APPLIED ECONOMICS

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Chapter 10: Linear Programming

1. Introduction

The theory of linear programming provides a good introduction to the study of constrained maximization (and minimization) problems where some or all of the constraints are in the form of inequalities rather than equalities. Many models in economics can be expressed as inequality constrained optimization problems. A linear program is a special case of this general class of problems where both the objective function and the constraint functions are linear in the decision variables.

Linear programming problems are important for a number of reasons:

- Many general constrained optimization problems can be approximated by a linear program.
- The mathematical prerequisites for studying linear programming are minimal; only a knowledge of matrix algebra is required.
- Linear programming theory provides a good introduction to the theory of duality in nonlinear programming.

Linear programs appear in many economic contexts but the exact form of the problems varies across applications. We shall present several equivalent formulations of the basic linear programming problem in this introductory section. In the following section, we provide a geometric interpretation of a linear program (LP) in activities space. In subsequent sections, we will present George Dantzig's (1963) *simplex algorithm* for solving an LP.[1]

Our *first formulation* of the basic linear programming problem is:

(1) $\min_x \{c^Tx : Ax = b \,;\, x \geq 0_N \}$

where $c^T \equiv [c_1,c_2,\ldots,c_N]$ and $b^T \equiv [b_1,b_2,\ldots,b_M]$ are N and M dimensional vectors of constants, $A \equiv [a_{mn}]$ is an M by N matrix of constants and $x^T \equiv [x_1,x_2,\ldots,x_N]$ is a *nonnegative* N dimensional vector of decision or choice variables. Thus this first form for a linear programming problem is the problem of minimizing a linear function c^Tx in the vector of nonnegative variables $x \geq 0_N$ subject to M linear equality constraints, which are written in the form $Ax = b$.

[1] "Linear programming was developed by George B. Dantzig in 1947 as a technique for planning the diversified activities of the U.S. Air Force." Robert Dorfman, Paul A. Samuelson and Robert M. Solow (1958; 3). Dorfman, Samuelson and Solow go on to note that Dantzig's fundamental paper was circulated privately for several years and finally published as Dantzig (1951). A complete listing of Dantzig's early contributions to developing the theory of linear programming can be found in Dantzig (1963; 597-597).

Our *second formulation* of an LP is:

(2) $\max_{x_0, x} \{x_0 : x_0 + c^T x = 0 ; Ax = b ; x \geq 0_N\}$

where x_0 is a new scalar variable, which is defined by the equality constraint in (2); i.e., $x_0 \equiv -c^T x$ and so minimizing $c^T x$ is equivalent to maximizing x_0. It can be seen that the first and second formulations of an LP are completely equivalent.

Our *third formulation* of an LP is the following problem:

(3) $\max_x \{c^T x : Ax \leq b ; x \geq 0_N\}$.

The above problem can be transformed into a problem of the type defined by (2) as follows:

(4) $\max_{x_0, x, s} \{x_0 : x_0 - c^T x = 0 ; Ax + I_M s = b ; x \geq 0_N ; s \geq 0_M\}$

where I_M is the M by M identity matrix. Note that we have converted the inequality constraints $Ax \leq b$ into equality constraints by adding an M dimensional vector of nonnegative *slack variables* s to Ax.

Note that equality constraints such as $Ax = b$ can be converted into inequality constraints by writing $Ax = b$ as two sets of inequality constraints: $Ax \leq b$ and $-Ax \leq -b$. Note also that it does not matter whether we are maximizing the objective function $c^T x$ or minimizing $c^T x$ since a problem involving the maximization of $c^T x$ is equivalent to a problem involving the minimization of $-c^T x$.

In the above problems, the vectors of variables x and s were restricted to be nonnegative. This is not an essential restriction, since if a variable, x_1 say, is allowed to be unrestricted, it may be written in terms of two nonnegative variables, say s_1 and s_2, as $x_1 = s_1 - s_2$. However, in most economic problems, restricting the decision variables x to be nonnegative will be a reasonable assumption and it will not be necessary to resort to the $x_1 = s_1 - s_2$ construction.

Thus the essence of a *linear programming problem* is that the objective function (the function that we are maximizing or minimizing) is *linear* and the constraint functions are *linear equalities or inequalities*.

It turns out that our second formulation of an LP is the most convenient one for actually solving an LP (the simplex algorithm due to Dantzig) but the third formulation is the most convenient one for proving the duality theorem for linear programs.

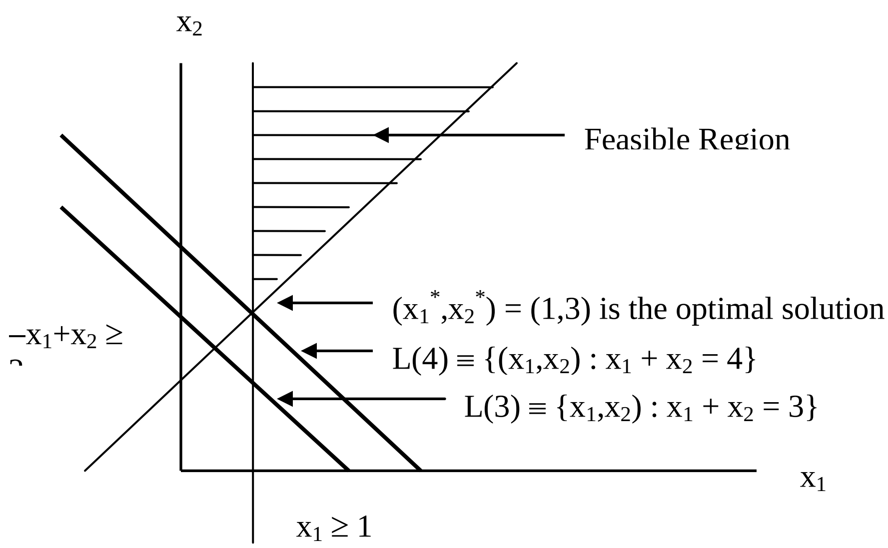
2. The Geometric Interpretation of a Linear Program in Activities Space

Consider the following LP:

(5) $\min_x \{c^Tx : Ax \geq b ; x \geq 0_2\}$

where $c^T \equiv [1,1]$, $b^T \equiv [1,2]$ and the two rows of A are $A_{1\bullet} \equiv [1,0]$ and $A_{2\bullet} \equiv [-1,1]$. Thus we wish to minimize $x_1 + x_2$ subject to the four inequality constraints: $x_1 \geq 1$; $-x_1 + x_2 \geq 2$; $x_1 \geq 0$; and $x_2 \geq 0$. We define the constraint set or *feasible region* in x space or *activities space* to be the set of x's which satisfy the constraints in (5). It is the shaded set in Figure 1 below. We also graph the level sets of the objective function $x_1 + x_2$; i.e., these are the family of straight lines indexed by k, $L(k) \equiv \{x : c^Tx = k\}$. These level sets form a system of parallel straight lines. Obviously, the optimal x solution to the LP problem will be that x which belongs to the feasible region and which also belongs to the lowest level set of the objective function.

Figure 1: A Linear Program in Activities Space



Note that the optimal solution to the LP lies on the boundary of the feasible region and that the optimal level set of the objective function is tangent to the feasible region. Finally, note that the optimal solution to the LP is at a *vertex* of the feasible region. This is typically what happens.

3. The Simplex Algorithm for Solving Linear Programs

In this section, we outline Dantzig's (1963; chapters 5-7) simplex algorithm for solving linear programming problems.[2] Dantzig's method is not only of interest from a computational point of view, but also from a theoretical point of view, since it enables us

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[2] Actually, we present a version of Dantzig's (1963; chapter 9) revised simplex algorithm.

to present an entirely algebraic proof of the duality theorem for linear programming problems as we shall see later. However, before the method can be explained, we need some additional definitions and results.

Definition 1: Any solution x^0 to the system of linear equations $Ax = b$ and inequalities $x \geq 0_N$ is called a *feasible solution* to the LP defined by (2).

We are considering the system of linear equations, $Ax = b$, which occurs in (2) above where A is an M by N matrix. In what follows, we assume that the number of equations M is less than the number of x_n variables, which is N. Hence A is a singular matrix.

Definition 2: Any nonnegative solution $x^0 \geq 0_N$ to the system of equations $Ax = b$ with at least N–M components equal to zero is called a *basic feasible solution* to the LP defined by (2).

Theorem 1; Carathéodory (1911; 200), Fenchel (1953; 37), Dantzig (1963; 113-114): If $Ax = b$, $x \geq 0_N$ has a feasible solution, then it has a basic feasible solution.

Proof: Assume that $M < N$ and that there exists an $x^* \equiv [x_1^*,\ldots,x_N^*]^T \geq 0_N$ such that $Ax^* = b$. If x^* has M or less nonzero components, then x^* is a basic feasible solution and we are done. Thus we assume that x^* has $K > M$ nonzero components. By reordering the variables if necessary, we assume that the first K components of x^* are the nonzero components. Now look at the following system of equations:

(6) $\sum_{k=1}^K A_{\bullet k}\lambda_k = 0_M$

where $A_{\bullet k}$ is the kth column of the A matrix. The matrix $[A_{\bullet 1}, A_{\bullet 2},\ldots, A_{\bullet K}]$ is M by K where $K > M$ and hence this matrix is singular or alternatively, the K columns in this matrix are linearly dependent. Hence, there is a nonzero $\lambda^* \equiv [\lambda_1^*,\ldots,\lambda_K^*]^T$ solution to (6). Without loss of generality, we can assume that at least one component of λ^* is positive.[3] We also have $x_k^* > 0$ for $k = 1,2,\ldots,K$. Hence the number α defined by (7) below must be positive:

(7) $\alpha \equiv \max_k \{\lambda_k^*/x_k^* : k = 1,2,\ldots,K\} > 0$.

Note that:

(8) $\alpha \geq \lambda_k^*/x_k^*$ or $\alpha x_k^* \geq \lambda_k^*$ for $k = 1,2,\ldots,K$.

Since $x^* \geq 0_N$ is a feasible solution for $Ax = b$, we have:

(9) $b = Ax^*$
$= \sum_{k=1}^K A_{\bullet k}x_k^*$
$= \sum_{k=1}^K A_{\bullet k}x_k^* - \alpha^{-1}\sum_{k=1}^K A_{\bullet k}\lambda_k^*$ using (6)

[3] If all components of λ^* were 0 or negative, then $-\lambda^*$ would satisfy (6), with all components 0 or positive.

$= \alpha^{-1}\sum_{k=1}^{K} A_{\bullet k}[\alpha x_k^* - \lambda_k^*]$.

But for all k which attain the max in (7), we will have $\alpha x_k^* - \lambda_k^*$ equal to 0. Thus $y_k^* \equiv \alpha^{-1}[\alpha x_k^* - \lambda_k^*]$ for k = 1,2,…,K is a feasible solution for Ax = b with at least one additional zero component compared to our initial feasible solution x^*.

We can continue this process of creating extra zero components as long as K, the number of nonzero variables, exceeds M, the number of equations. The procedure may stop when K = M, but at that stage, we have a basic feasible solution to Ax = b. Q.E.D.

In order to initiate the simplex algorithm, we need to start the algorithm with a basic feasible solution. But the above results say that given an arbitrary feasible solution, we can construct a basic feasible solution in a finite number of steps and thus initiate the simplex algorithm.

We can rewrite the basic linear program defined by (2) above as follows:

(10) $\max_{x_0, x_1 \geq 0, x_2 \geq 0, \ldots, x_N \geq 0} \{x_0 : e_0 x_0 + \sum_{n=1}^{N} A_{\bullet n}^* x_n = b^*\}$

where $c^T \equiv [c_1,\ldots,c_N]$, $A \equiv [A_{\bullet 1}, A_{\bullet 2},\ldots, A_{\bullet N}]$ is the original A matrix and

(11) $A_{\bullet n}^* \equiv \begin{bmatrix} c_n \\ A_{\bullet n} \end{bmatrix}$ for n = 1,…,N, $b^* \equiv \begin{bmatrix} 0 \\ b \end{bmatrix}$ and $e_0 \equiv \begin{bmatrix} 1 \\ 0_M \end{bmatrix}$.

In what follows, we also need to define the following M *unit vectors* of dimension M+1:

(12) $e_1^T \equiv [0, 1, 0,\ldots,0]$; $e_2^T \equiv [0, 0, 1,0,\ldots,0]$; … ; $e_M^T \equiv [0, 0, 0,\ldots,1]$.

Thus e_m is an M+1 dimensional vector consisting of 0 elements except entry m+1 is 1.

Assume that a feasible solution to (10) exists. Then by Theorem 1, a basic feasible solution exists. By relabelling variables if necessary, we can assume that the first M columns of A correspond to the nonzero variables in this basic feasible solution. Thus we have x_0^0, $x_1^0 \geq 0$, $x_2^0 \geq 0,\ldots,x_M^0 \geq 0$ such that:

(13) $e_0 x_0^0 + \sum_{m=1}^{M} A_{\bullet m}^* x_m^0 = b^*$.

At this point, we make two additional assumptions that will be relaxed at a later stage.

Nonsingular Basis Matrix Assumption: $[e_0, A_{\bullet 1}^*, A_{\bullet 2}^*,\ldots, A_{\bullet M}^*]^{-1} \equiv B^{-1}$ exists.

Nondegenerate Basis Matrix Assumption: $x_1^0 > 0$, $x_2^0 > 0,\ldots,x_M^0 > 0$; i.e., the x_m^0 which satisfy (13) for m = 1,2,…,M are all positive.

We define the initial M+1 by M+1 basis matrix B as $B \equiv [e_0, A_{\bullet 1}{}^*, A_{\bullet 2}{}^*,\ldots, A_{\bullet M}{}^*]$. By the nonsingular basis matrix assumption, we have the existence of B^{-1} and

(14) $B^{-1}B = B^{-1}[e_0, A_{\bullet 1}{}^*, A_{\bullet 2}{}^*,\ldots, A_{\bullet M}{}^*] = I_{M+1} = [e_0,e_1,\ldots,e_M]$.

Now premultiply both sides of the constraint equations in (10) by B^{-1}. In view of (14), the resulting system of M+1 equations is the following one:

(15) $$e_0x_0 + e_1x_1 + e_2x_2 + \ldots + e_Mx_M + B^{-1}A_{\bullet M+1}{}^*x_{M+1} + B^{-1}A_{\bullet M+2}{}^*x_{M+2} + \ldots + B^{-1}A_{\bullet N}{}^*x_N = B^{-1}b^*.$$

If we set $x_{M+1} = 0$, $x_{M+2} = 0$, … , $x_N = 0$ in (15), then we obtain our initial basic feasible solution to the LP problem:

(16) $x_m{}^0 = B_{m\bullet}{}^{-1}b^* > 0$ for $m = 1,2,\ldots,M$

where $B_{m\bullet}{}^{-1}$ is row m+1 of B^{-1} and the strict inequalities in (16) follow from the nondegenerate basis matrix assumption.

The question we now ask is: is our initial basic feasible solution to the LP (10) the optimal solution to the problem or not? To answer this question, look at equations (15) with $x_{M+1}, x_{M+2}, \ldots, x_N$ all equal to zero. If we try to increase any of these last N−M x_n variables from its initial 0 level, then obviously, we will increase x_0 only if $B_{0\bullet}{}^{-1}A_{\bullet n}{}^* < 0$ for some n > M. Thus we have the following:

(17) *Optimality Criterion*: If $B_{0\bullet}{}^{-1}A_{\bullet n}{}^* \geq 0$ for $n = M+1, M+2,\ldots,N$, then the current basis matrix and the corresponding solution (16) solve the LP (10). If $B_{0\bullet}{}^{-1}A_{\bullet n}{}^* > 0$ for $n = M+1, M+2,\ldots,N$, then the current basis matrix and the corresponding solution provide the unique solution to the LP (10).

Since $B_{0\bullet}{}^{-1}A_{\bullet n}{}^* = 0$ for $n = 1,2,\ldots,M$ using (14), the first part of the optimality criterion can be changed to:

(18) $B_{0\bullet}{}^{-1}A_{\bullet n}{}^* \geq 0$ for $n = 1, 2,\ldots,N$.

Suppose now that column s (not in the basis matrix) is such that

(19) $B_{0\bullet}{}^{-1}A_{\bullet s}{}^* < 0$.

In view of the nondegeneracy assumption, it can be seen that we can increase the value of our objective function x_0 and satisfy the inequality constraints in the LP (10) by increasing x_s from its initial 0 level. The question now is: which column should be dropped from the initial basis matrix B? To answer this question, look at the following system of equations:

(20) $e_0x_0 + e_1x_1 + e_2x_2 + \ldots + e_Mx_M + B^{-1}A_{\bullet s}{}^*x_s = B^{-1}b^*$.

Suppose that $B_{m\bullet}{}^{-1}A_{\bullet s}{}^* \leq 0$ for $m = 1,2,\ldots,M$. Then as we increase x_s from its initial 0 level, the last M components of $B^{-1}A_{\bullet s}{}^*$ are either 0 or negative and these nonpositive numbers can be offset by increasing $x_1, x_2, \ldots, x_M$ from their initial values $x_1^0, x_2^0, \ldots, x_M^0$. Thus we have a feasible solution to (10) which will allow the objective function to attain any large positive value. In this case, we obtain the following:

(21) *Unbounded Solution Criterion*: If we have a column s such that $B_{0\bullet}{}^{-1}A_{\bullet s}{}^* < 0$ and $B_{m\bullet}{}^{-1}A_{\bullet s}{}^* \leq 0$ for $m = 1,2,\ldots,M$, then we may increase x_s to $+\infty$ and obtain an unbounded solution to the LP (10).

Suppose now that (19) holds but that $B_{m\bullet}{}^{-1}A_{\bullet s}{}^* > 0$ for at least one index $m \geq 1$. In this case, as we *increase* x_s from its initial 0 level, in order to satisfy equations (20), we must *decrease* x_m from its initial positive level $x_m^0 = B_{m\bullet}{}^{-1}b^* > 0$. Thus an upper bound to the amount that we can increase x_s before we violate the nonnegativity constraints $x \geq 0_N$ is $B_{m\bullet}{}^{-1}b^*/B_{m\bullet}{}^{-1}A_{\bullet s}{}^*$. Now take the minimum of all such upper bounds over all indexes m such that $B_{m\bullet}{}^{-1}A_{\bullet s}{}^* > 0$:

(22) $x_s^* \equiv \min_m \{B_{m\bullet}{}^{-1}b^*/B_{m\bullet}{}^{-1}A_{\bullet s}{}^* : m \geq 1 \text{ and } m \text{ is such that } B_{m\bullet}{}^{-1}A_{\bullet s}{}^* > 0\}$.

The algebra in the above paragraph can be summarized as the following:

(23) *Dropping Criterion*: If $B_{0\bullet}{}^{-1}A_{\bullet s}{}^* < 0$ for some column index s not in the initial basis matrix and $B_{m\bullet}{}^{-1}A_{\bullet s}{}^* > 0$ for at least one $m \geq 1$, then add column s to the basis matrix and drop any column r such that column r attains the minimum in (22); i.e., r is such that $r \geq 1$ and

(24) $$B_{r\bullet}{}^{-1}b^*/B_{r\bullet}{}^{-1}A_{\bullet s}{}^* = \min_m \{B_{m\bullet}{}^{-1}b^*/B_{m\bullet}{}^{-1}A_{\bullet s}{}^* : m \geq 1 \text{ and } m \text{ is such that } B_{m\bullet}{}^{-1}A_{\bullet s}{}^* > 0\}.$$

Using the nondegeneracy assumption, it can be seen that x_s^* defined by (22) is positive and introducing column s into the basis matrix leads to a strict increase in the objective function for the LP (10).

If the minimum in (22) or (24) is attained by a unique column r, then it can be seen that the new basis matrix is uniquely determined. It is also possible to show that the new basis matrix will also satisfy the nondegeneracy assumption.

The above criteria form the core of an effective algorithm for solving the LP (10).[4] Obviously, a finite but tremendously inefficient algorithm for solving (10) would involve solving the following system of M equations in M unknowns:

[4] There are a few gaps but we will fill them in later.

(25) $\sum_{j=1}^{M} A_{n_j} x_{n_j} = b$

for all possible choices of M of the N columns of the A matrix, checking to see if the resulting x_{n_j} are nonnegative, evaluating the objective function at these basic feasible solutions and then the nonnegative solution that gave the lowest objective function $\sum_{j=1}^{M} c_{n_j} x_{n_j}$ would be picked. What the simplex algorithm does is to search through basis matrices in such a way that we always increase our objective function.[5] Experience with the simplex algorithm has shown that an optimal basis matrix is generally reached in the order of M to 3M iterations of the algorithm. *Thus the simplex algorithm is tremendously more efficient than simply searching through all of the basic feasible solutions for the linear program.*

4. An Example of the Simplex Algorithm

Let us use the simplex algorithm to solve the LP that was graphed in section 2 above. Rewriting the problem in the form (10) leads to the following problem:

(26) $\max_{x_0, x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0} \{x_0 : e_0 x_0 + A_{\bullet 1}{}^* x_1 + A_{\bullet 2}{}^* x_2 + A_{\bullet 3}{}^* s_1 + A_{\bullet 4}{}^* s_2 = b^*\}$

where $e_0 \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A_{\bullet 1}{}^* \equiv \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $A_{\bullet 2}{}^* \equiv \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $A_{\bullet 3}{}^* \equiv \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $A_{\bullet 4}{}^* \equiv \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ and $b^* \equiv \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Define the initial basis matrix to be

(27) $B \equiv [e_0, A_{\bullet 1}{}^*, A_{\bullet 2}{}^*] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & c^T \\ 0_2 & A \end{bmatrix}$

where A is a two by two matrix. Using determinants, if $A \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we know that the formula for A^{-1} is given by (provided that $a_{11}a_{22} - a_{12}a_{21} \neq 0$):

(28) $A^{-1} = \dfrac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ for our example.

Using (28), we can readily calculate B^{-1} as follows:

[5] In general, we either increase the objective function or leave it unchanged at each iteration of the simplex algorithm or we find an unbounded solution.

$$(29)\ B^{-1} = \begin{bmatrix} 1 & c^T \\ 0_2 & A \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -c^T A^{-1} \\ 0_2 & A^{-1} \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check for feasibility:

$$(30)\ \begin{bmatrix} x_0^* \\ x_1^* \\ x_2^* \end{bmatrix} = B^{-1}b^* = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}.$$

To check that we have a basic feasible solution that satisfies the nonnegativity constraints, we need to check that $x_1^* \geq 0$ and $x_2^* \geq 0$. Equations (30) show that these constraints are indeed satisfied.

Check for optimality:

We need to check that the inner product of the first row of B^{-1} with each column not in the basis matrix is positive or zero.

$$(31)\ B_{0\bullet}{}^{-1}A_{\bullet 3}{}^* = [1,-2,-1]\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = 2 > 0\ ; \quad B_{0\bullet}{}^{-1}A_{\bullet 4}{}^* = [1,-2,-1]\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = 1 > 0\ .$$

The optimality conditions are satisfied and thus $x_1^* = 1$ and $x_2^* = 3$ solve the LP.

In the above example, it was easy to find a starting basic feasible solution so that we could start the simplex algorithm. However, finding a starting basic feasible solution may not be all that easy in a complicated problem. In fact, it may be the case that a given LP may not even have a feasible solution. Thus in the following section, we discuss Dantzig's Phase I procedure for finding a starting basic feasible solution.

5. Finding a Starting Basic Feasible Solution

Consider again the first form (1) of a linear programming problem, $\min_x \{c^Tx : Ax = b\ ;\ x \geq 0_N\}$. Now multiply both sides of each constraint equation through by -1 if necessary so that the right hand side vector b is nonnegative; i.e., so that $b \geq 0_M$.

Now consider the following *Phase I* LP where we have introduced a vector of *artificial variables* s:

$$(32)\ \min_{x,s} \{1_M{}^Ts : Ax + I_Ms = b\ ;\ x \geq 0_N\ ;\ s \geq 0_M\}$$

where 1_M is an M dimensional column vector of ones and I_M is an M by M identity matrix. A basic feasible starting solution to (32) is $s = b \geq 0_M$ and $x = 0_N$. Note that this starting basis matrix is nonsingular. Now use the simplex algorithm to solve (32).[6]

Several cases can occur.

Case 1: The minimum for (32) is greater than 0.

In this case, there is no feasible solution for the original LP and we can stop at this point.

Case 2: The minimum for (32) equals 0.

In this case, the original LP has feasible solutions. However, we need to consider two subcases.

Case 2A: The minimum for (32) is 0 and no columns corresponding to the artificial variables are in the final basis matrix.

In this case, we have an $x^0 \geq 0_N$ such that $Ax^0 = b$ and at most M components of x^0 are nonzero. Thus we have a starting basic feasible solution for the original LP (1).

Case 2B: The minimum for (32) is 0 and one or more columns corresponding to the artificial variables are in the final basis matrix.

In this case, we have a starting basic feasible solution for the following linear programming problem, which turns out to be equivalent to the original problem (1):

(33) $\max_{x_0, x \geq 0_N, s \geq 0_M} \{x_0 : x_0 + c^T x = 0 \; ; \; Ax + I_M s = b \; ; \; 1_M{}^T s = 0\}$.

Note that (33) is the same as (1) except that the vector of artificial variables s has been inserted into the constraint equations $Ax = b$ and the extra constraint $1_M{}^T s = 0$ has been added to the problem, along with the nonnegativity restrictions on s, $s \geq 0_M$. However, note that the constraints $1_M{}^T s = 0$ and $s \geq 0_M$ together imply that $s = 0_M$. Thus the artificial variables will be kept at 0 levels for all iterations of the simplex algorithm applied to (33) and so solving (33) will also solve (1).

Thus solving the Phase I linear programming problem (32) will tell us whether a feasible solution to the original problem (1) exists or not and if a feasible solution to the original problem does exist, the solution to the Phase I problem will give us a starting basic feasible solution to solve the original linear programming problem (1), which is called the *Phase II problem* by Dantzig (1963).

[6] Note that the unbounded solution case cannot occur for this Phase I problem because the objective function is bounded from below by 0.

We conclude this section by solving the Phase I problem for the example in the previous section. Our Phase I problem is the following one:

(34) $\max_{x_0, x_1 \ge 0, x_2 \ge 0, s_1 \ge 0, s_2 \ge 0} \{x_0 :$
$$e_0x_0 + A_{\bullet 1}{}^*x_1 + A_{\bullet 2}{}^*x_2 + A_{\bullet 3}{}^*x_3 + A_{\bullet 4}{}^*x_4 + A_{\bullet 5}{}^*s_1 + A_{\bullet 6}{}^*s_2 = b^*\}$$

where $e_0 \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A_{\bullet 1}{}^* \equiv \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $A_{\bullet 2}{}^* \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $A_{\bullet 3}{}^* \equiv \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $A_{\bullet 4}{}^* \equiv \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, $A_{\bullet 5}{}^* \equiv \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$,
$A_{\bullet 6}{}^* \equiv \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $b^* \equiv \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Our starting basis matrix is B defined as follows:

$$(35)\quad B \equiv [e_0, A_{\bullet 5}{}^*, A_{\bullet 6}{}^*] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and so } B^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Feasibility Check:

$$(36)\quad \begin{bmatrix} x_0^{(0)} \\ s_1^{(0)} \\ s_2^{(0)} \end{bmatrix} = B^{-1}b^* = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}.$$

Since $s_1^{(0)} \ge 0$ and $s_2^{(0)} \ge 0$, the feasibility conditions are satisfied.

Optimality Check:

(37) $B_{0\bullet}{}^{-1}A_{\bullet 1}{}^* = 0$; OK
$B_{0\bullet}{}^{-1}A_{\bullet 2}{}^* = -1$; Can introduce $A_{\bullet 2}{}^*$ into the basis matrix
$B_{0\bullet}{}^{-1}A_{\bullet 3}{}^* = 1$; OK
$B_{0\bullet}{}^{-1}A_{\bullet 4}{}^* = 1$; OK
$B_{0\bullet}{}^{-1}A_{\bullet 5}{}^* = 0$; OK (must equal 0 since $A_{\bullet 5}{}^*$ is in the initial basis matrix)
$B_{0\bullet}{}^{-1}A_{\bullet 6}{}^* = 0$; OK (must equal 0 since $A_{\bullet 6}{}^*$ is in the initial basis matrix).

There is only one column $A_{\bullet s}{}^*$ such that $B_{0\bullet}{}^{-1}A_{\bullet s}{}^* < 0$, namely $A_{\bullet 2}{}^*$. If there were two or more columns with $B_{0\bullet}{}^{-1}A_{\bullet s}{}^* < 0$, then pick the s such that $B_{0\bullet}{}^{-1}A_{\bullet s}{}^*$ is the most negative. Thus column $A_{\bullet 2}{}^*$ will enter the new basis matrix but which column will leave? We need to calculate:

Dropping Criterion:

(38) $x_2^{(1)} \equiv \min_m \{B_{m\bullet}^{-1}b^*/B_{m\bullet}^{-1}A_{\bullet 2}^* : m \geq 1$ and m is such that $B_{m\bullet}^{-1}A_{\bullet 2}^* > 0\}$
$= B_{2\bullet}^{-1}b^*/B_{2\bullet}^{-1}A_{\bullet 2}^*$
$= 2/1$
$= 2$

since $B_{1\bullet}^{-1}A_{\bullet 2}^* = 0$, $B_{2\bullet}^{-1}A_{\bullet 2}^* = 1$ and $B_{2\bullet}^{-1}b^* = 2$. In other words, the minimum in (38) is attained for m = 2 and the corresponding column of the original B matrix leaves the basis matrix; i.e., the last column $A_{\bullet 6}^*$ leaves. Thus the new basis matrix becomes:

(39) $B^{(1)} \equiv [e_0, A_{\bullet 5}^*, A_{\bullet 2}^*] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and so $[B^{(1)}]^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Feasibility Check:

(40) $\begin{bmatrix} x_0^{(1)} \\ s_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = B^{(1)-1}b^* = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Since $s_1^{(1)} \geq 0$ and $x_2^{(1)} \geq 0$, the feasibility conditions are satisfied.

Optimality Check:

(41) $B_{0\bullet}^{(1)-1}A_{\bullet 1}^* = -1$; Can introduce $A_{\bullet 1}^*$ into the basis matrix
$B_{0\bullet}^{(1)-1}A_{\bullet 2}^* = 0$; OK (must equal 0 since $A_{\bullet 2}^*$ is in the basis matrix)
$B_{0\bullet}^{(1)-1}A_{\bullet 3}^* = 1$; OK
$B_{0\bullet}^{(1)-1}A_{\bullet 4}^* = 0$; OK
$B_{0\bullet}^{(1)-1}A_{\bullet 5}^* = 0$; OK (must equal 0 since $A_{\bullet 5}^*$ is in the basis matrix)
$B_{0\bullet}^{(1)-1}A_{\bullet 6}^* = 1$; OK.

There is only one column $A_{\bullet s}^*$ such that $B_{0\bullet}^{(1)-1}A_{\bullet s}^* < 0$, namely $A_{\bullet 1}^*$. Thus column $A_{\bullet 1}^*$ will enter the new basis matrix but which column will leave? We need to calculate:

Dropping Criterion:

(42) $x_1^{(2)} \equiv \min_m \{B_{m\bullet}^{(1)-1}b^*/B_{m\bullet}^{(1)-1}A_{\bullet 1}^* : m \geq 1$ and m is such that $B_{m\bullet}^{(1)-1}A_{\bullet 1}^* > 0\}$
$= B_{1\bullet}^{(1)-1}b^*/B_{1\bullet}^{(1)-1}A_{\bullet 1}^*$
$= 1/1$
$= 1$

since $B_{1\bullet}^{(1)-1}A_{\bullet 1}^* = 1$, $B_{2\bullet}^{(1)-1}A_{\bullet 1}^* = -1$ and $B_{1\bullet}^{(1)-1}b^* = 1$. In other words, the minimum in (42) is attained for m = 1 and the corresponding column of the $B^{(1)}$ matrix leaves the basis matrix; i.e., the second column $A_{\bullet 5}^*$ leaves and $A_{\bullet 1}^*$ enters. Thus the new basis matrix becomes:

$$(43)\ B^{(2)} \equiv \equiv [e_0, A_{\bullet 1}^*, A_{\bullet 2}^*] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \text{ and so } [B^{(2)}]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Feasibility Check:

$$(44)\ \begin{bmatrix} x_0^{(2)} \\ x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = B^{(2)-1}b^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

Since $x_1^{(2)} \geq 0$ and $x_2^{(2)} \geq 0$, the feasibility conditions are satisfied.

Optimality Check:

(45) $B_{0\bullet}^{(2)-1}A_{\bullet 1}^* = 0$; OK (must equal 0 since $A_{\bullet 2}^*$ is in the basis matrix)
$B_{0\bullet}^{(2)-1}A_{\bullet 2}^* = 0$; OK (must equal 0 since $A_{\bullet 2}^*$ is in the basis matrix)
$B_{0\bullet}^{(2)-1}A_{\bullet 3}^* = 0$; OK
$B_{0\bullet}^{(2)-1}A_{\bullet 4}^* = 0$; OK
$B_{0\bullet}^{(2)-1}A_{\bullet 5}^* = 1$; OK
$B_{0\bullet}^{(2)-1}A_{\bullet 6}^* = 1$; OK.

Thus the optimality conditions for solving the Phase I problem are satisfied with $A_{\bullet 1}^*$ and $A_{\bullet 2}^*$ the final columns in the optimal basis matrix. Note that the objective function x_0 increased at each iteration of the Phase I problem and we ended up with a basis matrix that we could use to start the Phase II problem. Note also that the basis matrix was invertible at each stage of the Phase I algorithm. In the following section, we will show that the basis matrix is always invertible, provided that we start with a basis matrix that is invertible, which we can always do. Thus our nonsingular basis matrix assumption made in section 2 is not a restrictive assumption.

6. Nonsingularity of the Basis Matrix in the Simplex Algorithm

Suppose that at some iteration of the simplex algorithm, the basis matrix is $B \equiv [e_0, A_{\bullet 1}^*, \ldots, A_{\bullet M}^*]$ is nonsingular; i.e., B^{-1} exists. If $B_{\bullet 0}^{-1}A_{\bullet n}^* \geq 0$ for n = 1,2,…,N, then we have an optimal solution and the algorithm stops. If $B_{\bullet 0}^{-1}A_{\bullet s}^* < 0$ for some s with $B_{\bullet m}^{-1}A_{\bullet s}^* \leq 0$ for m = 1,2,…,M, then we have an unbounded optimal solution and the algorithm stops. Now consider the remaining case; i.e., suppose that there exists a column index s such that

(46) $B_{\bullet 0}{}^{-1}A_{\bullet s}{}^{*} < 0$ and $B_{\bullet m}{}^{-1}A_{\bullet s}{}^{*} > 0$ for at least one $m \geq 1$.

Thus $A_{\bullet s}{}^{*}$ enters the basis matrix and according to the dropping criterion, $A_{\bullet r}{}^{*}$ leaves the basis matrix where r is such that $B_{r\bullet}{}^{-1}b^{*}/B_{r\bullet}{}^{-1}A_{\bullet s}{}^{*} = \min_m \{B_{m\bullet}{}^{-1}b^{*}/B_{m\bullet}{}^{-1}A_{\bullet s}{}^{*} : m \geq 1$ and m is such that $B_{m\bullet}{}^{-1}A_{\bullet s}{}^{*} > 0\}$. Thus we have:

(47) $B_{r\bullet}{}^{-1}A_{\bullet s}{}^{*} > 0$.

Since the initial basis matrix $B \equiv [e_0, A_{\bullet 1}{}^{*},\ldots, A_{\bullet M}{}^{*}]$ is nonsingular, it must be possible to express the entering column $A_{\bullet s}{}^{*}$ as a linear combination of the columns of B; i.e., there exists $\alpha \equiv [\alpha_0, \alpha_1, \ldots , \alpha_M]^T$ such that

(48) $A_{\bullet s}{}^{*} = B\alpha$ or $\alpha = B^{-1}A_{\bullet s}{}^{*}$.

Using (47) and (48), we have

(49) $\alpha_r = B_{r\bullet}{}^{-1}A_{\bullet s}{}^{*} > 0$.

Now form the new basis matrix $B^{(1)}$ defined as follows:

(50) $B^{(1)} \equiv [e_0, A_{\bullet 1}{}^{*}, A_{\bullet 2}{}^{*}, \ldots , A_{\bullet r-1}{}^{*}, A_{\bullet s}{}^{*}, A_{\bullet r+1}{}^{*}, A_{\bullet r+2}{}^{*}, \ldots , A_{\bullet M}{}^{*}]$.

It is easy to see that (48) and $\alpha_r > 0$ imply that $A_{\bullet r}{}^{*}$ can be expressed as a linear combination of the columns of $B^{(1)}$:

(51) $A_{\bullet r}{}^{*} = B^{(1)}\beta$; $\beta \equiv [\beta_0, \beta_1, \ldots ,\beta_M]^T$

where

(52) $\beta_r = 1/\alpha_r = 1/B_{r\bullet}{}^{-1}A_{\bullet s}{}^{*}$ and $\beta_m = -\alpha_m/\alpha_r = -B_{m\bullet}{}^{-1}A_{\bullet s}{}^{*}/B_{r\bullet}{}^{-1}A_{\bullet s}{}^{*}$ for $m \neq r$.

Comparing B with $B^{(1)}$, it can be seen that

(53)
$$\begin{aligned} B &\equiv [e_0, A_{\bullet 1}{}^{*}, \ldots , A_{\bullet r-1}{}^{*}, A_{\bullet r}{}^{*}, A_{\bullet r+1}{}^{*}, \ldots , A_{\bullet M}{}^{*}] \\ &= [e_0, A_{\bullet 1}{}^{*}, \ldots , A_{\bullet r-1}{}^{*}, A_{\bullet s}{}^{*}, A_{\bullet r+1}{}^{*}, \ldots , A_{\bullet M}{}^{*}][e_0, e_1,\ldots, e_{r-1}, \beta, e_{r+1},\ldots, e_M] \\ &= B^{(1)}E \end{aligned}$$

where $E \equiv [e_0, e_1,\ldots, e_{r-1}, \beta, e_{r+1},\ldots, e_M]$ and the components of β are defined by (52). Note that since $\beta_r = 1/\alpha_r > 0$, the determinant of E is equal to $\beta_r > 0$ and hence E^{-1} exists. Thus $B^{(1)} = BE^{-1}$ and

(54) $B^{(1)-1} = EB^{-1}$

and hence the inverse of the new basis matrix exists. Thus we have proven:

Theorem 2; Dantzig (1963): The basis matrix is nonsingular at each iteration of the simplex algorithm.

Note that the inverse of the new basis matrix $B^{(1)}$ is easy to compute using (54): we need only premultiply the inverse of the old basis matrix B^{-1} by the matrix $E \equiv [e_0, e_1,\ldots, e_{r-1}, \beta, e_{r+1},\ldots, e_M]$, which differs from the identity matrix in only one column.[7]

There is one remaining detail to be filled in to complete our discussion of Dantzig's simplex algorithm for solving a linear program.

7. The Degeneracy Problem

Recall the nondegeneracy assumption made in section 3. Recall that degeneracy occurs when the dropping criterion does not yield a *unique* column to be dropped; i.e., when we compute the minimum in (24), the minimum is attained by more than one column index m. When this situation occurs, then it can happen at the next stage of the simplex algorithm that we do not get a drop in the objective function and two or more columns cycle in and out of the basis matrix without causing a drop in the objective function. Thus there is the theoretical possibility that the simplex algorithm could enter an infinite cycle and the algorithm might fail to converge.

It seems from an a priori point of view that it would be extremely unlikely that this cycling phenomenon could occur. However, Dantzig reports that the degeneracy phenomenon is common:

"It is common experience, based on the solutions of thousands of practical linear programming problems by the simplex method, that nearly every problem at some stage of the process is degenerate." George Dantzig (1963; 231).

When degeneracy occurs, it is conceivable that we could have a sequence of basis matrices where the objective function does not change such that the initial basis matrix reappears at some stage, which could then lead to an endless cycle of the same sequence of basis matrices. There are at least two artificially constructed examples of linear programming problems where this cycling phenomenon occurred; see Hoffman (1953) and Beale (1955). Dantzig (1963; 228-230) presents these two examples but he comments on the phenomenon as follows:

"To date, there has not been one single case of circling, except in the specially constructed examples of Hoffman and Beale. Apparently, circling is a very rare phenomenon in practice. For this reason, most instruction codes for electronic computers use no special device for perturbing the problem to avoid degeneracy and the possibility of circling." George Dantzig (1963; 231).

[7] See (52) for the definition of the components of the β vector.

It turns out that cycling or circling can be avoided if we try dropping a different sequence of columns after going through a cycle of column changes where the objective function did not change the first time around. Dantzig (1963) develops various rules for choosing which column to drop which work and he has references to rules developed by others. For our purposes, the exact form of the dropping rule is not important; all we need to know is that a dropping rule exists such that the simplex algorithm will terminate in a finite number of iterations, even when nondegeneracy is not assumed.

We turn now to a topic of great economic interest.

8. The Dual Linear Program

In this section, we use the third formulation for a linear programming problem as our starting point. Thus we suppose that we are given as data an M by N matrix A, an N dimensional vector c and an M dimensional vector b. Then the *primal linear programming problem* is defined as the problem of maximizing the linear function $c^T x$ with respect to the vector of primal decision variables x subject to the inequality constraints $Ax \le b$ and the nonnegativity constraints $x \ge 0_N$:

(55) *Primal Problem*: $\max_x \{c^T x : Ax \le b\ ;\ x \ge 0_N\}$.

The *dual* to the above problem switches the roles of b and c, changes a maximization problem into a minimization problem, switches from a vector of primal variables x that operate on the columns of A to a vector of dual variables y which operate on the rows of A and writes the inequality constraints in the opposite direction:

(56) *Dual Problem*: $\min_y \{y^T b : y^T A \ge c^T\ ;\ y \ge 0_M\}$.

Note that the dual problem is also a linear programming problem. The primal and dual problems are related by the following theorem:

Theorem 3; von Neumann (1947), Dantzig (1963; chapter 6): If the primal problem has a finite optimal solution x^* say, then the dual problem also has a finite optimal solution y^* say and moreover:

(57) $c^T x^* = y^{*T} b$;

i.e., the optimal values of the primal and dual objective functions are equal.

Proof: We first put the primal problem into the standard simplex algorithm format. Define $x_0 \equiv c^T x$. After adding the vector of slack variables s to Ax, we find that the primal problem (55) is equivalent to the following problem:

(58) $\max_{x_0, x \ge 0_N, s \ge 0_M} \{x_0 : \begin{bmatrix} 1 \\ 0_M \end{bmatrix} x_0 + \begin{bmatrix} -c^T & 0_M^T \\ A & I_M \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}\}$.

Apply Phase I of the simplex algorithm to the LP defined by (58). Since we assume that the primal problem has a finite optimal solution, the optimal objective function for the Phase I problem will attain its lower bound of 0. Thus we will have:

$$(59)\ 0 = \max_{x_0, x\geq 0_N, s\geq 0_M, z\geq 0_M} \{x_0 : \begin{bmatrix} 1 \\ 0_M \end{bmatrix} x_0 + \begin{bmatrix} 0_N^T & 0_M^T & 1_M^T \\ A & I_M & -I_M \end{bmatrix} \begin{bmatrix} x \\ s \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}\}$$

where we have added the vector of artificial variables $z \geq 0_M$ to the constraints of (58). There are two cases that can occur at this point.

Case 1: In this case, the Phase I problem yields a nonsingular initial basis matrix for the Phase II problem (58), which does not have any columns in it that correspond to the artificial variables z. Thus in this case, we may apply the simplex algorithm to (58) without including any artificial variables and since by assumption, there is a finite optimal solution to (58), the simplex algorithm will terminate and there will exist an M+1 by M+1 nonsingular basis matrix B (where the first column of B is the unit vector e_0) such that the following *optimality conditions* hold:

$$(60)\ B_{0\bullet}^{-1} \begin{bmatrix} -c^T & 0_M^T \\ A & I_M \end{bmatrix} \geq [0_N^T, 0_M^T] \quad \text{and } B_{0\bullet}^{-1}e_0 = 1$$

where $B_{0\bullet}^{-1} \equiv [1, y^{*T}]$ say, is the first row of B^{-1}.[8] Thus the optimality conditions (60) imply that y^* satisfies the following inequalities:

$$(61)\ y^{*T}A \geq c^T \text{ and } y^{*T}I_M \geq 0_M^T.$$

Thus y^* is a feasible solution for the dual problem (56).

At this point, we need the following *auxiliary result*: if x^* and y^* are feasible solutions for the primal and dual problems respectively, then we have the following bounds for the objective functions for the two problems:

$$(62)\ c^Tx^* \leq y^{*T}b\ ;$$

i.e., the dual objective function evaluated at a feasible solution to the dual, $y^{*T}b$, is an upper bound for the primal objective function evaluated at any feasible solution and the primal objective function evaluated at a feasible solution for the primal, c^Tx^*, is a lower bound for the dual objective function evaluated at any feasible solution. To prove (62), note that x^* feasible for the primal problem and y^* feasible for the dual problem means that x^* and y^* satisfy the following inequalities:

[8] We can deduce that the first component of $B_{0\bullet}^{-1}$ is 1 using $B_{0\bullet}^{-1}e_0 = 1$ since e_0 is the first column of B.

(63) $x^* \geq 0_N$; $Ax^* \leq b$; $y^* \geq 0_M$; $y^{*T}A \geq c^T$.

Thus

$$
\begin{aligned}
(64)\ c^Tx^* &\leq c^Tx^* + y^{*T}(b - Ax^*) && \text{since } y^* \geq 0_M \text{ and } Ax^* \leq b \\
&= y^{*T}b + (c^T - y^{*T}A)x^* && \text{rearranging terms} \\
&\leq y^{*T}b && \text{since } x^* \geq 0_N \text{ and } y^{*T}A \geq c^T .
\end{aligned}
$$

Now (61) shows that the y^* defined by $B_{0\bullet}^{-1} \equiv [1, y^{*T}]$ is a feasible solution for the dual. In view of (64), we need only show that $y^{*T}b = c^Tx^*$ where x^* solves the primal, and we will have y^* as an optimal solution to the dual problem.

Substitute the optimal Phase II solution to (58), x_0^*, x^* and s^* into the constraints listed in (58) and then premultiply both sides of this matrix equation by $B_{0\bullet}^{-1} \equiv [1, y^{*T}]$. We obtain the following equation:

$$
(65)\ [1, y^{*T}]\begin{bmatrix} 1 \\ 0_M \end{bmatrix} x_0^* + [1, y^{*T}]\begin{bmatrix} -c^T & 0_M^T \\ A & I_M \end{bmatrix}\begin{bmatrix} x^* \\ s^* \end{bmatrix} = [1, y^{*T}]\begin{bmatrix} 0 \\ b \end{bmatrix}.
$$

Obviously, $[1, y^{*T}][1, 0_M{}^T]^T = 1$ and so the first term on the left hand side of (65) is simply equal to x_0^*. It turns out that the next set of terms on the left hand side of (65) is equal to 0 since if a particular column in the matrix $\begin{bmatrix} -c^T & 0_M^T \\ A & I_M \end{bmatrix}$ *is* in the optimal basis matrix, the inner product of this column with the first row of B^{-1}, $[1, y^{*T}]$, will be 0[9] and if a particular column in this matrix *is not* in the optimal basis matrix, then the corresponding x_n^* or s_m^* is zero and again the term will be 0. The term on the right hand side of (65) is equal to $y^{*T}b$. Thus (65) simplifies to:

(66) $x_0^* = y^{*T}b$.

Since $x_0^* = c^Tx^*$, we have $c^Tx^* = y^{*T}b$ and thus y^* is indeed an optimal solution to the dual problem.

Case 2: In this case, the solution to the Phase I problem contains one or more columns corresponding to the artificial variables z. In this case, the Phase I solution gives us a starting basic feasible solution to the following problem, which is equivalent to the primal problem (55):

---

[9] This follows from $B^{-1}B = I_{M+1}$.

$$(67)\ \max_{x_0, x\geq 0_N, s\geq 0_M, z\geq 0_M} \{x_0 : \begin{bmatrix} 1 \\ 0_M \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} -c^T \\ A \\ 0_N^T \end{bmatrix} x + \begin{bmatrix} 0_M^T \\ I_M \\ 0_M^T \end{bmatrix} s + \begin{bmatrix} 0_M^T \\ -I_M \\ 1_M^T \end{bmatrix} z = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \}.$$

We have set up the artificial variables in a slightly different way than was suggested in section 5 above. Since we already have added $+ I_M s$ to Ax, we added $- I_M z$ as the artificial variables. Between the s and z variables, we can obviously find a Phase I starting basic feasible solution to the constraints $Ax + I_M s - I_M z = b$, no matter what the signs are for the components of the b vector.

Now apply the Phase II simplex algorithm to (67). The constraints $1_M^T z = 0$ and $z \geq 0_M$ will force the z variables to be kept at 0 levels at all iterations of the Phase II algorithm. Since we assumed that a finite optimal solution to the original primal problem existed, the Phase II simplex algorithm will eventually terminate and there will exist an M+2 by M+2 basis matrix C such that the following optimality conditions will hold:

$$(68)\ C_{0\bullet}^{-1} \begin{bmatrix} -c^T & 0_M^T & 0_M^T \\ A & I_M & -I_M \\ 0_N^T & 0_M^T & 1_M^T \end{bmatrix} \geq [0_N^T, 0_M^T, 0_M^T]$$

where $C_{0\bullet}^{-1} \equiv [1, y^{*T}, y_0^*]$ is the first row of C^{-1}. The optimal solution to (67), x_0^*, x^*, s^*, z^* will satisfy the following relations:

$$(69)\ x_0^* = c^T x^* ;\ x^* \geq 0_N ;\ s^* \geq 0_M ;\ z^* = 0_M.$$

The first two sets of inequalities in (68) imply that y^* satisfies:

$$(70)\ y^{*T}A \geq c^T \ \text{ and } y^* \geq 0_M$$

which implies that y^* is feasible for the dual problem. Now evaluate the constraints in (67) at the optimal Phase II solution, x_0^*, x^*, s^*, z^* and premultiply the constraint equations through by $C_{0\bullet}^{-1} \equiv [1, y^{*T}, y_0^*]$. We find that as in Case 1:

$$(71)\ x_0^* + 0 = y^{*T}b$$

Where the second term is 0 because columns not in the final Phase II basis matrix are multiplied by 0 while the columns in the final basis matrix have a 0 inner product with the first row of the inverse of the basis matrix, $C_{0\bullet}^{-1} \equiv [1, y^{*T}, y_0^*]$. Since x^* is feasible (and optimal) for the primal problem and y^* is feasible for the dual problem, (69) and (71) and the auxiliary result imply that y^* is a solution to the dual with $y^{*T}b = c^T x^*$. Q.E.D.

Corollary: If the primal problem has a feasible solution and the dual problem has a feasible solution, then both the primal and dual problems have finite optimal solutions and the values of the optimal objective functions coincide.

Proof: The auxiliary result (62) shows that if the primal and dual both have feasible solutions, then the primal objective function is bounded from above. Hence the unbounded solution case for the primal cannot occur. Now repeat the proof of Theorem 3. Q.E.D.

Note that the above proof of the duality theorem for linear programs is entirely algebraic and rests on the mechanics of the simplex algorithm for solving linear programs. The proof of the above theorem is essentially due to Dantzig (1963).

A further implication of the above duality theorem is that if the primal solution has an unbounded optimal solution, then the dual cannot have a feasible solution.

The duality theorem for linear programs has many economic applications as will be seen by studying the problems at the end of the chapter.

We conclude this section by proving another result that will be helpful in providing an economic interpretation for the dual variables.

Theorem 4: *Basis Theorem for Linear Programs*; Dantzig (1963; 121): Suppose that a nonsingular basis matrix B is *optimal* for the problem:

(72) $\max_{x_0, x} \{x_0 : e_0 x_0 + A^* x = b^* \, ; \, x \geq 0_N\}$.

Suppose further that this basis matrix B generates a *feasible* solution for the following problem where the right hand side vector b^* has been replaced by b^1:

(73) $\max_{x_0, x} \{x_0 : e_0 x_0 + A^* x = b^1 \, ; \, x \geq 0_N\}$.

Then the basis matrix B is also optimal for the new problem (73).

Proof: Since B is an optimal basis matrix for (72), then we have:

(74) $B_{m\bullet}^{-1} b^* \geq 0$ for m = 1,2,…,M (feasibility conditions satisfied);
(75) $B_{0\bullet}^{-1} A_{\bullet n}^* \geq 0$ for n = 1,2,…,N (optimality conditions satisfied).

But by assumption, the initial optimal basis matrix B generates a feasible solution for the new b vector, b^1, so that we have:

(76) $B_{m\bullet}^{-1} b^1 \geq 0$ for m = 1,2,…,M.

But since the basis matrix has not changed, the optimality conditions (75) are still satisfied (these conditions do not depend directly on b) and so (75) and (76) imply that the old basis matrix B is still optimal for the new problem (73). Q.E.D.

Theorem 4 is helpful in providing an economic interpretation for the dual variables. However, to see the connection, we have to consider a slightly different primal and dual LP problems compared to (55) and (56). Thus consider an equality constrained primal problem of the following form:

(77) $\max_x \{ c^T x : Ax = b ; x \geq 0_N\}$.

In order to find the dual problem to (77), we need to write the equality constraints $Ax = b$ in an inequality format. Thus (77) is equivalent to the following problem:

(78) $\max_x \{ c^T x : \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix} ; x \geq 0_N\}$.

Using our rules for forming the dual problem, it can be seen that the dual to (78) is:

(79) $\min_{y^1, y^2} \{y^{1T}b - y^{2T}b : [y^{1T}, y^{2T}] \begin{bmatrix} A \\ -A \end{bmatrix} \geq c^T ; y^1 \geq 0_M ; y^2 \geq 0_M\}$.

Define the unrestricted vector of variables $y \equiv y^1 - y^2$ and using this definition, we can rewrite the dual problem (79) as follows:

(80) $\min_y \{y^T b : y^T A \geq c^T\}$.

Thus (80) is the dual problem to the primal problem (77): *equality constraints in the primal lead to a dual problem with unrestricted in sign dual variables* (instead of nonnegative dual variables as before).

Assume that there is a finite optimal solution to (77). Putting (77) into the standard simplex algorithm format leads to the following equivalent problem:

(81) $\max_{x_0, x} \{ x_0 : \begin{bmatrix} 1 \\ 0_M \end{bmatrix} x_0 + \begin{bmatrix} -c^T \\ A \end{bmatrix} x = \begin{bmatrix} 0 \\ b \end{bmatrix} ; x \geq 0_N\}$.

Suppose that the final optimal basis matrix for (81) turns out to include the first M columns of the A matrix. Thus we have

(82) $B = \begin{bmatrix} 1 & -c_1 & \ldots & -c_M \\ 0_M & A_{\bullet 1} & \ldots & A_{\bullet M} \end{bmatrix}$

and we assume that B is nonsingular. Our final assumption is that the final basis matrix B for (81) is *nondegenerate* so that

(83) $x_m^* = B_{m\bullet}^{-1}\begin{bmatrix}0\\ b\end{bmatrix} > 0$ for m = 1,2,…,M.

We may repeat the proof of the Duality Theorem for linear programs in this set up. As before, we find that

(84) $B_{m\bullet}^{-1} \equiv [1, y^{*T}]$

where y^* turns out to be a solution to the dual problem (80). The nondegeneracy assumptions (83) imply that y^* must satisfy the following equations:

(85) $[1, y^{*T}]\begin{bmatrix}-c_m\\ A_{\bullet m}\end{bmatrix} = 0$ for m = 1,2,…,M.

Equations (85) imply that the dual variables must be unique. Thus *nondegeneracy of the final basis matrix implies that the optimal solution to the dual problem is unique*.

Recall that the feasibility restrictions (83) are satisfied by our basis matrix, which we rewrite as follows:

(86) $B_{m\bullet}^{-1}\begin{bmatrix}0\\ b\end{bmatrix} > 0$ for m = 1,2,…,M.

The following optimality conditions are also satisfied by our basis matrix B:

(87) $B_{0\bullet}^{-1}\begin{bmatrix}-c_n\\ A_{\bullet n}\end{bmatrix} \geq 0$ for n = 1,2,…,N.

Now we can apply the Basis Theorem to the above setup. If b^1 is sufficient close to the initial b, the nonnegativity restrictions (86) will continue to hold using the old basis matrix B; i.e., we will have, b^1 close to b:

(88) $B_{m\bullet}^{-1}\begin{bmatrix}0\\ b^1\end{bmatrix} > 0$ for m = 1,2,…,M.

The optimality conditions (87) will continue to hold for this new primal problem where b^1 replaces b and hence the basis matrix B continues to be optimal for the new problem. Setting $x_{M+1} = x_{M+2} = \ldots = x_N = 0$ for the constraints in (81) and premultiplying both sides of the constraints by $[1, y^{*T}]$ where y^* is defined by (84) leads to the following equation, where we have also used (85):

(89) $x_0 = y^{*T}b$.

All of this shows that for b^1 close to b, the optimal primal and dual objective functions, regarded as functions of b^1, are given by

(90) $V(b^1) = y^{*T}b^1$.

Thus for $b^1 = b$, the vector of first order partial derivatives of the optimized primal and dual objective functions, with respect to the components of b is equal to:

(91) $\nabla_b V(b) = y^*$

where the unique vector of dual variables y^* is defined by (84). Thus y_m^* is the marginal increase in the primal (and dual) objective functions due to a small increase in the mth component of b, b_m, provided that we have a nondegenerate primal solution. Note that this interpretation for a dual variable is entirely analogous to the economic interpretation of a Lagrange multiplier in classical constrained optimization.[10]

9. The Geometric Interpretation of a Linear Program in Requirements Space

Recall the basic linear program defined by (2) above and recall that we rewrote in the form (10) above. For convenience, we repeat (10) below as problem (92):

(92) $\max_{x_0, x_1 \geq 0, x_2 \geq 0, \ldots, x_N \geq 0} \{x_0 : e_0 x_0 + \sum_{n=1}^{N} A_{\bullet n}^* x_n = b^*\}$

where $c^T \equiv [c_1,\ldots,c_N]$, $A \equiv [A_{\bullet 1}, A_{\bullet 2},\ldots, A_{\bullet N}]$ is the original A matrix and

(93) $A_{\bullet n}^* \equiv \begin{bmatrix} c_n \\ A_{\bullet n} \end{bmatrix}$ for $n = 1,\ldots,N$, $b^* \equiv \begin{bmatrix} 0 \\ b \end{bmatrix}$ and $e_0 \equiv \begin{bmatrix} 1 \\ 0_M \end{bmatrix}$.

Define the *super feasibility set* S as the following set in M+1 dimensional space:[11]

(94) $S \equiv \{z : z = \sum_{n=1}^{N} A_{\bullet n}^* x_n \,;\, x_1 \geq 0, x_2 \geq 0, \ldots , x_N \geq 0\}$.

The set S is the set of all nonnegative linear combinations of the N column vectors $A_{\bullet n}^*$. Thus S is a set in M+1 dimensional space.

A set S is a *cone* if and only if it has the following property:

[10] See Samuelson (1974; 65) and Diewert (1984; 148) for this interpretation for a Lagrange multiplier in classical (equality constrained) optimization theory. Dorfman, Samuelson and Solow (1958; 52-59) have a nice discussion on the meaning and interpretation of dual variables. Problem 16 at the end of the chapter provides an economic interpretation for the dual prices in the general case where conditions (86) hold only as weak inequalities rather than as strict inequalities.

[11] Contrast this definition with the feasible set of x's, which is the set of x such that $x \geq 0_N$ and $Ax = b$.

(95) $z \in S$, $\lambda \geq 0$ implies $\lambda z \in S$.

It is easy to verify that the super feasibility set S defined by (94) is a cone.[12] Note that this set S does not involve the right hand side vector b which appeared in the original constraints for the LP, $Ax = b$. The vector b is often called the *requirements vector*; i.e., a nonnegative vector x must be chosen so that "production" Ax meets the "requirements" b. The *requirements line* L is defined as follows:

(96) $L \equiv \{ \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_M \end{bmatrix} : \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_M \end{bmatrix} = \begin{bmatrix} z_0 \\ b_1 \\ \vdots \\ b_M \end{bmatrix}$ where z_0 is an unrestricted scalar variable$\}$.

Note that points z that belong to *both* L and S correspond to feasible solutions for the LP (92). An *optimal solution* for (92) will correspond to a point z such that:

- $z \in S$
- $z \in L$ and
- The first component of z is as small as possible subject to $z \in S$ and $z \in L$.

Thus to solve (92), *we look for the lowest point* (with respect to the first component of z) *on the requirements line that also belong to the super feasibility set S*. It also turns out that any hyperplane that supports the set S (is tangent to S) at this lowest point of L provides a set of optimal dual variables. An example will help to illustrate these points.

Suppose that we have 4 production units in a region or a country that uses varying amounts of two inputs and each production unit when run at unit scale produces output that is valued at 1 dollar. The input requirements of the 4 production units when run at unit scale are the $A_{\bullet n}$ listed below:

(97) $A_{\bullet 1} \equiv \begin{bmatrix} 0 \\ 2 \end{bmatrix}$; $A_{\bullet 2} \equiv \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$; $A_{\bullet 3} \equiv \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$; $A_{\bullet 4} \equiv \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

Thus the first production unit uses 0 units of the first input and 2 units of the second input in order to produce outputs worth 1 dollar, the second production unit uses 1/2 of a unit of the first input and 1 unit of the second input in order to produce outputs worth 1 dollar and so on. The 4 production units are controlled by a single firm which can allocate $b_1 > 0$ units of input 1 and $b_2 > 0$ units of input 2 across the 4 production units. Production in each plant is subject to constant returns to scale. The firm is interested in allocating the amounts of the two inputs across the 4 plants in order to maximize revenue; i.e., the firm would like to solve the following linear programming problem:

[12] It is also a convex set so it is a convex cone.

(98) $\max_x \{c^T x : Ax = b \; ; \; x \geq 0_4\}$

where $c^T \equiv [1,1,1,1]$, $b^T \equiv [b_1,b_2]$, $A \equiv [A_{\bullet 1}, A_{\bullet 2}, A_{\bullet 3}, A_{\bullet 4}]$ and $x^T \equiv [x_1,x_2,x_3,x_4]$ is the vector of nonnegative plant scales.

To put the problem defined by (97) into standard simplex algorithm format, define the $A_{\bullet n}{}^*$ by adding $-c_n$ as an extra component to the $A_{\bullet n}$:

$$(99)\; A_{\bullet 1}{}^* \equiv \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} ; A_{\bullet 2}{}^* \equiv \begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix} ; A_{\bullet 3}{}^* \equiv \begin{bmatrix} -1 \\ 1 \\ 1/2 \end{bmatrix} ; A_{\bullet 4}{}^* \equiv \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} ; b^* \equiv \begin{bmatrix} 0 \\ b_1 \\ b_2 \end{bmatrix}.$$

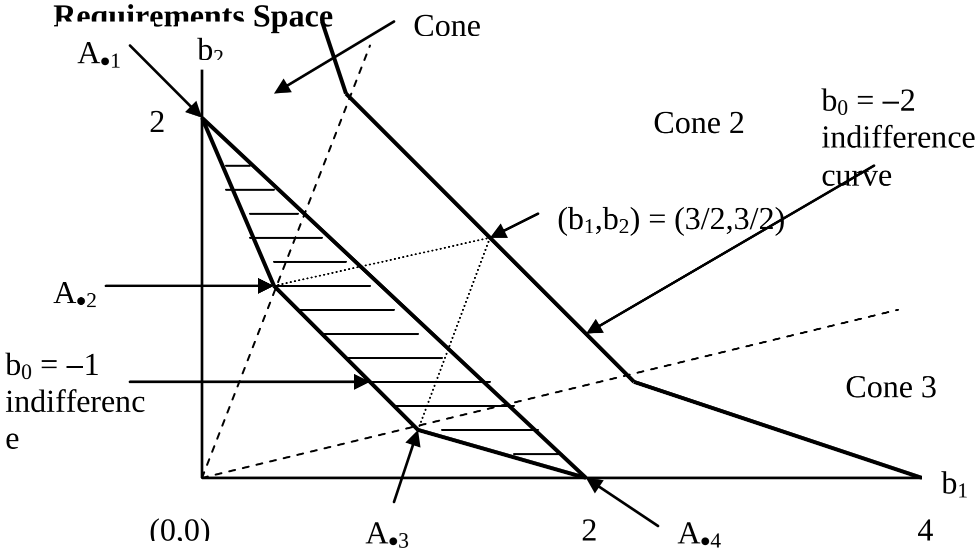
Using the above notation, the LP (98) can be rewritten as follows:

$$(100)\; \max_{x_0, x \geq 0} \{x_0 : e_0 x_0 + A_{\bullet 1}{}^* x_1 + A_{\bullet 2}{}^* x_2 + A_{\bullet 3}{}^* x_3 + A_{\bullet 4}{}^* x_4 = b^*\}.$$

Now recall the definition of the super feasibility set S, (94). For our example LP (100), this set will be a cone in three dimensional space. Now note that each of the 4 column vectors $A_{\bullet 1}{}^*$, $A_{\bullet 2}{}^*$, $A_{\bullet 3}{}^*$ and $A_{\bullet 4}{}^*$ will belong to S and the first component for each of these vectors is equal to -1. Hence if we intersect the cone S with the plane $\{(z_0,z_1,z_2): z_0 = -1$, z_1 and z_2 are unrestricted, we obtain the shaded set in Figure 2 below.[13]

[13] A three dimensional diagram would be more instructive but it is more difficult to graph. The reader should think of S as a 3 dimensional set in (b_0,b_1,b_2) space. What we are doing in Figure 2 is intersecting this three dimensional set with various (negative) heights or levels for b_0 and then we plot these level sets in (b_1,b_2) space.

Figure 2: The Simplex Algorithm in Requirements Space



The lower boundary of the shaded set is the counterpart to an indifference curve in consumer theory; this curve gives the set of (b_1,b_2) points which will allow the firm to earn one unit of revenue without wasting resources.[14] The point $A_{\bullet 1}$ is the top point on this curve and it is joined by a straight line to the point $A_{\bullet 2}$, which in turn is joined to the point $A_{\bullet 3}$, which in turn is joined to the lowest point on the curve, $A_{\bullet 4}$. Now suppose that the firm has $(b_1,b_2) = (3/2, 3/2)$ units of the two inputs available to use in the 4 production units. One can go through the algebra of the simplex algorithm in this case and one will find that $A_{\bullet 2}$ and $A_{\bullet 3}$ are the efficient plants to use in this case and the optimal x's are $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 1$ and $x_4^* = 0$. The optimal objective function in this case is $x_0^* = 2$. In terms of Figure 1, it can be seen that we get to the point $(b_1,b_2) = (3/2, 3/2)$ by moving up the dashed line through $A_{\bullet 2}$ until we hit $A_{\bullet 2}$ and by moving up the dashed line through $A_{\bullet 3}$ until we hit $A_{\bullet 3}$ and then we complete the parallelogram with the dotted lines until we hit $(b_1,b_2) = (3/2, 3/2)$. Note that the two dashed lines enclose a region of (b_1,b_2) that we called *cone 2*. Using the Basis Theorem for linear programs, it can be seen that we will use only production units 2 and 3 provided that (b_1,b_2) falls into this region. Using a similar argument, it can be seen only production units 1 and 2 will be used if (b_1,b_2) falls into the *cone 1* region and only production units 3 and 4 will be used if (b_1,b_2) falls into the *cone 3* region.

Suppose that (b_1,b_2) falls into the interior of the cone 2 region. The corresponding (unique) dual (input) prices can be calculated by solving the following two equations in two unknowns:

(101) $[1,y_1,y_2]A_{\bullet 2}^* = 0$; $[1,y_1,y_2]A_{\bullet 3}^* = 0$.

---

[14] In terms of the set S, this curve is the set of (b_1,b_2) points such that $(-1,b_1,b_2)$ belongs to the boundary of S.

Using our data listed in (99), we find that the cone 2 dual solution is:

(102) $y_1^{(2)} = 2/3$; $y_2^{(2)} = 2/3$.

Thus the input prices are equal in this case where the resource availability vector falls into the cone 2 region. This corresponds to the fact that the parallel "indifference" curves in this region all have slope equal to -1 (which is equal to $-y_1^{(2)}/y_2^{(2)}$).

Now suppose that (b_1,b_2) falls into the interior of the cone 1 region. The corresponding (unique) dual (input) prices can be calculated by solving the following two equations in two unknowns:

(103) $[1,y_1,y_2]A_{\bullet 1}^{*} = 0$; $[1,y_1,y_2]A_{\bullet 2}^{*} = 0$.

Using our data listed in (99), we find that the cone 1 dual solution is:

(104) $y_1^{(1)} = 1$; $y_2^{(1)} = 1/2$.

Thus the input 1 price is twice the size of the input 2 price in this region where input 1 is relatively scarce compared to input 2. This corresponds to the fact that the parallel "indifference" curves in this region all have slope equal to -2 (which is equal to $-y_1^{(1)}/y_2^{(1)}$).

Finally suppose that (b_1,b_2) falls into the interior of the cone 3 region. The corresponding (unique) dual (input) prices can be calculated by solving the following two equations in two unknowns:

(105) $[1,y_1,y_2]A_{\bullet 3}^{*} = 0$; $[1,y_1,y_2]A_{\bullet 4}^{*} = 0$.

Using our data listed in (99), we find that the cone 3 dual solution is:

(106) $y_1^{(3)} = 1/2$; $y_2^{(3)} = 1$.

Thus the input 2 price is twice the size of the input 1 price in this region where input 2 is relatively scarce compared to input 1. This corresponds to the fact that the parallel "indifference" curves in this region all have slope equal to $-1/2$ (which is equal to $-y_1^{(3)}/y_2^{(3)}$).

What happens if the endowment vector (b_1,b_2) happens to fall on the dashed line through the origin and $A_{\bullet 2}$ so that (b_1,b_2) is on the boundary of both cone 1 and cone 2? In this case, although the primal solution is unique (x_2^{*} solves the primal where $A_{\bullet 2}\, x_2^{*} = b$), the dual solution is not. In this case the dual solution set is:

(107) $(y_1,y_2) = \lambda\, (y_1^{(1)},y_2^{(1)}) + (1-\lambda)\, (y_1^{(2)},y_2^{(2)})$; $0 \le \lambda \le 1$;

i.e., it is the set of all convex combinations of the cone 1 and cone 2 dual prices. Similarly, if the endowment vector (b_1,b_2) happens to fall on the dashed line through the origin and $A_{\bullet 3}$ so that (b_1,b_2) is on the boundary of both cone 2 and cone 3, then in this case, although again the primal solution is unique (x_3^* solves the primal where $A_{\bullet 3}\, x_3^* = b$), the dual solution is not. In this case the dual solution set is:

(108) $(y_1,y_2) = \lambda\, (y_1^{(2)},y_2^{(2)}) + (1-\lambda)\, (y_1^{(3)},y_2^{(3)})\ ;\ \ 0 \leq \lambda \leq 1$;

i.e., it is the set of all convex combinations of the cone 2 and cone 3 dual prices.

The reader is now encouraged to visualize how the simplex algorithm would work if we had additional "inefficient' production units in the model. If our starting basis matrix had one or more inefficient columns in the basis matrix, then there would exist at least one more efficient column that would lie *below* the dual hyperplane that was generated by the starting columns. Introducing this more efficient column into the basis matrix would lead to a lower dual hyperplane at the next iteration and eventually, the dual hyperplane would lie on or below all of the columns in the model and the algorithm would come to a halt.

The geometry of the simplex algorithm in requirements space is illustrated in Dantzig (1963; 160-165) and Van Slyke (1968).[15] The representation of a linear programming problem in requirements space is a very useful one because we can simultaneously visualize the solution to the primal problem as well as to the dual problem. Also this requirements space geometric approach makes it easy to visualize the comparative statics effects of changing the right hand side vector b.[16]

10. The Saddlepoint Criterion for Solving a Linear Program

In this section, we develop some criteria for solving the primal dual problems that do not rely on the simplex algorithm. However, the material in this section is very closely related to the auxiliary result that appeared in the proof of Theorem 3, the duality theorem for linear programs.

Consider the following primal and dual problems:

(109) $P : \max_x \{c^Tx : Ax \leq b\ ;\ x \geq 0_N\}$;
(110) $D : \min_y \{y^Tb\ ;\ y^TA \geq c^T\ ;\ y \geq 0_M\}$

where A is an M by N matrix, b and y are M dimensional vectors and c and x are N dimensional vectors.

Theorem 3 above, the Duality Theorem for Linear Programs, developed optimality criteria for x^* to solve the primal problem (109) and for y^* to solve the dual problem (110), using the mechanics of the simplex algorithm as a tool for establishing the criteria. In

[15] Van Slyke was a student of Dantzig's.
[16] The Figure 1 geometry in activities space is useful for visualizing the comparative statics effects of changing the objective function vector c.

this section, we develop some alternative optimality criteria that do not rely on the simplex algorithm.

In order for x^* to solve P and y^* to solve D, it is obviously necessary that x^* and y^* satisfy the feasibility constraints for their respective problems:

(111) $Ax^* \leq b$; $x^* \geq 0_N$; $y^{*T}A \geq c^T$; $y^* \geq 0_M$.

Consider the following condition, which relates the primal and dual objective functions:

(112) $c^Tx^* = y^{*T}b$.

Theorem 5; Goldman and Tucker (1956): Conditions (111) and (112) are necessary and sufficient for x^* to solve P and for y^* to solve D.

Proof: Recall the auxiliary result that was proven in Theorem 3 above, which may be restated as follows: if x^* is feasible for the primal problem P and y^* is feasible for the dual problem D, then

$$
\begin{aligned}
(113)\ c^Tx^* &\leq c^Tx^* + y^{*T}(b - Ax^*) && \text{since } y^* \geq 0_M \text{ and } Ax^* \leq b \\
&= y^{*T}b + (c^T - y^{*T}A)x^* && \text{rearranging terms} \\
&\leq y^{*T}b && \text{since } x^* \geq 0_N \text{ and } y^{*T}A \geq c^T .
\end{aligned}
$$

Now suppose that x^* and y^* satisfy (111) and (112). Then repeating the proof of the auxiliary result in Theorem 3 shows that x^* solves P and y^* solves D.

Conversely, let x^* solve P and let y^* solve D. Then x^* and y^* satisfy (111) and (113) and so we have $c^Tx^* \leq y^{*T}b$. *Suppose* $c^Tx^* < y^{*T}b$, then we could repeat the proof of Theorem 3 and find a dual solution y^{**} such that $c^Tx^* = y^{**T}b < y^{*T}b$. But this last inequality contradicts the assumed optimality of y^* for the dual problem D. Thus our *supposition* is false and $c^Tx^* = y^{*T}b$, which is (112). Q.E.D.

Theorem 6; Goldman and Tucker (1956): Consider the following *complementary slackness conditions*:

(114) $y^{*T}(b - Ax^*) = 0$;
(115) $(c^T - y^{*T}A)x^* = 0$.

The feasibility conditions (111) and the complementary slackness conditions (114) and (114) are necessary and sufficient for x^* to solve P and for y^* to solve D.

Proof: By Theorem 5, we need only show that conditions (114) and (115) are equivalent to condition (112) when conditions (111) hold. But given that (111) holds, we know the inequality (113) holds. But now it can be seen that conditions (114) and (115) are precisely the conditions that are necessary and sufficient to convert the inequality (113) into the equality (112). Q.E.D.

In order to state the next theorem, it is first necessary to define the *Lagrangian* L that corresponds to the primal problem P:

(116) $L(x,y) \equiv c^Tx + y^T[b - Ax]$.

Note that the vector y in (116) plays the role of a vector of *Lagrange multipliers* for the constraints in the primal problem P.

Definition: (x^*,y^*) is a *saddle point* of the Lagrangian L if and only if

(117) $x^* \geq 0_N$; $y^* \geq 0_M$ and
(118) $L(x,y^*) \leq L(x^*,y^*) \leq L(x^*,y)$ for all $x \geq 0_N$ and $y \geq 0_M$.

Looking at conditions (118), we see that $L(x,y^*)$ attains a *maximum* with respect to x at $x = x^*$ over all $x \geq 0_N$. Conversely, $L(x^*,y)$ attains a *minimum* with respect to y at $y = y^*$ over all $y \geq 0_M$.

Theorem 7; Goldman and Tucker (1956; 77): The saddle point conditions (117) and (118) are necessary and sufficient for x^* to solve P and for y^* to solve D.

Proof: In view of Theorem 6, we need only show that (117) and (118) are equivalent to the feasibility restrictions (111) and the complementary slackness conditions (114) and (115).

Assume that (111), (114) and (115) hold. Conditions (111) imply that conditions (117) hold so we need only show that conditions (118) hold. Using the definition of L(x,y), conditions (118) can be rewritten as follows:

(119) $c^Tx + y^{*T}[b - Ax] \leq c^Tx^* + y^{*T}[b - Ax^*] \leq c^Tx^* + y^T[b - Ax^*]$
for all $x \geq 0_N$ and $y \geq 0_M$.

Using (114), the right hand set of inequalities in (119) is equivalent to:

(120) $0 \leq y^T[b - Ax^*]$ for all $y \geq 0_M$.

But the feasibility conditions (111) imply that $b - Ax^* \geq 0_M$ so (120) must hold.

Using (115), the left hand set of inequalities in (119) is equivalent to:

(121) $[c^T - y^{*T}A]x \leq [c^T - y^{*T}A]x^* = 0$ for all $x \geq 0_N$.

But the feasibility conditions (111) imply that $c^T - y^{*T}A \leq 0_N{}^T$ so that (121) holds. This completes the proof of the first half of the theorem.

Now assume that the saddle point conditions (117) and (118) hold. The right hand set of inequalities in (118) is equivalent to:

(122) $y^{*T}[b - Ax^*] \leq y^T[b - Ax^*]$ for all $y \geq 0_M$.

But it is easy to see that (122) implies that $b - Ax^* \geq 0_N$. In a similar manner, we can show that the left hand set of inequalities in (118) implies that $c^T - y^{*T}A \leq 0_N{}^T$. Thus using also (117), we have shown that x^* and y^* satisfy the feasibility conditions (111).

We have shown that $y^* \geq 0_M$ and $b - Ax^* \geq 0_M$. Hence $y^{*T}[b - Ax^*] \geq 0$. *Suppose* $y^{*T}[b - Ax^*] > 0$. Now set $y = 0_M$ and (122) becomes $0 < y^{*T}[b - Ax^*] \leq 0$, which is a contradiction. Hence our *supposition* is false and $y^{*T}[b - Ax^*] = 0$, which is the complementary slackness condition (114).

Finally, we have shown that $x^* \geq 0_N$ and $c^T - y^{*T}A \leq 0_N{}^T$. Hence $(c^T - y^{*T}A)x^* \leq 0$. Note that the left hand set of inequalities in (118) or (119) is equivalent to:

(123) $[c^T - y^{*T}A]x \leq [c^T - y^{*T}A]x^*$ for all $x \geq 0_N$.

Now *suppose* $(c^T - y^{*T}A)x^* < 0$. Set $x = 0_N$ and (123) becomes $0 \leq (c^T - y^{*T}A)x^* < 0$, which is a contradiction. Hence our *supposition* is false and $[c^T - y^{*T}A]x^*$, which is the complementary slackness condition (115). Q.E.D.

The saddle point characterization for optimal solutions to the primal and dual linear programs turns out to be very useful as will be seen in the problems at the end of the chapter.

11. Programming with Variable Coefficients

Consider the following *programming problem with variable coefficients* which has the structure (10) of a standard LP with one exception:

(124) $\max_{x_0, x_1 \geq 0, x_2 \geq 0, \ldots, x_N \geq 0, A_{\bullet 1}^*, \ldots, A_{\bullet N}} \{x_0 : e_0 x_0 + \sum_{n=1}^N A_{\bullet n}{}^* x_n = b^* \,;\, A_{\bullet 1}{}^* \in C^1, \ldots, A_{\bullet N}{}^* \in C^N\}$.

The only difference between (124) and our standard simplex algorithm primal LP problem (10) is that in (10), the columns $A_{\bullet n}{}^*$ were fixed but now columns are variable; i.e., $A_{\bullet n}{}^*$ may be chosen from a closed and bounded set[17] of columns C^n. Obviously, if each column set C^n has only a single member in it, then (124) boils down to our standard LP (10). Dantzig (1963; chapter 22) considered problems of the form (124) where the sets C^n consisted of convex combinations of a finite set of columns. Van Slyke (1968) considered general problems of the type (124).

[17] These restrictions on the sets C^n can be relaxed under certain additional hypotheses. It is sometimes assumed that the sets C^n are convex, but this additional assumption is not necessary for the simplex algorithm to work in this context.

Consider now how to solve a problem of the type (124). Suppose that $A_{\bullet 1}^{*} \in C^{1}, \ldots, A_{\bullet M}^{*} \in C^{M}$ exist such that $B \equiv [e_0, A_{\bullet 1}^{*}, \ldots, A_{\bullet M}^{*}]$ is an initial basis matrix for (124); i.e., B^{-1} exists and the last M components of the M+1 vector $B^{-1}b^{*}$ are nonnegative. Consider the following N *subproblems*:

$$(125)\ f_n(B) \equiv \min_{A_{\bullet n}^{*}} \{B_{0\bullet}^{-1}A_{\bullet n} : A_{\bullet n} \in C^{n}\}\ ; \qquad n = 1,2,\ldots,N.$$

Assuming that we can solve the subproblems (125) (a nontrivial assumption), then the variable coefficients simplex algorithm works as follows:

(i) If $f_n(B) \geq 0$ for $n = 1,2,\ldots,N$, then the present basis matrix is optimal and the algorithm terminates.
(ii) If $f_n(B) < 0$ for some n, then there exists $A_{\bullet n}^{*} \in C^{n}$ such that $f_n(B) = B_{0\bullet}^{-1}A_{\bullet n}^{*} < 0$. Introduce $A_{\bullet n}^{*}$ into the basis matrix using the usual unbounded solutions criterion or the dropping criterion. If we end up in the unbounded solutions criterion case, then the algorithm terminates. Otherwise, we solve the following LP:

$$(126)\ \max_{x_0, x_1 \geq 0, x_2 \geq 0, \ldots, x_M \geq 0, x_n \geq 0} \{x_0 : e_0x_0 + \textstyle\sum_{m=1}^{M} A_{\bullet m}^{*}x_m + A_{\bullet n}^{*}x_n = b^{*}\}$$

and we obtain a new basis matrix. We use this new basis matrix to solve the subproblems (126) and we return to (i) above and repeat the cycle.

We can no longer assert that the simplex algorithm will converge to a solution in a finite number of steps. However, Van Slyke (1968) proved that the algorithm will converge to an optimal solution of (124) under relatively weak regularity conditions.

The above algorithm has been used in the theory of economic planning; see Dorfman, Samuelson and Solow (1958; 59-63) and Malinvaud (1967). The basic idea is the following one. The central planner has a preliminary vector of resource prices $[1, y^{*T}]$, which is transmitted to the various sectors in the economy. Given these prices, the manager of sector n solves a (net) cost minimization problem similar to (126), where $B_{0\bullet}^{-1}$ is replaced by $[1, y^{*T}]$. Each manager then sends back a solution vector $A_{\bullet n}^{*}$ back to the center. The central planner then solves a simple linear program involving this sectoral data, obtains a new set of prices, transmits this new set of prices to the sectors and so on.[18]

The variable coefficients simplex algorithm does depend on the ease with which the subproblems (126) can be solved. If the sets C^{n} are unit scale production possibilities sets (these sets are dual to a unit cost function or to a unit profit function) and if the dual unit cost or profit functions are available to the programmer, then the subproblems (126) can

[18] Why has this theory of decentralized economic planning not worked in the real world? The assumption of constant returns to scale is somewhat problematic in this theory but probably the main reason it has not worked very well in applications is the *problem of dimensionality*; i.e., real life economic planning problems at the national scale involve thousands if not millions of variables and the resulting programming problems are just too difficult to solve. However, at the level of a firm which has several more or less independent divisions, the above theory of decentralized economic planning could work.

be solved by a simple application of Shephard's (1953) Lemma or Hotelling's (1932) Lemma. In this case, the variable coefficients simplex algorithm is just as easy to use as the ordinary simplex algorithm. Some applications along these lines may be found in Diewert (1975), Diewert and Woodland (1977) and Woodland (1982).

Problems

1. Consider the following linear program:

(i) $\max_x \{c^T x : A^{(1)}x \le b^{(1)} ; A^{(2)}x = b^{(2)} ; x \ge 0_N\}$.

Show that the dual to (i) is:

(ii) $\min_{y^{(1)},y^{(2)}} \{y^{(1)T}b^{(1)} + y^{(2)T}b^{(2)} : y^{(1)T}A^{(1)} + y^{(2)T}A^{(2)} \ge c^T ; y^{(1)} \ge 0_{M_1} ; y^{(2)}$ unrestricted$\}$.

Hint: Write $A^{(2)}x = b^{(2)}$ as $A^{(2)}x \le b^{(2)}$ and $-A^{(2)}x \le -b^{(2)}$. *Comment*: If $M_1 = 0$, then the primal problem has equality constraints and the corresponding dual variables are unrestricted.

2. Consider the following linear program:

(i) $\max_{x^{(1)},x^{(2)}} \{c^{(1)T}x^{(1)} + c^{(2)T}x^{(2)} : A^{(1)}x^{(1)} + A^{(2)}x^{(2)} \le b ; x^{(1)} \ge 0_{N_1} ; x^{(2)}$ unrestricted$\}$.

Show that the dual to (i) is

(ii) $\min_y \{y^T b : y^T A^{(1)} \ge c^{(1)T} ; y^T A^{(2)} = c^{(2)T} ; y \ge 0_M\}$.

3. Consider a firm that has 4 production processes (or activities) that can be used to produce two outputs.[19] Each process uses one primary input (call it labour) and produces one of the two outputs. Some processes use the other output as an intermediate input. The firm wishes to minimize the aggregate labour cost of producing given positive amounts, $b_1 > 0$ and $b_2 > 0$, of the two outputs. The linear program that the firm wishes to solve is:

$$\text{(i)}\ \min_{x_1\ge 0, x_2\ge 0, x_3\ge 0, x_4\ge 0} \{2x_1 + 1x_2 + 1x_3 + 2x_4 : \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \ge \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\}.$$

[19] Each process can be thought of as a Leontief production function that does not allow for substitution between inputs. Leontief (1936) (1941) was a pioneer in setting up production models of entire economies using Leontief type production functions. His influence lives on today: many national statistical agencies produce input-output tables based on his model of production.

The first two activities produce the finally demanded commodity b_1 while the last two activities produce the finally demanded commodity b_2.

(a) Solve the above LP when $b_1 = 1$ and $b_2 = 1$. *Hint*: rewrite (i) in standard simplex algorithm format as follows:

$$\text{(ii)}\ \max_{x_0, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, s_1 \geq 0, s_2 \geq 0} \{x_0 : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} x_3 + \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} x_4 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} s_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \}.$$

Note that the x_1 and x_4 columns can be used in a starting basis matrix.

The dual problem to (i) is:

$$\text{(iii)}\ \max_{y_1 \geq 0, y_2 \geq 0} \{y_1 b_1 + y_2 b_2 : [y_1, y_2]^T \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \leq [2, 1, 1, 2] \}.$$

(b) Find the dual prices y_1^* and y_2^* for problem (iii) when $b_1 = 1$ and $b_2 = 1$. *Hint*: You need only list the last two components of $B_{0\bullet}^{-1}$ where B is the optimal basis matrix for (ii).

(c) Finally, solve the LP (i) for a general $b_1 > 0$ and $b_2 > 0$ and show that the corresponding dual prices do not depend on b_1 and b_2. *Comment*: This problem provides a concrete illustration of the Samuelson (1951) Nonsubstitution Theorem; i.e., the optimal activities do not depend on the particular vector of final demands [b_1, b_2] that we are required to produce in this single input (labour theory of value) economy and neither do the dual prices, which can be interpreted as equilibrium prices for the outputs.

4. Consider the following *zero sum game theory problem*, where we list the payoff matrix of the student:

		Student's Strategy Space	
Professor's		Go to lecture	Sleep in
Strategy	Give a good lecture	2	1
Space	Give a terrible lecture	0	3

We suppose that the student wishes to choose a strategy, which consists of the probability of going to a lecture, $p_1 \geq 0$, and the probability of sleeping in, $p_2 \geq 0$, where $p_1 + p_2 = 1$. We further suppose that the somewhat psychotic student suspects that the professor is perversely attempting to minimize the student's payoff. Thus the student would be wise to choose a strategy which would ensure him or her a minimum amount of utility irrespective of the professor's actions; this is called a *minimax strategy*. In order to find the minimax strategy, the student will want to solve the following linear program:

(i) $\max_{v\geq 0, p_1\geq 0, p_2\geq 0} \{v : v \leq 2p_1 + 1p_1 ; v \leq 0p_1 + 3p_1 ; 1 = p_1 + p_2\}$.

Since the student's payoff matrix is nonnegative, we can define new variables x_1 and x_2 which will also be nonnegative as follows:

(ii) $x_1 \equiv p_1/v$; $x_2 \equiv p_2/v$.

Note that $p_1 + p_2 = vx_1 + vx_2 = 1$ so v can be expressed in terms *of* x_1 and x_2 as follows:

(iii) $v = 1/(x_1 + x_2)$.

Thus maximizing v is equivalent to minimizing $x_1 + x_2$ and we find that the LP (i) is equivalent to the following LP:

(iv) $\min_{x_1\geq 0, x_2\geq 0} \{x_1 + x_2 : 2x_1 + 1x_2 \geq 1 ; 0x_1 + 3x_2 \geq 1\}$.

(a) Solve the LP (iv) and use (ii) and (iii) to calculate the student's optimal strategy, p_1^* and p_2^*.

(b) What probability mix of good and terrible lectures should the professor use to ensure a minimum payoff to the student? *Hint*: We can transform the professor's minimization problem into the dual of the LP (iv). *Comment*: This problem provides an example of the *minimax theorem* for zero sum two person games. *Reference*: Dorfman, Samuelson and Solow (1958; chapters 15-16).

5. *Complementary Slackness in Linear Programming* ; Dantzig (1963; 136): Consider the following primal LP:

(i) $\max_{x\geq 0_N} \{c^Tx : Ax \leq b\}$ where A is an M by N matrix.

Suppose a finite optimal solution $x^0 \geq 0_N$ exists for the above problem. Then by the *duality theorem for linear programs*, an optimal vector of dual variables $y^{0T} \equiv [y_1^0,\ldots,y_M^0] \geq 0_M^T$ will also exist.

(a) Show that if the mth constraint in the primal LP is satisfied with a strict inequality when evaluated at the primal optimal solution x^0 so that $A_{m\bullet}\, x^0 < b_m$, then the corresponding dual variable y_m^0 must equal 0. *Hint*: If the mth constraint is satisfied with a strict inequality, then the corresponding slack variable must be in the optimal basis matrix.

(b) Give an economic interpretation of the above result.

(c) Prove that $y^{0T}[b - Ax^0] = 0$ and $[c^T - y^{0T}A]x^0 = 0$.

6. *Farkas* (1902) *Lemma*: Let A be a nonzero M by N matrix and c be an N dimensional vector. Then, *either* there exists an x such that

(i) $Ax \leq 0_M$ and $c^Tx > 0$

or there exists a y such that

(ii) $y^TA = c^T$ and $y \geq 0_M$.[20]

Prove the above result. *Hint*: Consider the following primal and dual problems:

(iii) $\max_x \{c^Tx : Ax \leq 0_M\}$;
(iv) $\min_y \{y^T0_M : y^TA = c^T ; y \geq 0_M\}$.

Using the results of Problems 1 and 2 above, it can be seen that (iv) is the dual to (iii). Since $x = 0_N$ is feasible for (iii), there are only two cases that can occur for (iii): *Case 1*: the optimal objective function for (iii) becomes unbounded and *Case 2*: (iii) has a finite optimal solution. Show that Case 1 corresponds to (i) and Case 2 corresponds to (ii). *Comment*: Farkas' Lemma can be used to prove Theorem 3, the duality theorem for linear programs (this is the path followed by Tucker (1956)) and as the above problem shows, the duality theorem established via the simplex algorithm can be used to prove Farkas' Lemma (this is the path followed by Dantzig (1963; 145)). Mangasarian (1969; chapter 2) calls Farkas' Lemma a *theorem of the alternative* and he lists several other similar theorems, which we will give as problems 7-9 below. These theorems of the alternative turn out to be very useful in the theory of nonlinear programming and in the theory of economic policy. In particular, theorems of the alternative can be used to establish the existence of Pareto improving changes in economic policy instruments. For examples of these applications to the theory of economic policy, see Diewert (1978; 281) (1983) (1987), Weymark (1979) and Diewert, Turunen-Red and Woodland (1989) (1991).

7. *Motzkin's (1936) Transposition Theorem*; see Mangasarian (1969; 28-29): Let E be an M_1 by N matrix, F be an M_2 by N matrix and G be an M_3 by N matrix where $M_1 > 0$, $N > 0$, $M_2 \geq 0$ and $M_3 \geq 0$.[21] Then *either* there exists x such that

(i) $Ex \gg 0_{M_1}$; $Fx \geq 0_{M_2}$; $Gx = 0_{M_3}$
or there exist y^1, y^2, y^3 such that

(ii) $y^{1T}E + y^{2T}F + y^{3T}G = 0_N{}^T$; $y^1 > 0_{M_1}$; $y^2 \geq 0_{M_2}$; y^3 unrestricted.

Hint: Consider the following LP which is closely related to (i):

[20] *Both* (i) and (ii) cannot hold.
[21] If $M_2 = 0$, then drop the matrix F from the problem; if $M_3 = 0$, then drop the matrix G from the problem.

(iii) $\max_x \{0_N^T x : -Ex \le -1_{M_1} ; -Fx \le 0_{M_2} ; -Gx = 0_{M_3} ; x \text{ unrestricted}\}$.

The dual to (iii) turns out to be the following LP, which is closely related to (ii):

(iv) $\min_{y^1,y^2,y^3} \{-y^{1T}1_{M_1} : y^{1T}E + y^{2T}F + y^{3T}G = 0_N^T ; y^1 \ge 0_{M_1} ; y^2 \ge 0_{M_2} ; y^3 \text{ unrestricted}\}$.

Note that the objective function for (iii) is bounded. Hence, there are two cases that can occur for (iii): Case 1: a feasible solution for (iii) exists and hence we have a bounded optimal solution for (iii) or Case 2: no feasible solution for (iii) exists.

8. *Tucker's* (1956; 14) *Theorem of the Alternative*; see Mangasarian (1969; 29): Let B be an M_1 by N matrix, C be an M_2 by N matrix and D be an M_3 by N matrix where $M_1 > 0$, $N > 0$, $M_2 \ge 0$ and $M_3 \ge 0$. Then *either* there exists x such that

(i) $Bx > 0_{M_1} ; Cx \ge 0_{M_2} ; Dx = 0_{M_3}$
or there exist y^1, y^2, y^3 such that

(ii) $y^{1T}B + y^{2T}C + y^{3T}D = 0_N^T ; y^1 \gg 0_{M_1} ; y^2 \ge 0_{M_2} ; y^3$ unrestricted.

Hint: Consider the following LP which is closely related to (i):

(iii) $\max_x \{1_{M_1}^T B x : -Bx \le 0_{M_1} ; -Cx \le 0_{M_2} ; -Dx = 0_{M_3} ; x \text{ unrestricted}\}$.

The dual to (iii) turns out to be the following LP, which is closely related to (ii):

(iv) $\min_{y^1,y^2,y^3} \{y^{1T}0_{M_1} + y^{2T}0_{M_2} + y^{3T}0_{M_3} : y^{1T}B + y^{2T}C + y^{3T}D = -1_{M_1}^T B ;$
$y^1 \ge 0_{M_1} ; y^2 \ge 0_{M_2} ; y^3 \text{ unrestricted}\}$.

Note that $x = 0_N$ is always feasible for (iii). Hence, there are two cases that can occur for (iii): Case 1: a feasible solution for (iii) exists but the corresponding optimal objective function is 0 and hence we have a bounded optimal solution for (iii) or Case 2: a feasible solution for (iii) exists and the objective function when evaluated at this feasible solution is positive and so in this case we obtain an unbounded solution for the primal and no feasible solution for the dual (iv).

9. *Slater's* (1951) *Theorem of the Alternative*: Let A be M_1 by N, B be M_2 by N, C be M_3 by N and D be M_4 by N with $M_1 > 0$, $M_2 > 0$ and $N > 0$. Then *either*

(i) $Ax \gg 0_{M_1} ; Bx > 0_{M_2} ; Cx \ge 0_{M_3} ; Dx = 0_{M_4}$ has a solution x *or*

(ii) $y^{1T}A + y^{2T}B + y^{3T}C + y^{4T}D = 0_N^T$
with $y^1 > 0_{M_1} ; y^2 \ge 0_{M_2} ; y^3 \ge 0_{M_3}$ *or*

$$y^1 \geq 0_{M_1} ;\; y^2 >> 0_{M_2} ;\; y^3 \geq 0_{M_3}$$

has a solution y^1, y^2, y^3, y^4.[22]

Hint: Consider the following LP which is closely related to (i):

(iii) $\max_x \{1_{M_1}^T B x : -Ax \leq -1_{M_1} ; -Bx \leq 0_{M_2} ; -Cx \leq 0_{M_3} ; -Dx = 0_{M_4}$; x unrestricted$\}$.

The dual to (iii) turns out to be the following LP, which is closely related to (ii):

$$\text{(iv)} \min_{y^1,y^2,y^3,y^4} \{ y^{1T}1_{M_1} + y^{2T}0_{M_2} + y^{3T}0_{M_3} + y^{4T}0_{M_4} : y^{2T}B + y^{2T}B + y^{3T}C + y^{4T}D$$
$$= -1_{M_1}^T B ;\; y^1 \geq 0_{M_1} ;\; y^2 \geq 0_{M_2} ;\; y^2 \geq 0_{M_3} ;\; y^4 \text{ unrestricted}\}.$$

For this problem, there are three cases to analyze: Case 1: a finite optimal solution for (iii) exists; Case 2: (iii) has an unbounded solution and Case 3: (iii) has no feasible solution.

10. *Factor Price Equalization Theorem*; Samuelson (1958), Diewert and Woodland (1977): Suppose that a country has N production functions of the Leontief no substitution variety where each of the N activities uses varying amounts of M primary inputs. Suppose that each production function produces a single output and suppose that all outputs are traded at the positive world price vector $p \equiv [p_1,p_2,\ldots p_N]^T$. Suppose further that the economy has an endowment of M primary inputs, $v \equiv [v_1,v_2,\ldots,v_M]^T >> 0_M$. The M by N matrix of unit output input requirements is $A \equiv [a_{mn}]$ where each $a_{mn} \geq 0$ for m = 1,…,M and n = 1,…,N. We assume that producers take output prices as fixed and they collectively attempt to maximize the value of their country's output subject to their country's M primary resource constraints; i.e., they attempt to solve the following LP (there are no intermediate inputs in this simple model):

(i) $\max_x \{p^T x : Ax \leq v ; x \geq 0_N\}$

where x is a nonnegative vector of industry outputs. The dual to (i) is:

(ii) $\min_w \{w^T v : w^T A \geq p^T ; w \geq 0_M\}$

where w can be interpreted as vector of primary input prices or resource costs. Thus in the dual problem, we attempt to choose input prices that will minimize the aggregate cost of production subject to the constraints that nonpositive profits are made in each industry. By the complementary slackness conditions for linear programs, we know that for industries that are operated at a positive scale in a solution to (i), then 0 profits will be made in that industry in equilibrium. Industries that are operated at 0 scale will make negative or 0 profits at the equilibrium input prices. Thus if producers collectively solve the above programming problems, then they will have chosen output levels that

[22] *Both* (i) and (ii) cannot hold.

maximize the country's value of output at international prices and they will have also determined competitive input prices for their country.

Consider the following example, which has 3 production activities (N = 3) and two primary inputs (M = 2):

(iii) $\max_{x_0, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0}$

$$\max_{x_0, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0} \{x_{0:} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} -p_1 \\ 2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} -p_2 \\ 0 \\ 2 \end{bmatrix} x_2$$

$$+ \begin{bmatrix} -p_3 \\ 3/2 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} s_2 = \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix} \}.$$

(a) Suppose $p_1 = p_2 = p_3 = 1$. Calculate the optimal dual prices to the LP (iii), w_1^* and w_2^*, and show that they do not depend on the particular values that $v_1 > 0$ and $v_2 > 0$ take on. *Hint*: Apply the Basis Theorem for Linear Programs. *Comment*: If each country in the world had access to the same technology matrix and faced the same vector of world prices[23], then each country which had an endowment vector which was a nonnegative linear combination of the same activity vectors $A_{\bullet n}^*$ in an optimal basis matrix would have the same vector of factor prices w^*. *Thus there is a tendency for trade in commodities to equalize factor prices across countries*. The above result can be generalized to deal with classical production functions and the existence of intermediate inputs; see Samuelson (1958), Diewert and Woodland (1977) and Woodland (1982).

11. Anderson (1958; 349): Let $A(\alpha) \equiv [a_{mn}(\alpha)]$ be an M by M matrix whose elements depend on a scalar parameter α. Suppose when $\alpha = \alpha^0$, $[A(\alpha^0)]^{-1} \equiv B(\alpha^0)$ exists. Let the matrix of derivatives of $A(\alpha)$ evaluated at α^0 be denoted as $A'(\alpha^0) \equiv [da_{mn}(\alpha^0)/d\alpha]$ and let the matrix of derivatives of $B(\alpha)$ evaluated at α^0 be denoted as $B'(\alpha^0) \equiv [db_{mn}(\alpha^0)/d\alpha]$. Show that

(i) $B'(\alpha^0) = -[A(\alpha^0)]^{-1}A'(\alpha^0)[A(\alpha^0)]^{-1}$.

Hint: Differentiate both sides of the matrix equation $A(\alpha^0)[A(\alpha^0)]^{-1} = I_M$ with respect to α.

12. *The Comparative Statics of the Strict Complementary Slackness Case*: Consider the following primal LP where the elements of the c and b vectors depend continuously on a scalar parameter α as do the elements of the A matrix, which is M by N where M < N:

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[23] Hence we are ignoring transport costs. We are also assuming that each producer behaves competitively so there are a lot of assumptions in this model that are unlikely to be completely satisfied in the real world. Moreover, factor price equalization becomes increasingly unlikely as the number of primary inputs (and domestically produced services which are not internationally traded) increases. But the factor price equalization theorem is a "pretty" theoretical result!

(i) $V(\alpha) \equiv \max_x \{c(\alpha)^T x : A(\alpha)x = b(\alpha) ; x \geq 0_N\}$.

Define the square M by M matrix, $\tilde{A}(\alpha^0) \equiv [A_{\bullet 1}(\alpha^0), A_{\bullet 2}(\alpha^0), \ldots , A_{\bullet M}(\alpha^0)]$ and we suppose that this matrix satisfies the following conditions:

(ii) $[\tilde{A}(\alpha^0)]^{-1}$ exists ;
(iii) $\tilde{x}(\alpha^0) \equiv [\tilde{A}(\alpha^0)]^{-1}b(\alpha^0) >> 0_M$; $x(\alpha^0)^T \equiv [\tilde{x}(\alpha^0)^T, 0_{N-M}{}^T]$;
(iv) $y(\alpha^0)^T \equiv [c_1(\alpha^0), c_2(\alpha^0), \ldots , c_M(\alpha^0)][\tilde{A}(\alpha^0)]^{-1} \equiv \tilde{c}(\alpha^0)^T[\tilde{A}(\alpha^0)]^{-1}$;
(v) $y(\alpha^0)^T A_{\bullet n}(\alpha^0) > c_n(\alpha^0)$ for $n = M+1, M+2, \ldots , N$.

From our knowledge of the simplex algorithm, it can be seen that conditions (ii)-(v) are sufficient to imply that the first M columns of the $A(\alpha^0)$ matrix form an optimal basis matrix for the LP defined by (i) when $\alpha = \alpha^0$. Conditions (v) imply that the $x(\alpha^0)$ solution defined in (iii) is the *unique solution* to the primal when $\alpha = \alpha^0$ and the strict inequalities in (iii) along with assumption (ii) implies that $y(\alpha^0)$ defined in (iv) is the *unique solution* to the following dual problem when $\alpha = \alpha^0$:

(vi) $V(\alpha) \equiv \min_y \{y^T b(\alpha) : y^T A(\alpha) \geq c(\alpha) ; y \text{ unrestricted}\}$.

The continuity of the function $A(\alpha)$, $b(\alpha)$ and $c(\alpha)$ means that for α sufficiently close to α^0, the primal solution $x(\alpha)$ to $V(\alpha)$ defined by (i) will be

(vii) $\tilde{x}(\alpha) \equiv [\tilde{A}(\alpha)]^{-1}b(\alpha)$; $x(\alpha)^T \equiv [\tilde{x}(\alpha)^T, 0_{N-M}{}^T]$

and the dual solution $y(\alpha)$ to$V(\alpha)$ defined by (vi) will be

(viii) $y(\alpha)^T \equiv [c_1(\alpha), c_2(\alpha), \ldots , c_M(\alpha)][\tilde{A}(\alpha)]^{-1} \equiv \tilde{c}(\alpha)^T[\tilde{A}(\alpha)]^{-1}$.

(a) Show that if $A(\alpha)$ and $b(\alpha)$ are differentiable at $\alpha = \alpha^0$, then the optimal x solution to (i) is differentiable at $\alpha = \alpha^0$, with

(ix) $x'(\alpha^0)^T \equiv [\tilde{x}'(\alpha^0)^T, 0_{N-M}{}^T]$ and
(x) $\tilde{x}'(\alpha^0) \equiv [\tilde{A}(\alpha^0)]^{-1}b'(\alpha^0) - [\tilde{A}(\alpha^0)]^{-1}\tilde{A}'(\alpha^0)[\tilde{A}(\alpha^0)]^{-1}b(\alpha^0)$.

(b) Show that if $A(\alpha)$ and $c(\alpha)$ are differentiable at $\alpha = \alpha^0$, then the optimal y solution to (vi) is differentiable at $\alpha = \alpha^0$, with

(xi) $y'(\alpha)^T = \tilde{c}'(\alpha)^T[\tilde{A}(\alpha)]^{-1} - \tilde{c}(\alpha)^T[\tilde{A}(\alpha^0)]^{-1}\tilde{A}'(\alpha^0)[\tilde{A}(\alpha^0)]^{-1}$.

(c) Show that if $A(\alpha)$, $b(\alpha)$ and $c(\alpha)$ are differentiable at $\alpha = \alpha^0$, then the optimal primal and dual objective functions $V(\alpha)$ defined by (i) or (vi) are differentiable at $\alpha = \alpha^0$ with

(xii) $V'(\alpha) = \tilde{c}\,'(\alpha)^T[\tilde{A}(\alpha)]^{-1}b(\alpha^0) - \tilde{c}(\alpha)^T[\tilde{A}(\alpha^0)]^{-1}\tilde{A}'(\alpha^0)[\tilde{A}(\alpha^0)]^{-1}b(\alpha^0)$
$+ \tilde{c}(\alpha)^T[\tilde{A}(\alpha)]^{-1}b'(\alpha^0)$.

(d) Specialize the results in (a)-(c) to the case where only A depends on α; i.e., assume b and c are constant vectors.

(e) Specialize the results in (a)-(c) to the case where only b depends on α.

(f) Specialize the results in (a)-(c) to the case where only c depends on α.

Comment: For similar comparative statics results for a general nonlinear programming problem, see Diewert (1984).

13. *Some Global Comparative Statics Results for Linear Programming Problems*; Beckmann (1955-56): Consider the following two linear programs:

(i) $\max_x \{c^{jT}x : A^jx \le b^j ; x \ge 0_N\}$; $\qquad$ j = 0,1

where A^j is an M by N matrix. The dual problems that correspond to the primal problems (i) are:

(ii) $\min_y \{y^Tb^j : y^TA^j \ge c^{jT} ; y \ge 0_M\}$; $\qquad$ j = 0,1.

Suppose that the problems in (i) and (ii) have finite optimal solutions, x^j and y^j respectively when j = 0,1. The *changes* ΔA, Δb, Δc, Δx and Δy are defined as follows:

(iii) $A^1 \equiv A^0 + \Delta A$; $b^1 \equiv b^0 + \Delta b$; $c^1 \equiv c^0 + \Delta c$; $x^1 \equiv x^0 + \Delta x$; $y^1 \equiv y^0 + \Delta y$.

(a) Show that the changes ΔA, Δb, Δc, Δx and Δy and x^0 and y^0 satisfy the following inequality:

(iv) $[\Delta c^T - y^{0T}\Delta A]\Delta x - \Delta y^T[\Delta b - \Delta Ax^0] \ge 0$.

Hint: Use the Saddlepoint inequalities (118) for the LP's in (i).

(b) Specialize (iv) to the case where only b changes. Further specialize (iv) to the case where only the first component of b changes. Provide an economic interpretation for the resulting formula.

(c) Specialize (iv) to the case where only c changes. Further specialize (iv) to the case where only the first component of c changes. Provide an economic interpretation for the resulting formula.

(d) Specialize (iv) to the case where only one component of A changes, say a_{mn} changes.

(e) *Bailey's* (1955-56) *Inequalities*: Show that under the above conditions, the following two inequalities are also valid:[24]

$$\begin{aligned}\text{(v) } \Delta[c^T - y^TA]\Delta x &\equiv [(c^{1T} - y^{1T}A^1) - (c^{0T} - y^{0T}A^0)]\Delta x \\ &= [\Delta c^T - y^{1T}A^1 + y^{0T}A^0]\Delta x \\ &= [\Delta c^T - \Delta y^TA^1 - y^{0T}\Delta A]\Delta x \\ &= [\Delta c^T - y^{1T}\Delta A - \Delta y^TA^0]\Delta x \\ &\geq 0.\end{aligned}$$

$$\begin{aligned}\text{(vi) } \Delta y^T\Delta[b - Ax] &\equiv \Delta y^T[(b^1 - A^1x^1) - (b^0 - A^0x^0)] \\ &= \Delta y^T[\Delta b - A^1x^1 + A^0x^0] \\ &= \Delta y^T[\Delta b - \Delta Ax^1 - A^0\Delta x] \\ &= \Delta y^T[\Delta b - A^1\Delta x^1 - \Delta Ax^0] \\ &\leq 0.\end{aligned}$$

Hint: Define the slack vectors for the primal problems as $s^j \equiv b^j - A^jx^j \geq 0_M$ for j = 0,1. Recall that $y^{jT}s^j = 0$ for j = 0,1. Now note that $\Delta y^T\Delta[b - Ax] = [y^{1T} - y^{0T}][s^1 - s^0] = y^{1T}s^1 + y^{0T}s^0 - y^{1T}s^0 - y^{0T}s^1$. Using the complementary slackness conditions for each primal LP, the inequality (vi) follows readily. A similar analysis works for (v).

14. Define the primal and dual LP's for two c vectors as follows:

(i) $V(c^j) \equiv \max_x \{c^{jT}x : Ax \leq b \,;\, x \geq 0_N\}$; j = 1,2 ;
(ii) $V(c^j) \equiv \min_y \{y^Tb : y^TA \geq c^{jT} \,;\, y \geq 0_M\}$; j = 1,2 .

Suppose that feasible solutions exist for both the primal and dual problems defined by (i) and (ii) so that each problem has a finite optimal solution. Let λ be a scalar such that $0 < \lambda < 1$ and define the following LP:

(iii) $V(\lambda c^1 + (1-\lambda)c^2) \equiv \max_x \{[\lambda c^1 + (1-\lambda)c^2]^Tx : Ax \leq b \,;\, x \geq 0_N\}$.

(a) Show that a finite optimal solution for the LP defined by (iii) exists. *Hint*: Show that a feasible solution exists for (iii) and its dual.

(b) Show that:

(iv) $V(\lambda c^1 + (1-\lambda)c^2) \leq \lambda V(c^1) + (1-\lambda)V(c^2)$.

Comment: This problem shows that the optimized objective function of a linear programming problem is a *convex function* in the c vector.

15. Define the primal and dual LP's for two b vectors as follows:

[24] These two inequalities imply Beckmann's inequalities.

(i) $V(b^j) \equiv \max_x \{c^Tx : Ax \le b^j ; x \ge 0_N\}$; $j = 1,2$;
(ii) $V(b^j) \equiv \min_y \{y^Tb^j : y^TA \ge c^T ; y \ge 0_M\}$; $j = 1,2$.

Suppose that feasible solutions exist for both the primal and dual problems defined by (i) and (ii) so that each problem has a finite optimal solution. Let λ be a scalar such that $0 < \lambda < 1$ and define the following LP:

(iii) $V(\lambda b^1 + (1-\lambda)b^2) \equiv \max_x \{c^Tx : Ax \le \lambda b^1 + (1-\lambda)b^2 ; x \ge 0_N\}$.

(a) Show that a finite optimal solution for the LP defined by (iii) exists. *Hint*: Show that a feasible solution exists for (iii) and its dual.

(b) Show that:

(iv) $V(\lambda b^1 + (1-\lambda)b^2) \ge \lambda V(b^1) + (1-\lambda)V(b^2)$.

Comment: This problem shows that the optimized objective function of a linear programming problem is a *concave function* in the right hand side vector b over the set of b's for which there is a feasible solution for the primal problem.

16. Consider the LP defined by (99) and (100) and suppose that $[b_1^*,b_2^*] = [2,1]$ so that the b vector is a multiple of $A_{\bullet 3}$ and b lies on the dashed line through $A_{\bullet 3}$ in Figure 2. Now suppose b_1 *increases* by a marginal unit from its initial value so that the new $[b_1,b_2^*]$ lies in the interior of the cone 3 region. Regard the optimized objective function of the LP defined by (100) as a function of b_1, say $V(b_1)$.

(a) Show that for $[b_1,b_2^*]$ in the interior of the cone 3 region, the derivative of the optimal objective function with respect to b_1 is

(i) $dV(b_1)/db_1 = y_1^{(3)} = 1/2$.

Hint: Recall (106) and use part (xii) of Problem 12 above.

Now suppose b_1 *decreases* by a marginal unit from its initial value of $[b_1^*,b_2^*] = [2,1]$ so that the new $[b_1,b_2^*]$ lies in the interior of the cone 2 region. Again, regard the optimized objective function of the LP defined by (100) as a function of b_1, say $V(b_1)$.

(b) Show that for $[b_1,b_2^*]$ in the interior of the cone 2 region, the derivative of the optimal objective function with respect to b_1 is

(ii) $dV(b_1)/db_1 = y_1^{(2)} = 2/3$.

Hint: Recall (102) and use part (xii) of Problem 12 above.

(c) Use parts (a) and (b) above to show that

(iii) $dV^+(b_1^*)/db_1 \equiv \lim_{h\to 0,\, h>0} [V(b_1^* + h) - V(b_1^*)]/h = y_1^{(3)}$ and
(iv) $dV^-(b_1^*)/db_1 \equiv \lim_{h\to 0,\, h<0} [V(b_1^* + h) - V(b_1^*)]/h = y_1^{(2)}$.

Comment: This problem shows that for the LP defined by (100), the optimized objective function is not always differentiable with respect to the components of the right hand side b vector. However, the *one sided derivatives* or *directional derivatives* of V(b) always exist. This problem also casts a bit more light on the meaning of the dual prices in the general case (as opposed to the special situation that we had in problem 12 where the dual price vector was locally unique).

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