APPLIED ECONOMICS

By W.E. Diewert.   December 13, 2008

Chapter 9: Flexible Functional Forms

1. Introduction

In this chapter, we will take an in depth look at the problems involved in choosing functional forms for estimating systems of consumer and producer demand functions and producer supply functions. We will attempt to find functional forms that are consistent with the restrictions on supply and demand functions that are implied by economic theory but are also sufficiently flexible that elasticities of supply and demand are not arbitrarily restricted by the choice of the functional form. We will make extensive use of duality theory[1] in this chapter in order to obtain systems of demand and supply functions that are consistent with economic theory but yet can be estimated by using linear regression techniques or "slightly" nonlinear regressions. Since many problems in applied economics depend on obtaining accurate estimates of elasticities, this topic is of considerable importance for the applied economist.

Section 2 below starts off by giving a formal definition of a flexible functional form for a production function and a cost function. Basically, flexible functional forms are functional forms that have a second order approximation property so that elasticities of supply and demand are not a priori restricted by using a flexible functional form. Sections 3-5 give three examples of flexible functional forms for cost functions: the Generalized Leontief cost function, the Translog cost function and the Normalized Quadratic cost function. The Normalized Quadratic functional form is our preferred functional form, because convexity or concavity restrictions can be imposed on this functional form in a parsimonious way without destroying the flexibility of the functional form. We do not know of any other flexible functional form that has this property.[2]

Section 6 shows how cost functions can be applied to the problems involved in estimating systems of consumer demand functions that are consistent with utility maximizing behavior. Sections 7 and 8 apply the general methodology to two specific functional forms: the Generalized Leontief cost function and the Normalized Quadratic cost function. Section 9 discusses the problems involved in cardinalizing a measure of utility. Section 10 discusses how nonhomothetic preferences can be estimated and section 11 extends this discussion by showing how the use of spline functions can add extra flexibility.

In section 12, we turn our attention to the problems involved in estimating multiple output, multiple input technologies.[3] The unit (capital) profit function is a key concept that is explained in this section. Sections 13-16 apply the general framework to a number of specific functional forms. Section 17 is a counterpart to section 11 and shows how spline functions can be used to add extra flexibility.

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[1] See chapter 3 of these notes. Some of the material in chapter 3 will be repeated in the present chapter.
[2] For a comparison of the Normalized Quadratic functional form with other flexible functional forms, see Diewert and Wales (1993).
[3] Sections 2-5 dealt with only single output, multiple input technologies.

Finally, sections 18 and 19 provide some generalizations of the basic normalized quadratic functional form. In section 18, we introduce a variant of the normalized quadratic profit function that can achieve flexibility at two points instead of the usual one point flexibility property.[4] In order to implement this model, the number of commodities cannot be too large, since having enough parameters to be flexible at two points instead of one point will double the number of parameters to be estimated. On the other hand, the generalization of the Normalized Quadratic functional form presented in section 19 is applicable to situations where the number of commodities is very large.[5]

2. The Definition of a Flexible Functional Form

Consider an N input, one output constant returns to scale *production function* f where $y = f(x_1,x_2,...,x_N) = f(x)$ and $y \geq 0$ denotes the output produced by the nonnegative input vector $x \geq 0_N$.

The constant returns to scale assumption means that f is *linearly homogeneous*; i.e., we have

(1) $f(\lambda x) = \lambda f(x)$ for all scalars $\lambda \geq 0$ and input vectors $x \geq 0_N$.

If in addition, f is twice continuously differentiable, then Euler's Theorem on homogeneous functions and Young's Theorem from calculus imply the following restrictions on the first and second order partial derivatives of f:

(2) $x^T \nabla f(x) = f(x)$;   (1 restriction)
(3) $\nabla^2 f(x)x = 0_N$;   (N restrictions)
(4) $\nabla^2 f(x) = [\nabla^2 f(x)]^T$ (N(N−1)/2 restrictions).

The restrictions given by (2) and (3) are implied by Euler's Theorem and the symmetry restrictions (4) are implied by Young's Theorem.

A *flexible functional form*[6] f is a functional form that has enough parameters in it so that f can approximate an arbitrary twice continuously differentiable function f* to the second order at an arbitrary point x* in the domain of definition of f and f*. Thus f must have enough free parameters in order to satisfy the following $1+N+N^2$ equations:

(5) $f(x^*) = f^*(x^*)$;   (1 equation)
(6) $\nabla f(x^*) = \nabla f^*(x^*)$;   (N equations)
(7) $\nabla^2 f(x^*) = \nabla^2 f^*(x^*)$;   (N^2 equations).

Of course, since both f and f* are assumed to be twice continuously differentiable, we do not have to satisfy all N^2 equations in (7) since Young's Theorem implies that $\partial^2 f(x^*)/\partial x_i \partial x_j = \partial^2 f(x^*)/\partial x_j \partial x_i$ and $\partial^2 f^*(x^*)/\partial x_i \partial x_j = \partial^2 f^*(x^*)/\partial x_j \partial x_i$ for all i and j. Thus the matrices of second order partial derivatives $\nabla^2 f(x^*)$ and $\nabla^2 f^*(x^*)$ are both symmetric matrices and so there are only N(N+1)/2

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[4] This section is based on Diewert and Lawrence (2002).
[5] This section is based on Diewert and Wales (1988b).
[6] This terminology was introduced by Diewert (1974a; 133).

independent equations to be satisfied in the restrictions (7). Thus a general flexible functional form must have at least $1+N+N(N+1)/2$ free parameters.

The simplest example of a flexible functional form is the following *quadratic function*:

$$(8)\ f(x) \equiv a_0 + a^T x + (1/2)x^T A x ; \qquad\qquad A = A^T$$

where a_0 is a scalar parameter, $a^T \equiv [a_1,...,a_N]$ is a vector of parameters and $A \equiv [a_{ij}]$ is a symmetric matrix of parameters. Thus the f defined by (8) has $1+N+N(N+1)/2$ parameters. To show that this f is flexible, we need to choose a_0, a and A to satisfy equations (5)-(7). Upon noting that $\nabla f(x) = a + Ax$ and $\nabla^2 f(x) = A$, equations (5)-(7) become the following equations:

$$(9)\ a_0 + a^T x^* + (1/2)x^{*T}Ax^* = f^*(x^*) ;$$
$$(10)\qquad\qquad a + Ax^* = \nabla f^*(x^*) ;$$
$$(11)\qquad\qquad A = \nabla^2 f^*(x^*).$$

To satisfy these equations, choose $A \equiv \nabla^2 f^*(x^*)$ (and A will be a symmetric matrix since f^* is assumed to be twice continuously differentiable); $a \equiv \nabla f^*(x^*) - Ax^*$ and finally, choose $a_0 \equiv f^*(x^*) -[\ a^T x^* + (1/2)x^{*T}Ax^*]$.

In many applications, we want to find a flexible functional form f that is also linearly homogeneous. For example, in production theory, if the minimum average cost plant size is small relative to the size of the market, then we can approximate the industry technology by means of a constant returns to scale production function. As another example, in the pure theory of international trade, we often assume that consumer preferences are *homothetic* [7] ; i.e., the consumer's utility function can be represented by g[f(x)] where f is linearly homogeneous and g is a monotonically increasing and continuous function of one variable. In this case, we can represent the consumer's preferences equally well by using the linearly homogeneous utility function g[f(x)].

If the production function f (or the utility function f) is linearly homogeneous, then the corresponding cost function C has the following structure: for $y > 0$ and $p \gg 0_N$,

$$
\begin{aligned}
(12)\quad C(y,p) &\equiv \min_x \{p^T x : f(x) \geq y\} \\
&= \min_x \{p^T x : f(x) = y\} \quad \text{if f is continuous and increasing in the components of x} \\
&= \min_x \{p^T x : (1/y)f(x) = 1\} \\
&= \min_x \{p^T x : f(\{1/y\}x) = 1\} \qquad \text{using the linear homogeneity of f} \\
&= \min_{x/y} \{yp^T(x/y) : f(x/y) = 1\} \\
&= y \min_z \{p^T z : f(z) = 1\} \qquad\qquad \text{letting } z \equiv x/y \\
&= y\ C(1,p) \\
&= y\ c(p)
\end{aligned}
$$

where we define the *unit cost function* c(p) as C(1,p), the minimum cost of producing one unit of output (or utility).

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[7] Shephard (1953) introduced this term.

It is straightforward to show that $C(1,p)$ and $c(p)$ are linearly homogeneous and concave in the components of the price vector p.

Problems.

1. Let $y \equiv [y_1,...,y_N]^T$ denote a vector of variable outputs and inputs that a firm produces or uses during a period; if the firm produces commodity i, then $y_i > 0$ while if the firm uses commodity i as an input, then $y_i < 0$ for $i = 1,...,N$. The vector y is called a net output vector or a netput vector. Given the net output vector y, the minimum amount of capital $k \geq 0$ that is required to produce the vector of net outputs y is $F(y)$, where F is the firm's capital requirements function.[8] Given a positive vector of variable input and output prices $p >> 0_N$ and a positive amount of capital $k > 0$, the firm's variable profit function $\Pi(k,p)$ is defined as follows[9]:

(i) $\Pi(k,p) \equiv \max_y \{p^T y : F(y) \leq k\}$.

Prove that for each $k > 0$, $\Pi(k,p)$ is a linearly homogeneous and convex function of p.

2. (Continuation of 1.) Let y* solve the variable profit maximization problem $\Pi(k^*,p^*) \equiv \max_y \{p^{*T} y : F(y) \leq k^*\}$ where $k^* > 0$ and $p^* >> 0_N$. Assume that $\Pi(k^*,p)$ is differentiable with respect to the components of p at the point p*; i.e., assume that the vector of first order partial derivatives $\nabla_p \Pi(k^*,p^*)$ exists. Show that $y^* = \nabla_p \Pi(k^*,p^*)$. This result is known as Hotelling's (1932; 594) Lemma. Hint: define $g(p) \equiv p^T y^* - \Pi(k^*,p)$ and show that $g(p) \leq 0$ and $g(p^*) = 0$.

3. Assume that the capital requirements function $F(y)$ is linearly homogeneous; i.e., $F(\lambda y) = \lambda F(y)$ for all $\lambda > 0$. (This means that the technology exhibits *constant returns to scale*.) Under this assumption, show that $\Pi(k,p)$ has the following decomposition: for $k > 0$ and $p >> 0_N$,

$\Pi(k,p) = k\Pi(1,p)$.

The function $\Pi(1,p) \equiv \pi(p)$ is known as the firm's *unit (capital) profit function*. By problem 1 above, it too will be a linearly homogeneous function.

4. Using problem 2 above, it can be seen that the firm's variable profit maximizing system of net supply functions, $y(k,p)$, is equal to the vector of first order partial derivatives $\nabla_p \Pi(k,p)$, provided that $\Pi(k,p)$ is differentiable with respect to the components of the variable price vector p. If $\Pi(k,p)$ is twice continuously differentiable with respect to the components of p, show that the N by N matrix of price derivatives of the net supply functions, $\nabla_p y(k,p) \equiv [\partial y_i(k,p)/\partial p_j]$, has the following properties:

(a) $\nabla_p y(k,p) = [\nabla_p y(k,p)]^T$;
(b) $[\nabla_p y(k,p)]p = 0_N$;

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[8] If there is no amount of capital that can produce a given vector of net outputs y, then we define $F(y) \equiv +\infty$. For more on the properties of factor requirements functions, see Diewert (1974b).
[9] We assume that for each $k > 0$, the lower level set of F defined as $\{y: F(y) \leq k \}$ is a nonempty, closed and bounded set so that the maximum in (i) exists.

(c) $z^T\nabla_p y(k,p)z \geq 0$ for every vector z;

(d) $e_i^T\nabla_p y(k,p)e_i \geq 0$ for i = 1,...,N where e_i is the ith unit vector. Provide an economic interpretation for these inequalities.

5. (Continuation of 4.) Commodities i and j are said to be *substitutes* in production if $\partial y_i(k,p)/\partial p_j < 0$ for i $\neq$ j. Commodities i and j are said to be *complements* in production if $\partial y_i(k,p)/\partial p_j > 0$ for i $\neq$ j. Commodities i and j are said to be *unrelated* in production if $\partial y_i(k,p)/\partial p_j = 0$. If N = 2, show that variable commodities 1 and 2 cannot be complements; i.e., they must be substitutes or be unrelated.

Linearly homogeneous functions arise naturally in a variety of economic applications. Moreover, even if we allow our production function or utility function f to be a general nonhomogeneous function, it is often of interest to allow f to have the capability to be flexible in the class of linearly homogeneous functions.

Consider what happens to the general quadratic function f defined by (8) if we attempt to specialize it to become a linearly homogeneous functional form. In order to make it homogeneous of degree one, we must set $a_0 = 0$ and set $A = 0_{NxN}$ and the resulting functional form collapses down to the following linear function:

(13) $f(x) = a^Tx$.

But the f defined by (13) is not a flexible linearly homogeneous functional form! Thus finding flexible linearly homogeneous functional forms is not completely straightforward.

Let us determine the minimal number of free parameters that a flexible linearly homogeneous functional form must have. If both f and f* are linearly homogeneous (and twice continuously differentiable), then both functions will satisfy the restrictions (2)-(4). In view of these restrictions, it can be seen that instead of f having to satisfy all $1+N+N^2$ of the equations (5)-(7), f need only satisfy the following $N+N(N-1)/2 = N(N+1)/2$ equations:

(14) $\nabla f(x^*) = \nabla f^*(x^*)$;   (N equations)
(15) $f_{ij}(x^*) = f^*_{ij}(x^*)$ for $1 \leq i < j \leq N$ (N(N−1)/2 equations)

where $f_{ij}(x^*) \equiv \partial^2 f(x^*)/\partial x_i \partial x_j$. Note that equations (15) are the equations in the upper triangle of the matrix equation (7) above. If the upper triangle equations in (7) are satisfied, then by Young's Theorem, the lower triangle equations will also be satisfied if equations (15) are satisfied. The main diagonal equations in (7) will also be satisfied if equations (15) are satisfied: the diagonal elements $f_{ii}(x^*)$ are determined by the restrictions $\nabla^2 f(x^*)x^* = 0_N$ and the $f^*_{ii}(x^*)$ are determined by the restrictions $\nabla^2 f^*(x^*)x^* = 0_N$.

Thus in order for f(x) (or c(p) or $\pi(p)$) to be a flexible linearly homogeneous functional form, it must have at least $N + N(N-1)/2 = N(N+1)/2$ free parameters. If it has exactly this number of free parameters, then we say that f is a *parsimonious flexible functional form*.

In the following sections, we shall give some examples of parsimonious flexible functional forms for unit cost functions. These same functional forms can be used as parsimonious flexible functional forms for unit profit functions.[10] Thus we are looking for linearly homogeneous functions c(p) that can satisfy the following $N(N+1)/2$ equations:

(16) $\nabla c(p^*) = \nabla c^*(p^*)$;   (N equations)

(17) $c_{ij}(p^*) = c^*_{ij}(p^*)$ for $1 \leq i < j \leq N$ ($N(N-1)/2$ equations).

Why is it important that functional forms used in applied economics be flexible? From Shephard's (1953; 11) Lemma, the producer's system of cost minimizing input demand functions, x(y,p), is equal to the vector of first order partial derivatives of the cost function with respect of input prices, $\nabla_p C(y,p)$. Thus the matrix of *first order input demand price derivatives* $\nabla_p x(y,p)$ is equal to the matrix of second order partial derivatives $\nabla^2_{pp} C(y,p)$. Hence, if the functional form for C is *not* flexible, *price elasticities of input demand will be a priori restricted in some arbitrary way*.[11] Many practical problems in applied economics depend crucially on estimates of elasticities and hence it is not appropriate to use estimates of elasticities that are restricted in some arbitrary manner.

In the following sections, we will exhibit some examples of flexible functional forms.

3. The Generalized Leontief Cost Function.

Define the *generalized Leontief unit cost function* c(p) as follows[12]:

(18) $c(p_1,...,p_N) \equiv \sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij} p_i^{1/2} p_j^{1/2}$; $b_{ij} = b_{ji}$ for $1 \leq i < j \leq N$.

Thus c is a quadratic form in the square roots of input prices and has $N(N+1)/2$ b_{ij} parameters.

We need to determine whether the unit cost function c(p) defined by (18) is flexible; i.e., whether we can choose the b_{ij} so as to satisfy equations (16) and (17). Upon differentiating (18), equations (16) and (17) become the following equations:

(19) $c_i(p^*) = \sum_{j=1}^{N} b_{ij} (p_i^*)^{-(1/2)}(p_j^*)^{1/2} = c^*_i(p^*)$; $i = 1,...,N$;

(20) $c_{ij}(p^*) = (1/2) b_{ij} (p_i^*)^{-(1/2)}(p_j^*)^{-(1/2)} = c^*_{ij}(p^*)$; $1 \leq i < j \leq N$.

Use equations (20) to determine b_{ij} for $1 \leq i < j \leq N$. Then use equations (19) to solve for the b_{ii} for $i = 1,...,N$. This proves that the c(p) defined by (18) is flexible. Since it has only $N(N+1)/2$ parameters, it is also parsimonious.

In a production study where there is only one output and N inputs and the assumption of competitive cost minimization is justified, given period t data on input demands, x_i^t, input prices, p_i^t

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[10] The only difference is that the concavity in prices property for unit cost functions must be replaced by the convexity in prices property for unit profit functions.

[11] A similar comment applies in the profit function context; unless the variable profit function $\Pi(k,p)$ is flexible, estimates of elasticities of net supply will be arbitrarily restricted.

[12] This functional form was introduced by Diewert (1971).

and on output produced, y^t, then the unknown parameters in (18) can be estimated by using the following N estimating equations:

(21) $x_i^t/y^t = \sum_{j=1}^{N} b_{ij} (p_j^t/p_i^t)^{1/2} + e_i^t$; $\qquad\qquad$ i = 1,...,N,

where the e_i^t are stochastic error terms for i = 1,...,N.

Note that b_{ij} in equation i should equal b_{ji} in equation j. These *cross equation symmetry restrictions* can be imposed in the estimation procedure or we could *test* for their validity.

After estimating the b_{ij}, it is necessary to check whether $\nabla^2 c(p^t)$ is *negative semidefinite* at each data point p^t.[13] Thus it will be necessary to calculate the second order derivatives of c at each data point. Differentiating the c(p) defined by (18) yields the following formulae for the derivatives:

(22) $c_{ij}(p^t) = (1/2) b_{ij} (p_i^t p_j^t)^{-(1/2)}$ $\qquad\qquad$ for i ≠ j ;
$\qquad c_{ii}(p^t) = -(1/2) \sum_{k \neq i, k=1}^{N} b_{ik} (p_i^t)^{-(3/2)}(p_k^t)^{(1/2)}$; $\qquad$ for i = 1,...,N.

Note that the b_{ii} do not appear in the formulae (22) for the second derivatives of the generalized Leontief unit cost function. Note also if all $b_{ij} = 0$ for i ≠ j, then the functional form defined by (18) collapses down to the no substitution Leontief (1941) functional form[14]. Under these restrictions, the input demand functions defined by (21) collapse down to the following system of equations:

(23) $x_i^t/y^t = b_{ii} + e_i^t$; $\qquad\qquad$ i = 1,...,N.

Thus input demands are not affected by changes in input prices if the producer's cost function has the Leontief functional form.

Problems.

6. Let N = 2 and try to determine necessary and sufficient conditions on the parameters b_{11}, b_{12} and b_{22} that will make the generalized Leontief unit cost function defined by (18), $c(p_1,p_2)$, concave in the input prices (p_1,p_2). Look at the system of estimating equations (21) when N = 2. Can you determine a simple method for making sure that your estimated generalized Leontief unit cost function will satisfy the concavity property?

7. Determine a simple set of sufficient conditions that will make the generalized Leontief unit cost function defined by (18) concave in p for an arbitrary N over the set $S \equiv \{p : p \gg 0_N\}$.

4. The Translog Unit Cost Function.

The translog unit cost function, c(p), is defined as follows:[15]

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[13] A necessary and sufficient condition for a twice continuously differentiable c(p) to be concave over a convex set S is that $\nabla^2 c(p)$ be negative semidefinite for all p belonging to S.
[14] This functional form was actually used by Walras (1954; 243); the first edition of this book was published in 1874.
[15] This functional form is due to Christensen, Jorgenson and Lau (1971) (1975).

(24) $\ln c(p) \equiv \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln p_i + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i \ln p_j$

where the parameters α_i and γ_{ij} satisfy the following restrictions:

(25) $\gamma_{ij} = \gamma_{ji}$; $\quad 1 \le i < j \le N$; $\qquad\qquad$ (N(N−1)/2 symmetry restrictions)
(26) $\sum_{i=1}^{N} \alpha_i = 1$; $\qquad\qquad\qquad\qquad$ (1 restriction)
(27) $\sum_{j=1}^{N} \gamma_{ij} = 0$; $\,$ i = 1,...,N $\qquad\qquad$ (N restrictions).

Note that the symmetry restrictions (25) and the restrictions (27) imply the following restrictions:

(28) $\sum_{i=1}^{N} \gamma_{ij} = 0$; $\,$ j = 1,...,N.

There are $1+N$ α_i parameters and N^2 γ_{ij} parameters. However, the restrictions (25)-(27) mean that there are only N independent α_i parameters and N(N−1)/2 independent γ_{ij} parameters, which is the minimal number of parameters required for a unit cost function to be flexible.

We show that the translog unit cost function c(p) defined by (24)-(27) is linearly homogeneous; i.e., we need to show that $c(\lambda p) = \lambda c(p)$ for $\lambda > 0$ and $p \gg 0_N$. Thus, we need to show that

(29) $\ln c(\lambda p) = \ln[\lambda c(p)] = \ln\lambda + \ln c(p)$ $\qquad$ for $\lambda > 0$ and $p \gg 0_N$.

Using definition (24), we have

(30) $\ln c(\lambda p_1,..., \lambda p_N) = \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln\lambda p_i + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln\lambda p_i \ln\lambda p_j$
$= \alpha_0 + \sum_{i=1}^{N} \alpha_i[\ln\lambda+\ln p_i] + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}[\ln\lambda+\ln p_i][\ln\lambda+\ln p_j]$
$= \alpha_0 + \sum_{i=1}^{N} \alpha_i[\ln\lambda]+\sum_{i=1}^{N} \alpha_i \ln p_i + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}[\ln\lambda+\ln p_i][\ln\lambda+\ln p_j]$
$= \alpha_0 + 1\,[\ln\lambda]+\sum_{i=1}^{N} \alpha_i \ln p_i + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}[\ln\lambda+\ln p_i][\ln\lambda+\ln p_j]$ $\qquad$ using (26)
$= \ln\lambda + \alpha_0+\sum_{i=1}^{N} \alpha_i \ln p_i + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}[\ln\lambda][\ln\lambda]$
$\quad + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}[\ln\lambda][\ln p_j]+ (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}[\ln p_i][\ln\lambda]$
$\quad + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}[\ln p_i][\ln p_j]$
$= \ln\lambda + \alpha_0+\sum_{i=1}^{N} \alpha_i \ln p_i + (1/2) \sum_{i=1}^{N} [\sum_{j=1}^{N} \gamma_{ij}][\ln\lambda][\ln\lambda]$
$\quad + (1/2) \sum_{j=1}^{N} [\sum_{i=1}^{N} \gamma_{ij}][\ln p_j][\ln\lambda] + (1/2) \sum_{i=1}^{N} [\sum_{j=1}^{N} \gamma_{ij}][\ln p_i][\ln\lambda]$
$\quad + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}[\ln p_i][\ln p_j]$
$= \ln\lambda + \alpha_0+\sum_{i=1}^{N} \alpha_i \ln p_i + (1/2) \sum_{i=1}^{N} [0][\ln\lambda][\ln\lambda]$
$\quad + (1/2) \sum_{j=1}^{N} [0][\ln p_j][\ln\lambda] + (1/2) \sum_{i=1}^{N} [0][\ln p_i][\ln\lambda]$
$\quad + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}[\ln p_i][\ln p_j]$ $\qquad$ using (27) and (28)
$= \ln\lambda + \alpha_0+\sum_{i=1}^{N} \alpha_i \ln p_i + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}[\ln p_i][\ln p_j]$
$= \ln\lambda + \ln c(p)$ $\qquad$ using definition (24)

which establishes the linear homogeneity property (29). Thus the restrictions (25)-(27) are just the right ones to imply the linear homogeneity of the translog unit cost function.

To establish the flexibility of the translog unit cost function c(p) defined by (24)-(27), we need only solve the following system of equations, which is equivalent to the N(N+1)/2 equations defined by (16) and (17):

(31) $\ln c(p^*) = \ln c^*(p^*)$;   (1 equation)
(32) $\partial \ln c(p^*)/\partial \ln p_i = \partial \ln c^*(p^*)/\partial \ln p_i$; $i = 1,2,...,N-1$;   (N−1 equations)
(33) $\partial^2 \ln c(p^*)/\partial \ln p_i \partial \ln p_j = \partial^2 \ln c^*(p^*)/\partial \ln p_i \partial \ln p_j$; $1 \leq i < j \leq N$;   (N(N−1)/2 equations).

Upon differentiating the translog unit cost function defined by (24), we see that equations (32) are equivalent to the following equations:

(34) $\alpha_i + \sum_{j=1}^N \gamma_{ij} \ln p_j = \partial \ln c^*(p^*)/\partial \ln p_i$; $i = 1,2,...,N-1$.

Differentiating the translog unit cost function again, we find that equations (33) are equivalent to the following equations:

(35) $\gamma_{ij} = \partial^2 \ln c^*(p^*)/\partial \ln p_i \partial \ln p_j$; $1 \leq i < j \leq N$.

Now use equations (35) to determine the γ_{ij} for $1 \leq i < j \leq N$. Now use the symmetry restrictions (25) to determine the γ_{ij} for $1 \leq j < i \leq N$. Now use equations (27) to determine the γ_{ii} for $i = 1,2,...,N$. With the entire N by N matrix of the γ_{ij} now determined, use equations (34) in order to determine the α_i for $i = 1,2,...,N-1$. Now use equation (26) to determine α_N. Finally, use equation (31) to determine α_0.

We turn our attention to the problems involved in obtaining estimates for the unknown parameters α_i and γ_{ij}, which occur in the definition of the translog unit cost function, c(p) defined by (24). In the producer context, the total cost function C(y,p) is defined in terms of the unit cost function c(p) as follows:

(36) $C(y,p) \equiv y c(p)$.

Taking logarithms on both sides of (36) yields:

(37) $\ln C(y,p) = \ln y + \ln c(p)$
           $= \ln y + \alpha_0 + \sum_{i=1}^N \alpha_i \ln p_i + (1/2) \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \ln p_i \ln p_j$

where we have replaced ln c(p) using (24). The corresponding system of cost minimizing input demand functions x(y,p) is obtained using Shephard's Lemma:

(38) $x(y,p) \equiv \nabla_p C(y,p) = y \nabla_p c(p)$.

Suppose that in period t, observed output is y^t, the vector of observed input prices is $p^t \gg 0_N$ and the vector of observed input demands is $x^t > 0_N$. Thus the *period t observed cost* is:

(39) $C^t \equiv p^{tT} x^t \equiv \sum_{i=1}^N p_i^t x_i^t$.

Now evaluate (37) at the period t data and add an error term, e_0^t. Using (39), (37) evaluated at the period t data becomes the following estimating equation:

(40) $\ln C^t = \ln y^t + \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln p_i^t + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i^t \ln p_j^t + e_0^t$; $t = 1,...,T$.

Note that (40) is linear in the unknown parameters.

In order to obtain additional estimating equations, we have to use the input demand functions, $x_i(y,p) \equiv y \partial c(p)/\partial p_i$ for $i = 1,...,N$; (see equations (38) above). The ith input share function, $s_i(y,p)$, is defined as:

$$
\begin{aligned}
(41)\ s_i(y,p) &\equiv p_i x_i(y,p)/C(y,p) && i = 1,...,N \\
&= p_i[y \partial c(p)/\partial p_i]/C(y,p) && \text{using (38)} \\
&= p_i[y \partial c(p)/\partial p_i]/yc(p) && \text{using (36)} \\
&= p_i[\partial c(p)/\partial p_i]/c(p) && \\
&= \partial \ln c(p)/\partial \ln p_i && \\
&= \alpha_i + \sum_{j=1}^{N} \gamma_{ij} \ln p_j && \text{upon differentiating the } c(p) \text{ defined by (24).}
\end{aligned}
$$

Now evaluate both sides of (41) at the period t data and add error terms e_i^t to obtain the following system of estimating equations:

(42) $s_i^t \equiv p_i^t x_i^t/C^t = \alpha_i + \sum_{j=1}^{N} \gamma_{ij} \ln p_j^t + e_i^t$; $\qquad i = 1,...,N.$

Note that equations (42) are also linear in the unknown parameters. Obviously, the N estimating equations in (42) could be added to the single estimating equation (40) in order to obtain N+1 estimating equations with cross equation equality constraints on the parameters α_i and γ_{ij}. However, since total cost in any period t, C^t, equals the sum of the individual expenditures on the inputs[16], $\sum_{i=1}^{N} p_i^t x_i^t$, the observed input shares $s_i^t \equiv p_i^t x_i^t/C^t$ will satisfy the following constraint for each period t:

(43) $\sum_{i=1}^{N} s_i^t = 1.$

Thus the stochastic error terms e_i^t in equations (42) cannot all be independent. Hence we must drop one estimating equation from (42). Thus equation (40) and any N−1 of the N equations in (42) may be used as a system of estimating equations in order to determine the parameters of the translog unit cost function.[17]

We now turn our attention to the problem of deriving a formula for the price elasticities of demand, $\partial x_i(y,p)/\partial p_j$, given that the unit cost function has the translog functional form defined by (24)-(27).

---

[16] This identity explains why we did not add the counterpart to (40) as an estimating equation to the estimating equations (21) in the previous section.

[17] In situations where N is large relative to the number of observations T, maximum likelihood estimation of equation (40) and N−1 of the equations (41) can fail if a general variance covariance matrix has to be estimated for the error terms in these equations. The problem is that all of the unknown economic parameters are contained in equation (40) and as a result, the estimated squared residuals in this equation will tend to be small relative to the estimated squared residuals in equations (41), where each equation has only a few unknown economic parameters. Hence equation (40) can suffer from multicollinearity problems and the small apparent variance of the residuals in this equation lead to the maximum likelihood estimation procedure giving too much weight to equation (40) relative to the other equations. Under these conditions, the resulting elasticities may be erratic and not satisfy the appropriate curvature conditions.

Using the equations in (41) above, we have the following expressions for the ith input share functions, $s_i(y,p)$:

$$(44)\ s_i(y,p) = p_i\, x_i(y,p)/C(y,p) = \partial lnc(p)/\partial lnp_i = \alpha_i + \sum_{j=1}^{N} \gamma_{ij}\, lnp_j\ ; \qquad i = 1,...,N.$$

For $j \neq i$, differentiate the ith equation in (44) with respect to the log of p_j and we obtain the following equations:

$$(45)\ \partial s_i(y,p)/\partial lnp_j = p_i\, \partial[x_i(y,p)/C(y,p)]/\partial lnp_j = \gamma_{ij}\ ; \qquad i \neq j.$$

Hence

$$(46)\ \gamma_{ij} = p_i\, \partial[x_i(y,p)/C(y,p)]/\partial lnp_j \qquad\qquad i \neq j$$
$$= p_i p_j\, \partial[x_i(y,p)/C(y,p)]/\partial p_j$$
$$= p_i p_j\, \{[1/C(y,p)][\partial x_i(y,p)/\partial p_j] - x_i(y,p)[1/C(y,p)]^2[\partial C(y,p)/\partial p_j]\}$$
$$= [p_i x_i(y,p)/C(y,p)]\{\partial ln\, x_i(y,p)/\partial lnp_j\} - [p_i x_i(y,p)/C(y,p)][\, p_j x_j(y,p)/C(y,p)]$$
$$\text{using Shephard's Lemma, } x_j(y,p) = \partial C(y,p)/\partial p_j$$
$$= s_i(y,p)\{\partial lnx_i(y,p)/\partial lnp_j\} - s_i(y,p)\, s_j(y,p).$$

Equations (46) can be rearranged to give us the following formula for the *cross price elasticities of input demand*:

$$(47)\ \partial lnx_i(y,p)/\partial lnp_j = [s_i(y,p)]^{-1} \gamma_{ij} + s_j(y,p)\ ; \qquad i \neq j.$$

Now differentiate the ith equation in (44) with respect to the log of p_i and get the following equations:

$$(48)\ \gamma_{ii} = p_i\, \partial[p_i x_i(y,p)/C(y,p)]/\partial p_i \qquad\qquad i = 1,...,N$$
$$= p_i\, \{[x_i(y,p)/C(y,p)] + [p_i/C(y,p)][\partial x_i(y,p)/\partial p_i] - [p_i x_i(y,p)/C(y,p)^2][\partial C(y,p)/\partial p_i]\}$$
$$= p_i\, \{[x_i(y,p)/C(y,p)] + [p_i/C(y,p)][\partial x_i(y,p)/\partial p_i] - [p_i x_i(y,p)/C(y,p)^2][x_i(y,p)]\}$$
$$\text{using Shephard's Lemma, } x_i(y,p) = \partial C(y,p)/\partial p_i$$
$$= p_i\, x_i(y,p)/C(y,p) + [p_i x_i(y,p)/C(y,p)][\partial lnx_i(y,p)/\partial lnp_i] - [p_i x_i(y,p)/C(y,p)]^2$$
$$= s_i(y,p) + s_i(y,p)[\partial lnx_i(y,p)/\partial lnp_i] - s_i(y,p)^2.$$

Equations (48) can be rearranged to give us the following formula for the *own price elasticities of input demand*:

$$(49)\ \partial lnx_i(y,p)/\partial lnp_i = [s_i(y,p)]^{-1} \gamma_{ii} + s_i(y,p) - 1\ ; \qquad i = 1,...,N.$$

Thus given econometric estimates for the α_i and γ_{ij}, which we denote by α_i^* and γ_{ij}^*, the estimated or fitted shares in period t, s_i^{t*} are defined using these estimates and equations (44) evaluated at the period t data:

$$(50)\ s_i^{t*} \equiv \alpha_i^* + \sum_{j=1}^{N} \gamma_{ij}^*\, lnp_j^t\ ; \qquad i = 1,...,N\ ;\ t = 1,...,T.$$

Now use equations (47) evaluated at the period t data and econometric estimates to obtain the following formula for the *period t cross elasticities of demand*, e_{ij}^t:

$$(51) \ e_{ij}^t \equiv \partial lnx_i(y^t,p^t)/\partial lnp_j = [s_i^{t*}]^{-1} \gamma_{ij}^* + s_j^{t*} \ ; \qquad\qquad i \neq j.$$

Similarly, use equations (49) evaluated at the period t data and econometric estimates to obtain the following formula for the *period t own elasticities of demand*, e_{ii}^t:

$$(52) \ e_{ii}^t \equiv \partial lnx_i(y^t,p^t)/\partial lnp_i = [s_i^{t*}]^{-1} \gamma_{ii}^* + s_i^{t*} - 1 \ ; \qquad\qquad i = 1,...,N.$$

We can also obtain an estimated or *fitted period t cost*, C^{t*}, by using our econometric estimates for the parameters and by exponentiating the right hand side of the equation t in (40):

$$(53) \ C^{t*} \equiv exp[lny^t + \alpha_0^* + \sum_{i=1}^N \alpha_i^* \ lnp_i^t + (1/2) \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij}^* \ lnp_i^t \ lnp_j^t] \ ; \qquad t = 1,...,T.$$

Finally, our fitted period t shares s_i^{t*} defined by (50) and our fitted period t costs C^{t*} defined by (53) can be used in order to obtain estimated or *fitted period t input demands*, x_i^{t*}, as follows:

$$(54) \ x_i^{t*} \equiv C^{t*} s_i^{t*}/p_i^t \ ; \qquad\qquad i = 1,...,N \ ; \ t = 1,...,T.$$

Given the matrix of period t estimated input price elasticities of demand, $[e_{ij}^t]$, we can readily calculate the matrix of period t *estimated input price derivatives*, $\nabla_p x(y^t,p^t) = \nabla^2_{pp} C(y^t,p^t)$. Our estimate for element ij of $\nabla^2_{pp} C(y^t,p^t)$ is:

$$(55) \ C_{ij}^{t*} \equiv e_{ij}^t \ x_i^{t*}/p_j^t \ ; \qquad\qquad i,j = 1,...,N \ ; \ t = 1,...,T$$

where the estimated period t elasticities e_{ij}^t are defined by (51) and (52) and the fitted period t input demands x_i^{t*} are defined by (54). Once the estimated input price derivative matrices $[C_{ij}^{t*}]$ have been calculated, then we may check whether each of them is negative semidefinite using determinantal conditions or by checking if all of the eigenvalues of each matrix are zero or negative. Unfortunately, *very frequently these negative semidefiniteness conditions will fail to be satisfied for both the translog and generalized Leontief functional forms*. Hence, in the following section, we study a functional form where these curvature conditions can be imposed without destroying the flexibility of the functional form.

5. The Normalized Quadratic Unit Cost Function.

The normalized quadratic unit cost function c(p) is defined as follows for $p \gg 0_N$:[18]

$$(56) \ c(p) \equiv b^Tp + (1/2)p^TBp/\alpha^Tp$$

where $b^T \equiv [b_1,...,b_N]$ and $\alpha^T \equiv [\alpha_1,...,\alpha_N]$ are parameter vectors and $B \equiv [b_{ij}]$ is a matrix of parameters. The vector α and the matrix B satisfy the following restrictions:

---

[18] This functional form was introduced by Diewert and Wales (1987; 53) where it was called the Symmetric Generalized McFadden functional form. Additional material on this functional form can be found in Diewert and Wales (1988a) (1988b) (1992) (1993).

(57) $\alpha > 0_N$;

(58) $B = B^T$; i.e., the matrix B is symmetric;

(59) $Bp^* = 0_N$ for some $p^* \gg 0_N$.

In most empirical applications, the vector of nonnegative but nonzero parameters α is fixed a priori. The two most frequent a priori choices for α are $\alpha \equiv 1_N$, a vector of ones or $\alpha \equiv (1/T) \sum_{t=1}^{T} x^t$, the sample mean of the observed input vectors in the producer context or the sample mean of the observed commodity vectors in the consumer context. The two most frequent choices for the reference price vector p^* are $p^* \equiv 1_N$ or $p^* \equiv p^t$ for some period t; i.e., in this second choice, we simply set p^* equal to the observed period t price vector.

Assuming that α has been predetermined, there are N unknown parameters in the b vector and $N(N-1)/2$ unknown parameters in the B matrix, taking into account the symmetry restrictions (58) and the N linear restrictions in (59). Note that the c(p) defined by (56) is linearly homogeneous in the components of the input price vector p.

Another possible way of defining the normalized quadratic unit cost function is as follows:

(60) $c(p) \equiv (1/2) p^T A p / \alpha^T p$

where the parameter matrix A is symmetric; i.e., $A = A^T \equiv [a_{ij}]$ and $\alpha > 0_N$ as before. Assuming that the vector of parameters α has been predetermined, the c(p) defined by (60) has $N(N+1)/2$ unknown a_{ij} parameters.

Comparing (56) with (60), it can be seen that (60) has dropped the b vector but has also dropped the N linear constraints (59). It can be shown that the model defined by (56) is a special case of the model defined by (60). To show this, given (56), define the matrix A in terms of B, b and α as follows:

(61) $A \equiv B + [b\alpha^T + \alpha b^T]$.

Substituting (61) into (60), (60) becomes:

(62) $c(p) = (1/2)p^T\{B + [b\alpha^T + \alpha b^T]\}p/\alpha^T p$
$= (1/2)p^T Bp/\alpha^T p + (1/2) p^T[b\alpha^T + \alpha b^T]p/\alpha^T p$
$= (1/2)p^T Bp/\alpha^T p + (1/2)\{p^T b\alpha^T p + p^T \alpha b^T p\}/\alpha^T p$
$= (1/2)p^T Bp/\alpha^T p + (1/2)\{2p^T b\alpha^T p\}/\alpha^T p$
$= (1/2)p^T Bp/\alpha^T p + p^T b$

which is the same functional form as (56). However, we prefer to work with the model (56) rather than with the seemingly more general model (60) for three reasons:

- The c(p) defined by (56) clearly contains the no substitution Leontief functional form as a special case (simply set $B = 0_{NxN}$);

- the estimating equations that correspond to (56) will contain constant terms and
- it is easier to establish the flexibility property for (56) than for (60).

The first and second order partial derivatives of the normalized quadratic unit cost function defined by (56) are given by:

(63) $\nabla_p c(p) = b + (\alpha^T p)^{-1} Bp - (1/2)(\alpha^T p)^{-2} p^T Bp\alpha$;
(64) $\nabla^2_{pp} c(p) = (\alpha^T p)^{-1} B - (\alpha^T p)^{-2} Bp\alpha^T - (\alpha^T p)^{-2} \alpha p^T B + (\alpha^T p)^{-3} p^T Bp\alpha\alpha^T$.

We now prove that the c(p) defined by (56)-(59) (with α predetermined) is a flexible functional form at the point p*. Using the restrictions (59), $Bp^* = 0_N$, we have $p^{*T} Bp^* = p^{*T} 0_N = 0$. Thus evaluating (63) and (64) at $p = p^*$ yields the following equations:

(65) $\nabla_p c(p^*) = b$;
(66) $\nabla^2_{pp} c(p^*) = (\alpha^T p^*)^{-1} B$.

We need to satisfy equations (16) and (17) above to show that the c(p) defined by (56)-(59) is flexible at p*. Using (65), we can satisfy equations (16) if we choose b as follows:

(67) $b \equiv \nabla c^*(p^*)$.

Using (66), we can satisfy equations (17) by choosing B as follows:

(68) $B \equiv (\alpha^T p^*) \nabla^2 c^*(p^*)$.

Since $\nabla^2 c^*(p^*)$ is a symmetric matrix, B will also be a symmetric matrix and so the symmetry restrictions (58) will be satisfied for the B defined by (68). Moreover, since c*(p) is assumed to be a linearly homogeneous function, Euler's Theorem implies that

(69) $\nabla^2 c^*(p^*)p^* = 0_N$.

Equations (68) and (69) imply that the B defined by (68) satisfies the linear restrictions (59). This completes the proof of the flexibility property for the normalized quadratic unit cost function.

It is convenient to define the vector of *normalized input prices*, $v^T \equiv [v_1,...,v_N]$ as follows:

(70) $v \equiv (p^T \alpha)^{-1} p$.

The system of input demand functions x(y,p) that corresponds to the normalized quadratic unit cost function c(p) defined by (56) can be obtained using Shephard's Lemma in the usual way:

(71) $x(y,p) = y \nabla c(p)$.

Using (71) and (63) evaluated at the period t data, we obtain the following system *of estimating equations*:

(72) $x^t/y^t = b + Bv^t - (1/2)v^{tT}Bv^t\alpha + e^t$; $t = 1,...,T$

where x^t is the observed period t input vector, y^t is the period t output, $v^t \equiv p^t/\alpha^T p^t$ is the vector of period t normalized input prices and $e^t \equiv [e_1^t,...,e_N^t]^T$ is a vector of stochastic error terms. Equations (72) can be used in order to statistically estimate the parameters in the b vector and the B matrix. Note that equations (72) are linear in the unknown parameters. Note also that the symmetry restrictions (58) can be imposed in (72) or their validity can be tested.

Once estimates for b and B have been obtained (denote these estimates by b^* and B^* respectively), then equations (72) can be used in order to generate a period t vector of fitted input demands, x^{t*} say:

(73) $x^{t*} \equiv y^t[b^* + B^*v^t - (1/2)v^{tT}B^*v^t\alpha]$; $t = 1,...,T$.

Equations (64) and (71) may be used in order to calculate the matrix of period t *estimated input price derivatives*, $\nabla_p x(y^t,p^t) = \nabla^2_{pp}C(y^t,p^t)$. Our estimate for $\nabla^2_{pp}C(y^t,p^t)$ is:

(74) $[C_{ij}^{t*}] \equiv y^t[(\alpha^T p^t)^{-1}B^* - (\alpha^T p^t)^{-2}B^*p^t\alpha^T - (\alpha^T p^t)^{-2}\alpha p^{tT}B^* + (\alpha^T p^t)^{-3} p^{tT}B^*p^t\alpha\alpha^T]$; $t = 1,...,T$.

Equations (73) and (74) may be used in order to obtain estimates for the matrix *of period t input demand price elasticities*, $[e_{ij}^t]$:

(75) $e_{ij}^t \equiv \partial \ln x_i(y^t,p^t)/\partial \ln p_j = p_j^t C_{ij}^{t*}/x_i^{t*}$; $i,j = 1,...,N$; $t = 1,...,T$

where x_i^{t*} is the ith component of the vector of fitted demands x^{t*} defined by (73).

There is one important additional topic that we have to cover in our discussion of the normalized quadratic functional form: what conditions on b and B are necessary and sufficient to ensure that c(p) defined by (56)-(59) is concave in the components of the price vector p?

The function c(p) will be concave in p if and only if $\nabla^2 c(p)$ is a negative semidefinite matrix for each p in the domain of definition of c. Evaluating (64) at $p = p^*$ and using the restrictions (59) yields:

(76) $\nabla^2 c(p^*) = (\alpha^T p^*)^{-1}B$.

Since $\alpha > 0_N$ and $p^* \gg 0_N$, $\alpha^T p^* > 0$. Thus in order for c(p) to be a concave function of p, the following necessary condition must be satisfied:

(77) B is a negative semidefinite matrix.

We now show that the *necessary condition* (77) is also *sufficient* to imply that c(p) is concave over the set of p such that $p \gg 0_N$. Unfortunately, the proof is somewhat involved.[19]

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[19] The method of proof is due to Diewert and Wales (1987).

Let $p \gg 0_N$. We assume that B is negative semidefinite and we want to show that $\nabla^2 c(p)$ is negative semidefinite or equivalently, that $-\nabla^2 c(p)$ is positive semidefinite. Thus for any vector z, we want to show that $-z^T \nabla^2 c(p)z \geq 0$. Using (64), this inequality is equivalent to:

$$(78) \quad -(\alpha^T p)^{-1} z^T B z + (\alpha^T p)^{-2} z^T B p \alpha^T z + (\alpha^T p)^{-2} z^T \alpha p^T B z - (\alpha^T p)^{-3} p^T B p z^T \alpha \alpha^T z \geq 0 \qquad \text{or}$$

$$(79) \quad -(\alpha^T p)^{-1} z^T B z - (\alpha^T p)^{-3} p^T B p (\alpha^T z)^2 \geq -2(\alpha^T p)^{-2} z^T B p \alpha^T z \qquad \text{using } B = B^T.$$

Define $A \equiv -B$. Since B is symmetric and negative semidefinite by assumption, A is symmetric and positive semidefinite. Thus there exists an orthonormal matrix U such that

$$(80) \quad U^T A U = \Lambda \ ;$$
$$(81) \quad U^T U \quad = I_N$$

where I_N is the N by N identity matrix and Λ is a diagonal matrix with the nonnegative eigenvalues of A, λ_i, $i = 1,...,N$, running down the main diagonal. Now premultiply both sides of (80) by U and postmultiply both sides by U^T. Using (81), $U^T = U^{-1}$, and the transformed equation (80) becomes the following equation:

$$(82) \quad \begin{aligned} A &= U \Lambda U^T \\ &= U \Lambda^{1/2} \Lambda^{1/2} U^T \\ &= U \Lambda^{1/2} U^T U \Lambda^{1/2} U^T \qquad\qquad \text{since } U^T U = I_N \\ &= S \, S \end{aligned}$$

where $\Lambda^{1/2}$ is the diagonal matrix that has the nonegative square roots $\lambda_i^{1/2}$ of the eigenvalues of A running down the main diagonal and the symmetric square root of A matrix S is defined as

$$(83) \quad S \equiv U \Lambda^{1/2} U^T.$$

If we replace $-B$ in (79) with A, the inequality that we want to establish becomes

$$(84) \quad 2(\alpha^T p)^{-1} z^T A p \alpha^T z \leq z^T A z + (\alpha^T p)^{-2} p^T A p (\alpha^T z)^2$$

 where we have also multiplied both sides of (79) by the positive number $\alpha^T p$ in order to derive (84) from (79).

Recall the Cauchy-Schwarz inequality for two vectors, x and y:

$$(85) \quad x^T y \leq (x^T x)^{1/2} (y^T y)^{1/2}.$$

Now we are ready to establish the inequality (84). Using (82), we have:

$$(86) \quad \begin{aligned} (\alpha^T p)^{-1} z^T A p \alpha^T z &= (\alpha^T p)^{-1} z^T S S p \alpha^T z \\ &\leq (z^T S S^T z)^{1/2} ([\alpha^T p]^{-2} [\alpha^T z]^2 \, p^T S^T S p)^{1/2} \\ &\qquad\qquad \text{using (85) with } x^T \equiv z^T S \text{ and } y \equiv (\alpha^T p)^{-1} (\alpha^T z) S p \end{aligned}$$

$$= (z^TSSz)^{1/2} ([\alpha^Tp]^{-2} [\alpha^Tz]^2 \, p^TSSp)^{1/2} \qquad\qquad \text{using } S = S^T$$
$$= (z^TAz)^{1/2} ([\alpha^Tp]^{-2} [\alpha^Tz]^2 \, p^TAp)^{1/2} \qquad\qquad \text{using (82), } A = SS$$
$$\leq (1/2)(z^TAz) + (1/2)[\alpha^Tp]^{-2} [\alpha^Tz]^2 \, (p^TAp)$$

using the nonnegativity of z^TAz, p^TAp, the positivity of α^Tz
and the Theorem of the Arithmetic and Geometric Mean.

The inequality (86) is equivalent to the desired inequality (84).

Thus the normalized quadratic unit cost function defined by (56)-(59) will be concave over the set of positive prices if and only if the symmetric matrix B is negative semidefinite. Thus after econometric estimates of the elements of B have been obtained using the system of estimating equations (72), we need only check that the resulting estimated B matrix is negative semidefinite.

However, suppose that the estimated B matrix is *not* negative semidefinite. How can one reestimate the model, impose negative semidefiniteness on B, but without destroying the flexibility of the normalized quadratic functional form?

The desired imposition of negative semidefiniteness can be accomplished using a technique due to Wiley, Schmidt and Bramble (1973): simply replace the matrix B by

$$(87) \quad B \equiv -\, AA^T$$

where A is an N by N lower triangular matrix; i.e., $a_{ij} = 0$ if $i < j$.[20]

We also need to take into account the restrictions (59), $Bp^* = 0_N$. These restrictions on B can be imposed if we impose the following restrictions on A:

$$(88) \quad A^Tp^* = 0_N.$$

To show how this curvature imposition technique works, let $p^* = 1_N$ and consider the case $N = 2$. In this case, we have:

$$A \equiv \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \text{ and } A^T = \begin{bmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{bmatrix}.$$

The restrictions (88) become: $A^T 1_2 = \begin{bmatrix} a_{11} + a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

and hence we must have $a_{21} = -\, a_{11}$ and $a_{22} = 0$. Thus in this case,

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[20] Since $z^TAA^Tz = (A^Tz)^T(A^Tz) = y^Ty \geq 0$ for all vectors z, AA^T is positive semidefinite and hence $-AA^T$ is negative semidefinite. Diewert and Wales (1987; 53) show that any positive semidefinite matrix can be written as AA^T where A is lower triangular. Hence, it is not restrictive to reparameterize an arbitrary negative semidefinite matrix B as $-AA^T$.

$$(89)\ B \equiv -\ AA^T = - \begin{bmatrix} a_{11} & 0 \\ -a_{11} & 0 \end{bmatrix} \begin{bmatrix} a_{11} & -a_{11} \\ 0 & 0 \end{bmatrix} = - \begin{bmatrix} a_{11}^2 & -a_{11}^2 \\ -a_{11}^2 & a_{11}^2 \end{bmatrix} = a_{11}^2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Equations (89) show how the elements of the B matrix can be defined in terms of the single parameter, a_{11}^2. Note that with this reparameterization of the B matrix, it will be necessary to use nonlinear regression techniques rather than modifications of linear regression techniques. This turns out to be the cost of imposing the correct curvature conditions on the unit cost function.

In the following section, we indicate how the functional forms described in sections 2-4 above can be adapted to estimate consumer preferences.

6. The Estimation of Consumer Preferences: The General Framework

The cost function and production function framework described in the previous sections can be readily adapted to the problem of estimating consumer preferences: simply replace output y by utility u, reinterpret the production function f as a utility function, reinterpret the input vector x as a vector of commodity demands and reinterpret the vector of input prices p as a vector of commodity prices. With these changes, the producer's cost minimization problem (12) becomes the following problem of *minimizing the cost or expenditure of attaining a given level of utility u*:

$$(90)\ C(u,p) \equiv \min_x \{p^T x : f(x) \geq u \}.$$

If the cost function is differentiable with respect to the components of the commodity price vector p, then Shephard's (1953; 11) Lemma applies and the consumer's system of commodity demand functions as functions of the chosen utility level u and the commodity price vector p, x(u,p), is equal to the vector of first order partial derivatives of the cost or expenditure function C(u,p) with respect to the components of p:

$$(91)\ x(u,p) = \nabla_p C(u,p).$$

The system of demand functions x(u,p) defined in (91) are known as *Hicksian*[21] *demand functions*.

Thus it seems that we can adapt the theory of cost and production functions used in sections 2-4 above in a very straightforward way and estimate consumer preferences in exactly the same way that we estimated production functions or their dual cost functions. Thus we need only replace period t output, y^t, by period t utility, u^t, in the estimating equations (21) (for the generalized Leontief cost function) and (72) (for the normalized quadratic cost function) and reinterpret the resulting equations. However, there is a problem: the period t output level y^t is an *observable* variable but the period t utility level u^t is *not observable*!

However, this problem can be solved. We need only equate the cost function C(u,p) to the consumer's *observable expenditure* in the period under consideration, Y say, and solve the resulting equation for u as a function of Y and p, say u = g(Y,p). Thus u = g(Y,p) is the solution to:

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[21] See Hicks (1946; 311-331).

(92) C(u,p) = Y

and the resulting solution function g(Y,p) is the *consumer's indirect utility function*. Now replace the u in the system of Hicksian demand functions (91) by g(Y,p) and we obtain the consumer's system of (observable) market demand functions:

(93) x = ∇_pC(g(Y,p),p).

We will illustrate how this general framework can be implemented in the context of several specific functional forms for the cost function.

7. The Generalized Leontief Cost Function for Homothetic Preferences.

We illustrate the above procedure for the generalized Leontief cost function defined in section 2 above. For this functional form, equation (92) becomes:

(94) u $\sum_{i=1}^{N}\sum_{j=1}^{N}$ b$_{ij}$ p$_i^{1/2}$ p$_j^{1/2}$ = Y ; $\qquad\qquad$ (b$_{ij}$ = b$_{ji}$ for all i and j)

and the u solution to this equation is:

(95) u = g(Y,p) = Y/[$\sum_{i=1}^{N}\sum_{j=1}^{N}$ b$_{ij}$ p$_i^{1/2}$ p$_j^{1/2}$].

Substituting (95) into (91) leads to the following system of market demand functions:

(96) x$_i$ = [$\sum_{j=1}^{N}$ b$_{ij}$ (p$_j$/p$_i$)$^{1/2}$] Y/[$\sum_{i=1}^{N}\sum_{j=1}^{N}$ b$_{ij}$ p$_i^{1/2}$ p$_j^{1/2}$] ; $\qquad\qquad$ i = 1,...,N.

Evaluating (96) at the period t data and adding a stochastic error term e$_i^t$ to equation i in (96) for i = 1,...,N leads to the following system of estimating equations:[22]

(97) x$_i^t$ = [$\sum_{j=1}^{N}$ b$_{ij}$ (p$_j^t$/p$_i^t$)$^{1/2}$] Y^t/[$\sum_{i=1}^{N}\sum_{j=1}^{N}$ b$_{ij}$ (p$_i^t$)$^{1/2}$ (p$_j^t$)$^{1/2}$] + e$_i^t$; $\qquad$ t = 1,...,T ; i = 1,...,N.

8. The Normalized Quadratic Cost Function for Homothetic Preferences.

We can also illustrate the above procedure for the normalized quadratic cost function defined in section 4 above. For this functional form, equation (92) becomes:

(98) u[b^{T}p + (1/2)(α^Tp)$^{-1}$p^{T}Bp] = Y

and the u solution to this equation is:

(99) u = g(Y,p) = Y/[b^{T}p + (1/2)(α^Tp)$^{-1}$p^{T}Bp].

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[22] Since Y^t will typically equal $\sum_{i=1}^{N}$ p$_i^t$x$_i^t$, it can be verified that the errors in (97) for any period t cannot be independently distributed since they must satisfy the restriction $\sum_{i=1}^{N}$ p$_i^t$e$_i^t$ = 0 for each t; see (104) below. It is also necessary to impose a normalization on the b$_{ij}$ since the right hand side of each equation in (97) is homogeneous of degree 0 in the b$_{ij}$. We will deal with the normalization problem in section 9 below.

Substituting (99) into (91) leads to the following system of market demand functions:

$$(100) \quad x = [b + Bv - (1/2)v^T Bv\alpha][(\alpha^T p)^{-1}Y]/[b^T v + (1/2)v^T Bv]$$

where $v \equiv (\alpha^T p)^{-1}p = p/\alpha^T p$ is the vector of normalized prices. Evaluating (100) at the period t data and adding a vector of stochastic error terms e^t to the system of equations (100) leads to the following system of estimating equations:

$$(101) \quad x^t = [b + Bv^t - (1/2)v^{tT}Bv^t\alpha][(\alpha^T p^t)^{-1}Y^t]/[b^T v^t + (1/2)v^{tT}Bv^t] + e^t; \quad t = 1,...,T$$

where $v^t \equiv p^t/\alpha^T p^t$ for $t = 1,...,T$.

In practice, period t "income" Y^t is defined to be period t expenditure, $p^{tT}x^t = \sum_{i=1}^{N}p_i^t x_i^t$; i.e., we have:

$$(102) \quad Y^t = p^{tT}x^t = \sum_{i=1}^{N}p_i^t x_i^t; \quad\quad\quad\quad\quad t = 1,...,T.$$

However, the identities (102) create some econometric difficulties: namely, we cannot assume that all of the error terms, e_i^t, in each period are independently distributed. Thus if we premultiply both sides of equation i for period t in (97) by p_i^t and sum over i, we obtain the following identity using (102):

$$(103) \quad Y^t = Y^t + \sum_{i=1}^{N} p_i^t e_i^t; \quad\quad\quad\quad t = 1,...,T$$

which in turn implies that the period t error terms e_i^t satisfy the following exact identity:

$$(104) \quad \sum_{i=1}^{N} p_i^t e_i^t = 0; \quad\quad\quad\quad t = 1,...,T.$$

In a similar fashion, premultiply both sides of the period t equation in (101) by p^{tT}, we obtain the following equations:

$$(105) \quad p^{tT}x^t = p^{tT}[b + Bv^t - (1/2)v^{tT}Bv^t\alpha][(\alpha^T p^t)^{-1}Y^t]/[b^T v^t + (1/2)v^{tT}Bv^t] + p^{tT}e^t; \quad t = 1,...,T \text{ or}$$
$$Y^t = p^{tT}\,\alpha^T p^t(\alpha^T p^t)^{-1}[b + Bv^t - (1/2)v^{tT}Bv^t\alpha][(\alpha^T p^t)^{-1}Y^t]/[b^T v^t + (1/2)v^{tT}Bv^t] + p^{tT}e^t \quad\quad \text{or}$$
$$Y^t = v^{tT}\,\alpha^T p^t[b + Bv^t - (1/2)v^{tT}Bv^t\alpha][(\alpha^T p^t)^{-1}Y^t]/[b^T v^t + (1/2)v^{tT}Bv^t] + p^{tT}e^t \quad\quad\quad \text{or}$$
$$Y^t = v^{tT}[b + Bv^t - (1/2)v^{tT}Bv^t\alpha][Y^t]/[b^T v^t + (1/2)v^{tT}Bv^t] + p^{tT}e^t \quad\quad\quad\quad\quad \text{or}$$
$$Y^t = [b^T v^t + (1/2)v^{tT}Bv^t][Y^t]/[b^T v^t + (1/2)v^{tT}Bv^t] + p^{tT}e^t \quad\quad\quad\quad\quad\quad \text{or}$$
$$Y^t = Y^t + p^{tT}e^t$$

which in turn implies that the period t error term vector e^t satisfies the following exact identity, $p^{tT}e^t = 0$ for $t = 1,...,T$, which is the same identity as (104).

Thus for both the generalized Leontief and the normalized quadratic cost function models the period t error vectors satisfy an exact identity and hence in both models, we must drop one estimating equation; i.e., we must drop one of the estimating equations in (97) and one of the estimating equations in (101). Thus there are some differences between the cost function models in the producer context and in the consumer context.

9. The Problem of Cardinalizing Utility.

There is another significant difference between the producer models discussed in the previous sections and the consumer models discussed in the present section. Look closely at the estimating equations (97) and (101). From (97), it can be seen that the right hand side explanatory variables are *homogeneous of degree 0* in the b_{ij} coefficients. Thus the regression will not be able to determine the *scale* of the b_{ij} parameters. Similarly, by looking at the right hand side of (101), it can be seen that the right hand side explanatory variables are *homogeneous of degree 0* in the components of the b vector and the B matrix. Thus the regression will not be able to determine the *scale* of the parameters in b and B. This indeterminacy means that we require at least one additional restriction or normalization on the parameters of each of these models. Basically, what we have to do is *cardinalize* our measure of utility in some way.

There are two simple ways of cardinalizing utility[23]:

- Pick a positive reference quantity vector $x^* \gg 0_N$. Let the period t consumption vector x^t be on the indifference surface $I(x^t) \equiv \{x: f(x) = f(x^t)\}$. Let $\lambda^t x^*$ be on the $I(x^t)$ indifference curve. Then measure period t utility as λ^t.
- Pick a positive reference price vector $p^* \gg 0_N$. Then normalize the consumer's cost function $C(u,p)$ so that it has the following property:

(106) $C(u,p^*) = u$ for all $u > 0$.

The meaning of (106) is that if the consumer faces the reference price vector p^*, then his or her utility will be equal to his or her "income" or expenditure on commodities at those reference prices. Thus if relative prices never changed, the consumer's utility is proportional to the size of the observed budget set. This serves to cardinalize utility for all consumption vectors. Samuelson (1974) called this type of cardinalization of utility, *money metric utility*.[24]

We will follow the money metric method of scaling utility. For the generalized Leontief model, the restriction (106) implies the following normalization of the b_{ij}:

(107) $\sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} p_i^{*1/2} p_j^{*1/2} = 1.$

For the normalized quadratic model, the restriction (106) implies the following normalization of the components of the b vector and the B matrix:

(108) $b^T p^* + (1/2) p^{*T} B p^* / \alpha^T p^* = 1.$

If we choose the reference vector p^* in (106) to be the same as the reference vector p^* which occurred in (59), then $B p^* = 0_N$ and the cardinalization restriction (108) becomes:

(109) $b^T p^* = 1.$

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[23] The two methods are equivalent in the case of homothetic preferences.
[24] The basic idea can be traced back to Hicks (1941-42).

Problems.

8. Adapt the translog unit cost function model presented in section 3 above to the consumer context. Hint: equations (42) do not depend on utility! However, you need to choose p* in a specific way in order to impose money metric utility scaling.

9. Suppose the consumer's cost function has the following form:

(i) $C(u,p) = uc(p)$

where c(p) is a well behaved unit cost function. Assuming that c(p) is differentiable, show that the consumer's system of market demand functions has the following form:

(ii) $x(Y,p) = Y \nabla_p c(p)/c(p)$.

Show that $\partial \ln x_i(Y,p)/\partial \ln Y = 1$ for $i = 1,...,N$; i.e., if the consumer has preferences given by (i), then all income elasticities of demand are one! This contradicts Engel's Law; i.e., that the income elasticity of demand for food is less than one.

10. Modeling Nonhomothetic Preferences.

Problem 9 above shows that the assumption that the utility function is linearly homogeneous (the homothetic preferences assumption) is not a good assumption from the empirical point of view. Hence we need to generalize our functional forms in order to accommodate nonhomothetic preferences.

Let C*(u,p) be an arbitrary twice continuously differentiable cost function that satisfies money metric scaling at the positive reference price vector $p* \gg 0_N$; i.e., C* satisfies:

(110) $C*(u,p*) = u$ for all $u > 0$.

Let c(p) be a flexible unit cost function. Then Diewert (1980; 597) showed that the following functional form could approximate C* to the second order at (u*,p*) where u* > 0:

(111) $C(u,p) \equiv a^T p + uc(p)$

where the vector of parameters a can be chosen to satisfy the following restriction:

(112) $a^T p* = 0$.

The parameters of the unit cost function also satisfy the following restriction:

(113) $c(p*) = 1$.

In order to derive the system of market demand functions that corresponds to the cost function defined by (111), we again set $C(u,p)$ equal to "income" Y and solve for the $u = g(Y,p)$ solution:

(114) $u = [Y - a^T p]/c(p)$.

The *system of Hicksian demand functions* that corresponds to the cost function defined by (111) is as usual obtained using Shephard's Lemma:

(115) $x(u,p) \equiv \nabla_p C(u,p) = a + u\nabla_p c(p)$.

Now replace u in the right hand side of (115) by the right hand side of (114) and we obtain the *consumer's system of market demand functions*:

(116) $x(Y,p) = a + \nabla_p c(p)[Y - a^T p]/c(p)$.

Letting $c(p) \equiv \sum_{i=1}^N \sum_{j=1}^N b_{ij}\, p_i^{1/2}\, p_j^{1/2}$ be the generalized Leontief unit cost function, the system of market demand functions (116) becomes, after adding stochastic error terms:

(117) $x_i^t = a_i + \{[\sum_{j=1}^N b_{ij} (p_j^t/p_i^t)^{1/2}][Y^t - \sum_{k=1}^N a_k p_k^t]/[\sum_{i=1}^N \sum_{j=1}^N b_{ij} (p_i^t)^{1/2} (p_j^t)^{1/2}]\} + e_i^t$;
$$t = 1,...,T ; i = 1,...,N.$$

One of the a_i needs to be eliminated from the estimating equations (117) using the restriction $a^T p^*$ $= 0$ and one of the b_{ij} needs to be eliminated using the restriction $c(p^*) = 1$ in order to obtain the final system of estimating equations. However, if period t "income" Y^t is equal to period t expenditure on the commodities, $p^{tT}x^t$, then as before, we can only use $N-1$ of the N equations in (117) as estimating equations.

Letting $c(p) \equiv b^T p + (1/2)(\alpha^T p)^{-1} p^T B p$ be the normalized quadratic unit cost function (with $b^T p^* = 1$ and $Bp^* = 0_N$), the system of market demand functions (116) becomes, after adding stochastic error terms:

(118) $x^t = a + \{[b + Bv^t - (1/2)v^{tT}Bv^t\alpha][(\alpha^T p^t)^{-1}][Y^t - a^T p^t]/[b^T v^t + (1/2)v^{tT}Bv^t]\} + e^t$; $\quad t = 1,...,T$

where $v^t \equiv p^t/\alpha^T p^t$ for $t = 1,...,T$. Obviously, nonlinear regression techniques have to be used in order to estimate the unknown parameters in the system of estimating equations (118). One of the a_i needs to be eliminated from the estimating equations (118) using the restriction $a^T p^* = 0$ and one of the b_i needs to be eliminated using the restriction $b^T p^* = 1$ in order to obtain the final system of estimating equations. However, if period t "income" Y^t is equal to period t expenditure on the commodities, $p^{tT}x^t$, then as before, we can only use $N-1$ of the N equations in (118) as estimating equations. If the estimated B matrix turns out to be *not* negative semidefinite, then we need to replace B by $-AA^T$ where A is a lower triangular matrix satisfying $Ap^* = 0_N$.

11. The Use of Linear Spline Functions to Achieve Greater Flexibility.

Although the above model is flexible around the point p^*, as we move away from p^*, the model (118) may not fit the data very well. If the plots of the actual and fitted values using the normalized

quadratic model defined by the estimating equations (118) have a zig-zag appearance, then it may be worthwhile to try a *linear spline model*. We will indicate below how a two segment linear spline model can be implemented. For more details (and an extension to 3 segments instead of 2), see Diewert and Wales (1993; 81-85).

We redefine the normalized quadratic cost function C(u,p) as follows:

(119) $C(u, p) = a^T p + u(1/2)(\alpha^T p)^{-1} p^T B p + d(u, p)$

where a satisfies $a^T p^* = 0$ and α and B satisfy the restrictions (57)-(59). The function $d(u, p)$ is defined as follows:

(120) $d(u,p) \equiv u b^T p$ for $0 \leq u \leq u^*$
           $\equiv u^* b^T p + (u - u^*) f^T p$ for $u^* \leq u$.

where $b^T \equiv [b_1,...,b_N]$ and $f^T \equiv [f_1,...,f_N]$ parameter vectors to be estimated and u^* is *a break point level of utility* to be chosen by the investigator. The vectors b and f satisfy the restrictions:

(121) $b^T p^* = 1$; $f^T p^* = 1$.

How should one pick the break point u^*? We examine the plots of the regression model defined by (118) and look for an observation number where the plot changes from a zig to a zag. Suppose that this observation number is t^*. Now compute index numbers of utility using the price and quantity data and determine what level of utility corresponds to the chosen observation and set this level equal to u^*. This choice of u^* will work satisfactorily if the observations which precede the chosen observation have estimated indirect utilities which are equal to or less than u^* and the remaining observations have indirect utilities that are greater than u^*.

The estimating equations for the first t^* observations will still be given by (118); i.e., for the first t^* observations, our estimating equations are:

(122) $x^t = a + \{[b + Bv^t - (1/2)v^{tT}Bv^t\alpha][(\alpha^T p^t)^{-1}][Y^t - a^T p^t]/[b^T v^t + (1/2)v^{tT}Bv^t]\} + e^t$; $t = 1,...,t^*$

where as usual, $v^t \equiv p^t/\alpha^T p^t$.

In order to obtain the estimating equations for the last $T - t^*$ observations, we need to form the Hicksian demand functions and calculate the indirect utility function. If $t > t^*$, then the Hicksian demand functions that correspond to the functional from defined by (119) and (120) are:

(123) $x(u,p) \equiv \nabla_p C(u,p) = a + u[(\alpha^T p)^{-1}Bp - (1/2)(\alpha^T p)^{-2}p^T Bp\alpha] + u^*b + (u - u^*)f$
                     $= a + u^*b - u^*f + u[f + (\alpha^T p)^{-1}Bp - (1/2)(\alpha^T p)^{-2}p^T Bp\alpha]$.

For $t > t^*$, the indirect utility function $u = g(Y,p)$ can be obtained by solving $C(u,p) = Y$. The solution is:

(124) $u = [Y - a^T p - u^*b^T p + u^*f^T p]/[f^T p + (1/2)(\alpha^T p)^{-1}p^T Bp]$.

Now substitute (124) into (123) in order to obtain the consumer's market demand functions for periods $t > t^*$. After adding stochastic error terms, we obtain the following estimating equations:

$$(125) \ x^t = a + u*b - u*f$$
$$+ \{[f + Bv^t - (1/2)v^{tT}Bv^t\alpha][(\alpha^Tp^t)^{-1}][Y^t - a^Tp^t - u*b^Tp^t + u*f\ ^Tp^t]/[f^Tv^t + (1/2)v^{tT}Bv^t]\} + e^t$$
$$\text{for } t^* < t \le T.$$

Although the estimating equations (125) look rather formidable, they can be programmed with a bit of effort. The most difficult part of implementing the above spline model is choosing the "right" observation at which the break point occurs.

As usual, if "income" Y^t in period t is equal to expenditure $p^{tT}x^t$, then we must drop one equation in the system of estimating equations (122) and (125). Finally, if the estimated B matrix is *not* negative semidefinite, then the model should be rerun, setting $B = - AA^T$, where A is lower triangular and satisfies the restrictions $A^Tp^* = 0_N$.

12. The Estimation of Unit Profit Functions: The General Framework.

Recall problems 1 to 5 above, which introduced the *capital requirements function*, F(y), which gives the minimum amount of capital k that is required to produce the vector of net outputs y. The corresponding variable profit (or operating profit) function V(k,p) can be defined as follows:

$$(126) \ V(p,k) \equiv \max_y\{p^Ty : k = F(y)\}$$

If commodity i is an output, then $y_i > 0$; if commodity i is an input, $y_i < 0$. The available capital is $k > 0$.

The function V(p,k) must be linearly homogeneous and convex in p for fixed k. The economy's system of profit maximizing supply and demand functions y(p,k) can be obtained by differentiating V(p,k) with respect to the components of p: (Hotelling's (1932) Lemma):

$$(127) \ y(p,k) = \nabla_p V(p,k).$$

The convexity property of V in p implies that:

$$(128) \ \nabla_p y(p,k) = \nabla^2_{pp} V(p,k) \text{ is a positive semidefinite matrix.}$$

If the capital requirements function F(y) is linearly homogeneous (so that the technology exhibits constant returns to scale), then V(p,k) has the following property:

$$(129) \ V(p,k) = V(p,1)k.$$

The unit profit function v(p) is the gross return to capital we can achieve using one unit of capital; i.e., define v as:

(130) $v(p) \equiv V(p,1)$.

With a constant returns to scale technology, we have $V(k,p) = kv(p)$ so that we need only pick a functional form for the unit profit function v. It turns out that we can use the functional forms for unit cost functions, $c(p)$, that we defined in sections 2-4 above as functional forms for the unit profit function $v(p)$. The only change that we need to make is that the concavity in p property for the unit cost function $c(p)$ must be replaced by a convexity in p property for the unit profit function $v(p)$. We illustrate the use of some of the unit cost functional forms in the sections below.

13. The Translog Variable Profit Function with Constant Returns to Scale.

The translog unit profit function, $v(p)$, is defined as follows:

(131) $\ln v(p) \equiv \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln p_i + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i \ln p_j$

where the parameters α_i and γ_{ij} satisfy the following restrictions:

(132) $\gamma_{ij} = \gamma_{ji}$; $1 \le i < j \le N$;   (N(N−1)/2 symmetry restrictions)
(133) $\sum_{i=1}^{N} \alpha_i = 1$;   (1 restriction)
(134) $\sum_{j=1}^{N} \gamma_{ij} = 0$; $i = 1,...,N$ (N restrictions).

Suppose that in period t, observed capital input is k^t, the vector of observed output and variable input prices is $p^t \gg 0_N$ and the vector of observed net output supplies $y^t > 0_N$. Thus the *period t observed variable profit or gross return to capital* is[25]:

(135) $V^t \equiv p^{tT} y^t \equiv \sum_{i=1}^{N} p_i^t y_i^t$.

The log of (129) can act as an estimating equation:

(136) $\ln V^t = \ln k^t + \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln p_i^t + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i^t \ln p_j^t + e_0^t$; $t = 1,...,T$ or
(137) $\ln[V^t/k^t] = \alpha_0 + \sum_{i=1}^{N} \alpha_i \ln p_i^t + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i^t \ln p_j^t + e_0^t$; $t = 1,...,T$.

Note that (137) is linear in the unknown parameters. As in section 3 above, the old estimating equations (42) can be adapted to yield the following estimating equations in the present context:

(138) $s_i^t \equiv p_i^t y_i^t / V^t = \alpha_i + \sum_{j=1}^{N} \gamma_{ij} \ln p_j^t + e_i^t$; $i = 1,...,N$.

As in section 3, only $N - 1$ of the N estimating equations in (138) are statistically independent.

Unfortunately, the above model is not adequate for empirical applications. The problem is that the economy becomes more efficient over time and more output is produced using the same amount of input; i.e., there is technical progress. Thus we generalize the translog unit profit function defined by (131) to include time trends to try and capture the effects of technical progress. Thus we now define the period t unit profit function $v(p,t)$ as follows:

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[25] It is important to check that $V^t > 0$ for each observation t.

(139) $\ln v(p,t) \equiv \alpha_0 + \beta_0 t + \sum_{i=1}^{N} \alpha_i \ln p_i + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i \ln p_j + \sum_{i=1}^{N} \beta_i t \ln p_i$

where the parameters α_i and γ_{ij} satisfy the restrictions (132)-(134) and the new β_i parameters satisfy the following restriction[26]:

(140) $\sum_{i=1}^{N} \beta_i = 0.$

Using this new definition for $v(p,t)$, defining $V(k,p,t) \equiv kv(p,t)$ and using the general methodology explained above, our initial estimating equations (137) and (138) are replaced by the following estimating equations:

(141) $\ln[V^t/k^t] = \alpha_0 + \beta_0 t + \sum_{i=1}^{N} \alpha_i \ln p_i^t + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i^t \ln p_j^t + \sum_{i=1}^{N} \beta_i t \ln p_i^t + e_0^t;$

(142) $s_i^t \equiv p_i^t y_i^t / V^t = \alpha_i + \sum_{j=1}^{N} \gamma_{ij} \ln p_j^t + \beta_i t + e_i^t ; \qquad i = 1,...,N ; t = 1,...,T.$

As in section 3, only $N - 1$ of the N equations in (142) can be used in the estimation.

We have not substituted the restrictions (132)-(134) and (140) into the estimating equations, (141) and (142). We now do this for the case $N = 4$ in order to show how explicit estimating equations can be derived. We use the restriction (133), $\sum_{i=1}^{4} \alpha_i = 1$, in order to eliminate the parameter α_4 and we use the restriction (140), $\sum_{i=1}^{4} \beta_i = 0$, in order to eliminate β_4. Finally, we use the restrictions (132) and (134) in order to eliminate the parameters γ_{i4} and γ_{4i} for $i = 1,...,4$. With these restrictions imposed, the estimating equation (141) becomes:

(143) $\ln[V^t/p_4^t k^t] = \alpha_0 + \beta_0 t + \sum_{i=1}^{3} \alpha_i \ln(p_i^t/p_4^t) + \sum_{i=1}^{3} \beta_i t \ln(p_i^t/p_4^t) + (1/2)\sum_{i=1}^{3} \gamma_{ii} [\ln(p_i^t/p_4^t)]^2$
$\qquad + \sum_{i=1}^{3} \sum_{j=1}^{3}{}_{;i<j} \gamma_{ij} \ln(p_i^t/p_4^t)\ln(p_j^t/p_4^t) + e_0^t; \qquad t = 1,...,T.$

Dropping the last equation in (142) and eliminating the γ_{i4} leads to the following system of estimating equations when $N = 4$:

(144) $s_i^t \equiv p_i^t y_i^t / V^t = \alpha_i + \sum_{j=1}^{3} \gamma_{ij} \ln(p_j^t/p_4^t) + \beta_i t + e_i^t ; \qquad i = 1,...,3 ; t = 1,...,T.$

The unknown parameters in (143) and (144) are $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3, \gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{12}, \gamma_{13}$ and γ_{23} or 14 parameters in all. Note that *all* of the unknown parameters occur in the estimating equation (143). This fact often creates econometric problems. With a great number of parameters in equation (143), the fit will tend to be good but due to multicollinearity, the parameters will not be very accurately determined using this equation. However, two stage estimation procedures (or maximum likelihood estimation) will tend to give the first equation undue weight in the system estimation procedure (due to the low variance in the first equation) and hence, very inaccurate estimates of the parameters can result.

We now return to the case of a model with a general N. Our old formulae (47) and (49) in section 3 above for obtaining elasticities of demand can be adapted in a straightforward manner to give us the

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[26] This restriction is required in order to ensure that $v(p)$ is linearly homogeneous in the components of p.

following formulae for the elasticities of net supply for variable inputs and outputs. The formulae for the *cross price elasticities of net supply* are given by:

(145) $\partial \ln y_i(k,p)/\partial \ln p_j = [s_i(y,p)]^{-1} \gamma_{ij} + s_j(y,p)$; $\qquad\qquad i \neq j$.

The formulae for the *own price elasticities of net supply* are given by:

(146) $\partial \ln y_i(k,p)/\partial \ln p_i = [s_i(y,p)]^{-1} \gamma_{ii} + s_i(y,p) - 1$; $\qquad\qquad i = 1,...,N$.

Thus given econometric estimates for the α_i, β_i and γ_{ij}, which we denote by α_i^*, β_i^* and γ_{ij}^*, the estimated or fitted shares in period t, s_i^{t*} are defined using these estimates and equations (142) evaluated at the period t data:

(147) $s_i^{t*} \equiv \alpha_i^* + \beta_i^* t + \sum_{j=1}^{N} \gamma_{ij}^* \ln p_j^t$; $\qquad\qquad i = 1,...,N$; $t = 1,...,T$.

Now use equations (147) and (145) evaluated at the period t data and econometric estimates to obtain the following formula for the *period t cross elasticities of net supply*, e_{ij}^t:

(148) $e_{ij}^t \equiv \partial \ln y_i(k^t,p^t)/\partial \ln p_j = [s_i^{t*}]^{-1} \gamma_{ij}^* + s_j^{t*}$; $\qquad\qquad i \neq j$.

Similarly, use equations (146) evaluated at the period t data and econometric estimates to obtain the following formula for the *period t own elasticities of net supply*, e_{ii}^t:

(149) $e_{ii}^t \equiv \partial \ln y_i(k^t,p^t)/\partial \ln p_i = [s_i^{t*}]^{-1} \gamma_{ii}^* + s_i^{t*} - 1$; $\qquad\qquad i = 1,...,N$.

We can also obtain an estimated or *fitted period t variable profits or gross return to capital*, V^{t*}, by using our econometric estimates for the parameters and by exponentiating the right hand side of the equation t in (141):

(150) $V^{t*} \equiv \exp[\ln k^t + \alpha_0^* + t\beta_0^* + \sum_{i=1}^{N} \alpha_i^* \ln p_i^t + \sum_{i=1}^{N} \beta_i^* t\ln p_i^t + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}^* \ln p_i^t \ln p_j^t]$;
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t = 1,...,T$.

Finally, our fitted period t shares s_i^{t*} defined by (147) and our fitted period t profits V^{t*} defined by (150) can be used in order to obtain estimated or *fitted period t net supplies*, y_i^{t*}, as follows:

(151) $y_i^{t*} \equiv V^{t*} s_i^{t*}/p_i^t$; $\qquad\qquad i = 1,...,N$; $t = 1,...,T$.

Given the matrix of period t estimated price elasticities of net supply, $[e_{ij}^t]$, we can readily calculate the matrix of period t *estimated net output price derivatives*, $\nabla_p y(k^t,p^t) = \nabla^2_{pp} V(k^t,p^t)$. Our estimate for element ij of $\nabla^2_{pp} V(k^t,p^t)$ is:

(152) $V_{ij}^{t*} \equiv e_{ij}^t y_i^{t*}/p_j^t$; $\qquad\qquad i,j = 1,...,N$; $t = 1,...,T$

where the estimated period t elasticities e_{ij}^t are defined by (148) and (149) and the fitted period t net output supplies y_i^{t*} are defined by (151). Once the estimated price derivative matrices $[V_{ij}^{t*}]$ have

been calculated, then we may check whether each of them is positive semidefinite using determinantal conditions or by checking if all of the eigenvalues of each matrix are zero or positive.

There remains the problem of measuring the effects of *technical progress*. Using (139) in order to define $V(k,p,t) \equiv kv(p,t)$, then differentiating $V(k,p,t)$ with respect to time t and evaluating the resulting expression at the period t data yields:

$$(153) \quad \partial \ln V(k,p,t)/\partial t = \beta_0^* + \sum_{i=1}^{N} \beta_i^* \ln p_i^t \equiv T^t \; ; \qquad\qquad t = 1,...,T.$$

The right hand side of (153), T^t, is our desired measure of technical progress going from period t−1 to period t: it gives us an estimate of the percentage increase in variable profits due to the improvements in technology going from period t−1 to period t.[27]

In the following section, we generalize the translog model to allow for nonconstant returns to scale.

14. The Translog Variable Profit Function with Nonconstant Returns to Scale.

The period t translog variable profit function is now defined as follows:

$$(154) \quad \ln V(k,p,t) \equiv \alpha_0 + \beta_0 t + \sum_{i=1}^{N} \alpha_i \ln p_i + \sum_{i=1}^{N} \beta_i t \ln p_i + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i \ln p_j$$
$$+ \delta_0 \ln k + \sum_{i=1}^{N} \delta_i \ln k \ln p_i + (1/2) \varepsilon [\ln k]^2$$

where the parameters α_i and γ_{ij} satisfy the restrictions (132)-(134), the β_i parameters satisfy (140) and the new δ_i parameters satisfy:

$$(155) \quad \sum_{i=1}^{N} \delta_i = 0.$$

The above restrictions ensure that the functional form is homogeneous of degree on in the prices p. Comparing our new more general translog with the constant returns to scale function defined in the previous section, we see that we have added N new independent δ_i parameters and one new ε parameter.[28]

Obviously, we can use (154) as an estimating equation. Defining $V^t \equiv p^{tT} y^t$ as in the previous section and evaluating (154) at the period t data, we obtain the following estimating equation:

$$(156) \quad \ln V^t \equiv \alpha_0 + \beta_0 t + \sum_{i=1}^{N} \alpha_i \ln p_i^t + \sum_{i=1}^{N} \beta_i t \ln p_i^t + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij} \ln p_i^t \ln p_j^t$$
$$+ \delta_0 \ln k^t + \sum_{i=1}^{N} \delta_i \ln k^t \ln p_i^t + (1/2) \varepsilon [\ln k^t]^2 \; ; \qquad t = 1,...,T.$$

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[27] Since payments to capital are typically only about one third the size of payments to labour, it will turn out that "reasonable" estimates of technical progress T^t will be about 3 times the size of our index number estimates of total factor productivity growth. The total factor productivity growth rate is the rate of growth of outputs divided by the rate of growth of inputs, where the inputs are taken to be labour and capital services. Hence the denominator in this estimator of technical progress is approximately three times as big as the implicit denominator in T^t which is just capital input.

[28] Thus if N = 4, we will have 19 independent parameters in all.

We need to eliminate the redundant parameters in (156) as was done in the previous section for the case N = 4. This leads to the following estimating equations: for t = 1,...,T:

(157) $\ln (V^t/p_N^t) \equiv \alpha_0 + \beta_0 t + \sum_{i=1}^{N-1} \alpha_i \ln(p_i^t/p_N^t) + \sum_{i=1}^{N-1} \beta_i t \ln(p_i^t/p_N^t) +$
$(1/2) \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\gamma_{ij} \ln(p_i^t/p_N^t)\ln(p_j^t/p_N^t) + \delta_0 \ln k^t + \sum_{i=1}^{N-1} \delta_i \ln k^t \ln(p_i^t/p_N^t) + (1/2) \varepsilon [\ln k^t]^2.$

Differentiating V(k,p,t) with respect to the components of the price vector p and using Hotelling's Lemma leads to the following share equation counterparts to equations (142) in the previous section:

(158) $s_i^t \equiv p_i^t y_i^t/V^t = \alpha_i + \beta_i t + \sum_{j=1}^{N-1} \gamma_{ij} \ln(p_j^t/p_N^t) + \delta_i \ln k^t + e_i^t ;$ \qquad i = 1,...,N−1 ; t = 1,...,T.

Note that we have only N−1 independent estimating equations in (158).

It turns out that the formulae (145) and (146) in the previous section are still valid formulae for the *cross price elasticities of net supply*. Hence given econometric estimates for the α_i, β_i, δ_i, ε and the γ_{ij}, which we denote by α_i^*, β_i^*, δ_i^*, ε^* and γ_{ij}^*, the estimated or fitted shares in period t, s_i^{t*} are defined using these estimates and equations (158) evaluated at the period t data:

(159) $s_i^{t*} \equiv \alpha_i^* + \beta_i^* t + \sum_{j=1}^{N} \gamma_{ij}^* \ln p_j^t + \delta_i \ln k^t;$ \qquad i = 1,...,N ; t = 1,...,T.

Now use equations (159) and (145) evaluated at the period t data and econometric estimates to obtain the following formula for the *period t cross elasticities of net supply*, e_{ij}^t:

(160) $e_{ij}^t \equiv \partial \ln y_i(k^t,p^t)/\partial \ln p_j = [s_i^{t*}]^{-1} \gamma_{ij}^* + s_j^{t*} ;$ \qquad i ≠ j.

Similarly, use equations (146) evaluated at the period t data and econometric estimates to obtain the following formula for the *period t own elasticities of net supply*, e_{ii}^t:

(161) $e_{ii}^t \equiv \partial \ln y_i(k^t,p^t)/\partial \ln p_i = [s_i^{t*}]^{-1} \gamma_{ii}^* + s_i^{t*} − 1 ;$ \qquad i = 1,...,N.

We can also obtain an estimated or *fitted period t variable profits or gross return to capital*, V^{t*}, by using our econometric estimates for the parameters and by exponentiating the right hand side of the equation t in (157):

(162) $V^{t*} \equiv \exp[\ln k^t + \alpha_0^* + t\beta_0^* + \sum_{i=1}^{N} \alpha_i^* \ln p_i^t + \sum_{i=1}^{N} \beta_i^* t\ln p_i^t + (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_{ij}^* \ln p_i^t \ln p_j^t$
$+ \delta_0 \ln k^t + \sum_{i=1}^{N} \delta_i \ln k^t \ln p_i^t + (1/2) \varepsilon [\ln k^t]^2]$ \qquad t = 1,...,T.

Our fitted period t shares s_i^{t*} defined by (159) and our fitted period t profits V^{t*} defined by (162) can be used in order to obtain estimated or *fitted period t net supplies*, y_i^{t*}, as follows:

(163) $y_i^{t*} \equiv V^{t*} s_i^{t*}/p_i^t ;$ \qquad i = 1,...,N ; t = 1,...,T.

Given the matrix of period t estimated price elasticities of net supply, $[e_{ij}^t]$, we can readily calculate the matrix of period t *estimated net output price derivatives*, $\nabla_p y(k^t,p^t) = \nabla^2_{pp} V(k^t,p^t)$. Our estimate for element ij of $\nabla^2_{pp} V(k^t,p^t)$ is:

(164) $V_{ij}^{t*} \equiv e_{ij}^{t} y_i^{t*}/p_j^{t}$; $i,j = 1,...,N$; $t = 1,...,T$

where the estimated period t elasticities e_{ij}^{t} are defined by (148) and (149) and the fitted period t net output supplies y_i^{t*} are defined by (163). Once the estimated price derivative matrices $[V_{ij}^{t*}]$ have been calculated, then we may check whether each of them is positive semidefinite using determinantal conditions or by checking if all of the eigenvalues of each matrix are zero or positive.

Differentiating V(k,p,t) with respect to time t and evaluating the resulting expression at the period t data yields:

(165) $\partial \ln V(k,p,t)/\partial t = \beta_0^{*} + \sum_{i=1}^{N} \beta_i^{*} \ln p_i^{t} \equiv T^{t}$; $t = 1,...,T$.

i.e., we obtain the same measure of technical progress that we obtained in the previous section.

There remains the problem of defining a measure of returns to scale. The measure we will use is one that calculates the percentage change in variable profits due to a one percent change in the use of capital. Thus our measure of returns to scale in period t is:

(166) $R^{t} \equiv \partial \ln V(k,p,t)/\partial \ln k = \delta_0 + \sum_{i=1}^{N} \delta_i \ln p_i^{t} + \varepsilon \ln k^{t}$; $t = 1,...,T$.

Note that if we set $\delta_0 = 1$, $\varepsilon = 0$ and $\delta_i = 0$ for $i = 1,...,N$, then the model in this section collapses down to the model presented in the previous section. Under these restrictions, it can be seen that $R^{t} = 1$; i.e., we have constant returns to scale.

When the above translog model is implemented, usually two things happen:

- The curvature conditions fail at one or more observations; i.e., the estimated period t substitution matrix $[V_{ij}^{t*}]$ defined by (164) above *fails to be positive semidefinite* at one or more periods t and
- The estimates for the returns to scale R^{t} and for technical progress T^{t} are *not reasonable*.

The reason why we cannot usually determine accurate estimates for returns to scale and for technical progress is that usually k grows fairly smoothly through the sample period and hence the variables k and t tend to be highly multicollinear and so our estimates for R^{t} and T^{t} are not very well determined.

Thus in subsequent sections, we will impose constant returns to scale. We will also turn to the normalized quadratic functional form where the correct curvature conditions can be imposed without destroying the flexibility of the functional form.

15. The Normalized Quadratic Unit Profit Function Model.

We adapt the normalized quadratic unit cost function defined by (56) in section 4 above into a unit profit function. Thus define the production unit's period t variable profit function V(k,p,t) as follows:

(167) $V(k,p,t) \equiv b^T pk + (1/2)[p^T Bp/\alpha^T p]k + c^T ptk$

where $b^T \equiv [b_1,...,b_N]$ and $c^T \equiv [c_1,...,c_N]$ are parameter vectors and $B \equiv [b_{ij}]$ is a matrix of parameters. The matrix B satisfies the following restrictions:

(168) $B = B^T$; i.e., the matrix B is symmetric;
(169) $Bp^* = 0_N$ for some $p^* >> 0_N$.

The vector of parameters $\alpha^T \equiv [\alpha_1,...,\alpha_N]$ is predetermined and satisfies $\alpha > 0_N$. We can adapt the analysis presented in section 4 and show that *a necessary and sufficient condition for V(k,p,t) defined by (167) above to be convex in prices is that the matrix B be positive semidefinite*.

Differentiating the normalized quadratic variable profit function defined by (167) with respect to the components of the price vector p leads to the following system of net supply functions using Hotelling's Lemma:

(170) $y(k,p,t) = \nabla_p V(k,p,t) = bk + [(\alpha^T p)^{-1} Bp - (1/2)(\alpha^T p)^{-2} p^T Bp\alpha]k + ckt.$

Evaluating (170) at the period t data, dividing both sides by k^t and adding a vector of errors e^t leads to the following system of estimating equations:

(171) $y^t/k^t = b + Bv^t - (1/2)v^{tT} Bv^t \alpha + ct + e^t$; $\qquad\qquad\qquad$ t = 1,...,T

where the vector of period t normalized prices is defined as $v^t \equiv (\alpha^T p^t)^{-1} p^t$.

We have not substituted the restrictions (169) into the estimating equations (171). We shall do this substitution below assuming that N = 4 and $p^* = 1_4$.

We use the restrictions (169) to solve for the b_{ii} in terms of the off diagonal b_{ij}. Thus we have, assuming that there are 4 variable commodities and $p^* = 1_4$ and using $B = B^T$:

(172) $b_{11} = - b_{12} - b_{13} - b_{14}$;

(174) $b_{22} = - b_{12} - b_{23} - b_{24}$;

(175) $b_{33} = - b_{13} - b_{23} - b_{34}$;

(176) $b_{44} = - b_{14} - b_{24} - b_{34}.$

Using (172)-(176), we can write Bv as follows:

(177) $[Bv]^T = [\sum_{j=1}^4 b_{1j}v_j, \sum_{j=1}^4 b_{2j}v_j , \sum_{j=1}^4 b_{3j}v_j, \sum_{j=1}^4 b_{4j}v_j]$
$= [-b_{12}w_{12}-b_{13}w_{13}-b_{14}w_{14}, \ b_{12}w_{12}-b_{23}w_{23}-b_{24}w_{24}, \ b_{13}w_{13}+b_{23}w_{23}-b_{34}w_{34}, \ b_{14}w_{14}+b_{24}w_{24}+b_{34}w_{34}]$

where

(178) $w_{ij} \equiv v_i - v_j$; $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ i, j = 1, 2, 3, 4.

Premultiplying Bv by v^T and using (177) and (178) yields the following formula:

(179) $v^T B v = - [b_{12}(w_{12})^2 + b_{13}(w_{13})^2 + b_{14}(w_{14})^2 + b_{23}(w_{23})^2 + b_{24}(w_{24})^2 + b_{34}(w_{34})^2]$.

Now substitute (177) and (179) into (171) and we obtain the following system of estimating equations[29] in the case where N = 4 and $p^* = 1_N$:

(180) $y_1^t/k^t = b_1 + c_1 t - b_{12}w_{12}^t - b_{13}w_{13}^t - b_{14}w_{14}^t - (1/2) v^{tT}Bv^t\alpha_1 + e_1^t$; $\qquad$ t = 1,...,T

(181) $y_2^t/k^t = b_2 + c_2 t + b_{12}w_{12}^t - b_{23}w_{23}^t - b_{24}w_{24}^t - (1/2) v^{tT}Bv^t\alpha_2 + e_2^t$; $\qquad$ t = 1,...,T

(182) $y_3^t/k^t = b_3 + c_3 t + b_{13}w_{13}^t + b_{23}w_{23}^t - b_{34}w_{34}^t - (1/2) v^{tT}Bv^t\alpha_3 + e_3^t$; $\qquad$ t = 1,...,T

(183) $y_4^t/k^t = b_4 + c_4 t + b_{14}w_{14}^t + b_{24}w_{24}^t + b_{34}w_{34}^t - (1/2) v^{tT}Bv^t\alpha_4 + e_4^t$; $\qquad$ t = 1,...,T.

We need to also replace $v^{tT}Bv^t$ in equations (180)-(181) by the right hand side of (179) evaluated at the period t data. The resulting estimating equations turn out to be linear in the unknown b_i, c_i and b_{ij} parameters (4 plus 4 plus 6 equals 14 parameters in all).

Returning to the case of a general number of variable commodities N, we need to calculate the matrix of net supply price derivatives. Differentiating (170) with respect to the components of p yields the following matrix of price derivatives at period t:

(184) $\nabla_p y(k^t,p^t,t) = \nabla_p^2 V(k^t,p^t,t) = (\alpha^T p^t)^{-1} [B - Bv^t\alpha^T - \alpha v^{tT}B + v^{tT}Bv^t\alpha\alpha^T]k^t$;   t = 1,...,T

where, as usual, the vector of period t normalized prices is defined as $v^t \equiv (\alpha^T p^t)^{-1} p^t$. Once estimates for b, c and B have been obtained (call these estimates b^*, c^* and B^* respectively), we can use equations (171) in order to obtain period t vectors of fitted net supply vectors y^{t*}:

(185) $y^{t*} \equiv k^t[b^* + B^* v^t - (1/2)v^{tT}B^* v^t\alpha + c^* t]$; $\qquad\qquad\qquad\qquad$ t = 1,...,T.

Equations (184) and (185) may be used to form estimated *period t price elasticity matrices*:

(186) $[e_{ij}^t] \equiv [\partial \ln y_i(k^t,p^t,t)/\partial \ln p_j] = [(p_j^t/y_i^{t*})\partial y_i(k^t,p^t,t)/\partial p_j]$; $\qquad$ t = 1,...,T

where the derivative estimates $\partial y_i(k^t,p^t,t)/\partial p_j$ can be obtained from (184).

An estimator of variable profits in period t, V^{t*}, can be obtained as the inner product of the period t fitted net supply vector y^{t*} defined by (185) and the period t vector of variable commodity prices, p^t:

(187) $V^{t*} \equiv p^{tT}y^{t*}$; $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ t = 1,...,T.

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[29] An alternative system of estimating equations multiplies both sides of (180)-(183) by k^t. This alternative system of estimating equations often performs "better" in the sense that it leads to more reasonable estimates of net supply elasticities. However, in theory, the original system of estimating equations (180)-(183) should have more homoskedastic variances.

Finally, a measure of *period t technical progress* T^t can be defined as follows:

(188) $T^t \equiv \partial \ln V(k^t, p^t, t)/\partial t = p^{tT} c^* k^t/V^{t*}$; $t = 1,...,T.$

Unfortunately, the estimated B^* matrix may fail to be positive semidefinite. Hence, in the following section, we adapt the technique used in section 4 above to impose the correct curvature conditions on the B matrix.

16. The Normalized Quadratic Unit Profit Function Model with Curvature Imposed.

If the estimated B matrix turns out to be not positive definite, then we can rerun the model in the previous section by replacing B by:

(189) $B = AA^T$

where A is a lower triangular matrix which satisfies:

(190) $A^T p^* = 0_N$.

For the case $N = 4$ and for $p^* = 1_4$, we can use the restrictions (190) and the lower triangular structure of A in order to eliminate the a_{ii} as follows:

(191) $a_{11} = - a_{21} - a_{31} - a_{41}$;
(192) $a_{22} = - a_{32} - a_{42}$;
(193) $a_{33} = - a_{43}$;
(194) $a_{44} = 0.$

If we substitute (191)-(194) into (189), we obtain the following formulae for the b_{ij} in terms of the a_{ij}:

(195) $b_{11} = a_{11}^2$ $= [a_{21} + a_{31} + a_{41}]^2$
(196) $b_{12} = a_{11} a_{21}$ $= - [a_{21} + a_{31} + a_{41}]a_{21}$
(197) $b_{13} = a_{11} a_{31}$ $= - [a_{21} + a_{31} + a_{41}]a_{31}$
(198) $b_{14} = a_{11} a_{41}$ $= - [a_{21} + a_{31} + a_{41}]a_{41}$
(199) $b_{22} = a_{21}^2 + a_{22}^2$ $= a_{21}^2 + [a_{32} + a_{42}]^2$
(200) $b_{23} = a_{21}a_{31} + a_{22}a_{32}$ $= a_{21}a_{31} - [a_{32} + a_{42}]a_{32}$
(201) $b_{24} = a_{21}a_{41} + a_{22}a_{42}$ $= a_{21}a_{41} - [a_{32} + a_{42}]a_{42}$
(202) $b_{33} = a_{31}^2 + a_{32}^2 + a_{33}^2$ $= a_{31}^2 + a_{32}^2 + a_{43}^2$
(203) $b_{34} = a_{31}a_{41} + a_{32}a_{42} + a_{33}a_{43} = a_{31}a_{41} + a_{32}a_{42} + a_{43}^2$
(204) $b_{44} = a_{41}^2 + a_{42}^2 + a_{43}^2$ $= a_{41}^2 + a_{42}^2 + a_{43}^2.$

Now we need only replace the b_{ij} parameters which occurred in the model of the previous section by the formulae on the right hand sides of (195)-(204) and run the previous model as a nonlinear regression. The parameters of the new model are the elements of the vectors b and c (as before) and the elements of the A matrix.

Once the a_{ij} have been estimated, the b_{ij} parameters can be computed using (189) (or (195)-(204) if $N = 4$ and $p^* = 1_4$) and the elasticity formulae (186) and the estimates of technical progress (188) in the previous section can be computed.

It turns out that we can use spline techniques in the production context as well as in the consumer context. In the following section, we indicate how technical progress can be modeled using spline techniques.

17. The Normalized Quadratic Unit Profit Function Model and the Use of Splines for Modeling Technical Progress.

We adapt the normalized quadratic profit function defined by (167) in section 15 above into a spline model. We illustrate the technique by developing the algebra for a model with two break points. Thus define the production unit's period t variable profit function $V(k,p,t)$ as follows:

(205) $V(k,p,t) \equiv b^T pk + (1/2)[p^T Bp / \alpha^T p]k + d(k,p,t)$

where $b^T \equiv [b_1,...,b_N]$ is a parameter vectors and $B \equiv [b_{ij}]$ is a matrix of parameters. The matrix B satisfies the restrictions (168) and (169) and is positive semidefinite. As usual, the vector of parameters $\alpha^T \equiv [\alpha_1,...,\alpha_N]$ is predetermined and satisfies $\alpha > 0_N$. The linear spline function $d(k,p,t)$ is defined as follows:

(206) $d(k,p,t) \equiv kc^T pt$ for $1 \leq t \leq t^*$
            $\equiv kc^T pt^* + (t - t^*)kf^T p$ for $t^* < t \leq t^{**}$
            $\equiv kc^T pt^* + (t^{**} - t^*)kf^T p + (t - t^{**})kg^T p$ for $t^{**} < t \leq T$

where $c^T \equiv [c_1,...,c_N]$, $f^T \equiv [f_1,...,f_N]$ and $g^T \equiv [g_1,...,g_N]$ are parameter vectors to be estimated. The periods t^* and t^{**} are *break points* where the rate of technological change shifts from one regime to another. These break points are to be chosen by the investigator.

Differentiating the normalized quadratic variable profit function defined by (205) with respect to the components of the price vector p leads to the following system of net supply functions using Hotelling's Lemma:

(207) $y(k,p,t) = \nabla_p V(k,p,t) = bk + [(\alpha^T p)^{-1} Bp - (1/2)(\alpha^T p)^{-2} p^T Bp\alpha]k + \nabla_p d(k,p,t)$.

Evaluating (207) at the period t data, dividing both sides by k^t and adding a vector of errors e^t leads to the following system of estimating equations:

(208) $y^t/k^t = b + Bv^t - (1/2)v^{tT}Bv^t \alpha + ct + e^t$; $t = 1,...,t^*$;
(209) $y^t/k^t = b + Bv^t - (1/2)v^{tT}Bv^t \alpha + ct^* + (t - t^*)f + e^t$; $t = t^*+1,...,t^{**}$;
(210) $y^t/k^t = b + Bv^t - (1/2)v^{tT}Bv^t \alpha + ct^* + (t^{**} - t^*)f + (t - t^{**})g + e^t$; $t = t^{**}+1,...,T$.

where, as usual, the vector of period t normalized prices is defined as $v^t \equiv (\alpha^T p^t)^{-1} p^t$. It can be seen that the model defined by the estimating equations (208)-(210) is linear in the unknown parameters (but their are cross equation equality constraints on the parameters in the B matrix).

It turns out that equations (184) are still valid in order to calculate the period t matrix of price derivatives of net supply; i.e., we have the following matrix of price derivatives at period t:

$$(211) \quad \nabla_p y(k^t, p^t, t) = \nabla^2_p V(k^t, p^t, t) = (\alpha^T p^t)^{-1} [B - Bv^t \alpha^T - \alpha v^{tT} B + v^{tT} Bv^t \alpha \alpha^T] k^t \; ; \quad t = 1,...,T.$$

Obviously equations (208)-(211) can be used to generate fitted net supply vectors y^{t*} and then equations (211) and (186) may be used to form estimated *period t price elasticity matrices*.

How should one pick the break points t* and t**? We examine the plots of the regression model defined by (208)-(210) and look for an observation numbers where the plot changes from a zig to a zag. Suppose that for most of the equations, these change of directions occur at periods t* and t**. This will determine the break points. Additional break points can be added if necessary.

The *period t measures of technical progress* T^t are defined as follows:

$$(212) \quad T^t \equiv \partial \ln V(k^t, p^t, t)/\partial t = p^{tT} c^* k^t / V^{t*} \; ; \qquad t = 1,...,t*;$$
$$(213) \quad T^t \equiv \partial \ln V(k^t, p^t, t)/\partial t = p^{tT} f^* k^t / V^{t*} \; ; \qquad t = t*+1,...,t**;$$
$$(214) \quad T^t \equiv \partial \ln V(k^t, p^t, t)/\partial t = p^{tT} g^* k^t / V^{t*} \; ; \qquad t = t**+1,...,T.$$

Our estimates of the rate of technological change will change discontinuously as we cross the break points, which is perhaps a disadvantage of this spline model.

Of course, if the estimated B matrix turns out to be *not* positive semidefinite, then we may replace B by AA^T as in the previous section.

18. Allowing for Flexibility at Two Sample Points

If we differentiate the normalized quadratic profit function defined by (167) above with respect to the mth component of the price vector p, we obtain the following equation that describes the net supply of commodity m as a function of the price vector p in period t:

$$(215) \quad y_m(k,p,t) = b_m k + c_m tk + \sum_{j=1}^N b_{mj} (p_j/\alpha^T p)k - (1/2) \alpha_m p^T Bp/(\alpha^T p)^2 k.$$

Now differentiate (215) with respect to p_n, the nth component of the price vector p:

$$(216) \quad \partial y_m(k,p,t)/\partial p_n = b_{mn} k/\alpha^T p - \sum_{j=1}^N b_{mj} \alpha_n p_j k/(\alpha^T p)^2 - \sum_{j=1}^N b_{nj} \alpha_m p_j k/(\alpha^T p)^2$$
$$+ (1/2) \alpha_m \alpha_n p^T Bp \, k/(\alpha^T p)^3..$$

Now turn (216) into the cross elasticity of net supply of commodity m with respect to a change in the price of commodity n, e_{mn}:

(217) $e_{mn}(p,t) \equiv [p_n/y_m] \; \partial y_m(k,p,t)/\partial p_n$

$\qquad = b_{mn} \, (p_n/\alpha^T p)(k/y_m) - [p_n /y_m] \sum_{j=1}^{N} b_{mj} \, \alpha_n \, p_j \, k/(\alpha^T p)^2$

$\qquad - [p_n/y_m]\sum_{j=1}^{N} b_{nj} \, \alpha_m \, p_j \, k/(\alpha^T p)^2 + [p_n/ym](1/2) \, \alpha_m \alpha_n \, p^T Bp \, k/(\alpha^T p)^3$.

Using the restrictions (169), the last three terms on the right hand side of (217) will be zero when p = p* and thus, empirically, these last three terms will typically be small in magnitude. Thus, the key determinant of the magnitude of the elasticity e_{mn} will typically be the first term on the right hand side of (217), namely, $b_{mn} \, (p_n/\alpha^T p)(k/y_m)$. Of course, the parameter b_{mn} will be constant over time but the other terms, p_n (the price of commodity n), y_m (the net output of commodity m), k (the amount of the "fixed" factor) and $\alpha^T p$ (a fixed basket price index of all N variable input and output prices) can all have substantial trends over our sample period. Thus, our chosen functional form has built in these possible trends in elasticities.

A solution to this problem is readily at hand but at a cost in terms of using up degrees of freedom. We have followed the example of most applied production function researchers and allowed technical progress to affect the constant terms in the system of net supply functions (215) but we have left the substitution matrix B unchanged over time. To solve the problem of trending elasticities, all we have to do is allow B to change over time as well. Thus, simply set the matrix B in (167) and (215) equal to a weighted average of a matrix C (which characterizes substitution possibilities at the beginning of the sample period) and a matrix D (which characterizes substitution possibilities at the end of the sample period); i.e., define B as follows in terms of C and D and the time variable t:

(218) $B^t = (1 - [t/T])C + [t/T]D$; $\qquad\qquad\qquad$ t = 0,1,2,…,T.

Note that there are T+1 sample observations. Essentially, we now let technical progress affect not only the constant terms in (167) but we also allow it to affect substitution possibilities as well. Another way of viewing our new functional form is that we allow the functional form to be flexible at two points (the first sample point and the last) instead of the usual one point.

As usual, the correct curvature conditions can be imposed globally (globally) by setting C and D equal to the product of UU^T and VV^T respectively, where U and V are lower triangular matrices; i.e., set:

(219) $C = UU^T$ and $D = VV^T$; $\qquad\qquad\qquad$ U and V lower triangular.

We can also impose the following normalizations on the matrices U and V:

(220) $U^T p* = 0_N$; $V^T p* = 0_N$.

This technique of imposing price flexibility at two points is due to Diewert and Lawrence (2002).

19. Semiflexible Functional Forms

Recall the basic normalized quadratic functional form for a unit profit function that was defined in section 16 above and recall that most of the unknown parameters for this functional form are in the

B equals AA^T matrix where A is an N by N lower triangular matrix which satisfies the restrictions (190), $A^T p^* = 0_N$.

In models where the number of commodities N is large, it can be difficult to estimate all of the parameters of the A matrix at one time. An effective way to estimate the A matrix is to estimate it one column at a time. Thus in the first stage, we use the estimating equations (180)-(183) or (215) with the A (and hence B) matrix set equal to zero. Then at the next stage we use the estimates for the parameters which are not in the B matrix as starting values for the stage 2 nonlinear regression model with B set equal to AA^T where A is a rank 1 lower triangular matrix; i.e., at this second stage, A is set equal to:[30]

$$(221) \quad A \equiv \begin{bmatrix} a_{11} & 0 & ... & 0 \\ a_{21} & 0 & ... & 0 \\ ... & ... & ... & ... \\ a_{N1} & 0 & ... & 0 \end{bmatrix}.$$

The estimated parameters from this stage 2 nonlinear regression are then used as starting values in a stage 3 nonlinear regression that fills in column 2 of the lower triangular matrix A; i.e., in the stage 3 regression, A is set equal to the following rank 2 lower triangular matrix:[31]

$$(222) \quad A \equiv \begin{bmatrix} a_{11} & 0 & ... & 0 \\ a_{21} & a_{22} & ... & 0 \\ ... & ... & ... & ... \\ a_{N1} & a_{N2} & ... & 0 \end{bmatrix}.$$

This procedure of gradually adding nonzero columns to the A matrix can be continued until the full number of N−1 nonzero columns have been added, provided that the number of time series observations T is large enough compared to N, the number of commodities in the model. However, in models where T is small relative to N, the above procedure of adding nonzero columns to A will have to be stopped well before the maximum number of N−1 nonzero columns has been added, due to the lack of degrees of freedom. Suppose that we stop the above procedure after K < N−1 nonzero columns have been added. Then Diewert and Wales (1988b; 330) call the resulting normalized quadratic functional form a *flexible of degree K* functional form or a *semiflexible functional form*. A flexible of degree K functional form for a profit or cost function can approximate an arbitrary twice continuously differentiable functional form to the second order at

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[30] We also need to use the restrictions (190) to express a_{11} in terms of a_{21}, ... , a_{N1}. Thus if p^* is a vector of ones, the a_{11} in (221) is replaced by $-a_{21} - a_{31} \ldots -a_{N1}$. If maximum likelihood estimation is used, then in the stage 2 nonlinear regression, the starting values for a_{21}, ... , a_{N1} are taken to be 0's so the starting log likelihood for the stage 2 nonlinear regression will be equal to the final log likelihood of the stage 1 regression. This provides a check on the programming code used. A similar strategy should be used with the subsequent stage 3, 4 and so on regressions.

[31] The starting values for the stage 3 nonlinear regression for the elements in the first column of A are the final estimated values from the stage 2 nonlinear regression and the starting values for the elements in the second column of A are 0's. Again, if p^* is a vector of ones, the a_{22} in (222) is replaced by $-a_{32} - a_{42} \ldots -a_{N2}$.

some point, except the matrix of second order partial derivatives of the functional form with respect to prices is restricted to have maximum rank K instead of the maximum possible rank, N−1.

What is the cost of estimating a semiflexible functional form for a profit function instead of a fully flexible functional form? When we estimate a fully flexible functional form, we need the B matrix to be able to approximate an arbitrary positive semidefinite symmetric matrix B^* of rank N−1. This arbitrary B^* can be represented as a sum of N−1 rank one positive semidefinite matrices as we now show.

Recall that any symmetric matrix can be diagonalized by means of an orthonormal transformation; i.e., there exists a matrix U equal to $[u^1, u^2, \dots, u^N]$, where the u^n are the columns of U, such that:

$$(223) \quad U^T B U = \Lambda \equiv \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_N \end{bmatrix}$$

where U satisfies

$$(224) \quad U^T U = I_N$$

and Λ is a diagonal matrix with the nonnegative eigenvalues of B, the λ_n, running down the main diagonal. We order these eigenvalues starting with the biggest and ending up with the smallest (which is equal to 0):

$$(225) \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{N-1} \geq \lambda_N = 0.$$

Now premultiply both sides of (223) by U and post multiply both sides of (223) by U^T. Using (224), we find that:

$$\begin{aligned}(226) \quad B &= U \Lambda U^T \\ &= [u^1 \lambda_1, u^2 \lambda_2, \dots, u^N \lambda_N] [u^1, u^2, \dots, u^N]^T \\ &= \sum_{n=1}^{N} \lambda_n u^n u^{nT} \\ &= \sum_{n=1}^{N-1} \lambda_n u^n u^{nT} \end{aligned}$$

where the last equality in (226) follows from the fact that $\lambda_N = 0$.

If we estimate a normalized quadratic that is flexible of degree K, then it turns out that the resulting AA^T matrix can approximate B defined by (226) as follows:

$$(227) \quad AA^T = \sum_{n=1}^{K} \lambda_n u^n u^{nT}.$$

Thus the cost of using a semiflexible functional form of degree K where K is less than N−1 is that we will miss out on the part of B that corresponds to the smallest eigenvalues of B; i.e., our

estimating AA^T will be too small by the positive semidefinite matrix $\sum_{n=K}^{N-1} \lambda_n u^n u^{nT}$. In many situations, this cost will be very small; i.e., as we go through the various stages of estimating A by adding an extra nonzero column to A at each stage, we can monitor the increase in the final log likelihood (if we use maximum likelihood estimation) and when the increase in stage k+1 over stage k is "small", we can stop adding extra columns, secure in the knowledge that we are not underestimating the size of B by a large amount.

This semiflexible technique has not been widely applied but it would seem to offer some big advantages in estimating substitution matrices in situations where there are a large number of commodities in the model.[32]

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[32] Diewert and Lawrence in some unpublished work have successfully estimated semiflexible models for profit functions for 40 to 45 commodities.

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