, a^{-1}\, m(a/b,1)$ using property (32) of m
$= m(1,x)m(x^{-1},1)$ letting $x \equiv b/a$
$= m(1,x)x^{-1}m(1,x)$ using property (32) of m.

Equation (37) can be rewritten as:

(38) $x = [m(1,x)]^2$.

[26] See Diewert (1992a; 218) for early references to this test. If we want our price index to have the same property as a single price ratio, then it is important to satisfy the time reversal test. However, other points of view are possible. For example, we may want to use our price index for compensation purposes in which case, satisfaction of the time reversal test is not so important.

Thus using (31), we can take the positive square root of both sides of (38) and obtain

(39) $m(1,x) = x^{1/2}$.

Using property (32) of m again, we have

(40) $m(a,b) = am(1,b/a)$
$= a[b/a]^{1/2}$ using (39)
$= a^{1/2}b^{1/2}$.

Now substitute (40) into (34) and we obtain the Fisher Index. Q.E.D.

It is interesting to note that this symmetric basket approach to index number theory dates back to one of the early pioneers of index number theory, Bowley, as the following quotations indicate:

"If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean … as a first approximation." A. L. Bowley (1901; 227).

"When estimating the factor necessary for the correction of a change found in money wages to obtain the change in real wages, statisticians have not been content to follow Method II only [to calculate a Laspeyres price index], but have worked the problem backwards [to calculate a Paasche price index] as well as forwards. … They have then taken the arithmetic, geometric or harmonic mean of the two numbers so found." A. L. Bowley (1919; 348).[27]

In section 5 below, we will study fixed basket indexes from a slightly different perspective.

We turn now to another approach to the index number problem.

3. The Unweighted Statistical or Stochastic Approach

"In drawing our averages the independent fluctuations will more or less destroy each other; the one required variation of gold will remain undiminished." W. Stanley Jevons (1884; 26).

The stochastic approach to the determination of the price index can be traced back to the work of Jevons and Edgeworth over a hundred years ago[28].

The basic idea behind the stochastic approach is that each price relative, p_n^1/p_n^0 for n = 1,2,…,N can be regarded as an estimate of a common inflation rate α between periods 0 and 1; i.e., it is assumed that

(41) $p_n^1/p_n^0 = \alpha + \varepsilon_n$; $n = 1,2,\ldots,N$

[27] Fisher (1911; 417-418) (1922) also considered the arithmetic, geometric and harmonic averages of the Paasche and Laspeyres indexes.
[28] For references to the literature, see Diewert (1993a; 37-38) (2010a) (2010b).

where α is the common inflation rate and the ε_n are random variables with mean 0 and variance σ^2. The least squares estimator for α is the Carli (1764) price index P_C defined as

(42) $P_C(p^0,p^1) \equiv \sum_{n=1}^N (1/N)\, p_n^1/p_n^0$.

Unfortunately, P_C does not satisfy the time reversal test, i.e., $P_C(p^1,p^0) \neq 1/P_C(p^0,p^1)$[29].

Let us change our stochastic specification as follows: assume that the logarithm of each price relative, $\ln(p_n^1/p_n^0)$, is an unbiased estimate of the logarithm of the inflation rate between periods 0 and 1, β say. Thus we have:

(43) $\ln(p_n^1/p_n^0) = \beta + \varepsilon_n$; $n = 1,2,\ldots,N$

where $\beta \equiv \ln\alpha$ and the ε_n are independently distributed random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for β is the logarithm of the geometric mean of the price relatives. Hence the corresponding estimate for the common inflation rate α[30] is the Jevons (1865) price index P_J defined as:

(44) $P_J(p^0,p^1) \equiv \prod_{n=1}^N (p_n^1/p_n^0)^{1/N}$.

The Jevons price index P_J does satisfy the time reversal test and hence is much more satisfactory than the Carli index P_C. However, both the Jevons and Carli price indexes suffer from a fatal flaw: each price relative p_n^1/p_n^0 is regarded as being equally important and is given an equal weight in the index number formulae (42) and (44). Keynes was particularly critical of this *unweighted stochastic approach* to index number theory. He directed the following criticism towards this approach, which was vigorously advocated by Edgeworth (1923):

"Nevertheless I venture to maintain that such ideas, which I have endeavoured to expound above as fairly and as plausibly as I can, are root-and-branch erroneous. The 'errors of observation', the 'faulty shots aimed at a single bull's eye' conception of the index number of prices, Edgeworth's 'objective mean variation of general prices', is the result of confusion of thought. There is no bull's eye. There is no moving but unique centre, to be called the general price level or the objective mean variation of general prices, round which are scattered the moving price levels of individual things. There are all the various, quite definite, conceptions of price levels of composite commodities appropriate for various purposes and inquiries which have been scheduled above, and many others too. There is nothing else. Jevons was pursuing a mirage.

What is the flaw in the argument? In the first place it assumed that the fluctuations of individual prices round the 'mean' are 'random' in the sense required by the theory of the combination of independent observations. In this theory the divergence of one 'observation' from the true position is assumed to have no influence on the divergences

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[29] In fact Fisher (1922; 66) noted that $P_C(p^0,p^1)\, P_C(p^1,p^0) \geq 1$ unless the period 1 price vector p^1 is proportional to the period 0 price vector p^0; i.e., Fisher showed that the Carli index has a definite upward bias. He urged statistical agencies not to use this formula.

[30] Greenlees (1998) pointed out that although $(1/N) \sum_{n=1}^N \ln(p_n^1/p_n^0)$ is an unbiased estimator for β, the corresponding exponential of this estimator, P_J defined by (32), will generally *not* be an unbiased estimator for α under our stochastic assumptions. To see this, let $x_n = \ln p_n^1/p_n^0$. Taking expectations, we have: $E\, x_n = \beta = \ln\alpha$. Define the positive, convex function f of one variable x by $f(x) \equiv e^x$. By Jensen's (1906) inequality, we have $Ef(x) \leq f(Ex)$. Letting x equal the random variable x_n, this inequality becomes: $E(p_n^1/p_n^0) = Ef(x_n) \leq f(E\, x_n) = f(\beta) = e^\beta = e^{\ln\alpha} = \alpha$. Thus for each n, we have $E(p_n^1/p_n^0) \leq \alpha$, and it can be seen that the Jevons price index defined by (32) will generally have a downward bias.

of other ‘observations’. But in the case of prices, a movement in the price of one commodity necessarily influences the movement in the prices of other commodities, whilst the magnitudes of these compensatory movements depend on the magnitude of the change in expenditure on the first commodity as compared with the importance of the expenditure on the commodities secondarily affected. Thus, instead of ‘independence’, there is between the ‘errors’ in the successive ‘observations’ what some writers on probability have called ‘connexity’, or, as Lexis expressed it, there is ‘sub-normal dispersion’.

We cannot, therefore, proceed further until we have enunciated the appropriate law of connexity. But the law of connexity cannot be enunciated without reference to the relative importance of the commodities affected—which brings us back to the problem that we have been trying to avoid, of weighting the items of a composite commodity.” John Maynard Keynes (1930; 76-77).

The main point Keynes seemed to be making in the above quotation is that prices in the economy are not independently distributed from each other and from quantities. In current macroeconomic terminology, we can interpret Keynes as saying that a macroeconomic shock will be distributed across all prices and quantities in the economy through the normal interaction between supply and demand; i.e., through the workings of the general equilibrium system. Thus Keynes seemed to be leaning towards the economic approach to index number theory (even before it was even developed to any great extent), where quantity movements are functionally related to price movements. A second point that Keynes made in the above quotation is that there is no such thing as the inflation rate; there are only price changes that pertain to well specified sets of commodities or transactions; i.e., the domain of definition of the price index must be carefully specified.[31] A final point that Keynes made is that price movements must be weighted by their economic importance; i.e., by quantities or expenditures.

In addition to the above theoretical criticisms, Keynes also made the following strong empirical attack on Edgeworth’s unweighted stochastic approach:

“The Jevons—Edgeworth “objective mean variation of general prices’, or ‘indefinite’ standard, has generally been identified, by those who were not as alive as Edgeworth himself was to the subtleties of the case, with the purchasing power of money—if only for the excellent reason that it was difficult to visualise it as anything else. And since any respectable index number, however weighted, which covered a fairly large number of commodities could, in accordance with the argument, be regarded as a fair approximation to the indefinite standard, it seemed natural to regard any such index as a fair approximation to the purchasing power of money also.

Finally, the conclusion that all the standards ‘come to much the same thing in the end’ has been reinforced ‘inductively’ by the fact that rival index numbers (all of them, however, of the wholesale type) have shown a considerable measure of agreement with one another in spite of their different compositions. … On the contrary, the tables give above (pp. 53,55) supply strong presumptive evidence that over long period as well as over short period the movements of the wholesale and of the consumption standards respectively are capable of being widely divergent.” John Maynard Keynes (1930; 80-81).

In the above quotation, Keynes noted that the proponents of the unweighted stochastic approach to price change measurement were comforted by the fact that all of the then existing (unweighted) indexes of wholesale prices showed broadly similar movements. However, Keynes showed empirically that these wholesale price indexes moved quite differently than his consumer price indexes.[32]

[31] We will return to this point in section 6 below.

[32] Using the OECD national accounts data for the last four decades, some broad trends in the rates of increase in prices for the various components of GDP can be observed: rates of increase for the prices of internationally traded goods have been the lowest, followed by the prices of reproducible capital goods, followed by consumer prices, followed by wage rates. From other sources, land prices have shown the highest rate of price increase over this

In order to overcome the Keynsian criticisms of the unweighted stochastic approach to index numbers, it is necessary to:

- have a definite domain of definition for the index number and
- weight the price relatives by their economic importance.[33]

In the following section, we review Theil's solution to the weighting problem, even though this solution came relatively recently compared to the remaining approaches to be discussed in this introductory chapter.

4. The Weighted Stochastic Approach of Theil

"It might seem at first sight as if simply every price quotation were a single item, and since every commodity (any kind of commodity) has one price-quotation attached to it, it would seem as if price-variations of every kind of commodity were the single item in question. This is the way the question struck the first inquirers into price-variations, wherefore they used simple averaging with even weighting. But a price-quotation is the quotation of the price of a generic name for many articles; and one such generic name covers a few articles, and another covers many. … A single price-quotation, therefore, may be the quotation of the price of a hundred, a thousand, or a million dollar's worths, of the articles that make up the commodity named. Its weight in the averaging, therefore, ought to be according to these money-unit's worth." Correa Moylan Walsh (1921; 82-83).

Theil (1967; 136-137) proposed a solution to the lack of weighting in (44). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the nth price relative is equal to $s_n^0 \equiv p_n^0 q_n^0/p^0 \cdot q^0$, the period 0 expenditure share for commodity n. Then the overall mean (period 0 weighted) logarithmic price change is $\sum_{n=1}^N s_n^0 \ln(p_n^1/p_n^0)$. Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) logarithmic price change of $\sum_{n=1}^N s_n^1 \ln(p_n^1/p_n^0)$. Each of these measures of overall logarithmic price change seems equally valid so we could argue for taking a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change[34]. Theil[35] argued that a nice symmetric index number formula can be

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period. Of course, if a country adjusts the price of computer related equipment for quality improvements, then the aggregate price of capital machinery and equipment tends to move *downwards* in recent years. Thus there are long term systematic differences in price movements over different domains of definition.

[33] Walsh (1901) (1921; 82-83) also objected to the lack of weighting in the unweighted stochastic approach to index number theory.

[34] "The [asymmetric] price index (1.6) has certain merits. It is, for example, independent of the units in which we measure the quantities of the various commodities (tons, gallons, etc.). It has the disadvantage, however, of being one sided in the sense that it is based on the distribution of expenditure in the *a*th region. We could equally well apply our random selection procedure to the *b*th region, in which case, w_{ia} is replaced by w_{ib} in (1.5) and (1.6). We must conclude that (6) is an asymmetric index number, which is a disadvantage because the question asked is symmetric: If the price level of the *b*th region exceeds that of the *a*th by a factor 1.2, say, we should expect that the price level of the latter region exceed that of the former by a factor 1/1.2." Henri Theil (1967; 137).

[35] "The price index number defined in (1.8) and (1.9) uses the n individual logarithmic price differences as the basic ingredients. They are combined linearly by means of a two stage random selection procedure: First, we give each region the same chance ½ of being selected, and second, we give each dollar spent in the selected region the same chance ($1/m_a$ or $1/m_b$) of being drawn." Henri Theil (1967; 138).

obtained if we make the probability of selection for the nth price relative equal to the arithmetic average of the period 0 and 1 expenditure shares for commodity n. Using these probabilities of selection, Theil's final measure of overall logarithmic price change was

(45) $\ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N (1/2)(s_n^0 + s_n^1) \ln(p_n^1/p_n^0)$.

We can give the following statistical interpretation of the right hand side of (45). Define the nth logarithmic price ratio r_n by:

(46) $r_n \equiv \ln(p_n^1/p_n^0)$ for $n = 1,\ldots,N$.

Now define the discrete random variable, R say, as the random variable which can take on the values r_n with probabilities $\rho_n \equiv (1/2)[\, s_n^0 + s_n^1]$ for $n = 1,\ldots,N$. Note that since each set of expenditure shares, s_n^0 and s_n^1, sums to one, the probabilities ρ_n will also sum to one. It can be seen that the expected value of the discrete random variable R is

(47) $E[R] \equiv \sum_{n=1}^N \rho_n r_n = \sum_{n=1}^N (1/2)(s_n^0 + s_n^1) \ln(p_n^1/p_n^0) = \ln P_T(p^0,p^1,q^0,q^1)$

using (45) and (46). Thus the logarithm of the index P_T can be interpreted as *the expected value of the distribution of the logarithmic price ratios* in the domain of definition under consideration, where the N discrete price ratios in this domain of definition are weighted according to Theil's probability weights, $\rho_n \equiv (1/2)[\, s_n^0 + s_n^1]$ for $n = 1,\ldots,N$.

Taking antilogs of both sides of (1.45), we obtain the Törnqvist (1936) (1937) Theil price index, P_T. This index number formula has a number of good properties. In particular, P_T satisfies the time reversal test (33). The price index P_T also satisfies the following *linear homogeneity test in current period prices*:

(48) $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$
for all positive numbers λ and strictly positive vectors p^0,p^1,q^0,q^1.

Thus if all period one prices increase by the same positive number λ and if the price index P satisfies the test (48), then the price index increases by this same scalar factor λ. This test can be justified in the context of the quantity theory of money.

The tests (33) and (48) can be used to justify Theil's (arithmetic) method of forming an average of the two sets of expenditure shares in order to obtain his probability weights, $\rho_n \equiv (1/2)[\, s_n^0 + s_n^1]$ for $n = 1,\ldots,N$. Consider the following *symmetric mean* class of Theil type *logarithmic index number formulae*:

(49) $\ln P_{ml}(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N m(s_n^0,s_n^1) \ln(p_n^1/p_n^0)$

where $m(s_n^0,s_n^1)$ is a homogeneous symmetric mean[36] of the period 0 and 1 expenditure shares, s_n^0 and s_n^1 respectively. In order for P_{ml} to satisfy the time reversal test, it is necessary that the

[36] Recall (26) to (32) above.

mean function m be symmetric. In order for the weights in (49) to sum to one so that the linear homogeneity test is satisfied and the weights can be interpreted as probability weights, it can be shown that the homogeneous symmetric mean function m(a,b) that appears in (49) *must* be the arithmetic mean; see the Appendix to this chapter.

The stochastic approach of Theil has another nice symmetry property. Instead of considering the distribution of the price ratios $r_n = \ln p_n^1/p_n^0$, we could also consider the distribution of the *reciprocals* of these price ratios, say:

$$(50)\quad \begin{aligned} t_n &\equiv \ln p_n^0/p_n^1 \quad \text{for } n = 1,\ldots,N \\ &= \ln(p_n^1/p_n^0)^{-1} \\ &= -\ln(p_n^1/p_n^0) \\ &= -r_n \end{aligned}$$

where the last equality follows using definitions (46). We can still associate the symmetric probability, $\rho_n \equiv (1/2)[s_n^0 + s_n^1]$, with the nth reciprocal logarithmic price ratio t_n for $n = 1,\ldots,N$. Now define the discrete random variable, T say, as the random variable which can take on the values t_n with probabilities $\rho_n \equiv (1/2)[s_n^0 + s_n^1]$ for $n = 1,\ldots,N$ It can be seen that the expected value of the discrete random variable T is

$$(51)\quad \begin{aligned} E[T] &\equiv \textstyle\sum_{n=1}^N \rho_n t_n \\ &= -\textstyle\sum_{n=1}^N \rho_n r_n && \text{using (50)} \\ &= -E[R] && \text{using (47)} \\ &= -\ln P_T(p^0,p^1,q^0,q^1). \end{aligned}$$

Thus it can be seen that the distribution of the random variable T is equal to minus the distribution of the random variable R. Hence it does not matter whether we consider the distribution of the original logarithmic price ratios, $r_n \equiv \ln p_n^1/p_n^0$, or the distribution of their reciprocals, $t_n \equiv \ln p_n^0/p_n^1$: we obtain essentially the same stochastic theory.

It is possible to consider weighted stochastic approaches to index number theory where we look at the distribution of price ratios, p_n^1/p_n^0, rather than the distribution of the logarithmic price ratios, $\ln p_n^1/p_n^0$. Thus, again following in the footsteps of Theil, suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the nth price relative is equal to $s_n^0 \equiv p_n^0 q_n^0/p^0 \cdot q^0$, the period 0 expenditure share for commodity n. Now the overall mean (period 0 weighted) price change is:

$$(52)\quad P_L(p^0,p^1,q^0,q^1) = \textstyle\sum_{n=1}^N s_n^0(p_n^1/p_n^0),$$

which turns out to be the Laspeyres price index, P_L (recall (23) above).

Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) price change equal to:

(53) $P_{Pal}(p^0,p^1,q^0,q^1) = \sum_{n=1}^N s_n^1(p_n^1/p_n^0)$.

The right hand side of (53) is known as the Palgrave (1886) index number formula, P_{Pal}.[37]

It can be verified that neither the Laspeyres nor Palgrave price indexes satisfy the time reversal test, (33). Thus, again following in the footsteps of Theil, we might try to obtain a formula that satisfied the time reversal test by taking a symmetric average of the two sets of shares. Thus we consider the following class of *symmetric mean index number formulae*:

(54) $P_m(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N m(s_n^0,s_n^1)\,(p_n^1/p_n^0)$

where $m(s_n^0,s_n^1)$ is a homogeneous symmetric mean of the period 0 and 1 expenditure shares, s_n^0 and s_n^1 respectively. However, in order to interpret the right hand side of (54) as an expected value of the price ratios p_n^1/p_n^0, it is necessary that

(55) $\sum_{n=1}^N m(s_n^0,s_n^1) = 1$.

However, in order to satisfy (55), m cannot be a symmetric geometric or harmonic mean or any of the commonly used homogeneous symmetric means. In fact, the only simple homogeneous symmetric mean that satisfies (55) is the arithmetic mean.[38] With this choice of m, (54) becomes the following (unnamed) index number formula, P_u:

(56) $P_u(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N (1/2)[s_n^0 + s_n^1](p_n^1/p_n^0)$.

Unfortunately, the unnamed index P_u does not satisfy the time reversal test either.

Instead of considering the distribution of the price ratios, p_n^1/p_n^0, we could also consider the distribution of the *reciprocals* of these price ratios. The counterparts to the asymmetric indexes defined earlier by (52) and (53) are now $\sum_{n=1}^N s_n^0(p_n^0/p_n^1)$ and $\sum_{n=1}^N s_n^1(p_n^0/p_n^1)$ respectively. These are (stochastic) price indexes going *backwards* from period 1 to 0. In order to make these indexes comparable with our previous forward looking indexes, we take the reciprocals of these indexes and obtain the following two indexes:

(57) $P_{13}(p^0,p^1,q^0,q^1) \equiv [\sum_{n=1}^N s_n^0(p_n^0/p_n^1)]^{-1}$;
(58) $P_P(p^0,p^1,q^0,q^1) \equiv [\sum_{n=1}^N s_n^1(p_n^0/p_n^1)]^{-1}$
$= [\sum_{n=1}^N s_n^1(p_n^1/p_n^0)^{-1}]^{-1}$
$= P_P(p^0,p^1,q^0,q^1)$ using (24) above.

Thus the reciprocal stochastic price index defined by (58) turns out to equal the fixed basket Paasche price index, P_P, defined earlier by (24). The other asymmetrically weighted reciprocal stochastic price index defined by (57) has no author's name associated with it but it was noted by

[37] It is formula number 9 in Fisher's (1922; 466) listing of index number formulae.
[38] For a proof of this assertion, see Diewert (2000).

Irving Fisher (1922; 467) as his index number formula 13. We can also consider the class of *symmetrically weighted reciprocal price indexes* defined as:

(59) $P_{mr}(p^0,p^1,q^0,q^1) \equiv [\sum_{n=1}^N m(s_n^0,s_n^1)\,(p_n^1/p_n^0)^{-1}\,]^{-1}$

where as usual, $m(s_n^0,s_n^1)$ is a homogeneous symmetric mean of the period 0 and 1 expenditure shares. However, it appears that none of the indexes defined by (57)–(59) satisfy the time reversal test.

The above considerations appear to explain why Theil's stochastic index number formula P_T seems to be the preferred member of this class of index number formula.

We will return to the stochastic approach to index number theory in chapters 10 and 11.

Additional material on stochastic approaches to index number theory and references to the literature can be found in Selvanathan and Rao (1994), Wynne (1997), Clements, Izan and Selvanathan (2006) and Diewert (2011).

5. The Approach of Divisia

"Nous définirons donc l'indice monétaire par la formule différentielle: $dI/I = \sum q_n\, dp_n / \sum q_n\, p_n$." François Divisia (1926; 40).

"As the elementary formula of the chaining, we may get Laspeyre's or Paasche's or Edgeworth's or nearly any other formula, according as we choose the approximation principle for the steps of the numerical integration." Ragnar Frisch (1936; 8).

Suppose that our price and quantity data on the N commodities in our chosen domain of definition can be regarded as continuous functions of time, say $p_n(t)$ and $q_n(t)$ for $n = 1,\ldots,N$. For the sake of concreteness, assume that we are in the consumer context and that the value of consumer expenditure at time t is V(t) defined in the obvious way as:

(60) $V(t) \equiv \sum_{n=1}^N p_n(t)q_n(t)$.

Now suppose that the functions $p_n(t)$ and $q_n(t)$ are differentiable. Then we can differentiate both sides of (60) to obtain:

(61) $V'(t) = \sum_{n=1}^N p_n'(t)q_n(t) + \sum_{n=1}^N p_n(t)q_n'(t)$.

Now divide both sides of (61) through by V(t) and using (60), we obtain the following equation using the chain rule from elementary calculus:

(62) $V'(t)/V(t) = [\sum_{n=1}^N p_n'(t)q_n(t) + \sum_{n=1}^N p_n(t)q_n'(t)] / \sum_{n=1}^N p_n(t)q_n(t)$

$= \{\sum_{n=1}^N [p_n'(t)/p_n(t)]\, p_n(t)q_n(t) + \sum_{n=1}^N [q_n'(t)/q_n(t)]q_n(t)p_n(t)\} / \sum_{n=1}^N p_n(t)q_n(t)$

(63) $= \sum_{n=1}^N [p_n'(t)/p_n(t)]\, s_n(t) + \sum_{n=1}^N [q_n'(t)/q_n(t)]s_n(t)$

where the time t expenditure share on commodity n, $s_n(t)$, is defined as:

(64) $s_n(t) \equiv p_n(t)q_n(t)\} / \sum_{m=1}^{N} p_m(t)q_m(t)$ for $n = 1,2,\ldots,N$.

Now Divisia (1926; 39) argued as follows: *suppose* the aggregate value at time t, V(t), can be written as the product of a time t price level function, P(t) say, times a time t quantity level function, Q(t) say; i.e., we have:

(65) $V(t) = P(t)Q(t)$.

Suppose further that the functions P(t) and Q(t) are differentiable. Then differentiating .65) yields:

(66) $V'(t) = P'(t)Q(t) + P(t)Q'(t)$.

Dividing both sides of (66) by V(t) and using (65) leads to the following equation:

(67) $V'(t)/V(t) = [P'(t)/P(t)] + [Q'(t)/Q(t)]$.

Divisia compared the two expressions for the logarithmic value derivative, $V'(t)/V(t)$, given by (63) and (67) and he simply *defined* the logarithmic rate of change of the aggregate price level, $P'(t)/P(t)$, as the first set of terms on the right hand side of (63) and he simply *defined* the logarithmic rate of change of the aggregate quantity level, $Q'(t)/Q(t)$, as the second set of terms on the right hand side of (63); i.e., he made the following definitions:

(68) $P'(t)/P(t) \equiv \sum_{n=1}^{N} s_n(t) [p_n'(t)/p_n(t)]$;
(69) $Q'(t)/Q(t) \equiv \sum_{n=1}^{N} s_n(t) [q_n'(t)/q_n(t)]$.

These are reasonable definitions for the proportional changes in the aggregate price and quantity levels, P(t) and Q(t). The problem with these definitions is that economic data are not collected in *continuous* time; they are collected in *discrete* time. More fundamentally, we have indicated above that as we make our discrete time interval smaller and smaller, we can expect our data to become increasingly erratic and generally meaningless.

Thus in order to make operational the continuous time Divisia price and quantity levels, P(t) and Q(t) defined by the differential equations (68) and (69), we have to convert to discrete time. Divisia (1926; 40) suggested a method for doing this conversion, which we now outline.

Define the following price and quantity (forward) differences:

(70) $\Delta P \equiv P(1) - P(0)$;
(71) $\Delta p_n \equiv p_n(1) - p_n(0)$; $n = 1,\ldots,N$.

Using the above definitions, we have:

(72) $P(1)/P(0) = [P(0) + \Delta P]/P(0)$
$= 1 + [\Delta P/P(0)]$

$$\approx 1 + [\sum_{n=1}^{N} \Delta p_n\, q_n(0)]/[\sum_{n=1}^{N} p_n(0)q_n(0)]$$
using (62) when t = 0 and approximating $p_n'(0)$ by the difference Δp_n
$$= [\sum_{n=1}^{N} \{p_n(0) + \Delta p_n\}\, q_n(0)]/[\sum_{n=1}^{N} p_n(0)q_n(0)]$$
$$= [\sum_{n=1}^{N} p_n(1)\, q_n(0)]/[\sum_{n=1}^{N} p_n(0)q_n(0)] \qquad \text{using } p_n(1) = p_n(0) + \Delta p_n$$
$$= p^1\cdot q^0/p^0\cdot q^0 \qquad \text{defining } p^t \equiv [p_1(t),\ldots,p_N(t)] \text{ and } q^t \equiv [q_1(t),\ldots,q_N(t)]$$
$$= P_L\,;$$

i.e., it can be seen that Divisia's discrete approximation to his continuous time price index is just our old friend, the Laspeyres price index. But now we run into the problem noted by Frisch (1936; 8): instead of approximating the derivatives by the discrete (forward) differences defined by (70) and (71), we could use other approximations and obtain a wide variety of discrete time approximations. For example, instead of using forward differences and evaluating the index at time t = 0, we could use backward differences and evaluate the index at time t = 1. These backward differences are defined as:

(73) $\Delta_b p_n \equiv p_n(0) - p_n(1)\,; \quad n = 1,\ldots,N.$

This use of backward differences leads to the following approximation for P(0)/P(1):

(74) $$P(0)/P(1) = [P(1) + \Delta_b P]/P(1)$$
$$= 1 + [\Delta_b P/P(1)]$$
$$\approx 1 + [\sum_{n=1}^{N} \Delta_b p_n\, q_n(1)]/[\sum_{n=1}^{N} p_n(1)q_n(1)]$$
using (62) when t = 1 and approximating $p_n'(1)$ by the difference $\Delta_b p_n$
$$= [\sum_{n=1}^{N} \{p_n(1) + \Delta_b p_n\}\, q_n(1)]/[\sum_{n=1}^{N} p_n(1)q_n(1)]$$
$$= [\sum_{n=1}^{N} p_n(0)\, q_n(1)]/[\sum_{n=1}^{N} p_n(1)q_n(1)] \qquad \text{using } p_n(0) = p_n(1) + \Delta_b p_n$$
$$= p^0\cdot q^1/p^1\cdot q^1 \qquad \text{defining } p^t \equiv [p_1(t),\ldots,p_N(t)] \text{ and } q^t \equiv [q_1(t),\ldots,q_N(t)]$$
$$= 1/P_P\,.$$

Taking reciprocals of both sides of (74) leads to the following discrete approximation to P(1)/P(0):

(75) $P(1)/P(0) \approx P_P$

where P_P is our old friend, the Paasche price index. Thus, as Frisch noted, both the Paasche and Laspeyres indexes can be regarded as (equally valid) approximations to the continuous time Divisia price index.[39] Since the Paasche and Laspeyres indexes can differ considerably in some empirical applications, it can be seen that Divisia's idea is not all that helpful in determining a unique discrete time index number formula.[40]

[39] Diewert (1980; 444) also obtained the Paasche and Laspeyres approximations to the Divisia index using a somewhat different approximation argument. He also showed how several other popular discrete time index number formulae could be regarded as approximations to the continuous time Divisia index.

[40] "We have now defined five reasonable looking discrete approximations to the Divisia quantity index. The problem is that the theory of Divisia indexes outlined above does not tell us which discrete index number formula should be used in empirical applications, even though it is known that the Laspeyres and Paasche quantity indexes can differ considerably from the other indexes." W.E. Diewert (1980; 445-446).

The above approach to index number theory relies on the theory of differentiation (and the numerical approximation of derivatives). Thus it does not appear to have any connection with economic theory. However, starting with Ville (1946), a number of economists[41] have established that the Divisia price and quantity indexes *do* have a connection with the economic approach to index number theory. We will conclude this section by indicating one of these connections.

As in section 1 above, we assume that a consumer is solving the cost minimization problem (10) where the utility function, f(q), is a positive, linearly homogeneous, concave function. The corresponding unit cost function is c(p) where $p \equiv [p_1,\ldots,p_N]$. We now think of the prices as being continuous, differentiable functions of time, $p_n(t)$ say, for n = 1,…,N. Thus the unit cost function can be regarded as a function of time t as well; i.e., we define this function as

(76) $c^*(t) \equiv c[p_1(t),p_2(t),\ldots,p_N(t)]$.

Assuming that the first order partial derivatives of the unit cost function c exist, we can calculate the logarithmic derivative of $c^*(t)$ as follows:

(77) $d\ln c^*(t)/dt \equiv [1/c^*(t)]\, dc^*(t)/dt$
$= [1/c^*(t)] \sum_{n=1}^N c_n[p_1(t),p_2(t),\ldots,p_N(t)]\, p_n'(t)$ using (76)

where $c_n[p_1(t),p_2(t),\ldots,p_N(t)] \equiv \partial c[p_1(t),p_2(t),\ldots,p_N(t)]/\partial p_n$ is the partial derivative of the unit cost function with respect to the nth price, p_n, and $p_n'(t) \equiv dp_n(t)/dt$ is the time derivative of the nth price function, $p_n(t)$. Using Shephard's (1953; 11) lemma, the cost minimizing demand for commodity n at time t is:

(78) $q_n(t) = u(t)\, c_n[p_1(t),p_2(t),\ldots,p_N(t)]$ for n = 1,…,N

where the utility level at time t is $u(t) = f[q_1(t),q_2(t),\ldots,q_N(t)]$. The continuous time counterpart to equations (15) in section 1 above is that total expenditure at time t is equal to total cost at time t which in turn is equal to the utility level, u(t), times the period t unit cost, $c^*(t)$; i.e., we have:

(79) $\sum_{n=1}^N p_n(t)q_n(t) = u(t)c^*(t) = u(t)\, c[p_1(t),p_2(t),\ldots,p_N(t)]$.

Now the logarithmic derivative of the Divisia price level P(t) can be written as (recall (62) and (68) above):

(80) $P'(t)/P(t) = \sum_{n=1}^N p_n'(t)q_n(t) / \sum_{n=1}^N p_n(t)q_n(t)$
$= \sum_{n=1}^N p_n'(t)q_n(t) / u(t)c^*(t)$ using (79)
$= \sum_{n=1}^N p_n'(t)\{u(t)\, c_n[p_1(t),p_2(t),\ldots,p_N(t)]\}/ u(t)c^*(t)$ using (78)
$= \sum_{n=1}^N c_n[p_1(t),p_2(t),\ldots,p_N(t)]\, p_n'(t) /c^*(t)$ rearranging terms
$= [1/c^*(t)]\, dc^*(t)/dt$ using (77)

[41] See for example Malmquist (1953; 227), Wold (1953; 134-147), Solow (1957), Jorgenson and Griliches (1967) and Hulten (1973). See Balk (2000) for a survey of work on Divisia price and quantity indexes.

$\equiv c^{*\prime}(t)/c^{*}(t)$.

Thus under the above continuous time cost minimizing assumptions, the Divisia price level, P(t), is essentially equal to the unit cost function evaluated at the time t prices, $c^{*}(t) \equiv c[p_1(t),p_2(t),\ldots,p_N(t)]$.

For more on the Divisia approach to index number theory, see Vogt (1978) and Balk (2000).

6. The Approaches of Bennet and Montgomery

"The fundamental idea is that in a short period the rate of increase of expenditure of a family can be divided into two parts, *x* and ℓ, where *x* measures the increase due to change in prices and ℓ measures the increase due to increase of consumption; …" T. L. Bennet (1920; 455)

Traditional bilateral index number theory decomposes a value ratio pertaining to the two periods under consideration into the product of a price index, $P(p^0,p^1,q^0,q^1)$, times a quantity index, $Q(p^0,p^1,q^0,q^1)$; i.e., we have:

$$(81)\ p^1\cdot q^1/p^0\cdot q^0 = P(p^0,p^1,q^0,q^1)\ Q(p^0,p^1,q^0,q^1).$$

If there is only one commodity in the aggregate, then the *price index* $P(p^0,p^1,q^0,q^1)$ collapses down to the single price ratio, p_1^1/p_1^0 and the *quantity index* $Q(p^0,p^1,q^0,q^1)$ collapses down to the single quantity ratio, q_1^1/q_1^0 . Thus traditional index number theory is based on a *ratio* principle.

Bennet (1920) and Montgomery (1929) (1937) pursued a branch of index number theory where *differences* replaced the *ratios* in (81). Thus, they looked for two functions of 4N variables, $\Delta P(p^0,p^1,q^0,q^1)$ and $\Delta Q(p^0,p^1,q^0,q^1)$, which added up to the value difference in the aggregate rather than the value ratio; i.e., these two functions were to satisfy the following equation:

$$(82)\ p^1\cdot q^1 - p^0\cdot q^0 = \Delta P(p^0,p^1,q^0,q^1) + \Delta Q(p^0,p^1,q^0,q^1).$$

The two functions, $\Delta P(p^0,p^1,q^0,q^1)$ and $\Delta Q(p^0,p^1,q^0,q^1)$, are to satisfy certain tests or properties that will allow us to identify $\Delta P(p^0,p^1,q^0,q^1)$ as a *measure of aggregate price change* and $\Delta Q(p^0,p^1,q^0,q^1)$ as a *measure of aggregate quantity or volume change*. Note that if either of these functions is determined, then the other function is also determined.

Where might one use the difference approach to analyzing value change? A natural home for this approach is in the business and accounting community. The usual ratio approach to the decomposition of value change is not one that the business and accounting community finds natural; a manager or owner of a firm is typically interested in analyzing profit *differences* rather than *ratios*. Thus interest centers on decomposing cost, revenue or profit changes into *price* and *quantity* (or volume) *effects*.[42] For example, the owner of an oil exploration company will generally be interested in knowing how much of the difference between current period profits

[42] In the accounting literature, this is known as variance analysis.

over the previous period profits is due to the change in the price of crude oil and how much of the profit change is due to improvements in the operating efficiency of the company.

Another natural area of application of the difference approach to index number theory is in *consumer surplus theory*. In this context, the problem is to decompose the change in a consumer's expenditures between two periods into a price change component, $\Delta P(p^0,p^1,q^0,q^1)$, plus a quantity change component, $\Delta Q(p^0,p^1,q^0,q^1)$, which can be interpreted as a constant dollar measure of utility change. This line of research was started by Marshall (1890) and Bennet (1920) and continued by Hotelling(1938; 253-254), Hicks (1941-42; 134) (1945-46) (1946; 330-333) and Harberger (1971).[43]

A final area of application of the difference approach to the analysis of value change is in accounting theory. For this use of the value decomposition, the period 0 prices and quantities are interpreted as period 1 *budgeted* or *standard* prices and quantities that are supposed to prevail in period 1. Then the value difference, $p^1\cdot q^1 - p^0\cdot q^0$, is the difference between the actual period 1 value, $p^1\cdot q^1$, and the budgeted performance, $p^0\cdot q^0$. The measure of price change, $\Delta P(p^0,p^1,q^0,q^1)$, is now interpreted as the contribution of price change between actual and budgeted prices to the value change. Similarly, the measure of aggregate quantity change, $\Delta Q(p^0,p^1,q^0,q^1)$, is now interpreted as the contribution of volume change between actual and budgeted quantities to the value change. This type of ex post decomposition is called *variance analysis* in the accounting literature and it can be traced back to the early accounting and industrial engineering literature.[44]

The notation $\Delta P(p^0,p^1,q^0,q^1)$ and $\Delta Q(p^0,p^1,q^0,q^1)$ in (82) as indicators of price and quantity change is a bit awkward. Hence, we will use the notation $I(p^0,p^1,q^0,q^1)$ (for *indicator* of price change) for $\Delta P(p^0,p^1,q^0,q^1)$ and $V(p^0,p^1,q^0,q^1)$ (for indicator of *volume* change) for $\Delta Q(p^0,p^1,q^0,q^1)$.[45] Using our new notation, (82) can be rewritten as follows:

(83) $p^1\cdot q^1 - p^0\cdot q^0 = \sum_{n=1}^N [p_n^1q_n^1 - p_n^0q_n^0] = I(p^0,p^1,q^0,q^1) + V(p^0,p^1,q^0,q^1)$.

Because the value difference in the aggregate decomposes into a sum of value differences over each commodity n, we can consider first the problem of appropriately decomposing the value change in a single commodity, n say, into an indicator of price change, $I_n(p_n^0,p_n^1,q_n^0,q_n^1)$ say, plus an indicator of quantity change, $V_n(p_n^0,p_n^1,q_n^0,q_n^1)$ say. Thus in this *separable approach*, we want to find I_n and V_n that satisfy the following equation:

(84) $p_n^1q_n^1 - p_n^0q_n^0 = I_n(p_n^0,p_n^1,q_n^0,q_n^1) + V_n(p_n^0,p_n^1,q_n^0,q_n^1)$.

Once the commodity specific indicators of price change have been determined, the overall indicator of price change can be defined as the sum of the specific indicators:

(85) $I(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N I_n(p_n^0,p_n^1,q_n^0,q_n^1)$.

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[43] For more recent developments, see Weitzman (1988), Diewert (1992b) and Diewert and Mizobuchi (2009).
[44] See Whitmore (1908) (1931), Harrison (1918) and Solomons (1968; 46-47) for the early history of this topic.
[45] This indicator terminology was introduced by Diewert (1992b) (1998) (2005a).

Of course, once the overall indicator of price change, $I(p^0,p^1,q^0,q^1)$, has been determined, the corresponding aggregate volume indicator $V(p^0,p^1,q^0,q^1)$ can be determined using equation (83).[46]

One way of viewing the decomposition of an overall value change into separate components of price and quantity change is that it provides a commodity by commodity explanation for an overall value change. It would be useful in many contexts to have a similar "micro" decomposition of an overall index number change into commodity specific "contributions" or effects. Diewert and Morrison (1986), Morrison and Diewert (1990), Kohli (1990) and Fox and Kohli (1998) have provided this type of decomposition into individual price and quantity effects for the Törnqvist index and we will cover this topic later when we study the output price index.

Before we define some specific examples of price and volume indicators, we need to address a preliminary problem: how should the period t prices and quantities for commodity n, p_n^t and q_n^t be defined? In real life applications of value decompositions into price and quantity parts, during any period t, there will typically be many transactions in commodity n at a number of different prices. Hence, there is a need to provide a more precise definition for the "average" or "representative" price for commodity n in period t, p_n^t. Irving Fisher (1922; 318) addressed this preliminary aggregation problem as follows:

"Essentially the same problem enters, however, whenever, as is usually the case, the data for prices and quantities with which we start are averages instead of being the original market quotations. Throughout this book, "the price" of any commodity or "the quantity" of it for any year was assumed given. But what is such a price or such a quantity? … The quantities sold will, of course, vary widely. What is needed is their sum for the year …. Or, if it is worth while to put any finer point on it, we may take the weighted arithmetic average for the prices, the weights being the quantities sold."

Thus Fisher more or less advocated the use of the *unit value* (total value transacted divided by total quantity) as the appropriate price p_n^t for commodity n and the total quantity transacted during period t as the appropriate quantity , q_n^t. As an aggregation formula at the first stage of aggregation, the unit value and total quantity transacted has been proposed by many economists and statisticians, perhaps starting with Walsh (1901; 96) (1921; 88) and Davies (1924; 183) (1932; 59) and including many other more recent writers. If we want q_n^t to equal the total quantity of commodity n transacted during period t and we also want the product of the price p_n^t times quantity q_n^t to equal the value of period t transactions in commodity n, then we are *forced* to define the aggregate period t price p_n^t to be the total value divided by the total quantity, or the unit value.[47]

We turn now to some specific examples of price and volume indicators. Bennet (1920; 457) proposed the following decomposition of a value change:

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[46] Note that once we solve the difference value decomposition problem for a single commodity, (84) above, then it is easy to solve the problem for an arbitrary N using (85). However, it is not easy to solve (84). Conversely, it is very easy to solve the ratio value decomposition problem (81) when N = 1 but it is very difficult to solve this problem for a general N.

[47] For additional discussion on this topic of preliminary aggregation over transactions within a time period, see Diewert (2010a) and Hill (1996). For a discussion on when it is not appropriate to use unit values, see Diewert and von der Lippe (2010).

(86) $p^1 \cdot q^1 - p^0 \cdot q^0 = (1/2)(q^0 + q^1) \cdot (p^1 - p^0) + (1/2)(p^0 + p^1) \cdot (q^1 - q^0)$.

The validity of (86) can be established simply by multiplying out the terms on the right hand side and rearranging terms. Bennet interpreted the first set of terms on the right hand side of (86) as a measure of price change and the second set of terms as a measure of quantity change. Thus the Bennet indicators of price and volume change are defined as:

(87) $I_B(p^0,p^1,q^0,q^1) \equiv (1/2)(q^0 + q^1) \cdot (p^1 - p^0)$;
(88) $V_B(p^0,p^1,q^0,q^1) \equiv (1/2)(p^0 + p^1) \cdot (q^1 - q^0)$.

Bennet (1920; 456-457) justified his volume indicator as a linear approximation to the area under a demand curve and his price indicator as a linear approximation to an area under an inverse demand curve. Hence, Bennet was following in Marshall's (1890) partial equilibrium consumer surplus footsteps. However, it is possible to derive Bennet's indicators by an alternative line of reasoning, which we now explain.

In the early industrial engineering literature, Harrison (1918; 393) made the following decomposition of a cost change into a price variation plus an efficiency variation:

(89) $p^1 \cdot q^1 - p^0 \cdot q^0 = q^1 \cdot (p^1 - p^0) + p^0 \cdot (q^1 - q^0)$.

Again, the proof that (1.89) is true is straightforward arithmetic. The reader familiar with index number theory will recognize that Harrison's indicator of price change, $q^1 \cdot (p^1 - p^0)$, is the difference counterpart to the Paasche price index, $p^1 \cdot q^1 / p^0 \cdot q^1$, defined earlier by (21). Similarly, Harrison's indicator of quantity or efficiency change, $p^0 \cdot (q^1 - q^0)$, is the difference counterpart to the Laspeyres quantity index, $p^0 \cdot q^1 / p^0 \cdot q^0$. Thus we define the *Paasche indicator of price change* I_P and the *Laspeyres indicator of quantity change* V_L as follows:

(90) $I_P(p^0,p^1,q^0,q^1) \equiv q^1 \cdot (p^1 - p^0)$;
(91) $V_L(p^0,p^1,q^0,q^1) \equiv p^0 \cdot (q^1 - q^0)$.

More recently in the accounting literature, it was recognized that the traditional variance analysis decomposition of a value change, (89) above, may not be as appropriate as the following decomposition:

(92) $p^1 \cdot q^1 - p^0 \cdot q^0 = q^0 \cdot (p^1 - p^0) + p^1 \cdot (q^1 - q^0)$.

The reason why the decomposition (92) may be preferable to (89) in the context of comparing actual performance to "standard" performance is that in the case of exogenous prices, the firm manager will have an incentive to maximize period 1 profits, $p^1 \cdot q^1$ with respect to q^1 and thus the efficiency change term, $p^1 \cdot (q^1 - q^0)$ in (92), is consistent with profit maximizing behavior. Again, the index number reader will recognize that the indicator of price change in (92), $q^0 \cdot (p^1 - p^0)$, is the difference analogue to the Laspeyres price index, $p^1 \cdot q^0 / p^0 \cdot q^0$ defined earlier by (20) above, and the indicator of quantity change in (92), $q^0\cdot(p^1 - p^0)$, is the difference counterpart to the Paasche quantity index, $p^1\cdot q^1/p^1\cdot q^0$. Thus we define the *Laspeyres and Paasche indicators of price and quantity change* respectively as follows:

(93) $I_L(p^0,p^1,q^0,q^1) \equiv q^0\cdot(p^1 - p^0)$;
(94) $V_P(p^0,p^1,q^0,q^1) \equiv p^1\cdot(q^1 - q^0)$.

In the producer theory context, the Bennet indicators of price and volume change for a single commodity n are illustrated below in Figure 1.

Figure 1: The Bennet Indicators in the Producer Context

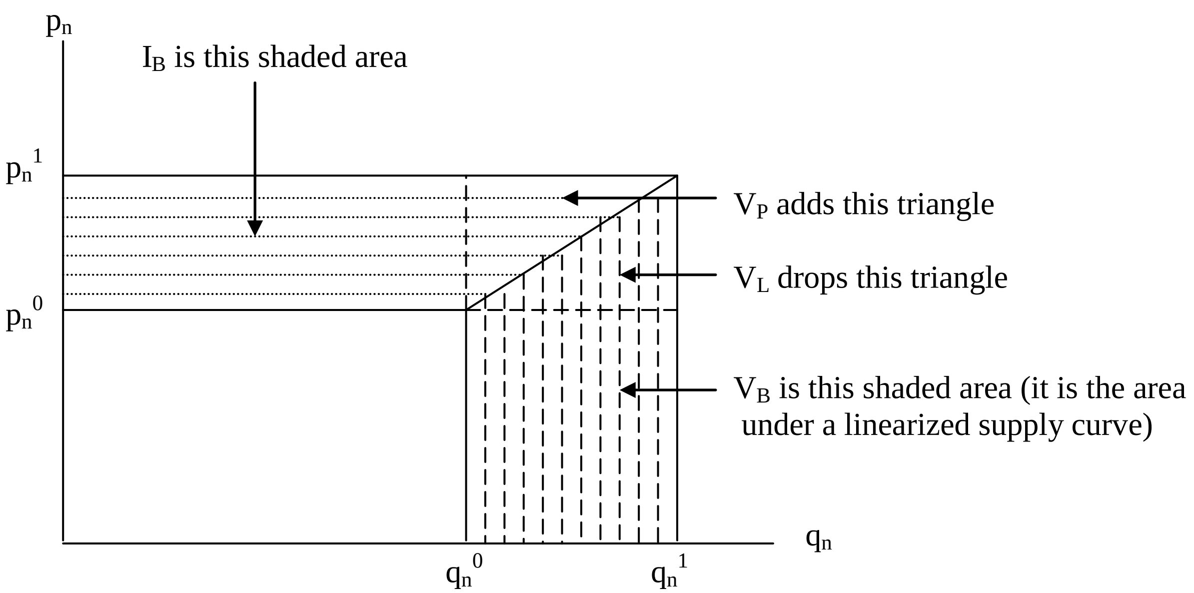


Figure 2: The Bennet Volume Indicator in the Consumer Context

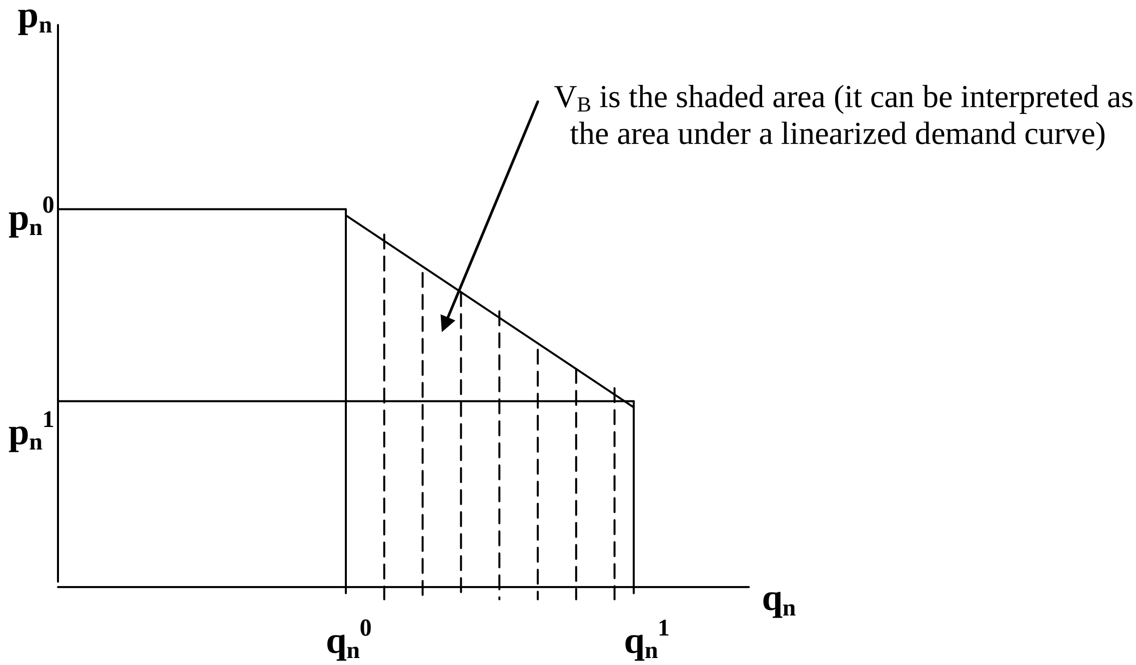


Figure 1 shows that the basic problem that we are trying to address is how to allocate the top right hand side rectangle into a part that should be attributed to the change in prices and a part that should be attributed to a change in quantities. The Paasche and Laspeyres indicators seem to be clearly biased.

In Figure 2, it can be verified that the Laspeyres indicator of quantity change, V_L, gives an *overestimate* of the change in consumer surplus while the Paasche indicator of quantity change, V_P, gives an *underestimate* of the change in consumer surplus.

We turn now to another possible decomposition of the top right rectangle in Figure 1.

In a rather obscure paper, Montgomery (1929) defined some interesting indicators of price and quantity change. The Paasche, Laspeyres and Bennet indicators of price and quantity change were well defined irrespective of the signs of the individual prices and quantities, p_n^t and q_n^t. However, in order to define the Montgomery indicators, we shall restrict all prices and quantities to be positive since it will be necessary to take natural logarithms of the individual prices and quantities. The restriction that all prices and quantities be positive is not restrictive in the context of computing revenue and cost indicators. Obviously, a profit indicator can be defined as the difference between the revenue and cost indicators and so, even in this context, the positivity restrictions are not too restrictive.

The Montgomery (1929; 5) indicators of price and volume change for the nth commodity are defined as follows:

(95) $I_M(p_n^0,p_n^1,q_n^0,q_n^1) \equiv \{[p_n^1q_n^1 - p_n^0q_n^0]/[\ln(p_n^1q_n^1) - \ln(p_n^0q_n^0)]\} \ln(p_n^1/p_n^0)$;
(96) $V_M(p_n^0,p_n^1,q_n^0,q_n^1) \equiv \{[p_n^1q_n^1 - p_n^0q_n^0]/[\ln(p_n^1q_n^1) - \ln(p_n^0q_n^0)]\} \ln(q_n^1/q_n^0)$.

Note that the functional form for the indicator of price change, I_M, is the same as the functional form for the indicator of quantity change, except that the roles of prices and quantities have been interchanged. Montgomery (1929; 3-9) also showed that

(97) $p_n^1q_n^1 - p_n^0q_n^0 = I_M(p_n^0,p_n^1,q_n^0,q_n^1) + V_M(p_n^0,p_n^1,q_n^0,q_n^1)$.

In order to understand (95) and (96) better, the reader should note that

(98) $L(a, b) \equiv [a - b]/[\ln a - \ln b]$

where $a > 0$, $b > 0$ is known in the economics literature as the Vartia (1976a) (1976b) mean and in the mathematics literature as the logarithmic mean. It can be shown that $L(a, b)$ is a linearly homogeneous symmetric mean. In (95) and (96), a is $p_n^1q_n^1$ and b is $p_n^0q_n^0$.

Montgomery (1929; 7-9) derived his indicators by using a very interesting argument (which parallels that of Bennet) which we shall repeat (without giving the details)[48] since it shows how a large number of "reasonable" price and quantity indicators can be derived. Suppose that q_n is functionally related to p_n by a "supply" function:

(99) $q_n = s_n(p_n)$.

Note that the "supply" function $s_n(p_n)$ is a partial equilibrium supply function since only the price of the nth good appears in (1.98) as an argument of the function. Montgomery (1929; 7) assumed the following functional form for $s_n(p_n)$:

(100) $s_n(p_n) \equiv \alpha\, p_n^\beta$; where $\alpha > 0$ and $\beta \neq 0$.

Now define a theoretical price change indicator as the area under the "supply" curve going from p_n^0 to p_n^1:

(101) $P^*(p_n^0,p_n^1,s_n) \equiv \int s_n(p)dp$ where the limits of integration are p_n^0 and p_n^1
$= \int \alpha\, p^\beta\, dp$
$= \alpha(1+\beta)^{-1}\, p^{(1+\beta)}$ evaluated at the upper and lower limits for $\beta \neq 1$
$= \alpha(1+\beta)^{-1} [(p_n^1)^{1+\beta} - (p_n^0)^{1+\beta}]$

The unknown parameters α and β which appear in (100) can be determined by assuming that the two data points p_n^0,q_n^0 and p_n^1,q_n^1 are on the "supply" function defined by (99). Thus we have:

[48] See Diewert (2005a) for the details.

(102) $q_n^0 = \alpha(p_n^0)^\beta$ and $q_n^1 = \alpha(p_n^1)^\beta$ or
(103) $q_n^0/\alpha = (p_n^0)^\beta$ and $q_n^1/\alpha = (p_n^1)^\beta$ or
(104) $p_n^0 q_n^0/\alpha = (p_n^0)^{\beta+1}$ and $p_n^1 q_n^1/\alpha = (p_n^1)^{\beta+1}$.

By taking ratios in (102), we can also deduce that

(105) $q_n^1/q_n^0 = [p_n^1/p_n^0]^\beta$ or
$\beta = \ln [q_n^1/q_n^0] / \ln [p_n^1/p_n^0]$ or
$1 + \beta = \{\ln [p_n^1/p_n^0] + \ln [q_n^1/q_n^0]\} / \ln [p_n^1/p_n^0]$ or
(106) $1 + \beta = \{\ln [p_n^1 q_n^1] - \ln [p_n^0 q_n^0]\} / \ln [p_n^1/p_n^0]$.

Now substitute (104) into (101) to get

(107) $P^*(p_n^0,p_n^1,s_n) = (1+\beta)^{-1} [p_n^1 q_n^1 - p_n^0 q_n^0]$
$= \ln [p_n^1/p_n^0] [p_n^1 q_n^1 - p_n^0 q_n^0] / \{\ln [p_n^1 q_n^1] - \ln [p_n^0 q_n^0]\}$ using (106)
$= I_M(p_n^0,p_n^1,q_n^0,q_n^1)$

where the Montgomery indicator of price change for good, $I_M(p_n^0,p_n^1,q_n^0,q_n^1)$, was defined by (95).

Montgomery (following Bennet (1920; 456)) also defined the theoretical indicator of quantity change going from q_n^0 to q_n^1 as the area under the *inverse "supply" curve* S_n where $p_n = S_n(q_n)$; i.e., define

(108) $Q^*_n(q_n^0,q_n^1,S_n) \equiv \int S_n(q)dq$ where the limits of integration are q_n^0 and q_n^1.

If the "supply" function $s_n(p)$ is defined by (100), then the corresponding inverse "supply" function $S_n(q)$ has the same functional form; i.e., we have

(109) $p_n = S_n(q_n) = [q_n/\alpha]^{1/\beta} \equiv \gamma q_n^\delta$

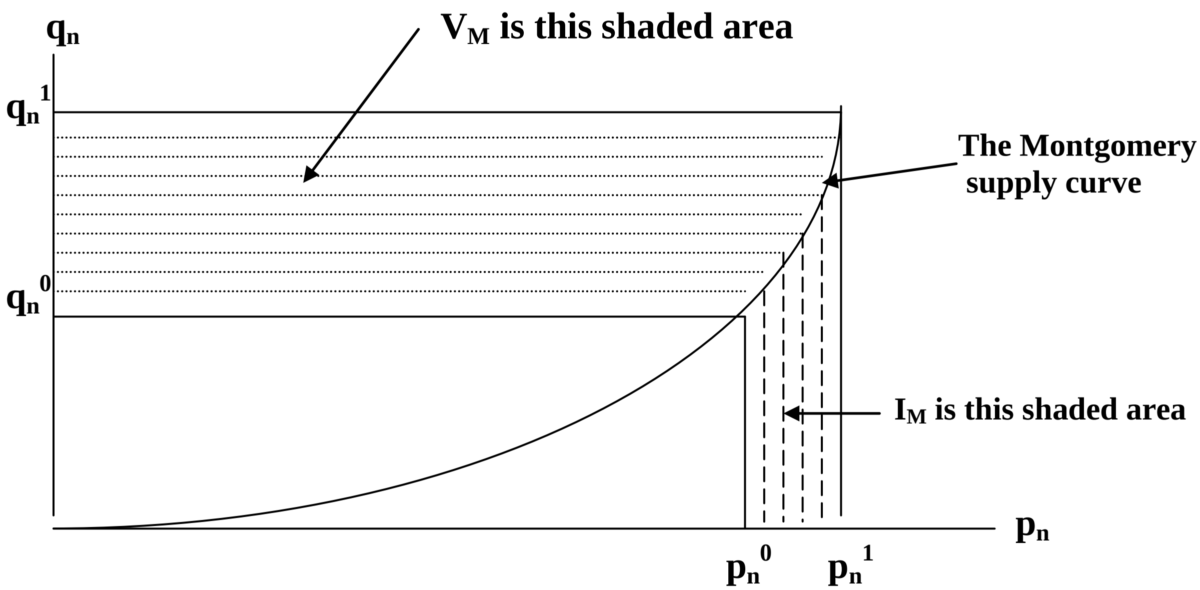
where $\delta \equiv 1/\beta$ (hence, we require $\beta \neq 0$ and $\gamma \equiv (1/\alpha)^{1/\beta}$ (hence, we also require $\alpha \neq 0$). Thus the same argument that we used to derive (107) can be adapted to prove that if S_n is defined by (109) or equivalently if s_n is defined by (100), then

(110) $Q^*_n(q_n^0,q_n^1,S_n) = V_M(p_n^0,p_n^1,q_n^0,q_n^1)$

where $V_M(p_n^0,p_n^1,q_n^0,q_n^1)$ is the *Montgomery indicator of quantity change* defined by (96).

Montgomery (1929; 13) gave a nice geometric interpretation of his method for the case where $0 < p_n^0 < p_n^1$ and $0 < q_n^0 < q_n^1$ which we repeat below in Figure 3.

Figure 3: The Montgomery Indicators of Price and Volume Change



Note that the Montgomery "supply" curve goes through the two observed price and quantity points, p_n^0, q_n^0 and p_n^1, q_n^1 and it also goes through the origin. Montgomery thought that this was an advantage of his nonlinear supply curve compared to Bennet's linear supply curve (recall Figure 1 above). Note that the period 1 value for commodity n, $p_n^1q_n^1$, can be interpreted as the area of the big rectangle in Figure 3, while the period 0 value, $p_n^0q_n^0$, can be interpreted as the area of the smaller inner rectangle. The part of the value change that is due to price change, $I_M(p_n^0,p_n^1,q_n^0,q_n^1)$, is the shaded area below the Montgomery "supply" curve and the part of the value change that is due to quantity change, $V_M(p_n^0,p_n^1,q_n^0,q_n^1)$, is the shaded area to the left of the Montgomery "supply" curve.

It is easy to see how Montgomery's idea could be generalized: instead of using the constant elasticity functional form defined by (100) or (109) to join the two observed price and quantity points, p_n^0, q_n^0 and p_n^1, q_n^1, *any* monotonic curve could be used to join (p_n^0,q_n^0) and (p_n^1,q_n^1) and the corresponding indicators of price and quantity change can be defined by (101) and (108).

This brings to the next problem: which indicator of price change is "best"? Unfortunately, we are not yet in a position to answer this question: we first need to study the test and economic approaches to index number theory in more detail.[49]

[49] For more on the differences approach to index number theory, see Diewert (2005a). There has been a recent surge of interest in the differences approach to index number theory and the measurement of welfare change: see Chambers and Färe (1998), Chambers (2001; 111), Balk, Färe and Grosskopf (2004), Balk (2007) (2008; Chapter 6) and Diewert and Mizobuchi (2009).

7. The Aggregation Theorems of Hicks and Leontief

"A simple measurement of the different quantities of the new composite commodity in terms of physical units will be possible only under the condition that its *composition remains constant*. Each unit of the composite good, I, must contain a fixed amount of B and C." Wassily Leontief (1936; 55).

"Thus we have demonstrated mathematically the very important principle, used extensively in the text, that if the prices of a group of goods change in the same proportion, that group of goods behaves just as if it were a single commodity." J.R. Hicks (1946; 312-313).

At one level, the aggregation theorems of Leontief and Hicks are intuitively obvious. Suppose we have price and quantity data for N commodities pertaining to an economic agent or a group of economic agents for T+1 periods, p^t and q^t, for $t = 0,1,2,\ldots,T$. Suppose further that the period t quantity vectors for $t \geq 1$ are all *proportional* to the period 0 quantity vector, q^0; i.e., we have:

(111) $q^t = \alpha_t q^0$ for $t = 1,2,\ldots,T$

where the proportionality factors α_t are all positive. This means that in period t, all of the long term quantity relatives, q_n^t/q_n^0 equal α_t. Under these circumstances, a very reasonable choice for the period t long term quantity index is α_t; i.e., if (111) is true, then the long term bilateral quantity index for period t should equal α_t:

(112) $Q(p^0,p^t,q^0,q^t) = \alpha_t$; $t = 1,2,\ldots,T.$

This is a simplified version of Leontief's (1936) aggregation theorem.

Similarly, suppose that the period t price vectors for $t \geq 1$ are all *proportional* to the period 0 price vector, p^0; i.e., we have:

(113) $p^t = \beta_t\, p^0$ for $t = 1,2,\ldots,T$

where the proportionality factors β_t are all positive. This means that in period t, all of the long term price relatives, p_n^t/p_n^0 equal β_t. Under these circumstances, a reasonable choice for the period t long term price index is β_t; i.e., if (113) is true, then the long term bilateral price index for period t should equal β_t:

(114) $P(p^0,p^t,q^0,q^t) = \beta_t$; $t = 1,2,\ldots,T.$

This is a simplified version of Hicks' (1946) aggregation theorem.[50]

However, it is of some interest to ask under what conditions on preference or production functions[51] the proportionality conditions (111) or (113) will hold.

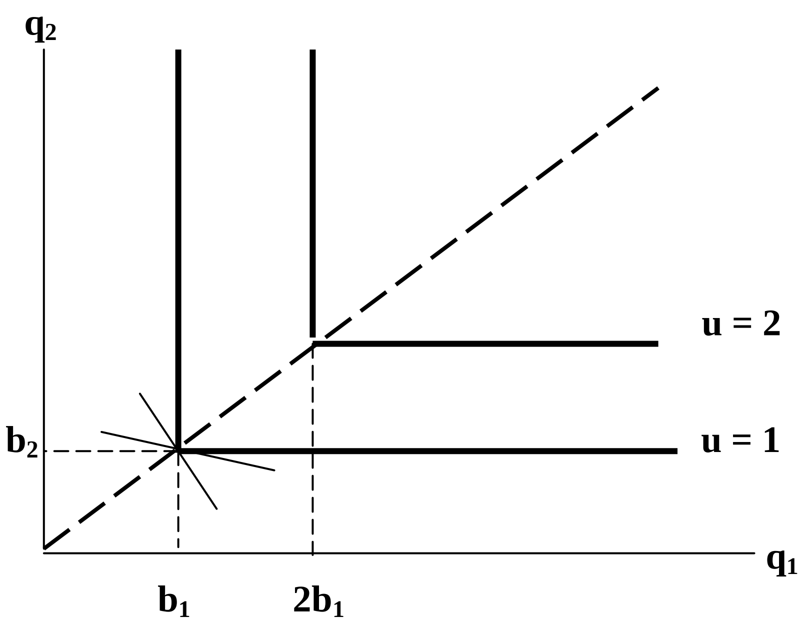
[50] Thus these aggregation theorems can also be regarded as bilateral index number *tests*; see chapter 2.
[51] We use the term *aggregator function* to cover both types of function, a term introduced by Diewert (1976).

Refer back to section 1 above where we defined f(q) to be a linearly homogeneous aggregator function and c(p) to be its unit cost function.[52] Consider the following functional form for f:

(115) $f(q_1,q_2,\ldots,q_N) \equiv \min_n \{ q_n/b_n : n = 1,2,\ldots,N \}$

where the b_n are positive parameters.[53] In the production function context, the f defined by (115) is known as the *Leontief production function*. In the utility function context, Figure 4 shows what the indifference curves or isoutility curves for this function look like for N = 2.

Figure 4: The Leontief Utility Function



If the aggregator function is defined by (115), it can be seen that the minimum cost of achieving one unit of utility is:

(116) $c(p) \equiv \min_q \{ p\bullet q : f(q) = 1 \}$
$= \sum_{n=1}^{N} p_n b_n$
$= p\cdot b$

where $b \equiv [b_1,\ldots,b_N]^T$ is the vector of input-output coefficients. From Figure 4, it can be seen that it does not matter what prices the consumer faces: in order to minimize the cost of achieving one unit of utility, he or she will purchase the vector b in order to do this. In view of this property, the Leontief utility function is sometimes called the *no substitution utility function*.

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[52] See equations (14) and (15).

[53] In the production function context, the b_n are called input-output coefficients; b_n is the minimum number of units of input n required to produce on unit of output.

Using Shephard's (1953) Lemma, the period t consumption vector q^t for the consumer is equal to the vector of derivatives of the period t cost function, $\nabla_p c(p^t)u^t$. Differentiating (116) gives us the following result:

(117) $q^t = bu^t$; $t = 0,1,2,\ldots,T$.

It can be seen that equations (117) are equivalent to equations (111). Thus Leontief preferences are consistent with Leontief's aggregation theorem.

There are also restrictions on preferences that give rise to the Hicks aggregation theorem conditions, (113). Consider the following *linear utility function*:

(118) $f(q_1,q_2,\ldots,q_N) \equiv \sum_{n=1}^{N} \alpha_n q_n = \alpha \cdot q$.

The unit cost function that corresponds to this utility function is the solution to the following problem:

(119) $c(p) \equiv \min_q \{ p \cdot q : \alpha \cdot q = 1 ; q \geq 0_N \}$.

The above unit cost minimization problem is a linear programming problem and generally, the q^* solution to it will not have all components positive; i.e., in general, we will not obtain an interior solution to (119). However, if consumers really do have linear preferences over the N commodities under consideration, then we could argue that if suppliers of commodities want to sell positive quantities, *they will adjust their prices so that they are proportional to the vector of taste parameters* α. Thus under these hypotheses, the period t price vectors will satisfy the following conditions:

(120) $p^t = \gamma_t \alpha$; $t = 0,1,2,\ldots,T$

where the γ_t are positive constants of proportionality to the α vector. It can be seen that conditions (120) imply the Hicksian price proportionality conditions (113).

Note that linear preferences assume perfect substitutability between the N commodities, which is the polar opposite case to the perfect lack of substitutability exhibited by Leontief preferences.

We conclude with a brief explanation of the quotation of Hicks that introduced this section.

Consider the following utility maximization problem over two vectors of commodities, x and y:

(121) $\max_{x,y} \{ f(x,y) : p \cdot x + w \cdot y = I \}$

where $p >> 0_N$ and $w >> 0_M$ are strictly positive vectors of prices facing the consumer and I is the "income" that the consumer has to allocate between the commodities. If the price vectors for the x commodities are expected to be always proportional to the positive vector of constants α, then we can define the following aggregated over x commodities utility function, F_α:

(122) $F_\alpha(y_0,y) \equiv \max_x \{ f(x,y) : \alpha\cdot x = y_0 \}$

where y_0 can be interpreted as a constant dollar allocation of income to the x commodities. We assume that the prices of the x commodities vary in strict proportion; i.e., we assume that

(123) $p = p_0 \alpha$

where p_0 is a scalar proportionality variable. Now consider the following aggregated utility maximization problem which uses the "macro" utility function F defined by (122) above:

(124) $\max_{y\text{'s}} \{ F_\alpha(y_0,y) : p_0 y_0 + w\cdot y = I \}$.

Under the proportionality assumption (123), we show that if x^*,y^* solves the micro utility maximization problem (121), then $y_0^* \equiv \alpha\cdot x^*$ and y^* solve the macro utility maximization problem (124). We start with the micro utility maximization problem, (121):

(125) $\max_{x,y} \{ f(x,y) : p\cdot x + w\cdot y = I \}$

$= \max_{x,y} \{ f(x,y) : p_0\alpha\cdot x + w\cdot y = I \}$ using (123)

$= \max_{x,y\text{'s}} \{ f(x,y) : p_0 y_0 + w\cdot y = I \,;\, \alpha\cdot x = y_0 \}$

where we have added the variable y_0 and a defining equation

$= \max_{y\text{'s}} \{ \{\max_x f(x,y) : \alpha\cdot x = y_0 \} : p_0 y_0 + w\cdot y = I \}$

$= \max_{y\text{'s}} \{ F_\alpha(y_0,y) : p_0 y_0 + w\cdot y = I \}$

using definition (122) for the macro function

which is the macro utility maximization problem. Thus under the price proportionality assumptions (123), the micro and macro utility maximization problems are essentially the same.[54]

Problems.

1. Prove that properties (26)-(29) imply property (30). *Hint*: let $\alpha \equiv \min \{a,b\}$ and $\beta \equiv \max \{a,b\}$. Then $\alpha 1_2 \le (a,b) \le \beta 1_2$ where 1_2 is a vector of ones of dimension 2.

2. Let a and b be positive numbers and define the following means of a and b:

(a) $m_A(a,b) \equiv (1/2)a + (1/2)b$ (the arithmetic mean);
(b) $m_G(a,b) \equiv (ab)^{1/2}$ (the geometric mean);
(c) $m_H(a,b) \equiv [(1/2)a^{-1} + (1/2)b^{-1}]^{-1}$ (the harmonic mean).

Prove that:

(d) $m_H(a,b) \le m_G(a,b) \le m_A(a,b)$.

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[54] For a more elaborate exposition of this point, see Diewert (1978b).

Under what conditions on a and b will strict inequalities hold in (d)?

3. When a = b > 0, show that the corresponding *first* order partial derivatives of m_H, m_G and m_A all coincide; i.e., show that :

(a) $\partial m_H(a,b)/\partial a = \partial m_G(a,b)/\partial a = \partial m_A(a,b)/\partial a$ if a = b;
(b) $\partial m_H(a,b)/\partial b = \partial m_G(a,b)/\partial b = \partial m_A(a,b)/\partial b$ if a = b.
(c) Are the second order partial derivatives also equal when a = b?

4. Instead of taking the arithmetic average of the expenditure shares in Theil's weighted stochastic approach, consider taking the geometric or harmonic average of these shares. Discuss the advantages or disadvantages of the resulting indexes.

5. We have defined the *logarithmic mean* of two positive numbers a and b as follows:

(a) $m_L(a,b) \equiv a$ if a = b
$\equiv [a - b]/[\ln a - \ln b]$ if $a \neq b$ where ln a is the natural logarithm of a.

Prove that m_L is a *homogeneous symmetric mean*; i.e., m_L satisfies (26)-(29) and (32).

6. Provide a proof for the Carli bias inequality in footnote 29 for the case when the number of commodities equals 2; i.e., you may assume that N = 2. *Hint*: use the results of problem 2 above.

7. Let f(q) be a general (nonhomothetic) utility function and define the corresponding cost function C(u,p) by (13) above. Suppose the consumer is engaging in cost minimizing behavior during periods 0 and 1 so that we have:

(a) $p^0 \cdot q^0 = C(u^0,p^0)$;
(b) $p^1 \cdot q^1 = C(u^1,p^1)$.

We assume that the observed quantity vectors for periods 0 and 1 are given by the following expressions using Shephard's Lemma:

(c) $q^0 = \nabla_p C(u^0,p^0)$;
(d) $q^1 = \nabla_p C(u^1,p^1)$.

Hicks (1941-42) defined the following (theoretical) measures of welfare change:

(e) $V(p^0,q^0,q^1) \equiv C(u^1,p^0) - C(u^0,p^0)$ (*the equivalent variation*) ;
(f) $V(p^1,q^0,q^1) \equiv C(u^1,p^1) - C(u^0,p^1)$ (*the compensating variation*).

(g) Obtain an observable first order approximation to the equivalent variation. Hint:

(h) $C(u^1,p^0) \approx C(u^1,p^1) + \nabla_p C(u^1,p^1) \cdot (p^0 - p^1)$.

(i) Does your first order approximation equal any of the indicators of volume change that we considered in section 6?

(j) Obtain an observable first order approximation to the compensating variation.

(k) Does your first order approximation obtained in part (j) equal any of the indicators of volume change that we considered in section 6?

(l) Take an arithmetic average of the two first order approximations that you obtained in parts (g) and (j) above. Does this average approximation equal any of the indicators of volume change that we considered in section 6?

8. Let q and b be N dimensional column vectors, let a be a scalar and let C be an N by N symmetric matrix. Define the quadratic function of q, f(q) as follows:

(a) $f(q) \equiv a + b \cdot q + (1/2)\, q^T C q$.

The second order Taylor series approximation to f(q) around the point q^0 is defined as:

(b) $F(q) \equiv f(q^0) + \nabla f(q^0)^T (q - q^0) + (1/2)(q - q^0)^T\, \nabla^2 f(q^0)(q - q^0)$.

(c) Show that f(q) = F(q) for all q when f(q) is defined by (a).

(d) Show that the following identity holds if f(q) is defined by (a):

(e) $f(q^1) - f(q^0) = (1/2)[\nabla f(q^0) + \nabla f(q^1)]^T [q^1 - q^0]$ for all q^0 and q^1.

(f) Consider the following two first order approximations:

(g) $f(q^1) - f(q^0) \approx \nabla f(q^0)^T [q^1 - q^0]$ and

(h) $f(q^0) - f(q^1) \approx \nabla f(q^1)^T [q^0 - q^1]$.

(i) Show that the right hand side of (e) is related to the two first order approximations in (g) and (h).

Appendix: A Note on the Uniqueness of the Theil Price Index Weights

On page 18 above, we indicated that the weighting function, $m(s_n^0, s_n^1)$, that appears in Theil's price index, $P_T(p^0,p^1,q^0,q^1)$, must be equal to the arithmetic mean of s_n^0 and s_n^1 in order that the Theil probability weights sum to one. In this Appendix, we will provide a proof of this result.

We want to find a homogeneous symmetric mean function, m(a,b), defined over positive numbers a and b such that the following functional equation is satisfied for all $N \geq 2$:

(A1) $\Sigma_{n=1}^{N}\, m(s_n^{0},s_n^{1}) = 1$ for all $s_n^{t} > 0$ such that
(A2) $\Sigma_{n=1}^{N}\, s_n^{0} = 1$ and $\Sigma_{n=1}^{N}\, s_n^{1} = 1$.

We will consider the case N = 2 and show that in this case, the mean function m(a,b) must be the arithmetic mean, $m(a,b) \equiv (1/2)a + (1/2)b$. But it is easy to show that this choice of mean satisfies the functional equation defined by (A1) and (A2) for all $N \geq 2$ and so the uniqueness of the Theil probability weights will be established.

Assume that m(a,b) is a symmetric homogeneous mean function and let N = 2. Let $a \equiv s_1^{0}$ and $b \equiv s_1^{1}$ where $0 < a < 1$ and $0 < b < 1$. Then using (A2) for N = 2, we see that $s_2^{0} = 1 - a$ and $s_2^{1} = 1 - b$. It can be verified that $0 < 1 - a < 1$ and $0 < 1 - b < 1$ using (A1) and (A2) for N = 2. Thus when N = 2, the functional equation defined by (A1) and (A2) becomes the following functional equation:

(A3) $m(a,b) + m(1-a,1-b) = 1$ for all a and b where $0 < a < 1$ and $0 < b < 1$.

Now let $b = 1 - a$ and equation (A3) becomes:

(A4) $1 = m(a,1-a) + m(1-a,a)$ using $1-b = a$
$= 2m(a,1-a)$ using the symmetry property of m(a,b)
$= 2am(1,a^{-1}-1)$ using $a > 0$ and the linear homogeneity property of m.

Equation (A4) can be solved for $m(1,a^{-1}-1)$:

(A5) $m(1,a^{-1}-1) = a^{-1}/2$; $0 < a < 1$.

Define $x \equiv a^{-1}-1$. It can be seen that as a approaches 0, x will approach $+\infty$ and as a approaches 1, x will approach 0. Thus as a ranges between 0 and 1, x will range between 0 and $+\infty$. Now let $x = a^{-1}-1$ and use this equation to eliminate a^{-1} from equation (A5). We find that the resulting equation is:

(A6) $m(1,x) = (x+1)/2$ for all $x > 0$.

Now multiply both sides of equation (A6) by an arbitrary $z > 0$. Using the linear homogeneity of m again, we find that m must satisfy the following functional equation:

(A7) $m(z,zx) = (z+zx)/2$ for all $z > 0$ and $x > 0$.

Finally define $y \equiv zx$ and use this equation to eliminate zx from equation (A7). We obtain the following functional equation that m must satisfy:

(A8) $m(z,y) = (z+y)/2$ for all $z > 0$ and $y > 0$.

Thus the mean function m that satisfies the functional equation defined by (A1) and (A2) must be the arithmetic mean.

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