APPLIED ECONOMICS

By W.E. Diewert, May, 2013

Chapter 5: Index Number Theory: Part I: Early Approaches [1]

1. Index Number Purpose and Overview.

"The answer to the question what is the *Mean* of a given set of magnitudes cannot in general be found, unless there is given also the object for the sake of which a mean value is required. There are as many kinds of average as there are purposes; and we may almost say in the matter of prices as many purposes as writers. Hence much vain controversy between persons who are literally at cross purposes." F.Y. Edgeworth (1888; 347).

The number of physically distinct goods and unique types of services that consumers can purchase is in the millions. On the business or production side of the economy, there are even more commodities that are actively traded. This is because firms not only produce commodities for final consumption, they also produce exports and intermediate commodities that are demanded by other producers. Firms collectively also use millions of imported goods and services, thousands of different types of labor services and hundreds of thousands of specific types of capital. If we further distinguish physical commodities by their geographic location or by the season or time of day that they are produced or consumed, then there are *billions* of commodities that are traded within each year of any advanced economy. Yet most macroeconomic models have only half a dozen quantity variables and many have only three: output, labor and capital. The models used in applied microeconomics generally have less than 20 or so quantity variables. The question that this Chapter addresses is: *how exactly should the microeconomic information involving possibly millions of prices and quantities be aggregated into a smaller number of price and quantity variables?* This is the basic *index number problem.*

Note that we have posed the index number problem in the context of microeconomic theory; i.e., given that we wish to implement some economic model based on producer or consumer theory, what is the "best" method for constructing a set of aggregates for the model? However, when constructing aggregate prices or quantities, other points of view (that do not rely on economics) are possible. We will also consider these alternative points of view in Part I of this Chapter but the primary focus will be on economic approaches to index number theory, which is Part II of this Chapter. Thus in sections 2 to 5 of Part I below, we consider some of the early noneconomic approaches to index number theory.

Section 2 will deal with the *levels approach to index number theory*, which will prove to be unsuccessful. In the following sections, we focus on determining functional forms for bilateral indexes. A *bilateral index* attempts to determine the rate of aggregate price and quantity change rather than aggregate price and quantity levels. We will consider three alternative approaches to the determination of the functional form for a bilateral price or quantity index. Section 3 below will consider fixed basket approaches; section 4 will consider stochastic or descriptive statistics approaches and section 5 will consider the axiomatic approach to the determination of the index number formula. Finally, section 6 will consider when chained indexes should be used.

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[1] The material for this Chapter is drawn from earlier versions of this Chapter and Diewert (2012).

2. Setting the Stage and the Levels Approach to the Index Number Problem

It will be useful to set the stage for the subsequent discussion of alternative approaches by defining more precisely what the index number problem is.

We specify two accounting periods, $t = 0,1$ for which we have micro price and quantity data for N commodities pertaining to transactions by a consumer (or a well defined group of consumers). Denote the price and quantity of commodity n in period t by p_n^t and q_n^t respectively for $n = 1,2,\ldots,N$ and $t = 0,1$. Before proceeding further, we need to discuss the exact meaning of the microeconomic prices and quantities if there are *multiple* transactions for say commodity n within period t. In this case, it is natural to interpret q_n^t as the *total* amount of commodity n transacted within period t. In order to conserve the value of transactions, it is necessary that p_n^t be defined as a *unit value* [2]; i.e., p_n^t must be equal to the value of transactions for commodity n during period t divided by the total quantity transacted, q_n^t. For $t = 0,1$, define *the value of transactions in period t* as:

$$(1)\ V^t \equiv \sum_{n=1}^{N} p_n^t\, q_n^t \equiv p^t \cdot q^t$$

where $p^t \equiv (p_1^t,\ldots, p_N^t)$ is the period t price vector, $q^t \equiv (q_1^t,\ldots, q_N^t)$ is the period t quantity vector and $p^t \cdot q^t$ denotes the inner product of these two vectors.

Using the above notation, we can now state the following *levels version of the index number problem using the test or axiomatic approach*: for $t = 0,1$, find scalar numbers P^t and Q^t such that

$$(2)\ V^t = P^t Q^t.$$

The number P^t is interpreted as an aggregate period t price level while the number Q^t is interpreted as an aggregate period t quantity level. The aggregate price level P^t is allowed to be a function of the period t price vector, p^t while the aggregate period t quantity level Q^t is allowed to be a function of the period t quantity vector, q^t; i.e., we have

$$(3)\ P^t = c(p^t) \text{ and } Q^t = f(q^t)\ ;\ t = 0,1.$$

However, from the viewpoint of the *test approach* to index number theory, the levels approach to finding aggregate quantities and prices comes to an abrupt halt: Eichhorn (1978; 144) showed that if the number of commodities N in the aggregate is equal to or greater than 2 and we restrict $c(p^t)$ and $f(q^t)$ to be positive if the micro prices and quantities p_n^t and q_n^t are positive, then there do not exist any functions c and f such that $c(p^t)f(q^t) = p^t \cdot q^t$ for all $p^t >> 0_N$ and $q^t >> 0_N$.[3]

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[2] The early index number theorists Walsh (1901; 96), Fisher (1922; 318) and Davies (1924; 96) (1932) all suggested unit values as the prices that should be inserted into an index number formula. This advice is followed in the *Consumer Price Index Manual: Theory and Practice* with the proviso that the unit value be a narrowly defined one; see the ILO (2004; 356).

[3] Notation: $p >> 0_N$ means all components of p are positive; $p \geq 0_N$ means all components of p are nonnegative and $p > 0_N$ means $p \geq 0_N$ but $p \neq 0_N$. Finally, $p \cdot q \equiv \sum_{n=1}^{N} p_n q_n$.

This negative result can be reversed if we take the *economic approach* to index number theory. This economic approach is due to Shephard (1953) (1970) and Samuelson and Swamy (1974). In this approach, we assume that the economic agent has a linearly homogeneous utility function, f(q), and when facing the prices p^t chooses q^t to solve the following cost minimization problem:

(4) $\min_q \{p^t \cdot q : p^t \cdot q = Y^t ; q \geq 0_N\}$; $t = 0,1$

where period t "income" Y^t is defined as $p^t \cdot q^t$. In this setup, it turns out that c(p) is the unit cost function that is dual[4] to the linearly homogeneous utility function f(q) and we can define P^t and Q^t as in (3) with $P^tQ^t = c(p^t)f(q^t) = p^t \cdot q^t$ for t = 0,1. Why does the economic approach work in the levels version of the index number problem whereas the test approach does not? In the test approach, both p^t and q^t are regarded as completely independent variables, whereas in the economic approach, p^t can vary independently but q^t cannot vary independently; it is a solution to the period t cost minimization problem (4).

Even though the economic approach to the index number problem as formulated above "works", it is not a *practical* solution that statistical agencies can implement and provide suitable aggregates to the public. In order to implement this solution, the statistical agency would have to hire hundreds of econometricians in order to estimate cost functions for all relevant macroeconomic aggregates and it is simply not feasible to do this. Thus we turn to our second formulation of the index number problem and it is this formulation that was initiated Walsh (1901) (1921a) and Fisher (1911) (1922) in their books on index number theory.

In the second approach to index number theory, instead of trying to decompose the value of the aggregate into price and quantity components for a single period, we instead attempt to decompose a *value ratio* for the two periods under consideration into a *price change component* P times a *quantity change component* Q.[5] Thus we now look for two functions of 4N variables, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ such that:[6]

(5) $p^1 \cdot q^1 / p^0 \cdot q^0 = P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1)$.

If we take the test approach, then we want equation (5) to hold for all positive price and quantity vectors pertaining to the two periods under consideration, p^0,p^1,q^0,q^1. If we take the economic approach, then only the price vectors p^0 and p^1 are regarded as independent variables while the quantity vectors, q^0 and q^1, are regarded as dependent variables.

In this second approach to index number theory, the *price index* $P(p^0,p^1,q^0,q^1)$ and the *quantity index* $Q(p^0,p^1,q^0,q^1)$ cannot be determined independently; i.e., if either one of these two functions

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[4] See Chapter 3 or Diewert (1974) for materials and references to the literature on duality theory.

[5] If we use the economic approach, P can be interpreted to be the ratio of unit cost functions, $c(p^1)/c(p^0)$, and Q can be interpreted to be the utility ratio, $f(q^1)/f(q^0)$. Note that the linear homogeneity assumption on the utility function f effectively cardinalizes utility.

[6] If N = 1, then we define $P(p_1^0,p_1^1,q_1^0,q_1^1) \equiv p_1^1/p_1^0$ and $Q(p_1^0,p_1^1,q_1^0,q_1^1) \equiv q_1^1/q_1^0$, the single price ratio and the single quantity ratio respectively. In the case of a general N, we think of $P(p_1^0,p_1^1,q_1^0,q_1^1)$ as being a weighted average of the price ratios p_1^1/p_1^0, p_2^1/p_2^0, ..., p_N^1/p_N^0. Thus we interpret $P(p_1^0,p_1^1,q_1^0,q_1^1)$ as an aggregate price ratio, P^1/P^0, where P^t is the aggregate price level for period t for t = 0,1.

is determined, then the remaining function is implicitly determined using equation (5). Historically, the focus has been on the determination of the price index but Fisher (1911; 388) was the first to realize that once the price index was determined, then equation (5) could be used to determine the companion quantity index.[7]

This value ratio decomposition approach to index number is called *bilateral index number theory* and its focus is the determination of "reasonable" functional forms for P and Q. Fisher's 1911 and 1922 books address this functional form issue using the test approach.

Once the functional forms for $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ have been determined, price and quantity *levels* for the two periods under consideration can be determined as follows:

- The price and quantity levels for period 0 are determined as $P^0 \equiv 1$ and $Q^0 \equiv V^0 \equiv p^0 \cdot q^0$;
- The price level for period 1 is set equal to $P^1 \equiv P(p^0,p^1,q^0,q^1)P^0 = P(p^0,p^1,q^0,q^1)$;
- The quantity level for period 1 is set equal to $Q^1 \equiv Q(p^0,p^1,q^0,q^1)Q^0 = Q(p^0,p^1,q^0,q^1)V^0$.

We turn now to a discussion of the various approaches that have been used to determine the functional form for the bilateral price index, $P(p^0,p^1,q^0,q^1)$.

3. Fixed Basket Approaches to Bilateral Index Number Theory

A very simple approach to the determination of a price index over a group of commodities is the *fixed basket approach*. In this approach, we are given a basket of commodities that is represented by the positive quantity vector q. Given the price vectors for periods 0 and 1, p^0 and p^1 respectively, we can calculate the cost of purchasing this same basket in the two periods, $p^0 \cdot q$ and $p^1 \cdot q$. Then the ratio of these costs is a very reasonable indicator of pure price change over the two periods under consideration, provided that the basket vector q is "representative". Thus define the *Lowe* (1823) *price index*, P_{Lo}, as follows:[8]

$$(6)\ P_{Lo}(p^0,p^1,q) \equiv p^1 \cdot q / p^0 \cdot q\ .$$

As time passed, economists and price statisticians demanded a bit more precision with respect to the specification of the basket vector q. There are two natural choices for the reference basket: the period 0 commodity vector q^0 or the period 1 commodity vector q^1. These two choices lead to the Laspeyres (1871) price index P_L defined by (7) and the Paasche (1874) price index P_P defined by (8):[9]

$$(7)\ P_L(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^0 / p^0 \cdot q^0 = \sum_{n=1}^N s_n^0 (p_n^1/p_n^0)\ ;$$

$$(8)\ P_P(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1 / p^0 \cdot q^1$$

[7] This approach to index number theory is due to Fisher (1911; 418) who called the implicitly determined Q, the *correlative formula*. Frisch (1930; 399) later called (5) the *product test*.
[8] See Ferger (1946) for the early history of the Lowe index.
[9] Note that $P_L(p^0,p^1,q^0,q^1)$ does not actually depend on q^1 and $P_P(p^0,p^1,q^0,q^1)$ does not actually depend on q^0. However, it does no harm to include these vectors and the notation indicates that we are in the realm of bilateral index number theory.

$$= [p^0 \cdot q^1 / p^1 \cdot q^1]^{-1}$$
$$= [\sum_{n=1}^{N} p_n^0 q_n^1 / p^1 \cdot q^1]^{-1}$$
$$= [\sum_{n=1}^{N} p_n^1 q_n^1 (p_n^0/p_n^1) / p^1 \cdot q^1]^{-1}$$
$$= [\sum_{n=1}^{N} p_n^1 q_n^1 (p_n^1/p_n^0)^{-1} / p^1 \cdot q^1]^{-1}$$
$$= [\sum_{n=1}^{N} s_n^1 (p_n^1/p_n^0)^{-1}]^{-1}$$

where the period t expenditure share on commodity n, s_n^t, is defined as $p_n^t q_n^t / p^t \cdot q^t$ for n = 1,…,N and t = 0,1. Thus the Laspeyres price index P_L can be written as a base period expenditure share weighted average of the N price ratios (or price relatives), p_n^1/p_n^0.[10] The last equation in (8) shows that the Paasche price index P_P can be written as a period 1 (or current period) expenditure share weighted *harmonic* average of the N price ratios.[11]

The problem with these index number formulae is that they are equally plausible but in general, they will give different answers. This suggests that if we require a single estimate for the price change between the two periods, then we should take some sort of evenly weighted average of the two indexes as our final estimate of price change between periods 0 and 1. Examples of such symmetric averages are the arithmetic mean, which leads to the Drobisch (1871) Sidgwick (1883; 68) Bowley (1901; 227)[12] index, $(1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the *Fisher* (1922) *ideal index*, P_F, defined as

(9) $P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)\, P_P(p^0,p^1,q^0,q^1)]^{1/2}$.

At this point, the fixed basket approach to index number theory has to draw on the *test approach* to index number theory; i.e., in order to determine which of these fixed basket indexes or which averages of them might be "best", we need *criteria* or *tests* or *properties* that we would like our indexes to satisfy.

What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical cost of living index? It is very desirable for a price index formula that depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test*.[13] We say that the index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

(10) $P(p^1,p^0,q^1,q^0) = 1/P(p^0,p^1,q^0,q^1)$;

i.e., if we interchange the period 0 and period 1 price and quantity data and evaluate the index, then this new index $P(p^1,p^0,q^1,q^0)$ is equal to the reciprocal of the original index $P(p^0,p^1,q^0,q^1)$.

[10] This result is due to Walsh (1901; 428 and 539).
[11] This expenditure share and price ratio representation of the Paasche index is described by Walsh (1901; 428) and derived explicitly by Fisher (1911; 365).
[12] See Diewert (1992) (1993) and Balk (2008) for additional references to the early history of index number theory.
[13] The concept of this test is due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test (and the commensurability test to be discussed later) that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 324) and Fisher (1922; 64).

Diewert (1997; 138) showed that the Fisher ideal price index defined by (9) above is the *only* index that is a homogeneous symmetric mean of the Laspeyres and Paasche price indexes, P_L and P_P, and satisfies the time reversal test (10) above. Thus our first *symmetric basket approach* to bilateral index number theory leads to the Fisher index (9) as being “best” from the perspective of this approach.[14]

Instead of looking for a “best” average of the two fixed basket indexes that correspond to the baskets chosen in either of the two periods being compared, we could instead look for a “best” average basket of the two baskets represented by the vectors q^0 and q^1 and then use this average basket to compare the price levels of periods 0 and 1.[15] Thus we ask that the nth quantity weight, q_n, be an average or *mean* of the base period quantity q_n^0 and the period 1 quantity for commodity n q_n^1, say $m(q_n^0,q_n^1)$, for n = 1,2,…,N.[16] Price statisticians refer to this type of index as a *pure price index* and it corresponds to Knibbs’ (1924; 43) *unequivocal price index*. Under these assumptions, the pure price index can be defined as a member of the following class of index numbers:

$$(11)\quad P_K(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N p_n^1 m(q_n^0,q_n^1) / \sum_{j=1}^N p_j^0 m(q_j^0,q_j^1).$$

In order to determine the functional form for the mean function m, it is necessary to impose some *tests* or *axioms* on the pure price index defined by (11). Suppose that we impose the time reversal test (10) and the following *invariance to proportional changes in current quantities test*:

$$(12)\quad P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1) \text{ for all } \lambda > 0.$$

Diewert (2001; 207) showed that these two tests determine the precise functional form for the pure price index P_K defined by (11) above: the pure price index P_K must be the *Walsh* (1901; 398) (1921a; 97) *price index*, P_W[17] defined by (13):

$$(13)\quad P_W(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N p_n^1(q_n^0q_n^1)^{1/2}/\sum_{j=1}^N p_j^0(q_j^0q_j^1)^{1/2} .$$

Thus the fixed basket approach to bilateral index number theory starts out with the Laspeyres and Paasche price indexes. Some form of averaging of these two indexes is called for since both indexes are equally plausible. Averaging these two indexes directly leads to the Fisher ideal

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[14] Bowley was an early advocate of taking a symmetric average of the Paasche and Laspeyres indexes: “If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean … as a first approximation.” Arthur L. Bowley (1901; 227). Fisher (1911; 418-419) (1922) considered taking the arithmetic, geometric and harmonic averages of the Paasche and Laspeyres indexes.

[15] Walsh (1901) (1921a) and Fisher (1922) considered both averaging strategies in their classic studies on index numbers.

[16] Note that we have chosen the mean function $m(q_n^0,q_n^1)$ to be the same for each commodity n.

[17] Walsh endorsed P_W as being the best index number formula: “We have seen reason to believe formula 6 better than formula 7. Perhaps formula 9 is the best of the rest, but between it and Nos. 6 and 8 it would be difficult to decide with assurance.” C.M. Walsh (1921a; 103). His formula 6 is P_W defined by (13) and his 9 is the Fisher ideal defined by (9) above. His formula 8 is the formula $p^1\cdot q^1/p^0\cdot q^0 Q_W(p^0,p^1,q^0,q^1)$, which is known as the implicit Walsh price index where $Q_W(p^0,p^1,q^0,q^1)$ is the Walsh quantity index defined by (13) except the role of prices and quantities is interchanged. Thus although Walsh thought that his Walsh price index was the best functional form, his implicit Walsh price index and the “Fisher” formula were not far behind.

index P_F defined by (9) as being "best" while a direct averaging of the two quantity baskets q^0 and q^1 leads to the Walsh price index P_W defined by (13) as being "best".

We turn now to another early approach to the index number problem.

4. Stochastic and Descriptive Statistics Approaches to Index Number Theory

The (unweighted) stochastic approach to the determination of the price index can be traced back to the work of Jevons (1865) (1884) and Edgeworth (1888) (1896) (1901) over a hundred years ago[18].

The basic idea behind the stochastic approach is that each price relative, p_n^1/p_n^0 for n = 1,2,…,N, can be regarded as an estimate of a common inflation rate α between periods 0 and 1; i.e., Jevons and Edgeworth essentially assumed that

(14) $p_n^1/p_n^0 = \alpha + \varepsilon_n$; $n = 1,2,\ldots,N$

where α is the common inflation rate and the ε_n are random variables with mean 0 and variance σ^2. The least squares estimator for α is the *Carli* (1804) *price index* P_C defined as

(15) $P_C(p^0,p^1) \equiv \sum_{n=1}^N (1/N)(p_n^1/p_n^0)$.

Unfortunately, P_C does not satisfy the time reversal test, i.e., $P_C(p^1,p^0) \neq 1/P_C(p^0,p^1)$[19].

Now assume that the logarithm of each price relative, $\ln(p_n^1/p_n^0)$, is an independent unbiased estimate of the logarithm of the inflation rate between periods 0 and 1, β say. Thus we have:

(16) $\ln(p_n^1/p_n^0) = \beta + \varepsilon_n$; $n = 1,2,\ldots,N$

where $\beta \equiv \ln\alpha$ and the ε_n are independently distributed random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for β is the logarithm of the geometric mean of the price relatives. Hence the corresponding estimate for the common inflation rate α is the *Jevons* (1865) *price index* P_J defined as:

(17) $P_J(p^0,p^1) \equiv \prod_{n=1}^N (p_n^1/p_n^0)^{1/N}$.

The Jevons price index P_J does satisfy the time reversal test and hence is much more satisfactory than the Carli index P_C. However, both the Jevons and Carli price indexes suffer from a fatal

[18] For additional references to the early literature, see Diewert (1993; 37-38) (1995) and Balk (2008; 32-36).

[19] In fact Fisher (1922; 66) noted that $P_C(p^0,p^1)P_C(p^1,p^0) \geq 1$ unless the period 1 price vector p^1 is proportional to the period 0 price vector p^0; i.e., Fisher showed that the Carli index has a definite upward bias. Walsh (1901; 327) established this inequality for the case N = 2. Fisher urged users to abandon the use of the Carli index but his advice was generally ignored by statistical agencies until recently: "In fields other than index numbers it is often the best form of average to use. But we shall see that the simple arithmetic average produces one of the very worst of index numbers. And if this book has no other effect than to lead to the total abandonment of the simple arithmetic type of index number, it will have served a useful purpose." Irving Fisher (1922; 29-30).

flaw: each price relative p_n^1/p_n^0 is regarded as being equally important and is given an equal weight in the index number formulae (15) and (17).[20] Keynes (1930; 76-81) also criticized the unweighted stochastic approach to index number theory on two other grounds: (i) price relatives are not distributed independently and (ii) there is no single inflation rate that can be applied to all parts of an economy; e.g., Keynes demonstrated empirically that wage rates, wholesale prices and final consumption prices all had different rates of inflation. In order to overcome the Keynesian criticisms of the unweighted stochastic approach to index numbers, it is necessary to:

- Have a definite domain of definition for the index number and
- Weight the price relatives by their economic importance.

Theil (1967; 136-137) proposed a solution to the lack of weighting in (15). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the nth price relative is equal to $s_n^0 \equiv p_n^0 q_n^0/p^0\cdot q^0$, the period 0 expenditure share for commodity n. Then the overall mean (period 0 weighted) logarithmic price change is $\sum_{n=1}^N s_n^0 \ln(p_n^1/p_n^0)$. Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) logarithmic price change of $\sum_{n=1}^N s_n^1 \ln(p_n^1/p_n^0)$. Each of these measures of overall logarithmic price change seems equally valid so we could argue for taking a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change. Theil (1967; 137) argued that a nice symmetric index number formula can be obtained if we make the probability of selection for the nth price relative equal to the arithmetic average of the period 0 and 1 expenditure shares for commodity n. Using these probabilities of selection, Theil's final measure of overall logarithmic price change is

(18) $\ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N (1/2)(s_n^0+s_n^1)\ln(p_n^1/p_n^0)$.

It is possible to give a *descriptive statistics* interpretation of the right hand side of (18). Define the nth logarithmic price ratio r_n by:

(19) $r_n \equiv \ln(p_n^1/p_n^0)$ for $n = 1,\ldots,N$.

Now define the discrete random variable, R say, as the random variable which can take on the values r_n with probabilities $\rho_n \equiv (1/2)(s_n^0+s_n^1)$ for $n = 1,\ldots,N$. Note that since each set of expenditure shares, s_n^0 and s_n^1, sums to one, the probabilities ρ_n will also sum to one. It can be seen that the expected value of the discrete random variable R is $\ln P_T(p^0,p^1,q^0,q^1)$ as defined by the right hand side of (18). Thus the logarithm of the index P_T can be interpreted as *the expected value of the distribution of the logarithmic price ratios* in the domain of definition under consideration, where the N discrete price ratios in this domain of definition are weighted according to Theil's probability weights, ρ_n.

[20] Walsh (1901) (1921a; 82-83), Fisher (1922; 43) and Keynes (1930; 76-77) all objected to the lack of weighting in the unweighted stochastic approach to index number theory.

Taking antilogs of both sides of (18), we obtain the Törnqvist Theil price index; P_T.[21] This index number formula has a number of good properties. In particular, P_T satisfies the time reversal test (10) and the linear homogeneity test (12).[22]

Additional material on stochastic approaches to index number theory and references to the literature can be found in Selvanathan and Rao (1994), Diewert (1995) (2004) (2005), Wynne (1997), Clements, Izan and Selvanathan (2006) and Balk (2008; 32-36).

5. Test Approaches to Index Number Theory [23]

Recall equation (5) above, which set the value ratio, V^1/V^0, equal to the product of the price index, $P(p^0,p^1,q^0,q^1)$, and the quantity index, $Q(p^0,p^1,q^0,q^1)$. This is called the Product Test and we assume that it is satisfied. This equation means that as soon as the functional form for the price index P is determined, then (5) can be used to determine the functional form for the quantity index Q. However, a further advantage of assuming that the product test holds is that we can assume that the quantity index Q satisfies a "reasonable" property and then use (5) to translate this test on the quantity index into a corresponding test on the price index P.[24]

If N = 1, so that there is only one price and quantity to be aggregated, then a natural candidate for P is p_1^1/p_1^0 , the single price ratio, and a natural candidate for Q is q_1^1/q_1^0 , the single quantity ratio. When the number of commodities or items to be aggregated is greater than 1, then what index number theorists have done over the years is propose properties or tests that the price index P should satisfy. These properties are generally multi-dimensional analogues to the one good price index formula, p_1^1/p_1^0. Below, we list twenty-one tests that turn out to characterize the Fisher ideal price index.

We shall assume that every component of each price and quantity vector is positive; i.e., $p^t >> 0_N$ and $q^t >> 0_N$ for t = 0,1. If we want to set $q^0 = q^1$, we call the common quantity vector q; if we want to set $p^0 = p^1$, we call the common price vector p.

Our first two tests are not very controversial and so we will not discuss them.

T1: *Positivity*: $P(p^0,p^1,q^0,q^1) > 0$.

T2: *Continuity*: $P(p^0,p^1,q^0,q^1)$ is a continuous function of its arguments.

Our next two tests are somewhat more controversial.

T3: *Identity or Constant Prices Test*: $P(p,p,q^0,q^1) = 1$.

[21] This index first appeared explicitly as formula 123 in Fisher (1922; 473). P_T is generally attributed to Törnqvist (1936) but this article did not have an explicit definition for P_T; it was defined explicitly in Törnqvist and Törnqvist (1937); see Balk (2008; 26).
[22] For a listing of some of the tests that P_T, P_F, and P_W satisfy, see Diewert (1992; 223). In Fisher (1922), these indexes were listed as numbers 123, 353 and 1153 respectively.
[23] The material in this section is based on Diewert (1992) where more detailed references to the literature on the origins of the various tests can be found.
[24] This observation was first made by Fisher (1911; 400-406). Vogt (1980) also pursued this idea.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. The controversial part of this test is that the two quantity vectors are allowed to be different in the above test.[25]

T4: *Fixed Basket or Constant Quantities Test*: $P(p^0,p^1,q,q) = \sum_{i=1}^N p_i^1 q_i / \sum_{i=1}^N p_i^0 q_i$.

That is, if quantities are constant during the two periods so that $q^0 = q^1 \equiv q$, then the price index should equal the expenditure on the constant basket in period 1, $\sum_{i=1}^N p_i^1 q_i$, divided by the expenditure on the basket in period 0, $\sum_{i=1}^N p_i^0 q_i$.

The following four tests are *homogeneity tests* and they restrict the behavior of the price index P as the scale of any one of the four vectors p^0,p^1,q^0,q^1 changes.

T5: *Proportionality in Current Prices*: $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. Most index number theorists regard this property as a very fundamental one that the index number formula should satisfy.

Walsh (1901) and Fisher (1911; 418) (1922; 420) proposed the related proportionality test $P(p,\lambda p,q^0,q^1) = \lambda$. This last test is a combination of T3 and T5; in fact Walsh (1901, 385) noted that this last test implies the identity test, T3.

In our next test, instead of multiplying all period 1 prices by the same number, we multiply all period 0 prices by the number λ.

T6: *Inverse Proportionality in Base Period Prices*: $P(\lambda p^0,p^1,q^0,q^1) = \lambda^{-1} P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0.

The following two homogeneity tests can also be regarded as invariance tests.

T7: *Invariance to Proportional Changes in Current Quantities*: $P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if current period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively)

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[25] Usually, economists assume that given a price vector p, the corresponding quantity vector q is uniquely determined. Here, we have the same price vector but the corresponding quantity vectors are allowed to be different.

homogeneous of degree zero in the components of the period 1 quantity vector q^1. Vogt (1980, 70) was the first to propose this test and his derivation of the test is of some interest. Suppose the quantity index Q satisfies the quantity analogue to the price test T5; i.e., suppose Q satisfies $Q(p^0,p^1,q^0,\lambda q^1) = \lambda Q(p^0,p^1,q^0,q^1)$ for $\lambda > 0$. Then using the product test (5), we see that P must satisfy T7.

T8: *Invariance to Proportional Changes in Base Quantities*: $P(p^0,p^1,\lambda q^0,q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if base period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 0 quantity vector q^0. If the quantity index Q satisfies the following counterpart to T8: $Q(p^0,p^1,\lambda q^0,q^1) = \lambda^{-1}Q(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$, then using (5), the corresponding price index P must satisfy T8. This argument provides some additional justification for assuming the validity of T8 for the price index function P.

T7 and T8 together impose the property that the price index P does not depend on the *absolute* magnitudes of the quantity vectors q^0 and q^1.

The next five tests are *invariance* or *symmetry tests*. Fisher (1922; 62-63, 458-460) and Walsh (1921b; 542) seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, 62-63) spoke of fairness but it is clear that he had symmetry properties in mind. It is perhaps unfortunate that he did not realize that there were more symmetry and invariance properties than the ones he proposed; if he had realized this, it is likely that he would have been able to provide an axiomatic characterization for his ideal price index. Our first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed:

T9: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p^0,p^1,q^0,q^1)$$

where p^{t*} denotes a permutation of the components of the vector p^t and q^{t*} denotes the same permutation of the components of q^t for t = 0,1. This test is due to Irving Fisher (1922), and it is one of his three famous reversal tests. The other two are the time reversal test and the factor reversal test which will be considered below.

T10: *Invariance to Changes in the Units of Measurement* (commensurability test):

$$P(\alpha_1 p_1^0,\ldots,\alpha_N p_N^0;\ \alpha_1 p_1^1,\ldots,\alpha_N p_N^1;\ \alpha_1^{-1}q_1^0,\ldots,\alpha_N^{-1}q_N^0;\ \alpha_1^{-1}q_1^1,\ldots,\alpha_N^{-1}q_N^1) =$$
$$P(p_1^0,\ldots,p_N^0;\ p_1^1,\ldots,p_N^1;\ q_1^0,\ldots,q_N^0;\ q_1^1,\ldots,q_N^1) \text{ for all } \alpha_1 > 0, \ldots, \alpha_N > 0.$$

That is, the price index does not change if the units of measurement for each commodity are changed. The concept of this test was due to Jevons (1884; 23) and the Dutch economist Pierson (1896; 131), who criticized several index number formula for not satisfying this fundamental test. Fisher (1911; 411) first called this test *the change of units test* and later, Fisher (1922; 420) called it the *commensurability test*.

T11: *Time Reversal Test*: $P(p^0,p^1,q^0,q^1) = 1/P(p^1,p^0,q^1,q^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio; this test is satisfied (as are all of the other tests listed in this section). When the number of goods is greater than one, many commonly used price indexes fail this test; e.g., the Laspeyres (1871) price index, P_L defined earlier by (7), and the Paasche (1874) price index, P_P defined earlier by (8), both *fail* this fundamental test. The concept of the test was due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test, that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 368) (1921b; 541) and Fisher (1911; 534) (1922; 64).

Our next two tests are more controversial, since they are not necessarily consistent with the economic approach to index number theory. However, these tests are quite consistent with the weighted stochastic approach to index number theory discussed earlier in section 4.

T12: *Quantity Reversal Test* (quantity weights symmetry test):
$P(p^0,p^1,q^0,q^1) = P(p^0,p^1,q^1,q^0)$.

That is, if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities are used to weight the prices in the index number formula, then the period 0 quantities q^0 and the period 1 quantities q^1 must enter the formula in a symmetric or even handed manner. Funke and Voeller (1978; 3) introduced this test; they called it the *weight property*.

The next test is the analogue to T12 applied to quantity indexes:

T13: *Price Reversal Test* (price weights symmetry test):

$$\{\sum_{i=1}^N p_i^1 q_i^1/ \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) = \{\sum_{i=1}^N p_i^0 q_i^1/ \sum_{i=1}^N p_i^1 q_i^0\}/P(p^1,p^0,q^0,q^1).$$

Thus if we use (5) to define the quantity index Q in terms of the price index P, then it can be seen that T13 is equivalent to the following property for the associated quantity index Q: $Q(p^0,p^1,q^0,q^1) = Q(p^1,p^0,q^0,q^1)$. That is, if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property T13 implies that these prices enter the quantity index in a symmetric manner.

The next three tests are mean value tests.

T14: *Mean Value Test for Prices*:
$$\min_i (p_i^1/p_i^0 : i=1,...,N) \le P(p^0,p^1,q^0,q^1) \le \max_i (p_i^1/p_i^0 : i = 1,...,N).$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is supposed to be some sort of an average of the N price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfy this test.

The next test is the analogue to T14 applied to quantity indexes:

T15: *Mean Value Test for Quantities*:

$$\min_i (q_i^1/q_i^0 : i=1,...,N) \le \{V^1/V^0\}/ P(p^0,p^1,q^0,q^1) \le \max_i (q_i^1/q_i^0 : i = 1,...,N)$$

where V^t is the period t value aggregate $V^t \equiv \sum_{n=1}^N p_n^t q_n^t$ for t = 0,1. Using (5) to define the quantity index Q in terms of the price index P, we see that T15 is equivalent to the following property for the associated quantity index Q:

$$(20)\ \min_i (q_i^1/q_i^0 : i=1,...,N) \le Q(p^0,p^1,q^0,q^1) \le \max_i (q_i^1/q_i^0 : i = 1,...,N).$$

That is, the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth q_i^1/q_i^0 of the individual quantities.

In section 3, we argued that it was very reasonable to take an average of the Laspeyres and Paasche price indexes as a single "best" measure of overall price change. This point of view can be turned into a test:

T16: *Paasche and Laspeyres Bounding Test*: The price index P lies between the Laspeyres and Paasche indices, P_L and P_P, defined earlier by (7) and (8) above.

The final four tests are monotonicity tests; i.e., how should the price index $P(p^0,p^1,q^0,q^1)$ change as any component of the two price vectors p^0 and p^1 increases or as any component of the two quantity vectors q^0 and q^1 increases.

T17: *Monotonicity in Current Prices*: $P(p^0,p^1,q^0,q^1) < P(p^0,p^2,q^0,q^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase, so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1. This property was proposed by Eichhorn and Voeller (1976; 23) and it is a very reasonable property for a price index to satisfy.

T18: *Monotonicity in Base Prices*: $P(p^0,p^1,q^0,q^1) > P(p^2,p^1,q^0,q^1)$ if $p^0 < p^2$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0 . This very reasonable property was also proposed by Eichhorn and Voeller (1976; 23).

T19: *Monotonicity in Current Quantities*: if $q^1 < q^2$, then

$$\{\sum_{i=1}^N p_i^1 q_i^1/ \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) < \{\sum_{i=1}^N p_i^1 q_i^2/ \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^2).$$

T20: *Monotonicity in Base Quantities*: if $q^0 < q^2$, then

$$\{\sum_{i=1}^N p_i^1 q_i^1/ \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) > \{\sum_{i=1}^N p_i^1 q_i^1/ \sum_{i=1}^N p_i^0 q_i^2\}/P(p^0,p^1,q^2,q^1).$$

If we define the implicit quantity index Q that corresponds to P using (5), we find that T19 translates into the following inequality involving Q:

(21) $Q(p^0,p^1,q^0,q^1) < Q(p^0,p^1,q^0,q^2)$ if $q^1 < q^2$.

That is, if any period 1 quantity increases, then the implicit quantity index Q that corresponds to the price index P must increase. Similarly, we find that T20 translates into:

(22) $Q(p^0,p^1,q^0,q^1) > Q(p^0,p^1,q^2,q^1)$ if $q^0 < q^2$.

That is, if any period 0 quantity increases, then the implicit quantity index Q must decrease. Tests T19 and T20 are due to Vogt (1980, 70).

The final test is Irving Fisher's (1921; 534) (1922; 72-81) third reversal test (the other two being T9 and T11):

T21: *Factor Reversal Test* (functional form symmetry test):
$$P(p^0,p^1,q^0,q^1)\, P(q^0,q^1,p^0,p^1) = \sum_{i=1}^N p_i^1q_i^1/ \sum_{i=1}^N p_i^0q_i^0 = V^1/V^0.$$

A justification for this test is the following one: if $P(p^0,p^1,q^0,q^1)$ is a good functional form for the price index, then if we reverse the roles of prices and quantities, $P(q^0,q^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument) and thus the product of the price index $P(p^0,p^1,q^0,q^1)$ and the quantity index $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ ought to equal the value ratio, V^1/V^0 . The second part of this argument does not seem to be valid and thus many researchers over the years have objected to the factor reversal test.

It is straightforward to show that the Fisher ideal price index P_F defined earlier by (9) satisfies all 21 tests. Is this the only index number formula that satisfies all of these tests? The answer is yes: Funke and Voeller (1978; 180) showed that the only index number function $P(p^0,p^1,q^0,q^1)$ which satisfies T1 (positivity), T11 (time reversal test), T12 (quantity reversal test) and T21 (factor reversal test) is the Fisher ideal index P_F defined by (9). Diewert (1992; 221) proved a similar result: namely that if P satisfied T1 and the three reversal tests T11-T13, then P must equal P_F. We will provide a proof of Diewert's result.

It is relatively straightforward to show that the Fisher index satisfies all of the above 21 tests. The more difficult part of the proof is to show that it is the *only* index number formula which satisfies these tests. This part of the proof follows from the fact that if P satisfies the positivity test T1 and the three reversal tests, T11-T13, then P must equal P_F. To see this, rearrange the terms in the statement of test T13 into the following equation:

$$\begin{aligned} (23)\ \{\textstyle\sum_{i=1}^N p_i^1\, q_i^1/ \sum_{i=1}^N p_i^0\, q_i^0\}/\{\sum_{i=1}^N p_i^0\, q_i^1/ \sum_{i=1}^N p_i^1\, q_i^0\} &= P(p^0,p^1,q^0,q^1)/P(p^1,p^0,q^0,q^1) \\ &= P(p^0,p^1,q^0,q^1)/P(p^1,p^0,q^1,q^0) \quad \text{using T12, the quantity reversal test} \\ &= P(p^0,p^1,q^0,q^1)\, P(p^0,p^1,q^0,q^1) \quad \text{using T11, the time reversal test.} \end{aligned}$$

Now take positive square roots on both sides of (23) and we see that the left hand side of the equation is the Fisher index $P_F(p^0,p^1,q^0,q^1)$ defined by (9) and the right hand side is $P(p^0,p^1,q^0,q^1)$. Thus if P satisfies T1, T11, T12 and T13, it must equal the Fisher ideal index P_F.

Thus it seems that from the perspective of the above test approach to index number theory, the Fisher ideal index satisfies more "reasonable" tests than competing indexes and hence can be regarded as "best" from the viewpoint of this perspective.

There is another perspective to the test approach to index number theory. The above approach looked at axioms or tests that pertained to situations where the price index was a function of the two price vectors, p^0 and p^1, and the two matching quantity vectors, q^0 and q^1. In this framework, the two quantity vectors essentially act as weights for the prices. However, there is an alternative framework where the price index, say $P^*(p^0,p^1,e^0,e^1)$, is regarded as a function of the two price vectors, p^0 and p^1, and the two matching *expenditure vectors*, e^0 and e^1.[26] An axiomatic approach to the determination of the functional form for indexes of this type is developed in the ILO (2004; 307-309) and the Törnqvist index defined earlier by (18) emerges as "best" from the perspective of this second test approach to index number theory. Thus both the Fisher and Törnqvist indexes can be given strong axiomatic justifications.

There is one final important test that should be added to the above list of tests and that is the following *circularity test*[27] which involves looking at the prices and quantities that pertain to three periods:

T22: *Circularity Test*: $P(p^0,p^1,q^0,q^1)\, P(p^1,p^2,q^1,q^2) = P(p^0,p^2,q^0,q^2)$.

If this test is satisfied, then the rate of price change going from period 0 to 1, $P(p^0,p^1,q^0,q^1)$, times the rate of price change going from period 1 to 2, $P(p^1,p^2,q^1,q^2)$, is equal to the rate of price change going from period 0 to 2 directly, $P(p^0,p^2,q^0,q^2)$. If there is only one commodity in the aggregate, then the price index $P(p^0,p^1,q^0,q^1)$ just becomes the single price ratio, p_1^1/p_1^0, and the circularity test T22 becomes the equation $[p_1^1/p_1^0][p_1^2/p_1^1] = [p_1^2/p_1^0]$, which is obviously satisfied. The equation in the circularity test illustrates the difference between chained index numbers and fixed base index numbers. The left hand side of T22 uses the *chain principle* to construct the overall inflation between periods 0 and 2 whereas the right hand side uses the *fixed base principle* to construct an estimate of the overall price change between periods 0 and 1.[28]

It would be good if our preferred index number formulae, the Fisher, Walsh and Törnqvist indexes (P_F, P_W and P_T), satisfied the circularity test but unfortunately, none of these indexes satisfy T22. An interesting problem is to determine exactly what class of indexes does satisfy the

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[26] Component n of the period t expenditure vector e^t is defined as $e_n^t \equiv p_n^t q_n^t$ for n = 1,...,N and t = 0,1. Thus if the price components p_n^t are known, then a knowledge of either the quantity components q_n^t or the expenditure components e_n^t will determine prices, quantities and expenditures in both periods.

[27] The test name is due to Fisher (1922; 413) and the concept was originally due to Westergaard (1890; 218-219).

[28] Thus when the chain principle is used, the price index $P(p^t,p^{t+1},q^t,q^{t+1})$ is used to update the period t index level to construct the period t+1 index level, whereas the fixed base system constructs the period t+1 index level relative to period 0 directly as $P(p^0,p^{t+1},q^0,q^{t+1})$, where the period 0 level is set equal to 1. Fisher (1911; 203) introduced this fixed base and chain terminology. The concept of chaining is due to Lehr (1885) and Marshall (1887; 373).

circularity test. The following proposition, due essentially to Eichhorn (1978; 167-168), helps to answer this question.

Proposition: Assume that the index number formula P satisfies the following tests: T1 (positivity), T2 (continuity), T3 (identity), T5 (proportionality in current prices), T10 (commensurability) and T17 (monotonicity in current prices) in addition to the circularity test above. Then P must have the following functional form due originally to Konüs and Byushgens[29] (1926; 163-166):[30]

(24) $P_{KB}(p^0,p^1,q^0,q^1) \equiv \prod_{i=1}^N [p_i^1/p_i^0]^{\alpha i}$

where the N constants α_i satisfy the following restrictions:

(25) $\sum_{i=1}^N \alpha_i = 1$ and $\alpha_i > 0$ for $i = 1,\ldots,N$.

Proof: Rewrite the circularity test T22 in the following form:

(26) $P(p^*,p,q^*,q) = P(p^*,p^0,q^*,q^0)P(p^0,p,q^0,q)$.

Using T1, we can rewrite (26) as follows:

(27) $P(p^0,p,q^0,q) = P(p^*,p,q^*,q)/P(p^*,p^0,q^*,q^0)$.

Now hold p^* and q^* constant at some fixed values and define the function f(p,q) as follows:

(28) $f(p,q) \equiv P(p^*,p,q^*,q) > 0$ for all $p>>0_N$ and $q >> 0_N$

where the positivity of f(p,q) follows from T1. Substituting definition (28) back into (27) gives us the following representation for $P(p^0,p,q^0,q)$:

(29) $P(p^0,p,q^0,q) = f(p,q)/f(p^0,q^0)$.

Now let $p^0 = p$ in (29) and apply the identity test T3 to the resulting equation. We obtain:

(30) $1 = P(p,p,q^0,q) = f(p,q)/f(p,q^0)$; $p >> 0_N$; $q >> 0_N$; $q^0 >> 0_N$.

Define the function g(p) as

(31) $g(p) \equiv f(p,1_N) > 0$ $p >> 0_N$.

Now set q^0 in (30) equal to a vector of ones, 1_N, and (30) becomes:

[29] Konüs and Byushgens show that the index defined by (24) is exact for Cobb-Douglas (1928) preferences; see also Pollak (1989; 23). The concept of an exact index number formula will be explained when we study the economic approach to index number theory in Part II of this Chapter.
[30] See also Eichhorn (1978; 167-168) and Vogt and Barta (1997; 47). Proofs of related results can be found in Funke, Hacker and Voeller (1979) and Balk (1995).

$$(32)\quad f(p,q) = f(p,1_N) = g(p) \qquad \text{using definition (31).}$$

Thus f(p,q) cannot depend on q. Now substitute (32) back into (29) and we find that P must have the following representation if P satisfies the circularity test and the tests T1 and T3:

$$(33)\quad P(p^0,p,q^0,q) = g(p)/g(p^0)\ ; \qquad p >> 0_N;\ p^0 >> 0_N;\ q >> 0_N;\ q^0 >> 0_N.$$

Now apply the commensurability test, T10, to the P that is defined by (33) where we set $\alpha_i = (p_i^0)^{-1}$ for i = 1,...,N. Using the representation for P given by (33), we find that g must satisfy the following functional equation:

$$(34)\quad g(p^1)/g(p^0) = g(p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0)/g(1_N)\ ; \qquad p^0 >> 0_N;\ p^1 >> 0_N.$$

Define h(p) as follows:

$$(35)\quad h(p) \equiv g(p)/g(1_N) > 0\ ; \qquad p >> 0_N$$

where the positivity of h follows from the positivity of g. Using definition (35), we have:

$$\begin{aligned}(36)\quad h(p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0) &= g(p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0)/g(1_N) && p^0 >> 0_N;\ p^1 >> 0_N \\ &= g(p^1)/g(p^0) && \text{using (34)} \\ &= [g(p^1)/g(1_N)]/[g(p^0)/g(1_N)] && \text{using T1} \\ &= h(p^1)/h(p^0) && \text{using (35) twice.}\end{aligned}$$

Thus h must satisfy the following functional equation:

$$(37)\quad h(p^0)h(p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0) = h(p^1)\ ; \qquad p^0 >> 0_N;\ p^1 >> 0_N.$$

Define the vector x as the vector p^0 and the vector y as $p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0$. Hence the product of the ith components of x and y is equal to the ith component of the vector p^1 and it can be seen that the functional equation (37) is equivalent to the following functional equation:

$$(38)\quad h(x_1y_1,x_2y_2,...,x_Ny_N) = h(x_1,x_2,...,x_N)h(y_1,y_2,...,y_N)\ ; \qquad x >> 0_N;\ y >> 0_N.$$

Equation (38) becomes the following equation if we allow x_1 and y_1 to vary freely but fix all x_i and y_i at 1 for i = 2,3,...,N:

$$(39)\quad h(x_1y_1,1,...,1) = h(x_1,1,...,1)h(y_1,1,...,1)\ ; \qquad x_1 > 0;\ y_1 > 0.$$

But (39) is an example of *Cauchy's* (1821) *fourth functional equation*.[31] Using the T1 (positivity) and T2 (continuity) properties of P, which carry over to h, we see that the solution to (39) is:

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[31] See Eichhorn (1978) for material on Cauchy's four fundamental functional equations.

(40) $h(x_1,1,...,1) = x_1^{c(1)}$

where c(1) is an arbitrary constant. In a similar fashion, (38) becomes the following equation if we allow x_2 and y_2 to vary freely but fix all other x_i and y_i at 1:

(41) $h(1,x_2y_2,1,...,1) = h(1,x_2,1,...,1)h(1,y_2,1,...,1)$; $x_2 > 0$; $y_2 > 0$.

The solution to (41) is:

(42) $h(1,x_2,1,...,1) = x_2^{c(2)}$

where c(2) is an arbitrary constant. In a similar fashion, we find that

(43) $h(1,1,x_3,1,...,1) = x_3^{c(3)}$; ... ; $h(1,1,...,1,x_N) = x_N^{c(N)}$

where the c(i) are arbitrary constants. Using (38) repeatedly, we can show:

(44)
$$\begin{aligned} h(x_1,x_2,...,x_N) &= h(x_1,1,...,1)h(1,x_2,...,x_N) \\ &= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,...,x_N) \\ &= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,1,...,1)h(1,1,1,x_4,...,x_N) \\ &\;\;... \\ &= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,1,...,1)...h(1,1,1,...,1,x_N) \\ &= \prod_{i=1}^N x_i^{c(i)} \qquad \text{using (40), (42) and (43).} \end{aligned}$$

Thus we have determined the functional form for the function h. Now use (35) to determine the function g(p) in terms of h(p):

(45)
$$\begin{aligned} g(p) &= g(1_N)h(p) \\ &= g(1_N) \prod_{i=1}^N p_i^{c(i)}. \end{aligned}$$

Using (33), we can express P in terms of g as follows:

(46)
$$\begin{aligned} P(p^0,p^1,q^0,q^1) &= g(p^1)/g(p^0) \\ &= g(1_N) \prod_{i=1}^N (p_i^1)^{c(i)}/g(1_N) \prod_{i=1}^N (p_i^0)^{c(i)} \qquad \text{using (45)} \\ &= \prod_{i=1}^N (p_i^1/p_i^0)^{c(i)}. \end{aligned}$$

Now apply test T5, proportionality in current prices, to the P defined by (46). It is easy to see that this test implies that the constants c(i) must sum to 1.

Finally, apply test T17, monotonicity in current prices, to conclude that the constants c(i) must be positive. Hence we can set the c(i) equal to the α_i and we have proved the Proposition. Q.E.D.

Thus under fairly weak regularity conditions, *the only price index satisfying the circularity test is a weighted geometric average of all the individual price ratios*, the weights being constant through time. This is a somewhat discouraging result!

Looking at the above proof, it is interesting to note that we arrive at the representation for the price index given by (33) using only the circularity test T22 and the two tests, T1 (Positivity) and T3 (Identity), which are rather weak tests. The representation (33) can be rewritten as follows:

(47) $P(p^0,p^1,q^0,q^1) = g(p^1)/g(p^0)$

where g(p) is a continuous function of p, assuming that $P(p^0,p^1,q^0,q^1)$ satisfies T2 (Continuity). Looking at (47), *it can be seen that the index number formula does not depend on the quantity vectors* q^0 *and* q^1. Thus any weighting of the prices (by quantities or expenditure shares) must be constant or nonexistent. Thus if the same formula is applied over long periods of time where relative quantities or expenditures are changing, the formula must lose its relevance for at least part of the sample period.

Fisher realized this difficulty with the circularity test as can be seen from the following quotation:

"The only formulae which conform perfectly to the circular test are index numbers which have *constant weights*; i.e., weights which are the same for all sides of the 'triangle' or segments of the 'circle'; i.e., for every pair of times or places compared. ... But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915 we need, theoretically at least, another set of weights. In the former case we need weights involving the quantities of the two years concerned, 1913 and 1914; in the second case we need weights involving the (somewhat different) quantities of the two years, 1913 and 1915. We cannot justify using the same weights for comparing the price level of 1913, not only with 1914 and 1915, but with 1969, 1776, 1492 and the times of Diocletian, Rameses II, and the Stone Age!" Irving Fisher (1922; 274-275).

Fisher did not have at his disposal a knowledge of functional equations so he was not able to prove the above result but his intuition was quite correct.

Thus far, the results of sections 3-5 have suggested that the "best" bilateral index number formulae for the price index are the Fisher or the Walsh (fixed basket approaches), the Törnqvist Theil (the stochastic or descriptive statistics approach) or the Fisher (the test approach). None of these indexes satisfy the circularity test and so there will be a difference if fixed base or chained indexes are used in empirical applications. The question now arises: should the sequence of index values be computed using fixed base indexes or chained indexes? We will address this question in the following section.

6. Fixed Base versus Chained Indexes

Fixed base indexes cannot be used for long periods of time in today's dynamic economy where new commodities appear and older ones become obsolete. Under these conditions, it becomes increasingly difficult to match commodity prices over long periods of time and index number theory is dependent on a high degree of matching of the prices between the two periods being compared. However, this possible lack of matching does not rule out using fixed base indexes for shorter periods of time, say over a year or two.

The main advantage of using chained indexes is that if prices and quantities are trending relatively smoothly, chaining will reduce the spread between the Paasche and Laspeyres indexes.[32] These two indexes each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration and it could be expected that a single point estimate of the aggregate price change should lie between these two estimates. Thus the use of either a chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth". Since annual data generally has smooth trends, the use of chained indexes is generally appropriate at this level of aggregation; see Hill (1988).

However, the story is different at subannual levels; i.e., if the index is to be produced at monthly or quarterly frequencies. Hill (1993; 388), drawing on the earlier research of Szulc (1983) and Hill (1988; 136-137), noted that it is not appropriate to use the chain system when prices oscillate or "bounce" to use Szulc's (1983; 548) term. This phenomenon can occur in the context of regular seasonal fluctuations or in the context of sales. The *price bouncing problem* or the problem of *chain drift* can be illustrated if we make use of the following test due to Walsh (1901; 389), (1921b; 540) (1924; 506):[33]

T23: *Multiperiod Identity Test*: $P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)P(p^2,p^0,q^2,q^0) = 1.$

Thus price change is calculated over consecutive periods but an artificial final period is introduced where the prices and quantities revert back to the prices and quantities in the very first period. The Walsh test T23 asks that the product of all of these price changes should equal unity. If prices have no definite trends but are simply bouncing up and down in a range, then the above test can be used to evaluate the amount of chain drift that occurs if chained indexes are used under these conditions. *Chain drift* occurs when an index does not return to unity when prices in the current period return to their levels in the base period; see the ILO (2004; 445). Fixed base indexes operating under these conditions will not be subject to chain drift.

It is possible to be a bit more precise under what conditions one should chain or not chain. Basically, one should chain if the prices and quantities pertaining to adjacent periods are *more similar* than the prices and quantities of more distant periods, since this strategy will lead to a narrowing of the spread between the Paasche and Laspeyres indexes at each link. Of course, one needs a measure of how similar are the prices and quantities pertaining to two periods. A practical problem with this *similarity linking approach* is: exactly how should the measure of price or quantity similarity be measured?[34] For *annual* time series data, it turns out that for various "reasonable" similarity measures, chained indexes are generally consistent with the

[32] See Diewert (1978; 895) and Hill (1988) (1993; 387-388). Chaining under these conditions will also reduce the spread between fixed base and chained indexes using P_F, P_W or P_T as the basic bilateral formula.

[33] This is Diewert's (1993; 40) term for the test. Walsh did not limit himself to just three periods as in T23; he considered an indefinite number of periods. If tests T3 and T22 are satisfied, then T23 will also be satisfied.

[34] This similarity approach to linking bilateral comparisons into a complete set of comparisons across all observations has been pioneered by Robert Hill (1999a) (1999b) (2001) (2004) (2009). For an axiomatic approach to similarity measures, see Diewert (2009).

similarity approach to linking observations. However, for subannual data, it is generally better to use fixed base indexes in order to eliminate the problem of chain drift.

We conclude this subsection with a discussion on how well our best indexes, P_F, P_W and P_T defined by (9), (13) and (18) above, satisfy the circularity test, T22. Fisher (1922; 277) found that for his annual data set, the Fisher ideal index P_F satisfied circularity to a reasonably high degree of approximation. It turns out that this result generally holds using annual data for P_W and P_T as well. It is possible to give a theoretical explanation for the approximate satisfaction of the circularity test for these three indexes. Alterman, Diewert and Feenstra (1999; 61) showed that if the logarithmic price ratios ln (p_n^t/p_n^{t-1}) trend linearly with time t and the expenditure shares s_i^t also trend linearly with time, then the Törnqvist index P_T will satisfy the circularity test *exactly*.[35] Since many economic time series on prices and quantities satisfy these assumptions approximately, the above exactness result will imply that the Törnqvist index P_T will satisfy the circularity test approximately. But Diewert (1978; 888) showed that P_T, P_F and P_W numerically approximate each other to the second order around an equal price and quantity point[36] and so these three indexes will generally be very close to each other using annual time series data. Hence since P_T will generally satisfy the circularity test to some degree of approximation, P_F and P_W will also satisfy circularity approximately in the time series context using annual data. Thus for *annual* economic time series, P_F, P_T and P_W will generally satisfy the circularity test to a high enough degree of approximation so that it will not matter whether we use the fixed base or chain principle. However, this same conclusion does *not* hold for *subannual* data that has substantial period to period fluctuations in prices. For fluctuating subannual data, chained indexes can give very unsatisfactory results; i.e., Walsh's multiperiod identity test can be far from being satisfied.[37] Under these conditions, fixed base indexes or multilateral methods should be used.[38]

Problems.

1. Let a and b be positive numbers and define the following *means* of a and b:

(a) $m_A(a,b) \equiv (1/2)a + (1/2)b$ (the arithmetic mean);
(b) $m_G(a,b) \equiv (ab)^{1/2}$ (the geometric mean);
(c) $m_H(a,b) \equiv [(1/2)a^{-1} + (1/2)b^{-1}]^{-1}$ (the harmonic mean).

Using elementary arguments, prove that:

(d) $m_H(a,b) \le m_G(a,b) \le m_A(a,b)$.

Under what conditions on a and b will strict inequalities hold in (d)?

[35] This exactness result can be extended to cover the case when there are monthly proportional variations in prices and the expenditure shares have constant seasonal effects in addition to linear trends; see Alterman, Diewert and Feenstra (1999; 65).
[36] See problems 8 and 9 below.
[37] See Szulc (1983) (1987), Feenstra and Shapiro (2003), Ivancic, Diewert and Fox (2011), de Haan and van der Grient (2011), de Haan and Krsinich (2012) and Diewert (2013) for evidence of chain drift using subannual data.
[38] See Szulc (1983) (1987) and Hill (1988) on this point. For solutions to the chain drift problem using subannual data, see and Ivancic, Diewert and Fox (2009) (2011), de Haan and van der Grient (2011) and de Haan and Krsinich (2012).

2. Instead of taking the arithmetic average of the expenditure shares in Theil's weighted stochastic approach, consider taking the geometric or harmonic average of these shares. Check whether the resulting indexes, P_{TG} and P_{TH}, and the Theil index P_T satisfy Tests T3, T4, T5, T8, T10 and T11. Provide proofs.

3. Prove that the Fisher index satisfies Tests T3-T18.

4. Which tests from Tests T3-T18 does the Walsh index P_W satisfy? Provide proofs.

5. If the price index P satisfies Test T4 (the Fixed Basket Test) and P and Q jointly satisfy the product test, (5) above, then show[39] that Q must satisfy the identity test $Q(p^0,p^1,q,q) = 1$ for all strictly positive vectors p^0,p^1,q. This *constant quantities test* for Q is somewhat controversial since p^0 and p^1 are allowed to be different.

6. Consider a test where the implicit quantity index Q that corresponds to P via (5) is to lie between the Laspeyres and Paasche quantity indexes, Q_L and Q_P, defined by (a) and (b) below:

(a) $Q_L \equiv p^0 \cdot q^1/p^0 \cdot q^0$;
(b) $Q_P \equiv p^1 \cdot q^1/p^1 \cdot q^0$.

Show that the resulting test turns out to be *equivalent* to test T16 on P.

7. Consider the following bilateral price index:

(a) $P_\alpha(p^0,p^1,q^0,q^1) \equiv p^1 \cdot \alpha/p^0 \cdot \alpha$

where $\alpha >> 0_N$ is vector of positive constants. (i) Which of the tests T3-T18 and T22 does P_α satisfy? (Provide proofs).

(ii) Note that the index defined by (a) is a simple index number formula that satisfies the Circularity Test, T22, but yet it is not equal to the Cobb-Douglas index defined by (46) above. Explain why this equality does not occur.

Comment: If the vector α is chosen to be an annual quantity vector pertaining to some year prior to period 0, then P_α becomes the Lowe index, which is widely used by statistical agencies as their target CPI index.

8. Consider the Laspeyres, Paasche, Fisher, Törnqvist and Walsh price indexes, P_L, P_P, P_F, P_T and P_W as functions of the four sets of variables, p^0,p^1,q^0,q^1. Show that all of the 4N first order partial derivatives of each of these 5 indexes are equal when evaluated at a point where the two price vectors are equal (so that $p^0 = p^1 \equiv p$) and where the two quantity vectors are equal (so that $q^0 = q^1 \equiv q$); i.e., show that

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[39] See Vogt (1980; 70).

(a) $\nabla P_L(p,p,q,q) = \nabla P_P(p,p,q,q) = \nabla P_F(p,p,q,q) = \nabla P_T(p,p,q,q) = \nabla P_W(p,p,q,q)$.

Hint: You need to calculate 20 vectors of first order partial derivatives and show that they are equal when evaluated at an equal price and equal quantity point. We calculate these derivatives for the case of the Laspeyres index to show what is required.

(b) $P_L(p^0,p^1,q^0,q^1) \equiv p^{1T}q^0/p^{0T}q^0$.

Differentiate P_L with respect to the components of p^0 and get the following vector of first order partial derivatives:

(c) $\nabla_{p^0}\ P_L(p^0,p^1,q^0,q^1) = -\ p^{1T}q^0(p^{0T}q^0)^{-2}q^0$.

Now evaluate these derivatives at $p^0 = p^1 = p$ and $q^0 = q^1 = q$ and we get the following expression:

(d) $\nabla_{p^0}\ P_L(p,p,q,q) = -\ q/p^Tq$

which is the answer for the first block of derivatives. Now differentiate $P_L(p^0,p^1,q^0,q^1)$ with respect to the components of p^1 and get the following vector of first order derivatives:

(e) $\nabla_{p^1}\ P_L(p^0,p^1,q^0,q^1) = q^0/p^{0T}q^0 = q/p^Tq$ when $p^0 = p^1 \equiv p$ and $q^0 = q^1 \equiv q$.

Similarly:

(f) $\nabla_{q^0}\ P_L(p^0,p^1,q^0,q^1) = [p^0/p^{0T}q^0] - p^{1T}q^0(p^{0T}q^0)^{-2}p^0 = 0_N$ when $p^0 = p^1 \equiv p$ and $q^0 = q^1 \equiv q$;

(g) $\nabla_{q^1}\ P_L(p^0,p^1,q^0,q^1) = 0_N$.

Calculating the derivatives of the Törnqvist Theil index is much more difficult. Note that the log of P_T is defined as

(h) $\ln P_T(p^0,p^1,q^0,q^1) \equiv (1/2)\ \sum_{n=1}^N (s_n^0 + s_n^1)\ln(p_n^1/p_n^0)$.

Letting z be equal to any one of the components in the vectors p^0, p^1, q^0 and q^1, we have:

(i) $\partial P_T(p^0,p^1,q^0,q^1)/\partial z = P_T(p^0,p^1,q^0,q^1)\partial \ln P_T(p^0,p^1,q^0,q^1)/\partial z$.

Now let $z = p_n^0$ and use formula (i) above along with definition (h) (and remembering that $s_n^t \equiv p_n^tq_n^t/p^t{\cdot}q^t$ for $t = 0,1$ and $n = 1,...,N$):

(j) $\partial P_T(p^0,p^1,q^0,q^1)/\partial p_n^0 = P_T(p^0,p^1,q^0,q^1)\partial \ln P_T(p^0,p^1,q^0,q^1)/\partial p_n^0$
$= P_T(p^0,p^1,q^0,q^1)(1/2)[q_n^0(p^{0T}q^0)^{-1}]\ln(p_n^1/p_n^0) - P_T(p^0,p^1,q^0,q^1)(1/2)[s_n^0 + s_n^1](p_n^0)^{-1}$
$- P_T(p^0,p^1,q^0,q^1)(1/2)[\sum_{j=1}^N p_j^0q_j^0q_j^0(p^{0T}q^0)^{-2}]\ln(p_j^1/p_j^0)$

$= 0 - 1(1/2)[(p_nq_n/p^Tq) + (p_nq_n/p^Tq)](p_n)^{-1} - 0$ when $p^0 = p^1 \equiv p$ and $q^0 = q^1 \equiv q$
$= - q_n/p^Tq$ for n = 1,...,N.

Thus equations (j) are equivalent to the equations $\nabla_{p^0} P_T(p,p,q,q) = - q/p^Tq$ which is the same result that we obtained for the Laspeyres price index in equations (d) above.

Comment: It is easy to show that

(k) $P_L(p,p,q,q) = P_P(p,p,q,q) = P_F(p,p,q,q) = P_T(p,p,q,q) = P_W(p,p,q,q) = 1$.

Equations (a) and (k) show that the Laspeyres, Paasche, Fisher, Törnqvist and Walsh indexes all approximate each other to the first order around an equal price and quantity point.

9. Now consider forming second order Taylor series approximations to the 5 indexes defined in the previous problem.

(a) Show that $\nabla^2 P_L(p,p,q,q) \neq \nabla^2 P_P(p,p,q,q)$ and hence the Laspeyres and Paasche indexes do *not* approximate each other to the second order around an equal price and quantity point; i.e., their 4N by 4N matrices of second order partial derivatives are not all equal when evaluated at an equal price and quantity point.
(b) Show that $\nabla^2 P_L(p,p,q,q) \neq \nabla^2 P_F(p,p,q,q)$ and hence the Laspeyres and Fisher indexes do *not* approximate each other to the second order around an equal price and quantity point.
(c) Show that $\nabla^2 P_P(p,p,q,q) \neq \nabla^2 P_F(p,p,q,q)$ and hence the Paasche and Fisher indexes do *not* approximate each other to the second order around an equal price and quantity point.
(d) Show that $\nabla^2 P_F(p,p,q,q) = \nabla^2 P_W(p,p,q,q)$ and hence the Fisher and Walsh indexes *do* approximate each other to the second order around an equal price and quantity point.
(e) Show that $\nabla^2 P_F(p,p,q,q) = \nabla^2 P_T(p,p,q,q)$ and hence the Fisher and Törnqvist indexes *do* approximate each other to the second order around an equal price and quantity point.

Comment: Problems 8 and 9 show that the Fisher, Törnqvist and Walsh price indexes all approximate each other to the second order around an equal price and quantity point and hence these indexes are likely to be numerically very close to each other provided prices and quantities do not change "too much" between the two periods under consideration. These problems also show that the Paasche and Laspeyres indexes do not approximate the other three indexes to the second order around an equal price and quantity point.

10. Let q and b be N dimensional column vectors, let a be a scalar and let C be an N by N symmetric matrix. Define the quadratic function of q, f(q) as follows:

(a) $f(q) \equiv a + b^Tq + (1/2)\, q^TCq$.

The second order Taylor series approximation to f(q) around the point q^0 is defined as:

(b) $F(q) \equiv f(q^0) + \nabla f(q^0)^T(q - q^0) + (1/2)(q - q^0)^T\, \nabla^2 f(q^0)(q - q^0)$.

(c) Show that f(q) = F(q) for all q when f(q) is defined by (a).

(d) Show that the following identity holds if f(q) is defined by (a):

(e) $f(q^1) - f(q^0) = (1/2)[\nabla f(q^0) + \nabla f(q^1)]^T[q^1 - q^0]$ for all q^0 and q^1.

(f) Consider the following two first order approximations:

(g) $f(q^1) - f(q^0) \approx \nabla f(q^0)^T[q^1 - q^0]$ and

(h) $f(q^0) - f(q^1) \approx \nabla f(q^1)^T[q^0 - q^1]$.

(i) Show that the right hand side of (e) is related to the two first order approximations in (g) and (h).

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