Preventing Self-fulfilling Debt Crises: The Role of Expectations

Appendix
(For Online Publication)

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This appendix contains the proofs of the results that have been stated in the paper and is divided into six sections. In Section A I solve the main model. This section contains the proofs of Lemma 1 and Lemma 2, the main uniqueness result (Proposition 1), and the proof of Lemma 3. Section B contains derivations of the direct and multiplier effects and the proofs of Propositions 2, 3, and 4 from the paper. Section C contains derivations of the total change in the default threshold when the agents expect the policy to be implemented with probability p, i.e., $dA^*/d\psi(p)$. In Section D I briefly discuss how the results would change if Assumption 4 was not imposed. Section E contains a discussion of the effect of an adjustment in the interest rate on the effects of policy changes while Section F contains several technical claims invoked in proofs throughout the Appendix.[1]

A Global Game model

A.1 Uniqueness Result

Proposition 1 *There exist $\overline{\varepsilon} > 0$ and $\overline{\sigma}_x > 0$ such that for all $\varepsilon \in (0, \overline{\varepsilon}]$ and all $\sigma_x \in (0, \overline{\sigma}_x]$ the model has a unique equilibrium in monotone strategies.*

To prove the above result, I first characterize the optimal households' and lenders' strategies in response to a monotone default strategy by the government. Then I show

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[1]The solution to the complete information version of the model, and detailed derivations of the multiplier and direct effects when agents are uncertain whether announced policies will be implemented, can be found in the "Additional Results" document available on the author's website (http://econ.sites.olt.ubc.ca/files/2016/01/pdf_szkup_debt_crises_additional.pdf)

that in response to these households' and lenders' strategies the government indeed finds it optimal to follow a monotone default strategy. Finally, I show that there exists a unique fixed-point of this argument. Before proceeding any further I introduce notation that will be useful when analyzing the model.

Notation 1 *I will use the following notation throughout the Appendix:*

1. *A^* denotes the default threshold used by the government.*

2. *A^{**} denotes the default threshold expected by the households and lenders.*

A.1.1 Households

Suppose that households expect the government to repay its debt if and only if $A \geq A^{**}$. Household i's optimal investment then solves the household's problem specified in Section 3.4. Each household receives a productivity shock A_i, where $A_i = A + \varepsilon_i$ and $\varepsilon_i \in [-\varepsilon, \varepsilon]$.

If $A_i > A^{**} + \varepsilon$, then household i expects no default; in that case,

$$k_2(A_i) = (1-\tau) e^{A_i} f(k_1) \frac{\alpha}{1+\alpha}.$$ [2]

If household i receives productivity $A_i < A^{**} - \varepsilon$, then household i believes that the government will always default and

$$k_2(A_i) = (1-\tau) e^{A_i} f(k_1) \frac{\alpha Z}{1+\alpha}.$$

Finally, in the case when $A_i \in (A^{**} - \varepsilon, A^{**} + \varepsilon)$ the household is uncertain as to whether the government will default. In that case,

$$k_2(A_i) = (1-\tau) e^{A_i} f(k_1) \Lambda(A_i; \varepsilon, A^{**})$$

where

$$\begin{aligned} \Lambda(A_i; \varepsilon, A^{**}) \quad = \quad & \frac{\alpha(1+Z) + P(A^{**}|A_i) + Z(1 - P(A^{**}|A_i))}{2(1+\alpha)} \\ & - \frac{\sqrt{[\alpha(1+Z) + P(A^{**}|A_i) + Z(1 - P(A^{**}|A_i))]^2 - 4\alpha Z(1+\alpha)}}{2(1+\alpha)} \end{aligned}$$

It is straightforward to show that $\Lambda(A_i; \varepsilon, A^{**})$ is increasing in A_i and decreasing in A^{**}. This establishes Lemma 1 in the paper.

[2] It is here that the assumption of full depreciation of households' capital simplifies the model. When the capital depreciates fully each period, the optimal choice of capital is linear. As we will see below, this will make the government's default condition near linear in e^A.

Next, I perform a change of variables $\kappa = \frac{\varepsilon_i}{\varepsilon}$, where $\varepsilon_i \in [-\varepsilon, \varepsilon]$ so that $\kappa \in [-1, 1]$. This change of variables turns out to be useful for computing the output in the limiting case as $\varepsilon \to 0$, and in general, when analyzing the effect of changes in ε. Define

$$\Lambda\left(A + \kappa\varepsilon; \kappa, A^{**}\right) \equiv \begin{cases} \frac{\alpha}{(1+\alpha)} & \text{when } A_i = A + \kappa\varepsilon > A^{**} + \varepsilon \\ \Lambda\left(A_i; \varepsilon, A^{**}\right) & \text{when } A_i = A + \kappa\varepsilon \in \left(A^{**} - \varepsilon, A^{**} + \varepsilon\right) \\ \frac{\alpha Z}{(1+\alpha)} & \text{when } A_i = A + \kappa\varepsilon < A^{**} - \varepsilon \end{cases}$$

In what follows I will denote the optimal choice of capital as $k_2^*\left(A, \kappa, A^{**}\right)$ to emphasize its dependence on A, κ and household's belief about the default threshold A^{**}.

A.1.2 Lenders[3]

Denote by $p_x = 1/\sigma_x^2$ and $p_A = 1/\sigma_A^2$ the precisions of the lenders' private signals and the prior, respectively. As usual, it is more convenient to work with precisions rather than standard deviations or variances.

Let $u\left(1, A; x^{**}, A^{**}\right)$ be the expected payoff to lender j from lending to the government when the average productivity is equal to A, the government uses a threshold strategy with cutoff A^{**}, and the other lenders use monotone strategies with cutoff x^{**}. Similarly, denote by $u\left(0, A; x^{**}, A^{**}\right)$ the payoff to lender j from investing in the risk-free asset. Then

$$\begin{aligned} u\left(1, A; x^{**}, A^{**}\right) &= \begin{cases} 1 + r \min\left\{\frac{B_2^{R,u}(A)}{S(A; x^{**})}, 1\right\} & \text{if } A \geq A^{**} \\ 0 & \text{otherwise} \end{cases} \\ u\left(0, A; x^{**}, A^{**}\right) &= 1 \end{aligned}$$

Define $\Delta u\left(A; x^{**}, A^{**}\right) \equiv u\left(1, A; x^{**}, A^{**}\right) - u\left(0, A; x^{**}, A^{**}\right)$.

It is immediate to see that for any pair $\left(A^{**}, x^{**}\right)$, and regardless of the government's desired borrowing function $B_2^{R,u}$, the function $\Delta u\left(A; x^{**}, A^{**}\right)$ satisfies a weak single crossing property in A.[4] Moreover, it is well-known that a family of normal density functions parameterized by x_j

$$\left\{ \left(p_x + p_A\right)^{1/2} \phi\left(\frac{A - \frac{p_x x_j + p_A A_{-1}}{p_x + p_A}}{\left(p_x + p_A\right)^{-1/2}} \right) \right\}_{x_j \in \mathbb{R}}$$

[3] In this section I make use of two results established in Athey (1996). The first of the results, Theorem 3.2 in Athey (1996), establishes that if g satisfies the weak single-crossing property, and if k is strictly log-supermodular and $k\left(s, \theta\right)$ has constant support in θ, then $G\left(\theta\right) \equiv \int_S g\left(s\right) k\left(s; \theta\right) ds$ satisfies the strict single-crossing property in θ. Theorem 3.4 in Athey (1996) extends this conclusion to the case where g also depends on θ under the additional assumption of piecewise continuity of g .

[4] A function $f\left(x\right)$, where $f\colon \mathbb{R} \to \mathbb{R}$, satisfies a weak single-crossing property in x if for all $x_H > x_L$, $f\left(x_L\right) > 0$ implies $f\left(x_H\right) \geq 0$.

satisfies the strict monotone likelihood ratio (MLR) property, implying that $(p_x + p_A)^{1/2} \phi \left(\frac{A - \frac{p_x x_j + p_A A_{-1}}{p_x + p_A}}{(p_x + p_A)^{-1/2}} \right)$ is strictly log-supermodular in (A, x_j) (see Athey, 1996). By Theorem 3.2 in Athey (1996),

$$\Delta U (x_j; x^*, A^{**}) \equiv \int_{A^{**}}^{\infty} \Delta u (A; x^{**}, A^{**}) (p_x + p_A)^{1/2} \phi \left(\frac{A - \frac{p_x x_j + p_A A_{-1}}{p_x + p_A}}{(p_x + p_A)^{-1/2}} \right) dA$$

satisfies the strict single-crossing property in A^{**}. Thus, in response to monotone strategies by the government and the other lenders, lender j finds it optimal to follow a monotone strategy.

Consider $\Delta U (x^*; x^*, A^{**})$, the expected utility difference from supplying the funds to the market versus not supplying them, evaluated at x^* and let $L (A^{**}, x^*) \equiv \Delta U (x^*; x^*, A^{**})$. I want to show that for each A^{**} there exists unique x^* such that $L (A^{**}, x^*) = 0$. First note that $\Delta u (A; x^*, A^{**})$ as defined above is increasing in x^*. This is because $S (A; x^*) = b \left(1 - \Phi \left(\frac{x^* - A}{p_x^{-1/2}} \right) \right)$ is decreasing in x^*. Moreover, for all $A \geq A^{**}$ $B_2^{R,u} (A)$ is differentiable in A and therefore $\Delta u (A; x^*, A^{**})$ is piecewise continuous. Second, note that $\Delta u (A; x^*, A^{**}) (p_x + p_A)^{1/2} \phi \left(\frac{A - \frac{p_x x^* + p_A A_{-1}}{p_x + p_A}}{(p_x + p_A)^{-1/2}} \right) \neq 0$, at least for all $A < A^{**}$. Then, by Theorem 3.4 in Athey (1996) it follows that $L (A^{**}, x^*)$ satisfies a strict single-crossing condition in x^*. This proves Lemma 2 in the text.

A.1.3 The Government's Monotone Default Strategy

Suppose that the households follow investment strategies as characterized above and the lenders use monotone strategies with a common threshold x^*. I show that $\Delta V (A, \mathbf{k}_2^*, S)$ is strictly increasing in A.

Define $\mathbf{k}_2^* (A, A^{**}) \equiv \{k_2 (A, \kappa, A^{**})\}_{\kappa \in [-1,1]}$, that is, $\mathbf{k}_2^* (A)$ denotes the households' investment choices when the average productivity is equal to A and when all households expect that the default threshold is A^{**}. Note that if the lenders follow monotone strategies, then $S = b \left[1 - \Phi \left(\frac{x^* - A}{\sigma_x} \right) \right]$. Thus, with a slight abuse of notation I will write $\Delta V (A, \mathbf{k}_2^* (A, A^{**}), S)$ as $\Delta V (A; \mathbf{k}_2^* (A, A^{**}), x^*)$. Finally, let $B_2^{R,u}$ denote the government optimal unconstrained borrowing (see also Section 3.2 in the paper).

Using the definition of $\Delta V (A, \mathbf{k}_2^* (A, A^{**}), x^*)$, substituting for $\mathbf{k}_2^* (A)$ the expression found in Section $A.1.1$ and rearranging, we get

$$\begin{aligned} \Delta V (A, \mathbf{k}_2^* (A, A^{**}), x^*) \quad = \quad & \int_{-1}^{1} \frac{1}{2} \log \left(\frac{1 - \Lambda (A + \kappa\varepsilon, \kappa, A^{**})}{Z - \Lambda (A + \kappa\varepsilon, \kappa, A^{**})} \right) d\kappa + \log \left(\frac{\tau Y_1^R - B_1 + B_2^{R^*}}{\tau Z Y_1^R + (1 - \xi) B_2^{D^*}} \right) \\ & + \log \left(\frac{1}{Z} \right) + \log \left(\frac{\tau Y_2^R - (1 + r) B_2^{R^*}}{Z \tau Y_2^R} \right), \end{aligned}$$

where

$$B_2^{R^*} = \begin{cases} B_2^{R,u}(A) & \text{if } B_2^{R,u} \leq S(A, x^*) \\ S(A, x^*) & \text{if } B_2^{R,u} > S(A, x^*) \end{cases} .$$

Differentiating with respect to A, simplifying, and taking the limit as $\xi \to 1$, we get

$$\frac{\partial \Delta V\left(A; \mathbf{k}_2^*\left(A, A^{**}\right), x^*, A^*\right)}{\partial A} \geq \frac{B_1 - B_2^{R^*}}{\tau Y_1^R - B_1 + B_2^{R^*}} + \frac{(1+\alpha) B_2^{R^*} (1+r)}{\tau Y_2^R - (1+r) B_2^{R^*}}. \qquad (1)$$

where I used the observation that if $B_2^{R^*} = B_2^{R,u}(A)$, then by the optimality of the government borrowing choices the terms containing $\partial B_2^{R^*}/\partial A$ add up to 0, while otherwise their sum is strictly positive.

Add the above fractionson the right-hand side of 1. The resulting numerator can be written as

$$2(1+r)\left(B_2^{R^*}\right)^2 - B_2^{R^*}\left(\tau Y_2^R + 2(1+r) B_1 - (1+r)\tau Y_1^R\right) + B_1 \tau Y_2^R.$$

This expression is quadratic in $B_2^{R^*}$. Let $B_2^{R^*,1}(A)$ and $B_2^{R^*,2}(A)$ be its two roots. Whether these roots are real or not depends on the parameters of the model. For all $A \in \left[\underline{A}, \overline{A}\right]$, define $\bar{b}(A) = \min\left\{B_2^{R^*,1}(A), B_2^{R^*,2}(A)\right\}$ if the roots are real, and $\bar{b}(A) = \infty$ if they are complex. Let $\bar{b} = \min_{A \in \left[\underline{A}, \overline{A}\right]} \bar{b}(A)$. It follows that if $b < \bar{b}$ then the government's best response to monotone strategies is itself monotone. I assume that the lenders' wealth b satisfies this constraint (Assumption 3 in the paper).[5]

A.1.4 Uniqueness of Equilibrium

In light of the above results, to establish uniqueness it is enough to show that

$$\Delta V\left(A^*, \mathbf{k}_2^*\left(A^*, A^*\right), x^*\left(A^*\right)\right)$$

is monotone in A^*, where $\mathbf{k}_2^*(A^*) \equiv \{k_2(A^*, \kappa, A^*)\}_{\kappa \in [-1,1]}$ is a vector whose components are the individual households' investment strategies when the households have the correct expectations about the default threshold (i.e., $A^{**} = A^*$), and x^* is the common signal threshold used by the lenders when households and lenders expect the default threshold to be A^*. I denote the optimal lender's threshold by $x^*(A^*)$, to emphasize that it depends on A^*.

[5] One may wonder how restrictive this assumption is. The answer is that it depends on the parameters. However, numerical simulations suggest that unless α or Z is very close to 1 both roots are complex, which means that the bound can be made arbitrarily large (though it has to be finite). In particular, this is the case for the calibration used in the paper.

Fix $\eta > 0$, where η is a small positive number. Differentiating $\Delta V\left(A^*; \mathbf{k}_2^*\left(A^*\right), x^*\left(A^*\right)\right)$ with respect to A^* and taking the limit as $\xi \to 1$ we get

$$\begin{aligned}\frac{d\Delta V}{dA^*} &= \int_{-1}^{1} \frac{-\frac{\partial \Lambda}{\partial A^*}\left[Z-\Lambda\right]+\left[1-\Lambda\right] Z \frac{\partial \Lambda}{\partial A^*}}{\left[1-\Lambda\right]\left[Z-\Lambda\right]} d\kappa \\ &\quad + \frac{\frac{dB_2^{R^*}}{dA^*}}{\tau Y_1^R - B_1 + B_2^{R^*}} - \frac{\left(1+r\right)\frac{\partial B_2^{R^*}}{\partial A^*}}{\tau Y_2^R - \left(1+r\right) B_2^{R^*}} \\ &\quad + \frac{B_1 - B_2^{R^*}}{\tau Y_1^R - B_1 + B_2^{R^*}} + \frac{\left(1+\Psi\right)\left(1+r\right) B_2^{R^*}}{\tau Y_2^R - \left(1+r\right) B_2^{R^*}},\end{aligned}$$

where

$$\Psi \equiv \frac{\int_{-1}^{1} \frac{1}{2} \frac{\partial}{\partial A^*} f\left(k_2\left(A^* + \kappa\varepsilon; \varepsilon, A^*\right)\right) d\kappa}{Y_2^R} \to \alpha \text{ as } \varepsilon \to 0.$$

Since

$$\lim_{\varepsilon \to 0} \frac{\partial \Pr\left(A^* | A^* + \kappa\varepsilon\right)}{\partial A^*} \to 0,$$

there exists $\overline{\varepsilon}$ such that for all $\varepsilon \in (0, \overline{\varepsilon}]$ we have

$$\int_{-1}^{1} \frac{-\frac{\partial \Lambda}{\partial A^*}\left[Z-\Lambda\right]+\left[1-\Lambda\right] Z \frac{\partial \Lambda}{\partial A^*}}{\left[1-\Lambda\right]\left[Z-\Lambda\right]} d\kappa < \frac{\eta}{2}$$

Next, since $\frac{\partial S(A^*)}{\partial A^*} > -b \frac{p_A}{p_x^{1/2}} \frac{1}{\sqrt{2\pi}} \to 0$ as $p_x \to \infty$, it follows that there exists a large enough $\overline{p}_x$ such that for all $p_x > \overline{p}_x$ we have

$$\frac{\frac{dB_2^{R^*}}{dA^*}}{\left[\tau Y_1^R - B_1 + B_2^{R^*}\right]} - \frac{\left(1+r\right)\frac{\partial B_2^{R^*}}{\partial A^*}}{\left[\tau Y_2^R - \left(1+r\right) B_2^{R^*}\right]} > -\frac{\eta}{2}.$$[6]

Finally, following the same argument as in Section $A.1.3$ one can show that there exists $\overline{\overline{b}}\left(\varepsilon\right)$ such that for all $b < \overline{\overline{b}}\left(\varepsilon\right)$ we have

$$\frac{B_1 - B_2^{R^*}}{\tau Y_1^R - B_1 + B_2^{R^*}} + \frac{\left(1+r\right) B_2^{R^*}}{\left[\tau Y_2^R - \left(1+r\right) B_2^{R^*}\right]} > \eta.$$

Therefore, for all ε with $0 < \varepsilon < \overline{\varepsilon}$ and all $p_x > \overline{p}_x$ we have

$$\frac{d\Delta V}{dA^*} > -\frac{\eta}{2} - \frac{\eta}{2} + \eta = 0$$

implying that there exists a unique default threshold A^* that satisfies all the equilibrium conditions.

The above analysis applies to a fixed value of A^*. However, since $A^* \in \left[\underline{A}, \overline{A}\right]$, which is a compact interval, there exists bounds $\overline{\varepsilon}$ and $\overline{p}_x$ which are independent of A^*, such that if $\varepsilon < \overline{\varepsilon}$ and $p_x < \overline{p}_x$, then $d\Delta V / dA^*$ is strictly positive for all $A^* \in \left[\underline{A}, \overline{A}\right]$. This completes the proof.

[6] If $\partial B_2^{R^*} / \partial A^* = \partial B_2^{R,u} / \partial A^*$, then the sum of these terms is 0.

A.2 Optimal Interest Rate

Proof of Lemma 3. Let $W(r)$ denote the expected payoff to the government from choosing interest rate r. Consider first $A_{-1} > A^*(0)$, and fix $r > 0$.[7] I show that for sufficiently low σ_A there exists r' with $0 < r' < r$ such that $W(r') > W(r)$. To see this, consider any $r' \in (0, r)$ such that $A^*(r') > A^*(0)$.[8] If $A^*(r') \leq A^*(r)$, then the result follows immediately (for any $\sigma_A > 0$). Thus, assume that $A^*(r') > A^*(r)$. Then

$$\begin{aligned} W(r) - W(r') &= \int_{A^*(r')}^{\infty} \left[V_1^R(A, \mathbf{k}_2, S; r) - V_1^R(A, \mathbf{k}_2, S; r')\right] f(A|A_{-1})\, dA \\ &\quad + \int_{A^*(r)}^{A^*(r')} \left[V_1^R(A, \mathbf{k}_2, S; r) - V_1^D(A, \mathbf{k}_2, S)\right] f(A|A_{-1})\, dA. \end{aligned}$$

Note that $V_1^R(A, \mathbf{k}_2, S; r) - V_1^R(A, \mathbf{k}_2, S; r') < 0$ for all $A > A^*(r')$. Let C be a positive constant. Since $A_{-1} > A^*(0) \geq A^*(r')$, we have

$$\begin{aligned} & \int_{A^*(r')}^{\infty} \left[V_1^R(A, \mathbf{k}_2, S; r) - V_1^R(A, \mathbf{k}_2, S; r')\right] f(A|A_{-1})\, dA \\ < & \int_{A^*(r')}^{A_{-1}+C} \left[V_1^R(A, \mathbf{k}_2, S; r) - V_1^R(A, \mathbf{k}_2, S; r')\right] f(A|A_{-1})\, dA \\ < & \max_{A \in [A^*(r), A_{-1}+C]} \left[V_1^R(A, \mathbf{k}_2, S; r) - V_1^R(A, \mathbf{k}_2, S; r')\right] \left[F(A_{-1}+C|A_{-1}) - F(A^*(r')|A_{-1})\right] \\ \to & \max_{A \in [A^*(r), A_{-1}+C]} \left[V_1^R(A, \mathbf{k}_2, S; r) - V_1^R(A, \mathbf{k}_2, S; r')\right] < 0 \quad \text{as } \sigma_A \to 0. \end{aligned}$$

On the other hand, for all $A \in (A^*(r), A^*(r'))$, the government defaults if the interest rate is r' and repays its debt when the interest rate is r. Moreover, we know that $V_1^R(A, \mathbf{k}_2, S; r) - V^D(A) > 0$ for all $A \in (A^*(r), A^*(r'))$. Therefore,

$$\begin{aligned} & \int_{A^*(r)}^{A^*(r')} \left[V_1^R(A, \mathbf{k}_2, S; r) - V^D(A, \mathbf{k}_2, S)\right] f(A|A_{-1})\, dA \\ < & \left[\max_{A \in [A^*(r'), A^*(r)]} \left\{V_1^R(A, \mathbf{k}_2, S; r) - V^D(A)\right\}\right] \left[F(A^*(r')|A_{-1}) - F(A^*(r)|A_{-1})\right] \\ \to & \; 0 \quad \text{as } \sigma_A \to 0. \end{aligned}$$

Thus, $\lim_{\sigma_A \to 0}[W(r) - W(r')] < 0$. Since r was arbitrary, the same argument holds for any $r > 0$; thus, for all $A_{-1} > A^*(0)$ we must have $r^*(\sigma_A) \to 0$ as $\sigma_A \to 0$.

Using an analogous argument, one can establish that for each $A_{-1} \in \left[\min_{r \in [0, \bar{r}]} A^*(r), A^*(0)\right]$ the optimal interest rate is equal to r^* where $r^* = \min\{r : A^*(r) = A_{-1}\}$. ■

[7] Note that $A^*(0) = \max_{r \in [0, \bar{r}]} A^*(r)$.

[8] Such r' exists since $W(r)$ is increasing in r in the neighborhood of r_F.

B Policy Analysis

Let ψ denote a parameter of the model (for concreteness, one can think of the tax rate, in which case $\psi = \tau$). Then, for given r^*, the equilibrium conditions can be written as

$$I\left(A^* + \kappa\varepsilon, A^{**}, k_2^*\left(\kappa\right), \psi\right) = 0,$$

which is the equilibrium condition for a households with productivity $A^* + \kappa\varepsilon$ and which determines the capital choice for a household with productivity shock $\kappa\varepsilon$;

$$L\left(A^{**}, x^*, \psi\right) = 0,$$

which is the equilibrium condition that describing the lenders' behavior and which determines x^*; and finally,

$$\Delta V\left(A^*, \left\{k_2^*\left(\kappa\right)\right\}_{\kappa \in [-1,1]}, x^*, \psi\right) = 0$$

which is the equilibrium condition that describes the government's default decision and determines A^*.[9]

Note that, for each $\kappa \in [-1, 1]$, the equation $I\left(A^* + \kappa\varepsilon, A^*, k_2^*\left(\kappa\right), \psi\right) = 0$ specifies $k_2^*\left(\kappa\right)$ as a function of household's productivity $A^* + \kappa\varepsilon$, household's belief about the default threshold A^{**}, and the policy parameter ψ. for each $\kappa \in [-1, 1]$. Similarly, the equation $L\left(A^*, x^*, \psi\right) = 0$ determines x^* as a function of the lenders' belief about the default threshold A^{**} and ψ. Without loss of generality, I assume that the households hold the same belief as the lenders in regard to the default threshold. In equilibrium, $A^{**} = A^*$, that is the households and lenders hold correct beliefs about the government's default decision. However, to derive the effect of a change in the households' and lenders' beliefs on the default threshold, we have to differentiate between the belief about the threshold held by the households and lenders and the actual default threshold, where the latter is defined as the level of productivity at which the government defaults.

B.1 The Effect of a Change in ψ on A^*

To compute the equilibrium change in A^* due to a change in ψ, I compute the total derivatives of the expressions on the both sides of equilibrium conditions and solve the

[9] Note that this condition implicitly assumes that the government's borrowing and spending decisions are optimal. In other words, $\Delta V = 0$ determines the productivity default threshold, given that the government behaves optimally in the case when it repays its debt as well as in the case when it chooses to default.

resulting linear system of equations for $dA^*/d\psi$:

$$I_1(\kappa)\frac{dA^*}{d\psi}+I_2(\kappa)\frac{dA^{**}}{d\psi}+I_3(\kappa)\frac{dk_2^*(\kappa)}{d\psi}+I_4(\kappa) = 0 \tag{2}$$

$$L_1\frac{dA^{**}}{d\psi}+L_2\frac{dx^*}{d\psi}+L_3 = 0 \tag{3}$$

$$\Delta V_1\frac{dA^*}{d\psi}+\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{dk_2^*(\kappa)}{d\psi}d\kappa+\Delta V_3\frac{dx^*}{d\psi}+\Delta V_4 = 0 \tag{4}$$

where I_n is the partial derivative of $I(A^*+\kappa\varepsilon, A^{**}, k_2^*(\kappa), \psi)$ with respect to its nth argument and similarly for L_n and ΔV_n. $dA^{**}/d\psi$ is the total change in agents' beliefs regarding the government default threshold implied by a change in ψ. In equilibrium, $dA^{**}/d\psi = dA^*/d\psi$, but for now it is important to keep the distinction between the two objects.

Solving for $dx^*/d\psi$ and $dk_2^*/d\psi$ using Equations (3) and (2) we get

$$\begin{aligned}\frac{dx^*}{d\psi} &= -\frac{L_1}{L_2}\frac{dA^{**}}{d\psi}-\frac{L_3}{L_2}\\ \frac{dk_2^*(\kappa)}{d\psi} &= -\frac{I_1(\kappa)}{I_3(\kappa)}\frac{dA^*}{d\psi}-\frac{I_2(\kappa)}{I_3(\kappa)}\frac{dA^{**}}{d\psi}-\frac{I_4(\kappa)}{I_3(\kappa)}\end{aligned}$$

or, recognizing that $\partial x^*/\partial A^{**} = -L_1/L_2$, $\partial k_2^*(\kappa)/\partial A^* = -I_1(\kappa)/I_3(\kappa)$, $\partial k_2^*(\kappa)/\partial A^{**} = -I_2(\kappa)/I_3(\kappa)$, and $\partial k_2^*(\kappa)/\partial\psi = -I_4(\kappa)/I_3(\kappa)$:

$$\begin{aligned}\frac{dx^*}{d\psi} &= \frac{\partial x^*}{\partial A^{**}}\frac{dA^{**}}{d\psi}+\frac{\partial x^*}{\partial\psi}\\ \frac{dk_2^*(\kappa)}{d\psi} &= \frac{\partial k_2^*(\kappa)}{\partial A^*}\frac{dA^*}{d\psi}+\frac{\partial k_2^*(\kappa)}{\partial A^{**}}\frac{dA^{**}}{d\psi}+\frac{\partial k_2^*(\kappa)}{\partial\psi}\end{aligned}$$

Substituting the above expressions into Equation (4) and rearranging, we get

$$\left[\Delta V_1+\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(\kappa)}{\partial A^*}d\kappa\right]\frac{dA^*}{d\psi} = \tag{5}$$

$$-\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\left[\frac{\partial k_2^*(\kappa)}{\partial A^{**}}\frac{dA^{**}}{d\psi}+\frac{\partial k_2^*(\kappa)}{\partial\psi}\right]d\kappa-\Delta V_3\left[\frac{\partial x^*}{\partial A^{**}}\frac{dA^{**}}{d\psi}+\frac{\partial x^*}{\partial\psi}\right]-\Delta V_4,$$

where $\left[\Delta V_1+\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(\kappa)}{\partial A^*}d\kappa\right]$ captures the effect of an increase in the productivity on the government's incentives to default.

At this point it is key to differentiate between a change in the households' investments due to a change in the households' strategies and a change in the households' investments due to merely a change in productivity holding households' strategies fixed. Recall that an individual household's investment strategy is a function that maps the individual productivity into an investment choice, that is it is a map $k_2^*: A_i \to \mathbb{R}$. Thus, a change in the household's strategy is defined as a shift in this mapping, that is a change in k_2^* for each A_i. On the other hand, holding household strategies constant, a change in A_i also affects household i's investments: It is simply a movement along the curve $k_2 : A_i \to \mathbb{R}$. Thus, the term $\Delta V_1 + \int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^*} d\kappa$ captures the effect of a change in the productivity on the government's incentives to default holding households' and lenders' strategies constant.

Using the above observation, divide Equation (5) by $\Delta V_1 + \int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^*} d\kappa$ to obtain

$$\begin{aligned}\frac{dA^*}{d\psi} = & \frac{-\int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial \psi} d\kappa}{\Delta V_1 + \int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^*} d\kappa} - \frac{\Delta V_3 \frac{\partial x^*}{\partial \psi}}{\Delta V_1 + \int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^*}} - \frac{\Delta V_4}{\Delta V_1 + \int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^*}} \\ & - \frac{\int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa}{\Delta V_1 - \int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^*}} \frac{dA^{**}}{d\psi} - \frac{\Delta V_3 \frac{\partial x^*}{\partial A^{**}}}{\Delta V_1 - \int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^*}} \frac{dA^{**}}{d\psi}\end{aligned}$$

The first three terms capture the direct effects of a change in ψ on the equilibrium strategies of the households', the lenders' and the government, respectively, holding households' and lenders' beliefs about the default threshold constant (i.e., holding A^{**} constant). The two remaining terms capture the effect of a change in ψ has on the the households' and lenders' beliefs. In particular, note that

$$\frac{\partial A^*}{\partial \psi} = -\frac{\Delta V_4}{\Delta V_1 + \int_{-1}^{1} \Delta V_2 \frac{\partial k_2^*(\kappa)}{\partial A^*} d\kappa},$$

that is, the third term captures the partial effect of a change in ψ on the government's default incentives holding households' and lenders' strategies and beliefs constant. Similarly,

$$\frac{\partial A^*}{\partial x^*} \frac{\partial x^*}{\partial A^{**}} = \frac{\Delta V_3}{\Delta V_1 + \int_{-1}^{1} \Delta V_2 \frac{\partial k_2^*(\kappa)}{\partial A^*}} \frac{\partial x^*}{\partial A^{**}}$$

and, slightly abusing notation,

$$\int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa = -\int_{-1}^{1} \frac{\frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa}{\Delta V_1 + \int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^*}},$$

where this term captures the effect of a change in the households' beliefs on the government's incentives to default. In a similar fashion,

$$\int_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi} d\kappa = -\int_{-1}^{1} \frac{\frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial \psi} d\kappa}{\Delta V_1 + \int_{-1}^{1} \frac{1}{2}\Delta V_2(\kappa) \frac{\partial k_2^*(\kappa)}{\partial A^*} d\kappa},$$

where $\int_{-1}^{1}\frac{1}{2}\frac{\partial A^*}{\partial k_2^*(\kappa)}\frac{\partial k_2^*(\kappa)}{\partial\psi}d\kappa$ captures the effect of a change in the households' strategies caused by a change in ψ holding the households' beliefs about the default threshold, A^{**}, constant.

Using the above notation, we obtain

$$\frac{dA^*}{d\psi}=\int_{-1}^{1}\frac{1}{2}\frac{\partial A^*}{\partial k_2^*(\kappa)}\frac{\partial k_2^*(\kappa)}{\partial\psi}d\kappa+\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial\psi}+\frac{\partial A^*}{\partial\psi}+\int_{-1}^{1}\frac{1}{2}\frac{\partial A^*}{\partial k_2^*(\kappa)}\frac{\partial k_2^*(\kappa)}{\partial A^{**}}d\kappa\frac{\partial A^{**}}{\partial\psi}+\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}}\frac{\partial A^{**}}{\partial\psi}$$

In equilibrium, $A^{**}=A^*$, and so it has to be the case that $\partial A^{**}/\partial\psi=dA^*/d\psi$. Thus, after rearranging,

$$\frac{dA^*}{d\psi}=\frac{\frac{\partial A^*}{\partial\psi}+\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial\psi}+\int_{-1}^{1}\frac{1}{2}\frac{\partial A^*}{\partial k_2^*(\kappa)}\frac{\partial k_2^*(\kappa)}{\partial\psi}d\kappa}{1-\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^*}-\int_{-1}^{1}\frac{1}{2}\frac{\partial A^*}{\partial k_2^*(\kappa)}\frac{\partial k_2^*(\kappa)}{\partial A^{**}}d\kappa}\tag{6}$$

Finally, note that $\int_{-1}^{1}\frac{1}{2}\frac{\partial A^*}{\partial k_2^*(\kappa)}\frac{\partial k_2^*(\kappa)}{\partial\psi}d\kappa$ corresponds simply to $\int_0^1\frac{\partial A^*}{\partial k_2^{i,*}}\frac{\partial k_2^{i,*}}{\partial\psi}di$, while the term $\int_{-1}^{1}\frac{1}{2}\frac{\partial A^*}{\partial k_2^*(\kappa)}\frac{\partial k_2^*(\kappa)}{\partial A^{**}}d\kappa$ corresponds to $\int_0^1\frac{1}{2}\frac{\partial A^*}{\partial k_2^{i,*}}\frac{\partial k_2^{i,*}}{\partial A^{**}}di$. Thus, we obtain

$$\frac{dA^*}{d\psi}=\frac{\frac{\partial A^*}{\partial\psi}+\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial\psi}+\int_0^1\frac{\partial A^*}{\partial k_2^{i,*}}\frac{\partial k_2^{i,*}}{\partial\psi}di}{1-\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^*}-\int_0^1\frac{\partial A^*}{\partial k_2^{i,*}}\frac{\partial k_2^{i,*}}{\partial A^{**}}di},$$

which corresponds to Equation (2) in the paper.

B.2 Proofs of Propositions 2, 3 and 4

I first establish a preliminary result that will be useful in the proof of Proposition 2.

Lemma B.1 *Consider* $\left.\frac{\partial P(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}\Big/\left.\frac{\partial S}{\partial A^{**}}\right|_{A^{**}=A^*}$.

1. *If σ_x is fixed and $\varepsilon\to 0$, then*

$$\lim_{\varepsilon\to 0}\frac{\left.\frac{\partial P(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}}{\left.\frac{\partial S}{\partial A^{**}}\right|_{A^{**}=A^*}}=+\infty$$

2. *If ε is fixed and $\sigma_x\to 0$, then*

$$\lim_{\sigma_x\to 0}\frac{\left.\frac{\partial P(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}}{\left.\frac{\partial S}{\partial A^{**}}\right|_{A^{**}=A^*}}=0$$

3. *If $\varepsilon\to 0$ and $\sigma_x=c\varepsilon^\theta$ for some $c,\theta\in\mathbb{R}_{++}$, then*

$$\lim_{\substack{\varepsilon\to 0\\ \sigma_x=c\varepsilon^\theta}}\frac{\left.\frac{\partial P(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}}{\left.\frac{\partial S}{\partial A^{**}}\right|_{A^{**}=A^*}}=\begin{cases}0 & \text{if }\theta\in(0,1)\\ \frac{-c}{\phi\left(\Phi^{-1}\left(\frac{1}{1+r}\right)\right)} & \text{if }\theta=1\\ -\infty & \text{if }\ \theta>1\end{cases}.$$

Proof. First, note that under Assumption 4 we have $\frac{B_2^{R,u}(A)}{S(A;x^{**})} = 1$, and hence it can be shown that

$$x^* = \frac{p_x + p_A}{p_x} A^{**} - \frac{p_A}{p_x} A_{-1} + \frac{\sqrt{p_x + p_A}}{p_x} \Phi^{-1}\left(\frac{1}{1+r}\right),$$

where $p_x = \frac{1}{\sigma_x^2}$ and $p_A = \frac{1}{\sigma_A^2}$. Therefore,

$$\left.\frac{\partial S}{\partial A^{**}}\right|_{A^{**}=A^*} = -b p_x^{1/2} \phi\left(\frac{x^* - A^*}{p_x^{-1/2}}\right) \frac{p_x + p_A}{p_x}.$$

Moreover,

$$\frac{\partial P\left(A^{**} | A^* + \kappa\varepsilon\right)}{\partial A^{**}} = \frac{\frac{1}{\sigma_A}\phi\left(\frac{A^{**} - A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^* + (1+\kappa)\varepsilon - A_{-1}}{\sigma_A}\right) - \left(\frac{A - (1-\kappa)\varepsilon - A_{-1}}{\sigma_A}\right)}$$

Thus,

$$\frac{\left.\frac{\partial P(A^{**}|A^* + \kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}}{\left.\frac{\partial S}{\partial A^{**}}\right|_{A^{**}=A^*}} = \frac{\frac{\frac{1}{\sigma_A}\phi\left(\frac{A^* - A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^* + (1+\kappa)\varepsilon - A_{-1}}{\sigma_A}\right) - \Phi\left(\frac{A^* - (1-\kappa)\varepsilon - A_{-1}}{\sigma_A}\right)}}{-b p_x^{1/2} \phi\left(\frac{x^* - A^*}{p_x^{-1/2}}\right) \frac{p_x + p_A}{p_x}}.$$

Taking the limit as $\varepsilon \to 0$, we see that the above expression tends to ∞. On the other hand, if ε is fixed and we take the limit as $\sigma_x \to 0$ (i.e., $p_x \to \infty$), then the above expression tends to 0 $\left(\text{since at } A^{**} = A^* \text{ we have } p_x^{1/2}\left(x^* - A^*\right) \to \Phi^{-1}\left(\frac{1}{1+r}\right)\right)$.

Finally, consider the case when $\varepsilon \to 0$, $\sigma_x \to 0$, and $\sigma_x = c\varepsilon^\theta$. Then

$$\lim_{\substack{\varepsilon,\sigma_x \to 0 \\ \sigma_x = c\varepsilon^\theta}} \frac{\frac{\frac{1}{\sigma_A}\phi\left(\frac{A^* - A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^* + \varepsilon - A_{-1}}{\sigma_A}\right) - \left(\frac{A^* - \varepsilon - A_{-1}}{\sigma_A}\right)}}{-p_x^{1/2} \phi\left(\frac{x^* - A^*}{p_x^{-1/2}}\right) \frac{p_x + p_A}{p_x}} = \lim_{\varepsilon \to 0} \frac{c\varepsilon^\theta \frac{1}{\sigma_A}\phi\left(\frac{A^* - A_{-1}}{\sigma_A}\right)}{-\left[\Phi\left(\frac{A^* + \varepsilon - A_{-1}}{\sigma_A}\right) - \Phi\left(\frac{A^* - \varepsilon - A_{-1}}{\sigma_A}\right)\right] \phi\left(\frac{x^* - A^*}{p_x^{-1/2}}\right) \left[1 + \frac{c^2\varepsilon^{2\theta}}{\sigma_A^2}\right]},$$

where I used the substitution $p_x = \frac{1}{c^2\varepsilon^{2\theta}}$ to eliminate p_x. As $\varepsilon \to 0$, both the numerator and denominator tend to 0. Thus, to compute the limit we can apply l'Hôpital's Rule. Differentiating the numerator with respect to εwe get

$$\theta c \varepsilon^{\theta - 1} \frac{1}{\sigma_A} \phi\left(\frac{A^* - A_{-1}}{\sigma_A}\right)$$

From this it follow that as $\varepsilon \to 0$ the numerator tends to ∞ if $0 < \theta < 1$, to $c\frac{1}{\sigma_A}\phi\left(\frac{A^* - A_{-1}}{\sigma_A}\right)$ if $\theta = 1$ and to 0 if $\theta > 1$.

The derivative of the denominator with respect to ε is given by

$$\left[\frac{(1+\kappa)}{2\sigma_A}\phi\left(\frac{A^* + \varepsilon - A_{-1}}{\sigma_A}\right) + \frac{(1-\kappa)}{2\sigma_A}\phi\left(\frac{A^* - \varepsilon - A_{-1}}{\sigma_A}\right)\right] \phi\left(\frac{x^* - A^*}{p_x^{-1/2}}\right) \left[1 + \frac{c^2\varepsilon^{2\theta}}{\sigma_A^2}\right]$$
$$+ \left[\Phi\left(\frac{A^* + \varepsilon - A_{-1}}{\sigma_A}\right) - \Phi\left(\frac{A^* - \varepsilon - A_{-1}}{\sigma_A}\right)\right] \phi\left(\frac{x^* - A^*}{p_x^{-1/2}}\right) \frac{2\theta c^2 \varepsilon^{2\theta - 1}}{\sigma_A^2}$$

For any $\theta > 0$ and any $c > 0$, the above expression tends to

$$\frac{1}{2\sigma_A}\phi\left(\frac{A^* - A_{-1}}{\sigma_A}\right)\phi\left(\Phi^{-1}\left(\frac{1}{1+r}\right)\right) > 0,$$

which proves the result. ■

Proof of Proposition 2. **(1)** Recall from the proof of uniqueness that the government default condition, after taking into account the dual role of A^* as the average value of productivity in the economy and the default threshold, is strictly increasing in A^*. Thus,

$$\Delta V_1 + \int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(\kappa)}{\partial A^*}d\kappa + \int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\left.\frac{\partial k_2^*(\kappa)}{\partial A^{**}}\right|_{A^{**}=A^*}d\kappa + \Delta V_3\left.\frac{\partial x^*}{\partial A^{**}}\right|_{A^{**}=A^*} > 0,$$

where the third and fourth terms capture the effect of a change in the households' and lenders' beliefs, respectively. Dividing the above expression by $\Delta V_1 + \int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(\kappa)}{\partial A^*}d\kappa$ establishes the non-negativity of the multiplier effect.

Under Assumption 4 we have $\frac{B_2^{R,u}(A)}{S(A;x^{**})} = 1$ for all A, and hence it can be shown that $x^* = \frac{p_x+p_A}{p_x}A^{**} - \frac{p_A}{p_x}A_{-1} + \frac{\sqrt{p_x+p_A}}{p_x}\Phi^{-1}\left(\frac{1}{1+r}\right)$, implying that $\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}} > 0$. Similarly, it is straightforward to show that $\partial k_2^*/\partial A^{**} < 0$. Since a higher investment by all households decreases the government's incentives to default ($\int_{-1}^{1}\frac{1}{2}\partial A^*/\partial k_2^*(\kappa)\,d\kappa < 0$), we have $\int_{-1}^{1}\frac{1}{2}\frac{\partial A^*}{\partial k_2^*(\kappa)}\frac{\partial k_2^*(\kappa)}{\partial A^{**}}d\kappa > 0$. It follows that the denominator of the multiplier effect is less than 1, so that the multiplier effect is greater than 1.

(2) I show first that $D \to 0$ as $\varepsilon \to 0$. Note that $\lim_{\varepsilon\to 0}\left.\frac{\partial \Pr(A^{**}|A^*+\kappa\varepsilon)}{\partial A^*}\right|_{A^{**}=A^*} = \infty$, and thus, $\lim_{\varepsilon\to 0}\left.\frac{\partial k_2^*(A^*+\kappa\varepsilon,\kappa,A^{**})}{\partial A^*}\right|_{A^{**}=A^*} = \infty$.[10] From this it follows that $\Delta V_1 + \int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(\kappa)}{\partial A^*}d\kappa \to \infty$, that is the effect of higher productivity on government's default incentive tends to infinity. On the other hand, for any parameter of the model ψ both $\frac{\partial k_2}{\partial \psi}$ and $\frac{\partial S}{\partial \psi}$ exist, and thus are finite. Moreover, the only dependence of these derivatives on ε and σ_x is though $P(A^*|A_i)$ and $S(A,x^*)$. Since both $P(A^*|A_i)$ and $S(A,x^*)$ converge to a constant as the households' and lenders' information becomes infinitely precise, $\frac{\partial k_2}{\partial \psi}$ and $\frac{\partial S}{\partial \psi}$ are finite in the limit as $\varepsilon, \sigma_x \to 0$, implying that $\frac{\partial A^*}{\partial k_2^*}\frac{\partial k_2^*}{\partial \psi} + \frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial \psi} \to 0$. Similarly, $\Delta V_4 = \partial \Delta V/\partial \psi$ is well-defined for each parameter of the model. By the same argument, we know that $\lim_{\varepsilon,\sigma_A\to 0}\Delta V_4$ is finite and, thus $\frac{\partial A^*}{\partial \psi} \to 0$. It follows that $D \to 0$ as $\varepsilon \to 0$.

Next, consider the case when $\sigma_x \to 0$. Note that ΔV_1 includes the effect of an increase in S holding x^* constant. In particular, at the default threshold we have $S = b\left(1 - \Phi\left(\frac{x^*-A^*}{\sigma_x}\right)\right)$. Now holding x^* constant, we have that $\partial S/\partial A^* = b\phi\left(\frac{x^*-A^*}{\sigma_x}\right)\frac{1}{\sigma_x}$. Then $\lim_{\sigma_x\to 0}\frac{\partial}{\partial A^*}\left.S(A^*,x^*)\right|_{A^{**}=A^*} = \infty$, so that $\lim_{\sigma_x\to 0}\left[\Delta V_1 + \int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(\kappa)}{\partial A^*}d\kappa\right] \to$

[10] For the proof of this statement and other statements regarding the limiting behavior of $P(A^{**}|A^* + \kappa\varepsilon)$, see Section F of this appendix.

∞. Thus, by the same argument as in the case of $\varepsilon \to 0$, we conclude $D \to 0$ as $\sigma_x \to 0$. Finally, the case when $\varepsilon, \sigma_x \to 0$ follows from the above observations.

Next, I show that the multiplier effect tends to infinity as $\varepsilon \to 0$ or $\sigma_x \to 0$ or both. Consider first the case when $\varepsilon \to 0$ and $\sigma_x > 0$ is fixed. Note that in this case (as argued above) $\lim_{\varepsilon\to 0} \frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}} = 0$, since $\lim_{\varepsilon\to 0}\left[\Delta V_1 + \int_{-1}^{1} \frac{1}{2}\Delta V_2 \left.\frac{\partial k_2^*(\kappa)}{\partial A}\right|_{A=A^*} d\kappa\right] = \infty$. Next, consider the remaining term in the definition of the multiplier effect:

$$\frac{\partial A^*}{\partial k_2^*}\frac{\partial k_2^*}{\partial A^{**}} = -\frac{\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\left.\frac{\partial k_2^*(A^*+\kappa\varepsilon,\kappa,A^{**})}{\partial A^{**}}\right|_{A^{**}=A^*} d\kappa}{\Delta V_1 + \int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(A^*+\kappa\varepsilon,\kappa,A^{**})}{\partial A^*} d\widehat{\kappa}}$$

Recall that

$$\frac{\partial k_2^*(A^*+\kappa\varepsilon,\kappa,A^{**})}{\partial A^{**}} = (1-\tau)\, e^{A^*+\kappa\varepsilon} f(k_1)\frac{\partial P(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}} C_\Lambda(A^*+\kappa\varepsilon,\kappa,A^{**}),$$

where

$$C_\Lambda(A^*+\kappa\varepsilon,\kappa,A^{**}) \equiv (1-Z)\left(1-\frac{\alpha(1+Z)+P(A^{**}|A^*+\kappa\varepsilon)(1-Z)+Z}{\sqrt{[\alpha(1+Z)+P(A^{**}|A^*+\kappa\varepsilon)(1-Z)+Z]^2-4\alpha Z(1+Z)}}\right) < 0.$$

Define

$$\begin{aligned} T_1(\kappa) &\equiv \frac{1}{2}\frac{(1-Z)\frac{C_\Lambda(A^*+\kappa\varepsilon,\kappa,A^{**})}{\Lambda(A^*+\kappa\varepsilon,\kappa,A^{**})}k_2^*(A^*+\kappa\varepsilon,\kappa,A^*)}{(Z-\Lambda(A^*+\kappa\varepsilon,\kappa,A^{**}))(1-\Lambda(A^*+\kappa\varepsilon,\kappa,A^{**}))} < 0 \\ T_2(\kappa) &\equiv \frac{1}{2}\frac{\alpha(1+r)S}{Y_2(\tau Y_2-(1+r)S)} e^{A^*+\kappa\varepsilon}\frac{C_\Lambda(A^*+\kappa\varepsilon,\kappa,A^{**})f(k_2^*(\kappa))}{\Lambda(A^*+\kappa\varepsilon,\kappa,A^{**})} < 0, \end{aligned}$$

where

$$\frac{1}{2}\Delta V_2(\kappa) = [T_1(\kappa)+T_2(\kappa)]\left.\frac{\partial k_2^*(A^*+\kappa\varepsilon,\kappa,A^{**})}{\partial A^{**}}\right|_{A^{**}=A^*}\frac{1}{\left.\frac{\partial P(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}},$$

so that we can concisely write

$$\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\left.\frac{\partial k_2^*(A^*+\kappa\varepsilon,\kappa,A^{**})}{\partial A^{**}}\right|_{A^{**}=A^*} d\kappa = \int_{-1}^{1}[T_1(\kappa)+T_2(\kappa)]\left.\frac{\partial P(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*} d\kappa$$

Recall that

$$\int_{-1}^{1}\frac{1}{2}\frac{\partial A^*}{\partial k_2(\kappa)}\frac{\partial k_2(\kappa)}{\partial A^{**}} d\kappa = \frac{\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\left.\frac{\partial k_2^*(A^*+\kappa\varepsilon,\kappa,A^{**})}{\partial A^{**}}\right|_{A^{**}=A^*} d\kappa}{\Delta V_1 + \int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(\kappa)}{\partial A^*} d\kappa}$$

Now

$$\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\frac{\partial k_2^*(\kappa)}{\partial A^*} d\kappa = \int_{-1}^{1}[T_1(\kappa)+T_2(\kappa)]\left.\frac{\partial \Pr(A^{**}|A^*+\kappa\varepsilon)}{\partial A^*}\right| d\kappa$$

and

$$\Delta V_1 = \frac{B_1-S}{\tau Y_1-B_1+S}+\frac{S(1+r)}{\tau Y_2-(1+r)S}+\frac{\partial S}{\partial A^*}\left[\frac{1}{\tau Y_1-B_1+S}+\frac{(1+r)}{\tau Y_2-(1+r)S}\right] \equiv T_3 > 0,$$

so that ΔV_1 captures the effect of an increase in A^* on the government's incentive to default through an increase in output (the first two terms) and through an increase in the supply of funds. Thus,

$$\frac{\partial A^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}} = -\frac{\int_{-1}^{1}\left[T_1(\kappa)+T_2(\kappa)\right]\left.\frac{\partial \Pr(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*} d\kappa}{\int_{-1}^{1}\left[T_1(\widehat{\kappa})+T_2(\widehat{\kappa})\right]\left.\frac{\partial \Pr(A^{**}|A^*+\widehat{\kappa}\varepsilon)}{\partial A^*}\right|_{A^{**}=A^*} d\widehat{\kappa}+T_3}$$

Moreover,

$$\lim_{\varepsilon\to 0}\frac{\partial \Pr\left(A^{**}|A^*+\kappa\varepsilon\right)}{\partial A^{**}} = \infty \text{ and } \lim_{\varepsilon\to 0}\frac{\left.\frac{\partial \Pr(A^*|A+\widehat{\kappa}\varepsilon)}{\partial A}\right|_{A=A^*}}{\left.\frac{\partial \Pr(A^{**}|A^*+\widehat{\kappa}\varepsilon)}{\partial A^*}\right|_{A^{**}=A^*}} = -1$$

where both limits are computed in the proof of Claim 4 in Section F of this appendix. Given the above limits, it follows that

$$\begin{aligned}
\frac{\partial A^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}} &= -\frac{\int_{-1}^{1}\left[T_1(\kappa)+T_2(\kappa)\right]\left.\frac{\partial \Pr(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*} d\kappa}{\int_{-1}^{1}\left[T_1(\widehat{\kappa})+T_2(\widehat{\kappa})\right]\left.\frac{\partial \Pr(A^{**}|A^*+\widehat{\kappa}\varepsilon)}{\partial A^*}\right|_{A^{**}=A^*} d\widehat{\kappa}+T_3} \\
&= -\int_{-1}^{1}\frac{\left[T_1(\kappa)+T_2(\kappa)\right]d\kappa}{\int_{-1}^{1}\left[T_1(\widehat{\kappa})+T_2(\widehat{\kappa})\right]\frac{\left.\frac{\partial \Pr(A^{**}|A^*+\widehat{\kappa}\varepsilon)}{\partial A^*}\right|_{A^{**}=A^*}}{\left.\frac{\partial \Pr(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}}+\frac{T_3}{\left.\frac{\partial \Pr(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}}d\widehat{\kappa}} \\
&\to -\int_{-1}^{1}\frac{\left[T_1(\kappa)+T_2(\kappa)\right]d\kappa}{\int_{-1}^{1}(-1)\left[T_1(\widehat{\kappa})+T_2(\widehat{\kappa})\right]d\widehat{\kappa}} = 1
\end{aligned}$$

Therefore, $\frac{\partial A^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}} \to 1$, implying that $M \to \infty$. The remaining cases follow by a similar argument.

Finally, to see that $\lim_{\varepsilon,\sigma_x} M \times D$ is finite, recall that

$$M \times D = \frac{\frac{\partial A^*}{\partial \mathbf{k}_2^*}\frac{\partial \mathbf{k}_2^*}{\partial \psi}+\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial \psi}+\frac{\partial A^*}{\partial \psi}}{1-\frac{\partial A^*}{\partial \mathbf{k}_2^*}\frac{\partial \mathbf{k}_2^*}{\partial A^*}-\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^*}}$$

To see that the above product is finite multiply both the numerator and denominator by $\Delta V_1+\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\left.\frac{\partial k_2^*(\kappa)}{\partial A}\right|_{A=A^*}d\kappa$ and then take the limit of the resulting expression.

(3) Consider the limiting behavior of

$$\frac{\frac{\partial A^*}{\partial k_2}\frac{\partial k_2}{\partial A^*}}{\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^*}} = \frac{\int_{-1}^{1}\frac{1}{2}\Delta V_2(\kappa)\left.\frac{\partial k_2^*(A^*+\kappa\varepsilon,\kappa,A^{**})}{\partial A^{**}}\right|_{A^{**}=A^*}d\kappa}{\Delta V_3\frac{\partial x^*}{\partial A^*}}$$

Recall that

$$\Delta V_3\frac{\partial x^*}{\partial A^*} = (-1)\left[\frac{1}{\tau Y_1-B_1+S}-\frac{(1+r)}{\tau Y_2-(1+r)S}\right]bp_x^{1/2}\phi\left(\frac{x^*-A^*}{p_x^{-1/2}}\right)\frac{p_x+p_A}{p_x}$$

and (from the proof of part (2) of this proposition) that

$$\left.\frac{\partial k_2^*\left(A^*+\kappa\varepsilon,\kappa,A^{**}\right)}{\partial A^{**}}\right|_{A^{**}=A^*}=\left[T_1\left(\kappa\right)+T_2\left(\kappa\right)\right]\left.\frac{\partial P\left(A^{**}|A^*+\kappa\varepsilon\right)}{\partial A^{**}}\right|_{A^{**}=A^*}$$

I consider first the case when σ_x is fixed and $\varepsilon\to0$. Then

$$\begin{aligned}\frac{\frac{\partial A^*}{\partial k_2}\frac{\partial k_2}{\partial A^*}}{\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^*}} &= \lim_{\varepsilon\to0}\frac{\int_{\kappa=-1}^{1}\left[T_1\left(\kappa\right)+T_2\left(\kappa\right)\right]\left.\frac{\partial\Pr\left(A^{**}|A^*+\kappa\varepsilon\right)}{\partial A^{**}}\right|_{A^{**}=A^*}d\kappa}{\left(-1\right)\left[\frac{1}{\tau Y_1-B_1+S}-\frac{(1+r)}{\tau Y_2-(1+r)S}\right]bp_x^{1/2}\phi\left(\frac{x^*-A^*}{p_x^{-1/2}}\right)\frac{p_x+p_A}{p_x}}\\ &= \infty\end{aligned}$$

since $T_1\left(\kappa\right)$ and $T_2\left(\kappa\right)$ converge to negative constants, $\lim_{\varepsilon\to0}\left.\frac{\partial\Pr\left(A^{**}|A^*+\kappa\varepsilon\right)}{\partial A^{**}}\right|_{A^{**}=A^*}=\infty$ (see Lemma B.1), and the denominator is always finite and negative.[11] An analogous argument establishes that when ε is fixed and $\sigma_x\to0$ (i.e., $p_x\to\infty$), then $\frac{\partial A^*}{\partial k_2}\frac{\partial k_2}{\partial A^*}\Big/\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^*}$ tends to 0.

Finally, consider the case where both $\varepsilon\to0$ and $\sigma_x\to0$ and $\sigma_x=c\varepsilon^\theta$. Then,

$$\frac{\frac{\partial A^*}{\partial k_2^*}\frac{\partial k_2^*}{\partial A^{**}}}{\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}}}=\lim_{\substack{\varepsilon,\sigma_x\to0\\ \sigma_x=c\varepsilon^\theta}}\frac{\int_{\kappa=-1}^{1}\left[T_1\left(\kappa\right)+T_2\left(\kappa\right)\right]\left.\frac{\partial\Pr\left(A^{**}|A^*+\kappa\varepsilon\right)}{\partial A^{**}}\right|_{A^{**}=A^*}d\kappa}{\left(-1\right)\left[\frac{1}{\tau Y_1-B_1+S}-\frac{(1+r)}{\tau Y_2-(1+r)S}\right]bp_x^{1/2}\phi\left(\frac{x^*\left(A^*\right)-A^*}{p_x^{-1/2}}\right)\frac{p_x+p_A}{p_x}}$$

The above limit is determined by the limiting behavior of

$$\frac{\left.\frac{\partial\Pr\left(A^{**}|A^*+\kappa\varepsilon\right)}{\partial A^{**}}\right|_{A^{**}=A^*}}{p_x^{1/2}\phi\left(\frac{x^*\left(A^*\right)-A^*}{p_x^{-1/2}}\right)\frac{p_x+p_A}{p_x}}$$

The result then follows from Lemma B.1.[12] ■

B.3 Policies

The default threshold is determined by the following condition:

$$\begin{aligned}0 \;=\; \Delta V\left(A^*,\mathbf{k}_2^*,x^*\right) &= \int_{-1}^{1}\log\left(c_1^R\right)d\kappa+\log\left(\tau Y_1^R-B_1+B_2^{R^*}\right)\\ &+\int_{-1}^{1}\log\left(c_2^R\right)d\kappa+\log\left(\tau Y_2^R-\left(1+r\right)B_2^{R^*}\right)\\ &-\int_{-1}^{1}\log\left(c_1^D\right)d\kappa-\log\left(\tau ZY_1^R\right)\\ &-\int_{-1}^{1}\log\left(c_2^D\right)d\kappa-\log\left(\tau ZY_2^R\right),\end{aligned}\tag{7}$$

[11] That $\frac{1}{\tau Y_1-B_1+S}-\frac{(1+r)}{\tau Y_2-(1+r)S}>0$ follows from the fact that in equilibrium the government is unable to borrow the unconstrained optimal amount.

[12] That we can pass the limit through the integral sign follows from the observation that the integrand is bounded for $\theta\geq1$. For the case of $\theta<1$ the results follows by Fatou's lemma.

where c_t^R and c_t^D are the consumption in period t in repayment and default, respectively, Y_t^R is the total output of the economy in period t, and $B_2^{R^*}$ is the equilibrium borrowing by the government, all evaluated at the threshold productivity A^* Before proceeding further, note that $\int \log\left(\frac{c_t^R}{c_t^D}\right)$ is independent of τ and k_1 for $t = 1, 2$, and thus policy change will affect the government's incentive to default only through its effect on government spending in repayment and in default[13]. Equation (7) plays a key role in establishing Propositions 3 and 4.[14]

B.3.1 Proof of Proposition 3

Differentiate $\Delta V\left(A^*, \mathbf{k}_2^*, x^*\right)$ with respect to τ to obtain

$$u_{g_1}^R Y_1^R + u_{g_2}^R Y_2^R + u_{g_2}^R \tau \frac{\partial Y_2^R}{\partial \tau} - u_{g_1}^D Z Y_1^R + u_{g_2}^D Y_2^R + u_{g_2}^D Z \tau \frac{\partial Y_2^R}{\partial \tau},$$

where $u_{g_t}^R$ and $u_{g_t}^D$ are the marginal utility from government spending in period t in repayment and default, respectively, and Y_1^R the total output of the economy in period t in repayment, all evaluated at A^*.[15] Given households' investment choices, $\partial Y_2^R / \partial \tau = \frac{\alpha}{1-\tau}$. Thus, rearranging the terms in the above expression, we obtain

$$\underbrace{Y_1^R (1-Z) u_{g_1^*}^D + Y_2^R (1-Z) u_{g_1^*}^D}_{\text{Differential increase in tax revenues}} + \underbrace{Y_1^R \left[u_{g_1^*}^R - u_{g_1^*}^D\right] + Y_2^R \left[u_{g_2^*}^R - u_{g_2^*}^D\right]}_{\text{Concavity effect}} - \underbrace{\frac{\partial Y_2^R}{\partial \tau} \tau \left[u_{g_2^*}^R - Z u_{g_2^*}^D\right]}_{\text{Investment distortion}}$$

which corresponds to the expression (3) in the paper.

Consider the concavity effect. Note that

$$\begin{aligned}
Y_1^R \left[u_{g_1^*}^R - u_{g_1^*}^D\right] &= Y_1^R \left[\frac{1}{\tau Y_1^R - B_1 + B_2^{R^*}} - \frac{1}{\tau Z Y_1^R}\right] = Y_1^R \left[\frac{(Z-1)\tau Y_1^R + B_1 - B_2^{R^*}}{\tau Z Y_1^R \left(\tau Y_1^R - B_1 + B_2^{R^*}\right)}\right] \\
&= Y_1^R \left[\frac{(Z-1)\left(\tau Y_1^R - B_1 + B_2^{R^*}\right) + Z\left(B_1 - B_2^{R^*}\right)}{\tau Z Y_1^R \left(\tau Y_1^R - B_1 + B_2^{R^*}\right)}\right] \\
&= Y_1^R (Z-1) u_{g_1^*}^D + \frac{\left(B_1 - B_2^{R^*}\right)}{\tau\left(\tau Y_1^R - B_1 + B_2^{R^*}\right)} \\
&= Y_1^R (Z-1) u_{g_1^*}^D + \frac{1}{\tau}\left(B_1 - B_2^{R^*}\right) u_{g_1^*}^R
\end{aligned}$$

Similarly, following analogous steps, one can show that

$$Y_2^R \left[u_{g_2^*}^R - u_{g_2^*}^D\right] = Y_2^R (Z-1) u_{g_2^*}^D + \frac{1}{\tau}(1+r) B_2^{R^*} u_{g_2^*}^R$$

[13] This is because $c_2^D = Z c_2^R$, $c_1^D = Z(1-\tau) e^{A_i} f(k_1) - k_2$, $c_1^R = (1-\tau) e^{A_i} f(k_1) - k_2$, and k_2 is linear in $f(k_1)$ and τ.

[14] Equations (3) and (4) can be computed directly from Equation 7.

[15] There is no effect of a change of τ on $B_2^{R^*}$, the equilibrium level of borrowing, since under Assumption 4, $B_2^{R^*} = S(A, x^*)$ and $\partial S(A, x^*)/\partial \tau = 0$. If Assumption 4 were relaxed there would be an additional term capturing the potential impact of a change in taxes on government borrowing in equilibrium (via the competition effect among lenders).

Thus, the concavity effect can be written as

$$\frac{1}{\tau}\left(B_1 - B_2^{R^*}\right)u_{g_1^*}^R + \frac{1}{\tau}(1+r)B_2^{R^*}u_{g_2^*}^R - Y_1^R(1-Z)u_{g_1^*}^D - (1-Z)Y_2^R u_{g_2^*}^D$$

It then follows that the sum of the concavity effect and differential increase in tax revenues is simply

$$\frac{1}{\tau}\left(B_1 - B_2^{R^*}\right)u_{g_1^*}^R + \frac{1}{\tau}(1+r)B_2^{R^*}u_{g_2^*}^R$$

Finally, consider the investment distortion effect. Note that

$$\begin{aligned}
\frac{\partial Y_2^R}{\partial \tau}\tau\left[u_{g_2^*}^R - Zu_{g_2^*}^D\right] &= \frac{\alpha\tau Y_2^R}{1-\tau}\left[\frac{1}{\left(\tau Y_2^R - (1+r)B_2^{R^*}\right)} - \frac{Z}{\tau Z Y_2^R}\right] \\
&= \frac{\alpha\tau Y_2^R}{1-\tau}\left[\frac{\tau Y_2^R - \tau Y_2^R + (1+r)B_2^{R^*}}{\tau Y_2^R\left(\tau Y_2^R - (1+r)B_2^{R^*}\right)}\right] \\
&= \frac{\alpha\tau}{1-\tau}\frac{1}{\tau}(1+r)B_2^{R^*}u_{g_2^*}^R
\end{aligned}$$

Proposition 3 then follows from summing the terms that capture the concavity effect, differential increase in tax revenues and the investment distortion (ς_τ defined in Proposition 3 is proportional to their sum).

B.3.2 Proof of Proposition 4

The proof of Proposition 4 is similar to the proof of Proposition 3. I consider only a stimulus financed with short-term debt. The case of a stimulus financed with long-term debt is analogous.

Note first that when the government engages in a fiscal stimulus financed with short-term debt that matures at the end of period 1, government spending in repayment in period 1 becomes $\tau Y_1^R - B_1 + B_2^{R^*} - \left(1+r^{ST}\right)sk_1$, where sk_1 is the size of stimulus. The positive effect of such a stimulus is that it leads to expansion of output. Differentiating both sides of the government indifference condition with respect to s, we get

$$\tau\frac{\partial Y_1}{\partial s}u_{g_1}^R - \left(1+r^{ST}\right)k_1 u_{g_1}^R + \tau\frac{\partial Y_2}{\partial s}u_{g_2}^R - \tau Z\frac{\partial Y_1}{\partial s}u_{g_1}^D - \tau Z\frac{\partial Y_2}{\partial s}u_{g_2}^D$$

Rearranging, we get

$$\underbrace{\frac{\partial Y_1}{\partial s}\tau(1-Z)u_{g_1}^D + \frac{\partial Y_2}{\partial s}\tau(1-Z)u_{g_2}^D}_{\text{Differential increase in tax revenues}} + \underbrace{\tau\frac{\partial Y_1}{\partial s}\left[u_{g_1}^R - u_{g_1}^D\right] + \tau\frac{\partial Y_2}{\partial s}\left[u_{g_2}^R - u_{g_2}^D\right]}_{\text{Concavity effect}} - \underbrace{\left(1+r^{ST}\right)k_1 u_{g_1}^R}_{\text{Increase in debt}}$$

When the government engages in a stimulus, $k_2^i = (1-\tau)e^{A_i}f\left(k_1(1+s)\right)\Lambda\left(A_i;\varepsilon,A^*\right)$, and thus

$$\frac{\partial Y_1}{\partial s} = \frac{\alpha}{1+s}Y_1^R \text{ and } \frac{\partial Y_2}{\partial s} = \frac{\alpha^2}{1+s}Y_2^R$$

As in the case of a tax increase, the concavity effect of a stimulus can be written as

$$\tau\frac{\partial Y_1}{\partial s}\frac{1}{\tau Y_1}\left(B_1-B_2\right)u_{g_1}^R+\tau\frac{\partial Y_2}{\partial s}\frac{1}{\tau Y_2}\left(1+r\right)B_2u_{g_2}^R-\frac{\partial Y_1}{\partial s}\tau\left(1-Z\right)u_{g_1}^D-\frac{\partial Y_2}{\partial s}\tau\left(1-Z\right)u_{g_2}^D$$

Thus, the sum of the concavity effect and the differential increase in tax revenues is equal to

$$\frac{\alpha}{1+s}\left(B_1-B_2\right)u_{g_1}^R+\frac{\alpha^2}{1+s}\left(1+r\right)B_2u_{g_2}^R,$$

and thus a further stimulus, from the level s, leads to a decrease in the probability of default if and only if

$$\frac{\alpha}{1+s}\left(B_1-B_2\right)u_{g_1}^R+\frac{\alpha^2}{1+s}\left(1+r\right)B_2u_{g_2}^R-\left(1+r^{ST}\right)k_1u_{g_1}^R>0$$

In particular, if the government is considering engaging in a small stimulus (when the alternative is not to engage in stimulus at all, so that $s=0$), then the condition becomes

$$\alpha\left(B_1-B_2\right)u_{g_1}^R+\alpha^2\left(1+r\right)B_2u_{g_2}^R-\left(1+r^{ST}\right)k_1u_{g_1}^R>0$$

which is the condition reported in Proposition 4.

B.4 Discussion of Assumption 4

The above analysis was conducted under the following assumption:

Assumption 4 *B_1 is large enough so that for all $A>\underline{A}(0)$ the government's desired borrowing in repayment exceeds the supply of funds in the market.*

To determine a bound on B_1, which is assumed implicitly in Assumption 4, assume that interest rate r is less than $\widehat{r}$ for some arbitrarily high $\widehat{r}$. Recall that

$$B_2^{R,u}=\frac{\left(1+r\right)B_1+\tau Y_2^R-\left(1+r\right)\tau Y_1^R}{2\left(1+r\right)}.$$

For a fixed $r<\widehat{r}$, a higher B_1 increases B_2, not only directly, but also indirectly by shifting the lower bound of the fragility region, $\underline{A}\left(r\right)$, upwards. For sufficiently high $\underline{A}\left(r\right)$, we have $Y_2^R\underline{A}\left(r\right)>Y_1^R\underline{A}\left(r\right)$. Moreover, $\partial Y_2^R/\partial A=\left(1+\alpha\right)Y_2^R$ and $\partial Y_1^R/\partial A=Y_1^R$, implying that once $\underline{A}\left(r\right)$ is high enough so that $Y_2^R\underline{A}\left(r\right)>Y_1^R\underline{A}\left(r\right)$, a further increase in $\underline{A}\left(r\right)$ leads to an increase in $\tau Y_2^R-\left(1+r\right)\tau Y_1^R$, and hence in the desired borrowing. It follows that for a fixed b and a fixed r, there exists a high enough B_1 such that $B_2^{R,u}>b$. Since $[0,\widehat{r}]$ is a compact interval there exists a high enough B_1, call it $\widehat{B}_1$, such that if $B_1>\widehat{B}_1$ then $B_2^{R,u}>b$ for all $r\in[0,\widehat{r}]$.[16]

Assumption 4 simplifies the lender's problem. The difficult part of the lender's problem is the competition effect: Ceteris paribus, a higher supply of funds in the bond

[16]Assuming that Z is high enough would have the same effect.

market decreases the lenders' expected return from lending. This effect, however, is not present when $B_2^{R,u} > b$, in which case there exists a closed-form solution for x^*. In particular, under Assumption 4, we have

$$x^* = \frac{p_x + p_A}{p_x} A^{**} - \frac{p_A}{p_x} A_{-1} + \frac{\sqrt{p_x + p_A}}{p_x} \Phi^{-1}\left(\frac{1}{1+r}\right).$$ [17]

This in turn substantially simplifies the analysis presented in Sections 4 and 5 of the paper.[18] In Section F of this appendix, I discuss briefly how the result change if Assumption 4 is not imposed in Section F of this Appendix.

C Policy Adjustments under Uncertainty

In this section I derive the change in the default threshold when households and lenders are uncertain as to whether the policy change will be implemented and assign probability p to the government implementing the new policy. As in Section B of this appendix, I am interested in understanding the effect of an announcement of a small policy change on the default threshold. To do so, I start by considering a situation where with probability $(1-p)$ the policy parameter takes value ψ (which I associate with the case when the policy change is not implemented) and with probability p the policy parameter takes value ψ' (which I associate with the new level of the policy parameter if the policy is implemented). I then compute the effect of a further change in ψ' and I impose the condition that initially $\psi' = \psi$. By following these steps, I obtain the effect of an announcement of a change in the policy parameter when such a change will take place with probability p.

Let A^* be the threshold if the policy parameter takes value ψ (i.e., the policy change is not implemented) and $A^{*\prime}$ the policy threshold when the policy parameter takes value ψ' (i.e., the policy change is implemented).[19] Then the equilibrium conditions can be written as

$$(1-p)\, I\left(A^* + \kappa\varepsilon, A^*, k_2^*(\kappa), \psi\right) + pI\left(A^* + \kappa\varepsilon, A^{*\prime}, k_2^*(\kappa), \psi'\right) = 0 \quad (8)$$

$$(1-p)\, I\left(A^{*\prime} + \kappa\varepsilon, A^*, k_2^{*\prime}(\kappa), \psi\right) + pI\left(A^{*\prime} + \kappa\varepsilon, A^{*\prime}, k_2^{*\prime}(\kappa), \psi'\right) = 0 \quad (9)$$

$$(1-p)\, L\left(A^*, x^*, \psi\right) + pL\left(A^{*\prime}, x^*, \psi'\right) = 0 \quad (10)$$

$$\Delta V\left(A^*; \{k_2^*(\kappa)\}_{\kappa\in[0,1]}, x^*, A^*, \psi\right) = 0 \quad (11)$$

$$\Delta V\left(A^{*\prime}; \{k_2^*(\kappa)\}_{\kappa\in[0,1]}, x^*, A^{*\prime}, \psi'\right) = 0, \quad (12)$$

[17] Derivations of the threshold x^* when there is no competition effect are standard and can be found, for example, in Szkup and Trevino (2015).

[18] Assuming that lenders ignore the competition effect would have the same implications.

[19] For example, if the relevant policy parameter is a tax rate τ and the government contemplates increasing the tax rate to $\tau' > \tau$ then $\psi = \tau$ while $\psi' = \tau'$.

where $k_2^*(\kappa)$ denotes an individual household's equilibrium investment when that household's productivity is equal to $A^* + \kappa\varepsilon$, while $k_2^{*\prime}$ denotes the individual household's equilibrium investment when that household's productivity is equal $A^{*\prime} + \kappa\varepsilon$.

When households and lenders are uncertain whether an announced policy will be implemented there are additional equilibrium equations compared to the case considered in Section B of this appendix. This is because we need to determine the default threshold both when the policy in implemented and when it is not (the possibility of a policy change also affects the threshold even if in the end the policy is not implemented). In particular, to compute the equilibrium default threshold when the policy parameter takes value ψ, we need both the government's default condition and the household investment decisions evaluated both evaluated at ψ (Equations 8 and 11). Similarly, to compute the equilibrium default threshold when the policy parameter takes value ψ', we need both the government's default condition and the household investment conditions evaluated both evaluated at ψ' (Equations 9 and 12).

To compute the effect of a policy announcement when the policy is expected to be implemented with probability p, one can follow an approach similar to the one in Section B of this appendix, that is consider the total derivatives of both sides of all equilibrium condition with respect to ψ'. Solving the resulting system of equations for $dA^*/d\psi$ and $dA^{*\prime}/d\psi'$ and evaluating all derivatives at $\psi = \psi'$ (since we consider a small policy change from its initial level at ψ) yields the desired result.[20]

C.1 Proofs of Propositions 5 and 6

Proposition 5 follows immediately from the discussion in the paper and part (2) of Proposition 2. Proposition 6 follows directly from Equation (5) in the paper which states that $\frac{dA^*}{d\psi}(p) = p\frac{dA^*}{d\psi}(1) + (1-p)\frac{\partial A^*}{\partial \psi}$, and part (2) of Proposition 2.

D Discussion of Assumptions 1–4

D.1 Assumptions 1–3

To solve the model described in Section 2 of the paper, I imposed Assumptions 1–3 (Section 3.1 in the paper). Assumption 1, which states that $B_1 \geq \underline{B}_1$ is needed to make the problem interesting. It is straightforward to show that the unconstrained optimal borrowing by the government when the interest rate is $r = 0$ is given by

$$B_2^{R,u} = \frac{B_1 + \tau Y_2 - \tau Y_1}{2}$$

[20] The detailed derivations can be found in the "Additional Results" document available on author's website (http://economics.ubc.ca/faculty-and-staff/michal-szkup/).

If B_1 is low, then the government might have no incentives to borrow in the fragility region (low B_1 means that the fragility region contains low values of productivity A, for which Y_2 tends to be substantially smaller than Y_1). But in this case lenders' expectations stop playing role in the model. By imposing an appropriate lower bound on B_1, I can ensure that the government will always want to borrow in the fragility region.[21]

Assumption 2 consists of two parts. First, it imposes a bound on the total wealth of the lenders. This is needed for two reasons. First of all, an individual lender's wealth has to be bounded, since (given the assumption of risk-neutrality) after receiving a good signal he always supplies all his funds to the market. Thus, if lenders had an infinite amount of funds, the government would always be able to borrow funds from the few agents that receive high signals.[22] Second, a bound on b is needed to ensure that $\Delta V\left(A^*, \mathbf{k}_2^*, x^*\right)$. The details of establishing the bound on b can be found in sections $A.1.3$ and $A.1.4$ in this appendix. The role of the assumption that $\xi \rightarrow 1$ is discussed in Footnote 14 in the paper and in Section $A.3$ in the "Additional Results" document on the author's website. To reiterate here, I need a high ξ in order to ensure that the government's incentives to default satisfies single-crossing as the supply of funds in the market, S, increases. If that is not the case, then no equilibrium in monotone strageies exists. In the limit as $\xi \rightarrow 1$, government's incentive to default is monotonically decreasing in S. Such monotonicity, while probably not necessary, greatly simplifies the analysis.

Finally, Assumption 3 implies that $B_2^{R,u}$ is increasing in the fragility region. This simplifies the analysis of the lender's problem (when the stronger Assumption 4 is not imposed), and I use it to establish that x^* is increasing in A^*. Under Assumption 3, a lender who observes a higher signal not only believes that default is less likely but also that he will be able to lend more to the government. The details of the derivations of the bound on Z can be found in Section $A.5$ in the "Additional Results" document on the author's website. Numerical simulations suggest that this assumption is not crucial for the model to have a unique equilibrium in monotone strategies.

[21] The details of deriving a sufficient bound on B_1 can be in "Addtional Results" document on the author's website (http://econ.sites.olt.ubc.ca/files/2016/01/pdf_szkup_debt_crises_additional.pdf)

[22] The second reason is to ensure that the government incentives to default decrease as A increases. As shown in Section B of this appendix $\partial \Delta V / \partial A$ depends on B_2, the amount that the government can borrow. A bound on b, and hence on B_2, ensures that $\partial \Delta V / \partial A > 0$ for all A in the fragility region and for all possible choices of B_2, that is for all $B_2 \in [0, b]$. As numerical simulations suggest, unless parameters are extreme (Z is close to 1 or α close to 1) this is not an issue. However, analytically this is hard to show and hence I take care of this issue by imposing appropriate bound on b.

D.2 Policy Analysis without Assumption 4

Assumption 4 is useful, since it simplifies the lender's problem. However, one can obtain a similar decomposition of $dA^*/d\psi$ when Assumption 4 is not imposed.

Without Assumption 4, a change in households' investment strategies will affect the lenders' equilibrium behavior. This is becase the government's desired unconstrained borrowing, $B_2^{R,u}$, depends on Y_2, and a change in $B_2^{R,u}$ translate into a change in x^*. Thus, the lenders' indifference condition has to be written as

$$L\left(A^{**}, x^*, \psi, k_2\right) = 0$$

rather than as $L\left(A^{**}, x^*, \psi\right) = 0$. This is the only change compared to the case when Assumption 4 is imposed. Following the same steps, one can show that

$$\frac{dA^*}{d\psi} = \frac{\frac{\partial A^*}{\partial \psi} + \int\limits_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi} d\kappa + \frac{\partial A^*}{\partial x^*} \left[\frac{\partial x^*}{\partial \psi} + \int\limits_{-1}^{1} \frac{\partial x^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi} d\kappa \right]}{1 - \int\limits_{-1}^{1} \frac{1}{2} \frac{\partial A^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa - \frac{\partial A^*}{\partial x^*} \left[\frac{\partial x^*}{\partial A^*} + \int\limits_{-1}^{1} \frac{\partial x^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa \right]}$$

Thus, compared to the case when Assumption 4 holds, there is an additional term in the expression for the direct effect, $\frac{\partial A^*}{\partial x^*} \int\limits_{-1}^{1} \frac{\partial x^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial \psi} d\kappa$. This is because a change in ψ leads to an adjustment in the households' investment which affects the government's desired borrowing. Without Assumption 4 there is "competition effect", higher supply of funds to the bond market tends to mean less lending per lender, hence a change in the households' investment strategies leads to an adjustment in x^*. Similarly, the multiplier effect has an additional term equal to $\frac{\partial A^*}{\partial x^*} \int\limits_{-1}^{1} \frac{\partial x^*}{\partial k_2^*(\kappa)} \frac{\partial k_2^*(\kappa)}{\partial A^{**}} d\kappa$, since now a change in households' expectations affects the lenders' behavior through its impact on the government's desired borrowing.

There are two main reasons why in the paper I consider a case when Assumption 4 holds. First of all, Assumption 4 substantially simplifies the subsequent analysis. This is particularly true when considering effects of an increase in taxes and of a fiscal stimulus, or when deriving an expression for $dA^*/d\psi$, since $\int_{-1}^{1} \left[\partial x^* / \partial k_2^*(\kappa)\right] \left[\partial k_2^*(\kappa) / \partial \psi\right]$ is a complicated object and can be computed only implicitly. Second, numerical simulations suggests that the competition effect, which is assumed away when Assumption 4 is imposed, plays only a minor role when determining the desirability of a particular policy.

E The Effect of the Interest Rate on Policy Adjustments

Above I analyzed the case when the policy change takes place after the interest rate has been set, and thus the change in the policy and the resulting change in the default threshold A^* do not affect the interest rate r. In this section I analyze what happens when the policy change is announced before the government chooses the interest rate, in which case we have to take into account how a policy change affects the choice of interest rate and how this change in the interest rate affects the default threshold.

Recall that the government chooses the interest rate to maximize the ex-ante welfare. The optimal interest rate is then the solution to the first-order condition associated with this problem, which can be written as

$$R\left(A^*, k_2, x^*, \psi, r^*\right) = 0$$

Here, we recognize that r^* depends on the government's future decisions, households' investment choices, and lenders' supply decisions. The choice of r^* is also affected by the policy parameters, since ψ affects the gains and costs associated with a higher r.

Following the same approach as in Section $B.1$ of this appendix I find that the total effect of a change in policy ψ on the default threshold is given by

$$\begin{aligned}\frac{dA^*}{d\psi} &= \frac{1-\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}}-\int_{i=0}^{1}\frac{\partial A^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}}di}{M_{Total}}\left[\frac{\frac{\partial A^*}{\partial \psi}+\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial \psi}+\int_{i=0}^{1}\frac{\partial A^*}{\partial k_2}\frac{\partial k_2^i}{\partial \psi}di}{1-\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}}-\int_{i=0}^{1}\frac{\partial A^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}}di}\right]\\ &+\frac{1-\frac{\partial r^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}}-\int_{i=0}^{1}\frac{\partial r^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}}di}{M_{Total}}\left[\frac{\left(\frac{\partial A^*}{\partial r}+\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial r}\right)\left(\frac{\partial r^*}{\partial \psi}+\frac{\partial r^*}{\partial x^*}\frac{\partial x^*}{\partial \psi}+\int_{i=0}^{1}\frac{\partial r^*}{\partial k_2}\frac{\partial k_2}{\partial \psi}di\right)}{1-\frac{\partial r^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}}-\int_{i=0}^{1}\frac{\partial r^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}}di}\right],\end{aligned}$$

where M_{Total} is the (total) multiplier effect that is present in the model when r can adjust; is given by

$$M_{Total} = \frac{1}{1-\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}}-\int_{i=0}^{1}\frac{\partial A^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}}di-\frac{\left(\frac{\partial A^*}{\partial r}+\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial r}\right)}{1-\frac{\partial r^*}{\partial x^*}\frac{\partial x^*}{\partial r}}\left(\frac{\partial r^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}}+\int_{i=0}^{1}\frac{\partial r^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}}di\right)}.[23]$$

To understand the above expression, note first that $\left[1-\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}}-\int_{i=0}^{1}\frac{\partial A^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}}\right]^{-1}$ is the multiplier effect in the case when we hold the interest constant, and $\left[1-\frac{\partial r^*}{\partial x^*}\frac{\partial x^*}{\partial A^{**}}-\int_{i=0}^{1}\frac{\partial r^*}{\partial k_2}\frac{\partial k_2}{\partial A^{**}}\right]^{-1}$ is the multiplier effect in the case when the government's default decision is affected by the change in households beliefs through only an implied adjustment in the interest rate. Then the first term in the expression for $dA^*/d\psi$ captures the change in the default threshold implied by a change in the policy holding the interest rate constant (the expression in the square brackets) weighted by the relative importance of the "partial" multiplier effect (i.e., multiplier effect when r is kept constant as in Section 4 of the paper) compared to the total multiplier effect, M_{Total}. This effect is familiar from the earlier analysis. The second term captures the total change in the default threshold implied by the adjustment in r^*. Here, $\left(\frac{\partial A^*}{\partial r}+\frac{\partial A^*}{\partial x^*}\frac{\partial x^*}{\partial r}\right)\left(\frac{\partial r^*}{\partial \psi}+\frac{\partial r^*}{\partial x^*}\frac{\partial x^*}{\partial \psi}+\int_{i=0}^{1}\frac{\partial r^*}{\partial k_2}\frac{\partial k_2}{\partial \psi}di\right)$

[23] The above expression can be derived by following the same steps as in Section $B.1$.

captures the effect that an adjustment in ψ has on r^* (and hence on A^*) holding households' and lenders' expectations constant: A change in ψ leads to a change in r^*, which then affects A^*. This effect is then reinforced by the associated multiplier effect that results from the initial adjustment in r^* and is adjusted by the relative importance of its partial multiplier effect.

How does an adjustment in r^* alter the effectiveness of various government policies compared to the case when r^* is constant? While it is difficult to answer this question analytically, intuition suggests that an adjustment in r^* tends to decrease the magnitude of the change in A^* implied by ψ as long as the default threshold A^* is lower than the prior of the mean belief about A, A_{-1}.[24] To understand this, note that a decrease in A^* decreases the benefit of a higher r (since a lower A^* means that a further decrease in A^* driven by an adjustment in r translates into a lower decrease in the probability of default) and increases the cost of a higher r (since a fall in A^* implies that the government has to incur the cost of a higher r for a larger set of productivity values). The opposite is true when A^* increases. This suggests that a policy change that leads to a decrease in A^* is accompanied by a decrease r^*, which decreases the positive effect of the policy adjustment. On the other hand, a policy change that leads to an increase in A^* is accompanied by an increase in r^*, which tends to partially offset the negative effect that such a policy has on the probability of default.

Finally, one may wonder how an adjustment in r^* affects the relative importance of the direct and the multiplier effects. The next proposition states that the main result of Section 4.1 in the paper still holds.[25]

Proposition A *Let M_{Total} denote the total multiplier effect and D_{Total} denote the total direct effect in the model when the policy is announced before the interest r is set. If either $\varepsilon \to 0$ or $\sigma_x \to 0$ (or both) then $M_{Total} \to \infty$ and $D_{Total} \to 0$ with $\lim_{\varepsilon,\sigma_x \to 0} (M_{Total} \times D_{Total}) \in \mathbb{R}$.*

The discussion above suggests that in absolute terms the multiplier effect is lower when the policy is announced before the interest rate is set. However, it is still true that when households and lenders have very precise information about the current state of the economy, the change in the probability of default is driven mainly by the multiplier effect. The intuition for this result is exactly the same as before: In the limit when there is no fundamental uncertainty, households and lenders care only about the actions (and hence beliefs) of others. Thus, if an agent expects that the beliefs of others do not

[24] This is also confirmed by numerical simulations.

[25] The proof of Proposition A is analogous to the proof of Proposition 2 provided in Section B in this appendix.

change, he will not change his action in response to a policy adjustment, implying that the direct effect will be equal to 0.

F Auxiliary Results

In this section I provide proofs of several results that have been invoked throughout this appendix. First, I show that $\partial x^*/\partial A^* < \frac{p_x+p_A}{p_x}$. Then I compute limits of several expressions as $\varepsilon, \sigma_x \to 0$ and which where used in the proof of Proposition 2.

Lemma 2 *The derivative of x^* with respect to A^* is bounded from above by $\frac{\sigma_x^2+\sigma_A^2}{\sigma_A^2}$.*

Proof. Applying the implicit function theorem to the lenders' indifference condition, we get

$$\frac{dx^*}{dA^*} = -\frac{(-1)\left(1+r\min\left\{1, \frac{B_2^u(A^*)}{S(A^*,x^*)}\right\}\right) f\left(A^*|x^*\right)}{\frac{\partial}{\partial x^*}\left[\int\limits_{A^*}^{\infty}\left(1+r\min\left\{1, \frac{B_2^u(A)}{S(A.x^*)}\right\}\right) f\left(A|x^*\right) dA\right]},$$

where $f(A|x)$ is the conditional density of A given lender j observed signal $x_j = x^*$. Define $\mathcal{A}^u = \left\{A \geq A^* | B_2^{R,u}(A) < S(A)\right\}$ and $\mathcal{A}^c = \left\{A \geq A^* | B_2^{R,u}(A) \geq S(A)\right\}$, and note that $B_2^{R,u}(A)$ and $S(A)$ intersect at most finitely many times. Without loss of generality, I assume that $B_2^{R,u}(A)$ and $S(A)$ intersect at least once (otherwise, the result follows immediately). Then we can write $\mathcal{A}^u$ and $\mathcal{A}^c$ as $\mathcal{A}^u = \cup_{i=1}^{N_u}\left[A_{i_0}^u, A_{i_1}^u\right]$ and $\mathcal{A}^c = \cup_{i=1}^{N_c}\left(A_{i_0}^c, A_{i_1}^c\right)$, where $N_u, N_c \in \mathbb{N}$, $\left\{A_{i0}^u\right\}_{i=1}^{N_u}$ are the values of the productivity at which $B_2^{R,u}(A)$ intersects $S(A)$ from above and $\left\{A_{i1}^u\right\}_{i=1}^{N_u}$ are the values of productivity at which $B_2^{R,u}(A)$ intersects $S(A)$ from below.[26] With these definitions, we can write the above derivative as

$$\frac{dx^*}{dA^*} = \frac{\left(1+r\min\left\{1, \frac{BB_2^{R,u}(A)}{S(A^*,x^*)}\right\}\right) f\left(A^*|x^*\right)}{\sum \frac{\partial}{\partial x^*}\int\limits_{A_{i_0}^c}^{A_{i_1}^c}(1+r) f\left(A|x^*\right) dA + \sum \frac{\partial}{\partial x^*}\int\limits_{A_{i_0}^u}^{A_{i_1}^u}\left\{1+r\frac{B_2^{R,u}(A)}{S(A,x^*)} f\left(A|x^*\right)\right\} dA}$$

Consider the case where at $A = A^*$ we have $B_2^u(A^*) \geq S(A^*, x^*)$. Then the denominator becomes:

$$\sum_{i=1}^{N_u}\int\limits_{A_{i_0}^c}^{A_{i_1}^c}\frac{\partial}{\partial x^*}(1+r) f\left(A|x^*\right) dA + \sum_{i=1}^{N_c}\int\limits_{A_{i_0}^u}^{A_{i_1}^u}\frac{\partial}{\partial x^*}\left\{1+r\frac{B_2^{R,u}(A)}{S(A,x^*)} f\left(A|x^*\right)\right\} dA$$

$$= \int\limits_{A^*}^{\infty}\frac{\partial}{\partial x^*}(1+r) f\left(A|x^*\right) dA + \sum_{i=1}^{N_c}\int\limits_{A_{i_0}^u}^{A_{i_1}^u}\frac{\partial}{\partial x^*}\left\{r\left(\frac{B_2^{R,u}(A)}{S(A,x^*)} - 1\right) f\left(A|x^*\right)\right\} dA$$

[26] If at A^* we have $S(A,x^*) > B_2^{R,u}(A)$, then $A_{1_0}^u = A^*$, $A_{1_1}^u = A_{1_0}^c$, $A_{i_1}^c = A_{2_0}^u$, and so on. If at A^* we have $S(A,x^*) < B_2^{R,u}(A)$ then $A_{1_0}^c = A^*$, $A_{1_1}^c = A_{1_0}^u$, $A_{i_1}^u = A_{2_0}^c$, and so on.

It remains to show that the second of the above terms is positive. Intuitively, that is what we expect, since a higher x^* makes high values of A more likely and $B_2^u(A)$ is increasing in A. The remainder of this proof is devoted to establishing it analytically.

The idea of the next few steps is to change differentiation with respect to x^* with the differentiation with respect to A. First, note that, since $f(A|x^*) = (p_A + p_x)^{1/2} \phi \left(\frac{A - \frac{p_x x^* + p_A A_{-1}}{p_x + p_A}}{(p_A + p_x)^{-1/2}} \right)$, we have

$$\int_{A^*}^{\infty} \frac{\partial}{\partial x^*} (1+r) f(A|x^*) \, dA = -\frac{p_x}{p_x + p_A} \int_{A^*}^{\infty} \frac{\partial}{\partial A} (1+r) f(A|x^*) \, dA = \frac{p_x}{p_x + p_A} (1+r) f(A^*|x^*)$$

Next, let $H(A, x^*) = \left(\frac{B_2^u(A)}{S(A,x^*)} - 1 \right) f(A|x^*)$. Then,

$$\frac{\partial}{\partial x^*} H(A, x^*) = -\frac{p_x}{p_x + p_A} \frac{\partial}{\partial A} H(A, x^*) + \frac{\partial B_2^{R,u}(A)}{\partial A} \frac{1}{S(A, x^*)} f(A|x^*) - \frac{p_A}{p_x + p_A} \frac{B_2^{R,u}(A)}{S(A, x^*)} \frac{\frac{\partial}{\partial x^*} S(A, x^*)}{S(A, x^*)},$$

where, since $\partial B_2^{R,u}(A) / \partial A > 0$ and $\frac{\partial}{\partial x^*} S(A, x^*) < 0$, the last two terms are strictly positive.[27] Moreover, note that for $i = 1, ..., N_c$ we have $H\left(A_{i_1}^u, x^*\right) = H\left(A_{i_0}^u, x^*\right) = 0$. Therefore,

$$\begin{aligned}
& \sum_{i=1}^{N_c} \int_{A_{i_0}^u}^{A_{i_1}^u} \frac{\partial}{\partial x^*} \left\{ r \left(\frac{B_2^{R,u}(A)}{S(A, x^*)} - 1 \right) f(A|x^*) \right\} dA \\
> & \sum_{i=1}^{N_c} \int_{A_{i_0}^u}^{A_{i_1}^u} -\frac{p_x}{p_x + p_A} \frac{\partial}{\partial A} H(A, x^*) \, dA \\
= & -\frac{p_x}{p_x + p_A} \sum_{i=1}^{N_c} \left[H\left(A_{i_1}^u, x^*\right) - H\left(A_{i_0}^u, x^*\right) \right] = 0
\end{aligned}$$

This establishes the claim for the conclusion of the Lemma when at $A = A^*$ we have $B_2^u(A^*) \geq S(A^*, x^*)$. The case when $B_2^u(A^*) < S(A^*, x^*)$ is established in an analogous way. ■

The next claim has been used in Section $A.1.4$ to establish uniqueness of equilibrium in monotone strategies.

Claim 3 $\lim_{\varepsilon \to 0} \frac{\partial \Pr(A^*|A^* + \kappa\varepsilon)}{\partial A^*} = 0$

[27] The second and third terms "correct" for the fact that

$$\frac{\partial}{\partial x^*} H(A, x^*) \neq -\frac{p_x}{p_x + p_A} \frac{\partial}{\partial A} H(A, x^*)$$

Proof. Note that

$$\frac{\partial \Pr\left(A^*|A^*+\kappa\varepsilon\right)}{\partial A^*}$$

$$= \frac{\frac{1}{\sigma_A}\phi\left(\frac{A^*-A_{-1}}{\sigma_A}\right)-\frac{1}{\sigma_A}\phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^*+(1+\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)}+$$

$$+\frac{\left[\frac{1}{\sigma_A}\phi\left(\frac{A^*+(1+\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)-\frac{1}{\sigma_A}\phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)\right]\left[\Phi\left(\frac{A^*-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)\right]}{\left[\Phi\left(\frac{A^*+(1+\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)\right]^2}$$

Taking the limit as $\varepsilon \to 0$ and using l'Hôpital's rule one can show that the first term converges to $\frac{A^*-A_{-1}}{\sigma_A}\frac{(1-\kappa)}{2}$ while the second term converges to $-\frac{A^*-A_{-1}}{\sigma_A}\frac{(1-\kappa)}{2}$. It follows that $\lim_{\varepsilon\to 0}\frac{\partial \Pr(A^*|A^*+\kappa\varepsilon)}{\partial A^*}=0$.

■

The next two claims have been used in the proof of Proposition 2.

Claim 4 $\lim_{\varepsilon\to 0}\left.\frac{\partial \Pr(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}=\infty$ *and* $\lim_{\varepsilon\to 0}\frac{\left.\frac{\partial \Pr(A^{**}|A^*+\widehat{\kappa}\varepsilon)}{\partial A^*}\right|_{A^{**}=A^*}}{\left.\frac{\partial \Pr(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}}=-1$

Proof. If $A^{**}\in\left(A^*-(1-\kappa)\varepsilon, A^*+(1+\kappa)\varepsilon\right)$, then

$$\Pr\left(A^{**}|A^*+\kappa\varepsilon\right)=\frac{\Phi\left(\frac{A^{**}-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^*+(1+\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)}$$

Differentiating with respect to A^{**}, we get

$$\frac{\partial \Pr\left(A^{**}|A^*+\kappa\varepsilon\right)}{\partial A^{**}}=\frac{\phi\left(\frac{A^{**}-A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^*+(1+\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)}$$

Taking the limit as $\varepsilon\to 0$ at $A^*=A^{**}$, we get

$$\lim_{\varepsilon\to 0}\frac{\partial \Pr\left(A^{**}|A^{**}+\kappa\varepsilon\right)}{\partial A^{**}}=\infty$$

Next, consider

$$\frac{\left.\frac{\partial \Pr(A^{**}|A^*+\widehat{\kappa}\varepsilon)}{\partial A^*}\right|_{A^{**}=A^*}}{\left.\frac{\partial \Pr(A^{**}|A^*+\kappa\varepsilon)}{\partial A^{**}}\right|_{A^{**}=A^*}} = -\frac{\frac{\frac{1}{\sigma_A}\phi\left(\frac{A^*-(1-\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^*+(1+\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)}}{\frac{\frac{1}{\sigma_A}\phi\left(\frac{A^*-A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^*+(1+\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)}} - \frac{\frac{\left[\Phi\left(\frac{A^*-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)\right]\left[\frac{1}{\sigma_A}\phi\left(\frac{A^*+(1+\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)-\frac{1}{\sigma_A}\phi\left(\frac{A^*-(1-\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)\right]}{\left[\Phi\left(\frac{A^*+(1+\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)\right]^2}}{\frac{\frac{1}{\sigma_A}\phi\left(\frac{A^*-A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^*+(1+\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)}}$$

Using l'Hôpital's rule, one can establish that

$$\lim_{\varepsilon\to 0} -\frac{\frac{\frac{1}{\sigma_A}\phi\left(\frac{A^*-(1-\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^*+(1+\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\widehat{\kappa})\varepsilon-A_{-1}}{\sigma_A}\right)}}{\frac{\phi\left(\frac{A^*-A_{-1}}{\sigma_A}\right)}{\Phi\left(\frac{A^*+(1+\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)-\Phi\left(\frac{A^*-(1-\kappa)\varepsilon-A_{-1}}{\sigma_A}\right)}} = -1$$

Similarly, using l'Hôpital's rule, one can show that the second term converges to 0. ■

Claim 5 $\lim_{\sigma_x\to 0}\frac{\partial}{\partial A}\, S\left(A,x^*\right)\big|_{A=A^*}=\infty$

Proof. Note that

$$S\left(A,x^*\right)=b\left[1-\Phi\left(\frac{x^*-A}{\sigma_x}\right)\right]$$

Taking the derivative with respect to A, we get

$$\frac{\partial S\left(A,x^*\right)}{\partial A}=\frac{1}{\sigma_x}\phi\left(\frac{x^*-A}{\sigma_x}\right).$$

Under Assumption 4, we have $x^*=\frac{\sigma_x^2+\sigma_A^2}{\sigma_A^2}A^*-\frac{\sigma_x^2}{\sigma_A^2}A_{-1}+\sigma_x^2\sqrt{\frac{1}{\sigma_x^2}+\frac{1}{\sigma_A^2}}\Phi^{-1}\left(\frac{1}{1+r}\right)$, and thus $\lim_{\sigma_x\to 0}\frac{x^*-A}{\sigma_x}=\phi\left(\Phi^{-1}\left(\frac{1}{1+r}\right)\right)$. The Claim follows immediately from this observation. ■

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