IDENTIFYING MULTIDIMENSIONAL ADVERSE SELECTION MODELS∗

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Abstract. In this paper, I study the nonparametric identification of a multidimensional adverse selection model. In particular, I consider the screening model of Rochet and Choné (1998), where products have multiple characteristics and consumers have private information about their multidimensional taste for these characteristics, and determine the data features and additional condition(s) that identify model parameters. The parameters include the nonparametric joint density of consumer taste, the cost function and the utility function, and the data includes individual level data on choices and prices paid from one market. When the utility is nonlinear in product characteristics, however, data from one market is not enough, but if data from at least two markets, or over two periods is sufficient as long as they have different marginal prices. Therefore, it is sufficient to have data with exogenous and binary cost shifter that affects the marginal prices. I also characterize all (testable) restrictions imposed by the model on the data.

Keywords: identification, multidimensional adverse selection, rationalizability.

Date: This Version: June 23, 2015.

∗ Previously, the paper was circulated under the title of, “Identifying a Screening Model with Multidimensional Private Information.” Thanks to be added.

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1. Introduction

At least since Akerlof (1970); Spence (1973) and Rothschild and Stiglitz (1976), economists believe that information asymmetry is a universal phenomenon, and that it always results in a significant loss of welfare. This belief has influenced the design of various regulatory policies; see Baron (1989); Joskow and Rose (1989); Laffont and Tirole (1993) and Laffont (1994). However, Chiappori and Salanié (2000) find no evidence of asymmetric information in automobile insurance, and Einav, Finkelstein, and Cullen (2010) estimate the welfare loss due to asymmetric information to be insignificant. One reason for this lack of empirical support for the theory could be that the most widely used theoretical models for asymmetric information assume one dimensional informational asymmetry, while in reality informational asymmetry is multidimensional, which cannot be sorted out in a satisfactory manner according to only one dimension. Ignoring multidimensionality would then lead to incorrect analysis and flawed welfare conclusion(s).\(^1\) Stiglitz (1977) assumes insurees have private information only about their accident risk, but Finkelstein and McGarry (2006) and Cohen and Einav (2007) have shown that insurers have private information about their risk and risk preference, and if anything, risk preferences are more likely to be privately known than the risk. Aryal, Perrigne, and Vuong (2015) study the nonparametric identification of insurance market where insurers have asymmetric information about risk and risk preferences. Most of the literature is focused on insurance and annuities markets, and very little is known about other markets where multidimensionality is potentially equally important. In this paper I attempt to contribute in this line of research by studying the identification of multidimensional adverse selection model, also known as multidimensional screening model, which lays the foundation for estimating such a model.

In particular, I study an environment where a seller sells a product with multiple characteristics to consumers with heterogeneous taste for each of those characteristics, similar to Lancaster (1971), but where only the consumer is privy to her taste profile. The seller knows only the (market wide) joint density of the taste profile, and offers a menu of options (product characteristics and prices) to maximize the expected profit. In equilibrium a seller offers a menu of options – configuration of product characteristics and prices – that maximizes the expected profit, subject to the truth-telling and participation constraints for

\(^1\) It is instructive to note that for third-degree price discrimination the sufficient condition for price discrimination to improve social welfare is known (Varian, 1985).
the consumer. Rochet and Choné (1998) (henceforth, Rochet-Choné ) showed that there exists a unique equilibrium, and the (first-order) conditions that characterize the equilibrium provide mappings from the (unobserved) structural parameters to the (observed) choices.

I study the condition(s) under which I can use data on consumer choices and payments from a single market to nonparametrically identify the joint density of taste profile, the cost function, and the (common) utility function when product characteristics enter the utility function nonlinearly. Identification is tantamount to showing that the equilibrium mapping implied by Rochet-Choné is invertible. Even though I use the multiple characteristics interpretation, the results can be applied to several other environments such as multi-product monopoly and a labor contract for multiple tasks where workers differ in their task-specific skills.

In a sharp contrast with one dimensional asymmetric information, with multi-dimensional asymmetric information, bunching (two distinct types of consumers choose the same bundle) is inevitable with possibly a very different welfare implications. In equilibrium the seller partitions the consumers’ type into three categories: the high-types who are perfectly screened (where each type is offered a unique bundle of qualities); the medium-types who are further divided into different categories where all types of the same category are bunched and get the same bundle; and the low-types who are always excluded or offered the outside option. I consider these sub-groups sequentially, first when the product characteristics enter the utility function linearly and, then, when they enter nonlinearly.

Consider the high-types, who are perfectly screened, which means their incentive compatibility constraints are satisfied. This translates into the restriction that the marginal utility for each characteristics is equal to the marginal price. With linear utility, marginal utility is nothing but the consumers’ type, and the marginal price is the gradient (slope) of the observed price function, which identifies the truncated joint density of high-types. Once we identify the types, I show that we can use the first-order condition that characterizes optimal allocation rule (a map from types to product characteristics) to nonparametrically identify the cost function over an appropriate domain.

This identification strategy does not work for other types because they are bunched. So I have to use some other form of exogenous variation. In particular I show that we can use variation in the observed consumer characteristics.
Suppose there are as many consumer characteristics as product characteristics which are independent of the consumer types, then I can identify the truncated joint density of medium types. The intuition behind this strategy is to represent the conditional choice density for a given consumer characteristics as a Radon transform of truncated density of consumer types, which is invertible (Helgason, 1999).\textsuperscript{2}

Next, I consider the case when the product characteristic enter the utility function nonlinearly. Then the marginal utility is a product of consumer type and the slope of the nonlinear function. I show that because of this substitution between the nonlinear function and the type, the model cannot be identified, even for the high-types. I then show that if we have data from either two markets or over two-periods, where the two markets differ in terms of some exogenous (binary) change in cost then we can identify the nonlinear utility and the joint density of high-types as long as the cost affect the marginal prices. In particular, if the cost is exogenous and is independent of the consumer type then we can compare the multivariate quantiles of choices across these two cost regimes to identify the multivariate quantile, Koltchinskii (1997), of the high-types. The incentive compatibility constraints for the high-type implies that a median type (among the high-types), say, will choose the median bundle (meant for the high-types) under both costs. Once we have identified the quantiles for the high-types, we can “control” the type and use the consumers’ optimality condition to identify the utility function. This method of matching-quantiles under appropriate exclusion restriction draws insights from Matzkin (2003) and Guerre, Perrigne, and Vuong (2009), and as a result is also related to D’Haultfoeuille and Février (2011, 2015) and Torgovitsky (2015).\textsuperscript{3}

Even though I do not estimate my model, it might be instructive to consider a concrete example of a market where my results would be applicable. Consider the optimal bundling problem faced by a telecommunication company. For example, consider a dominant telecommunication company in a city in China that offers multiple cellphone plans that have different rates for voices data and instant text messaging. For more on this market see Luo, Perrigne, and Vuong (2012). A scatter plot of the choices and payments are in Figure 1. As can be seen, there is substantial heterogeneity among consumers in terms of their

\textsuperscript{2} See Gautier and Kitamura (2013) for application to random coefficient model.

\textsuperscript{3} The working paper version D’Haultfoeuille and Février (2011) is more relevant than the published version. Also see Aryal, Grundl, Kim, and Zhu (2015) for application to auctions with ambiguity.
final usage. The payments are not in a ray as function of either the voice or the messages. Profit for the seller is highest for the plans that are designed for high-type consumers, those who have higher willingness to pay and have higher usage. These high-types, however, cannot be prevented from choosing plans that are meant for the medium-types or the low-types. So higher profit can only be realized if the seller distorts the product characteristics meant for the latter types in the direction that makes them relatively unattractive for the high-types. But the optimal amount of distortion, and therefore the welfare estimation, depends on the joint density of consumer types. To determine the level of distortion we have to solve multidimensional adverse selection (screening) problem, which would mean that the observed product characteristics or bundles are endogenous. This means the approach used in the classic demand estimation, pioneered by Berry (1994); and Berry, Levinsohn, and Pakes (1995),
and more recently by Berry, Gandhi, and Haile (2013), is inapplicable, not to mention the fact that those models do not allow asymmetric information.\footnote{Fan (2013) considers endogenous product characteristics, but with perfect information.}

Next I explore the questions of over-identification and rationalizability of the models. I show that if the truncated joint density of high-types is over-identified when the utility is nonlinear and we use data on exogenous consumer characteristics. Over-identification affects both the efficiency of an estimator and the refutability of the model (Koopmans and Riersol, 1950; Romano, 2004). The logic behind this result is as follows: Since we can identify consumers’ for every bundle (meant for the high-types), and since this mapping is independent of the consumer characteristics, if the data is generated by Rochet-Choné equilibrium then it must be the case that the this mapping between type and bundle must be such that it maximizes the expected correlation between the two for all consumer characteristics. In other words, I ask if there is a way to determine the best way to “transport” the identified types to the observed bundles. The data is an equilibrium outcome if and only if such transport maximizes the correlation between the two, where the correlation is computed with respect to the density of choices and the density of high-types. Such a problem is known as the optimal transport problem and has long history in economics; see Kantorovich (1960). And we know from Brenier (1991) and McCann (1995) that such a mapping exists and is unique, allowing me to conclude that the model is over-identified. Given the complexity of the model used, I also study the empirical content of the model. To that end, I determine all the testable restrictions implied by the models on the data that can possibly be used in specification tests for model validity.

Pioner (2009) studies semiparametric identification of the Rochet-Choné model, but he assumes that: a) there are only two dimensional private information; and b) the econometrician observes one of those types. Between the seller and the econometrician, it is most like the seller who knows more than the econometrician. These are very strong and restrictive assumptions and my results show neither are necessary. The paper also is related to Perrigne and Vuong (2011) and Gayle and Miller (2014) who study identification of the pure and hybrid moral hazard, respectively; Luo, Perrigne, and Vuong (2012) who study the telecommunication data; Ivaldi and Martimort (1994) and Aryal (2013) who study competitive nonlinear pricing; and Ekeland, Heckman, and Nesheim
MULTIDIMENSIONAL ADVERSE SELECTION


The paper is organized as follows. Notations and definitions are in Section 2, Section 3 describes the model, while Section 4 contains the identification results, and Section 5 provides the rationalizability lemmas. Section 6 considers measurement error and unobserved heterogeneity before concluding.

2. NOTATIONS AND DEFINITIONS

I will use the notation \( \zeta \in S_\zeta \subset \mathbb{R}^{d_\zeta} \) to mean that it is \( d_\zeta \) dimensional vector that can take value in the set \( S_\zeta \), and use \( \partial S_\zeta \) to denote the boundary values for \( S_\zeta \). If \( \gamma \in S_\gamma \subset \mathbb{R}^{d_\gamma} \) and \( d_\gamma = d_\zeta = d \) then \( \zeta \cdot \gamma := \sum_{j=1}^{d} \zeta_j \gamma_j \) denotes the inner product. Let \( \kappa : S_\zeta \to \mathbb{R}^{d_\kappa} \) define a \( d_\kappa \) dimensional vector of functions

\[
\kappa(\zeta) = \begin{pmatrix}
\kappa_1(\zeta_1, \ldots, \zeta_{d_\zeta}) \\
\vdots \\
\kappa_{d_\kappa}(\zeta_1, \ldots, \zeta_{d_\zeta})
\end{pmatrix}.
\]

For a scalar function \( \kappa(\zeta_1, \ldots, \zeta_{d_\zeta}) \in \mathbb{R} \), \( \nabla \kappa = \left( \frac{\partial \kappa}{\partial \zeta_1}, \ldots, \frac{\partial \kappa}{\partial \zeta_{d_\zeta}} \right) \) denotes the gradient of \( \kappa(\cdot) \), and \( \nabla_j \kappa(\cdot) \) denotes the \( j \)th element of the gradient vector. The divergence of a scalar function \( \kappa(\zeta) \) is defined as \( \text{div} \kappa = \sum_{j=1}^{d_\kappa} \frac{\partial \kappa(\zeta)}{\partial \zeta_j} \).

**Definition 2.1.** A scalar function \( \kappa : \mathbb{R}^{d_\kappa} \to \mathbb{R} \) is a real analytic function at \( \tilde{\zeta} \in \mathbb{R}^{d_\zeta} \) if there \( \exists \delta > 0 \) and open ball \( B(\tilde{\zeta}, \delta) \subset \mathbb{R}^{d_\zeta}, 0 \leq r < \delta \) with \( \sum_{k_1, \ldots, k_J} |a_{k_1, \ldots, a_{K_J}}| r^{k_1 + \cdots + k_J} < \infty \) such that

\[
\kappa(\tilde{\zeta}) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_J=0}^{\infty} a_{k_1, \ldots, a_{K_J}} (\tilde{\zeta}_1 - \zeta_1)^{k_1} \cdots (\tilde{\zeta}_J - \zeta_J)^{k_J}, \tilde{\zeta} \in B(\tilde{\zeta}, \delta).
\]

One of the properties of analytic functions is that if two real analytic convex functions coincide on an open set then they coincide on any connected open subset of \( \mathbb{R}^{d_\kappa} \).

**Multivariate Quantiles (Koltchinskii (1997)):** Let \( (\mathcal{S}, B, L) \) be a probability space with probability measure \( L \). Let \( g : \mathbb{R}^J \times S \to \mathbb{R} \) be a function such that \( g(q, \cdot) \) is integrable function \( L- \) almost everywhere and \( g(\cdot, s) \) is strictly convex. Let

\[
g_L(q) := \int_S g(q, s) L(ds), \quad q \in \mathbb{R}^J.
\]

be an integral transform of \( L \). Let the minimal point of the functional

\[
g_{L,t}(q) := g_L(q) - \langle q, t \rangle, \quad q \in \mathbb{R}^J
\]
be called an \((M,t)\)-parameter of \(L\) with respect to \(g\), where \(< \cdot, \cdot \>\) is the inner product in \(\mathbb{R}^J\). The subdifferential of \(g\) at a point \(s \in \mathbb{R}^J\) is denoted by \(\partial g(s) = \{ t \in \mathbb{R}^J | g(s') \geq g(s) + \langle s' - s, t \rangle \}\). Since the kernel \(g(\cdot, s)\) is strictly convex, \(gL\) is convex and the subdifferential map \(\partial gL\) is well defined. The inverse of this map \(\partial gL^{-1}(t)\) is the quantile function and is the set of all \((M,t)\)-parameters of \(L\). Since \(g\) is strictly convex, \(\partial gL^{-1}\) is a single-valued map, and hence we get a unique quantile.\(^5\)

One can choose any kernel function \(g\) as long as it satisfies the conditions mentioned above to define multivariate quantile. Then from Proposition 2.6 and Corollary 2.9 in Koltchinskii (1997) we know that \(\partial gL\) is a strictly monotone homeomorphism from \(\mathbb{R}^J\) onto \(\mathbb{R}^J\) and for any two probability measures \(L_1\) and \(L_2\), the equality \(\partial gL_1 = \partial gL_2\) implies \(L_1 = L_2\). For this paper, we choose \(g(q; s) := |q - s| - |s|\), so that \(gL(q) = \int_{\mathbb{R}^J} (|q - s| - |s|)L(ds), s \in \mathbb{R}^J,\) and

\[
\partial gL(q) := \int_{\{s \neq q\}} \frac{(q - s)}{|q - s|}L(ds),
\]

with the inverse \(\partial gL^{-1}(\cdot)\) as the (unique) quantile function. See Chernozhukov, Galichon, Hallin, and Henry (2015) for an alternative definition.

3. The Model

In this section I will present the model of multidimensional adverse selection of Rochet-Choné. My main goal is to introduce the environment and the conditions that characterize the solution that are essential for identification. I refer readers to the main paper for any missing detail including the proofs.

A seller offers a menu of options that includes a product line \(Q \subseteq \mathbb{R}^d_+\), set of all feasible characteristics, and a corresponding price function \(P : Q \rightarrow \mathbb{R}_+\), to consumers who have different tastes for the product characteristics. Let \(\theta \in S_\theta \subseteq \mathbb{R}^{d_\theta}_+\) denote this taste profile (or simply type), that is independently and identically distributed (across consumers) as \(F_\theta(\cdot)\). Let \(X \in S_x \subseteq \mathbb{R}^{d_x}\) denote consumer’s observed socioeconomic or demographic characteristics. If a type \(\theta\) chooses \(q \in Q\) and pays \(P(q)\), let his net utility be given by

\[
V(q; \theta, X) := u(q, \theta, X) - P(q).
\]

Therefore, we assume the utility function is quasilinear in prices. Let \(C : \mathbb{R}^d_+ \rightarrow \mathbb{R}_+\) be the cost function. The objective of the seller is to choose a (convex) set of product varieties (after suppressing the dependence on \(X\) \(Q\) and a price

\(^5\) For example, with one-dimensional case, for any \(t \in (0, 1)\) the set of all \(t^{th}\) quantiles of a cdf \(M\) is exactly the set of all minimal points of \(gL_1(q) := 1/2 \int_{\mathbb{R}} (|q - s| - |s| + q) L(ds) - qt\).
function $P(\cdot)$ that maximizes her expected profit given the cost function $C(\cdot)$ and $\theta \sim F_\theta(\cdot)$. I begin with the following assumptions:

**Assumption 1.** Let

(i) $d_\theta = d_q = J$.

(ii) $\theta \overset{i.i.d}{\sim} F_\theta(\cdot)$ which has a square integrable density $f(\cdot) > 0$ a.e. on $\mathcal{S}_\theta$.

(iii) The net utility be an element of a Sobolev space

$$V(q; X) \in \mathcal{V}(\mathcal{S}_\theta) = \{V(q; X) | \int_{S_\theta} V^2(\theta) d\theta < \infty, \int_{S_\theta} (\nabla V(q, \theta, X))^2 d\theta < \infty\}.$$

with the norm $|V| := \left(\int_{S_\theta} \{V^2(\theta) + \|\nabla V(\theta)\|^2\} d\theta\right)^{\frac{1}{2}}$.

(iv) The gross utility function is multiplicative in $\theta$:

$$u(q, \theta, X) := \theta \cdot v(q, X) = \sum_{j=1}^J \theta_j v_j(q_j; X),$$

where each $v_j(\cdot; X)$ is differentiable, strictly increasing and is either:

(iv-a) **linear utility**: $v_j(q_j, X) = q_j$.

(iv-b) **bilinear utility**: $d_x \geq J$ such that $X \equiv (X_1, X_2) \in \mathbb{R}^{J+d_x}$ with $d_x \geq 0$, such that $X_1 = (X_{11}, \ldots, X_{1J})$ denote those consumer characteristics that interact multiplicatively with the corresponding product characteristics so that for $j = 1, \ldots, J$, $v_j(q_j, X) = q_j \cdot X_{1j}$.

(iv-c) **nonlinear utility**: everything is same as the bilinear case, except that $v_j(q_j, X) = v_j(q_j, X_2) \cdot X_{1j}$, where $v_j(\cdot, X_2)$ is twice continuously differentiable and strictly quasi concave function, with full rank Jacobian matrix $Dv(q; X_2)$ for all $q \in \mathbb{R}^J_+$, $v_j(0; \cdot) = 0$ and $\lim_{q \to \infty} v_j(q) = \infty$.

(v) $C(\cdot)$ is a strongly convex function with parameter $\epsilon'$, i.e. the minimum eigenvalues of the Hessian matrix is $\epsilon'$.

Assumption 1-(i) assumes that agents differ in as many dimensions as the attributes of a contract. It means that consumers’ unobserved preference heterogeneity is exactly as rich as dimension of product characteristics. Assumption 1-(ii)-(iii) are standard assumptions in the literature in mechanism design; for more see Rochet-Choné. Assumption 1-(v) is the standard convexity of cost assumption.

Assumption 1, however, needs more explanation. The first part of Assumption 1-(iv) assumes that the utility function is multiplicatively separable in consumer type $\theta$ and some function of product characteristics. This multiplicative separability assumption is an important assumption, and is used universally
in mechanism design literature; see Wilson (1993); and Laffont and Martimort (2001). The second part of the assumption puts more stricture on the way in which the product characteristics enter the utility function (from linear to nonlinear). Assumption 1-(iv-a) assumes that the product characteristics enters linearly and does not depend on consumer characteristics \(X\). Assumption 1-(iv-b) assumes that some consumer characteristics might interact with the product characteristics. In particular it assumes that there be at least as many consumer characteristics as product characteristics, i.e. \(d_x \geq J\), such that we can divide the vector \(X\) into two parts: \(X_1\), a \(J\)-dimensional vector, and \(X_2\), a \(d_x - J\) dimensional vector, depending on whether or not they interact with the product characteristics. The gross utility from a product characteristics \(q_j\) is then simply \(\theta \cdot q_j \cdot X_1\). So under this specification the product characteristics still enter linearly and is independent of \(X_2\). Assumption 1-(iv-c) generalizes the previous assumption and allows product characteristics to enter the utility function nonlinearly and allows this function to also depend on \(X_2\) as long as it is bilinear in type and \(X_1\).

It is important to note that for the theoretical model it suffices that the utility function \(u(\theta, q; X)\) be multiplicatively separable in \(\theta\), because we can redefine the units of measurement for the product characteristics and substitute \(\tilde{q} := v(q, X)\) in the place of \(q\) because the seller observes \(X\) and knows \(v(\cdot, \cdot)\). Therefore the only purpose of these three assumptions is from the point of view of empirical application. Since there is no intrinsic unit of measurement when it comes to the product characteristic, I consider three different cases and study the identification problem for each of them. So, even though assumption 1-(iv-c) allow for the most general functional form, studying the identification of linear and bilinear utility will allow me to isolate the data feature that identifies the model parameter.

For notational simplification I will suppress the dependence on \(X\) until next section. A menu \(\{Q, P\}\) is feasible if there exists an allocation rule \(\rho : S_\theta \rightarrow Q\) that satisfies incentive compatibility (IC) condition,

\[
\forall \theta \in S_\theta, V(\rho(\theta), \theta) = \max_{\tilde{q} \in Q} \{\theta \cdot v(\tilde{q}) - P(\tilde{q})\} \equiv U(\theta), \tag{2}
\]

and individual rationality (IR) condition, \(U(\theta) \geq U_0 := \theta \cdot v(q_0) - P_0\). \(\{q_0\}\) denotes the outside option available to everyone at a fixed price \(P_0\). To ensure the principal’s optimization problem is convex, we assume that \(P_0 \geq C(q_0)\), so that the seller will always offer \(q_0\), i.e. \(Q \ni q_0\). The seller chooses a feasible
menu \((Q, \rho(\cdot), P(\mathbf{q}))\), that maximizes expected profit

\[
\mathbb{E} \Pi = \int_{S_\theta} \pi(\theta) dF(\theta) := \int_{S_\theta} 1\{U(\theta) \geq U_0\}(P(\mathbf{q}(\theta)) - C(\mathbf{q}(\theta)))dF(\theta),
\]

where \(1\{\cdot\}\) is an indicator function. Let \(S(\rho(\theta), \theta)\) be the social surplus when \(\theta\) type is allocated \(\mathbf{q}(\theta)\) so that

\[
S(\rho(\theta), \theta) = U(\theta) + \pi(\theta),
\]

or, equivalently

\[
S(\rho(\theta), \theta) = \{\theta \cdot \mathbf{v}(\rho(\theta)) - P(\rho(\theta))\} + \{P(\rho(\theta)) - C(\rho(\theta))\}.
\]

Equating these two definitions, allows us to express type-specific profit as

\[
\pi(\theta) = \theta \cdot \mathbf{v}(\rho(\theta)) - C(\rho(\theta)) - U(\theta).
\]

Rochet (1987) showed that under Assumption 1, a menu \(\{Q, \rho(\cdot), P(\cdot)\}\) is such that \(U(\theta)\) solves Equation (2) (satisfies IC) if and only if: (i) \(\rho(\theta) = \mathbf{v}^{-1}(\nabla U(\theta))\); and (ii) \(U(\cdot)\) is convex on \(\Theta\). This means that choosing an optimal contract \(\{Q, \rho(\cdot), P(\cdot)\}\) is equivalent to determining the net utility (or the information rent) \(U(\theta)\) that each \(\theta\) gets by participating. \(U(\theta)\) also determines a the corresponding optimal allocation rule as \(\rho(\theta) = \mathbf{v}^{-1}(\nabla U(\theta))\).

Using this result we can pose the seller’s problem as, to choose \(U(\theta) \in H^1(S_\theta)\) that maximizes the expected profit

\[
\mathbb{E} \Pi(U) = \int_{S_\theta} \{\theta \cdot \nabla U(\theta) - U(\theta) - C(\mathbf{v}^{-1}(\nabla U(\theta)))\}dF(\theta),
\]

subject to the IC and the IR constraints. As mentioned above, the global IC constraint is equivalent to convexity of \(U(\cdot)\), i.e. \(D^2U(\theta) \geq 0\), and IR is equivalent to \(U(\theta) \geq U_0(\theta)\) for all \(\theta \in S_\theta\). Rochet-Choné showed that Assumption 1 is sufficient to guarantee existence of a unique maximizer \(U^*(\cdot)\). In what follows we will characterize some key properties of the solution. This, however, requires us to solve the variational problem with inequality constraints.

When there is only one dimensional asymmetric information \((J = 1)\), we can ignore the inequality constraints to find an unconstrained maximizer, and verify ex post that the solution satisfied these inequality constraints are satisfied. The assumption that the type distribution is regular (the inverse hazard rate \([1 - F(\cdot)]/[f(\cdot)]\) is strictly decreasing) was sufficient to guarantee that the constraints where satisfied and equilibrium always had perfect screening. When \(J > 1\), however, Rochet-Choné showed that this approach of ex-post verification does not work, and bunching can never be ruled out. In an important paper,
Armstrong (1996) proposes various assumptions that were sufficient to ensure perfect screening. Rochet-Choné have shown that those assumptions are very restrictive and are seldom satisfied. Moreover, it is my opinion that imposing such restrictions to simplify the problem would have been at odds with the nonparametric identification objective of this paper.

One of the key results of Rochet-Choné is that with multidimensional type, the seller will always find it profitable not to perfectly screen consumers and hence bunching (for some subsets of types) were inevitable. And since, under bunching two distinct types of consumers choose the same option, it makes identification all that more difficult. In equilibrium the consumers will be divided into three types: the lowest-types $S^0_\theta$ who are screened out and offered only $\{q_0\}$, the medium-types $S^1_\theta$ who are bunched and offered “medium type” of bundles and the high-types $S^2_\theta$ who are perfectly screened. The next step is to determine these subsets, which will depend on the model parameters.

If an indirect utility function $U^*(\cdot)$ is optimal then offering any other feasible function $(U^* + h)(\cdot)$, where $h$ is non-negative and convex, must lower expected profit for the seller, i.e., $\mathbb{E}\Pi(U^*) \geq \mathbb{E}\Pi(U^* + h)$. This means the directional derivative of the expected profit, in the direction of $h$, must be nonnegative so $U^*(\cdot)$ is the solution iff: (a) $U^*(\cdot)$ is convex function and for all convex, non negative function $h$, $\mathbb{E}\Pi'(U^*)h \geq 0$; and (b) $\mathbb{E}\Pi'(U^*)(U^* - U_0) = 0$ with $(U^* - U_0) \geq 0$. The Euler-Lagrange condition for the (unconstrained) problem is

$$\frac{\partial \pi}{\partial U^*} - \sum_{j=1}^J \frac{\partial}{\partial \theta_j} \left[ \frac{\partial \pi}{\partial (\nabla_j U^*)} \right] = 0,$$

or, $\alpha(\theta) := [-f(\theta) + \text{div} \{f(\theta)(\theta - \nabla C(\nabla U^*))\}] = 0$. (4)

Intuitively, $\alpha(\theta)$ measures the marginal loss of the seller when the indirect utility (information rent) of type $\theta$ is increased marginally from $U^*$ to $U^* + h$. Alternatively, define $\nu(\theta) := \frac{\partial S(\theta, q(\theta))}{\partial q}$, the marginal distortion vector, then $\alpha(\theta) = 0$ is equivalent to $\text{div}(\nu(\theta)) = -f(\theta)$, which is the optimal tradeoff between distortion and information rent. Let $L(h) = -\mathbb{E}\Pi'(U^*)h$ be the loss of the seller at $U^*$ for the variation $h$. So if the seller increases $U^*$ in the direction of some $h$ then the seller’s marginal loss can be expressed as

$$L(h) = \int_{S_\theta} h(\theta)\alpha(\theta)d\theta + \int_{\partial S_\theta} h(\theta) (-\nu(\theta) \cdot \hat{n}(\theta)) d\sigma(\theta) := \int_{S_\theta} h(\theta)d\mu(\theta), \quad (5)$$
where $d\sigma(\theta)$ is the Lebesgue measure on the boundary $\partial S_\theta$, $\hat{n}(\theta)$ is an outward normal and $d\mu(\theta) := \alpha(\theta)d\theta + \beta(\theta)d\sigma(\theta)$. If we consider those who participate, i.e., $U^*(\theta) \geq U_0(\theta)$, this marginal loss $L(h)$ must be zero. Since $h \geq 0$ it means $\mu(\theta) = 0$, so that both $\alpha(\theta)$ and $\beta(\theta) := -\nu(\theta) \cdot \hat{n}(\theta)$ must be equal to zero. For those who do not participate, it must mean the loss is positive.

**Lemma 1.** Rochet-Choné Proposition 4: The optimal $U^*$ satisfies

$$\forall \theta \in S_\theta, \quad \alpha(\theta) \begin{cases} > 0, & U^*(\theta) \leq U_0(\theta) \\ = 0, & U^*(\theta) > U_0(\theta) \end{cases},$$

and $\forall \theta \in \partial S_\theta, \quad \beta(\theta) \begin{cases} > 0, & U^*(\theta) \leq U_0(\theta) \\ = 0, & U^*(\theta) > U_0(\theta) \end{cases}.$

The global incentive compatibility condition is important because it determines the optimal bunching in the equilibrium by requiring $(U^* - U_0)(\theta)$ be convex. This corresponds to determination of the subset $S^1_\theta$ where the optimal allocation rule $\rho(\cdot)$ will be such that some types are allotted same quantity $q$. Let $S^1_\theta(q)$ be the types that gets the same $q$, i.e. $S^1_\theta(q) = \{ \theta \in \Theta : \rho(\theta) = q \} = \{ \theta \in \Theta : U^*(\theta) = \theta \cdot q - P(q) \}$. If $U^*(\theta)$ is convex for all $\theta$, that is if the global incentive compatibility constraint is satisfied then there is no bunching and $S^1_\theta$ would be an empty set. In most of the cases, however, the convexity condition fails and hence there will be non-trivial bunching. So, $U^*$ is affine on all the bunches, and the incentive compatibility constraint is binding for any two types $\theta', \theta$ if and only if they both belong to $S^1_\theta(q)$, i.e. if $\theta' \not\in S^1_\theta(q)$ but $\theta \in S^1_\theta(q)$ then $U^*(\theta') > U^*(\theta) + (\theta - \theta')^Tq$.

**Theorem 3.1.** Rochet-Choné , Theorem 2’: Under the Assumptions 1-(i)–(iv-a) and (v) the optimal solution $U^*$ to the problem is characterized by three subsets $S^0_\theta, S^1_\theta$ and $S^2_\theta$ such that:

1. A positive mass of types $S^0_\theta$ do not participate because $U^*(\theta) = U_0(\theta)$. This set is characterized by $\mu(S^0_\theta) = 1$, i.e. $\int_{S^0_\theta} \alpha(\theta)d\theta + \int_{\partial S^0_\theta} \beta(\theta)d\theta = 1$.
2. $S^1_\theta$ is a set of “medium types” known as the bunching region, which is further subdivided into subset $S^1_\theta(q)$ such that all types in this subset get one type $q$, $U^*$ is affine. $\mu$ restricted to $S^1_\theta(q)$ satisfies: $\int_{S^1_\theta(q)} d\mu(\theta) = 0$ and $\int_{S^1_\theta(q)} \theta d\mu(\theta) = 0$.
3. $S^2_\theta$ is the perfect screening region where $U^*$ satisfies the Euler condition $\alpha(\theta) = 0$, or equivalently $\text{div}(\nu(\theta)) = -f(\theta)$, for all $\theta \in S^1_\theta \cap S_\theta$, and there is no distortion in the optimal allocation on the boundary, i.e. $\beta(\theta) = 0$ on $S^1_\theta \cap \partial S_\theta$. 

Summary: the type space is (endogenously) divided into three parts: those who are excluded $S_0^\theta$ and get the outside option $q_0$; those who are bunched $S_1^\theta$ and are allocated some intermediate quality $q \in Q_1$ such that all $\theta \in S_1^\theta(q)$ get the same quantity $q$; and finally those who are perfectly screened $S_2^\theta$ and are allocated some unique (customized) $q \in Q_2$. An example shown in Fig. 2. It is also important to note that the allocation rule $\rho(\cdot)$ is continuous.

**Corollary 1.** $\rho(\cdot)$ is continuous and $\frac{\partial \rho(\theta_j, \theta_{-j})}{\partial \theta_j} > 0, \forall (\theta_j, \theta_{-j}) \in \Theta_2$.

**Proof.** For $\theta \in \Theta_2$, since $D^2 U^*(\theta) > 0$ and because $\rho(\theta) = \nabla U^*(\theta)$ it is also continuous. Likewise, for all $\theta \in \Theta_0, \rho(\theta) = q_0$ and hence continuous. Similar arguments show that $\rho(\theta)$ is continuous for all $\theta \in S_1^\theta$. $\square$

4. Identification

In this section we study the identification of the distribution of types $F_\theta(\cdot)$ and the cost function $C(\cdot)$ from the observables that include the triplet $\{q_i, p_i, X_i\}$ for every consumer. Let $(q, p, X) \overset{i.i.d}{\sim} \Psi_{p,q,X}(\cdot, \cdot, \cdot) = \Psi_{p,q,X}(\cdot, \cdot) \times \psi_X(\cdot)$. The seller offers $(Q(x), P(x)(\cdot))$ to agent $i$ with observed characteristics $X_i = x \sim \Psi_X(\cdot)$ and unobserved type $\theta_i \sim F_\theta(\cdot)$, who then chooses $q_i \in Q(x)$ and pays $p_i = P(x)(q_i)$ that maximizes the net utility. The seller chooses the menu optimally, which from the revelation principle is equivalent to saying that there exists a direct mechanism, a unique pair of allocation rule $\rho(\cdot): S_\theta \rightarrow Q_{(x,z)}$ and pricing function $P_x(\cdot): Q(x) \rightarrow \mathbb{R}_+$, such that $q_i = \rho(\theta_i)$ and $p_i = P_x(\rho(\theta_i))$. Henceforth, $\rho(\cdot)$ will stand for optimal allocation rule.

Thus assuming that: a) consumers have private information about $\theta$; b) the seller only knows the $F_\theta(\cdot)$ and $C(\cdot)$, and designs a $\{Q, P(\cdot)\}$ to maximize profit; and c) consumers optimize, leads to the following model:

\[
\begin{align*}
p_i &= P[q_i, F_\theta(\cdot), C(\cdot); X] \\
q_i &= \rho[\theta_i, F_\theta(\cdot), C(\cdot); X], \quad i \in [N], k = 1, 2. \tag{8}
\end{align*}
\]

The model parameters $[F_\theta(\cdot), C(\cdot)]$ are said to be identified if for any different parameters $[\tilde{F}_\theta(\cdot), \tilde{C}(\cdot)]$, the implied data distributions are also different, i.e., $\Psi_{p,q,X}(\cdot, \cdot, \cdot) \neq \tilde{\Psi}_{p,q,X}(\cdot, \cdot, \cdot)$. Since the equilibrium is unique, there is a unique distribution of the observable $\Psi_{p,q,X}(\cdot, \cdot, \cdot)$ for every parameter, Jovanovic (1989). My objective is to determine some low-level conditions under which the model
is globally identified which is tantamount to showing that Equations in (8) are globally invertible.  

Following the equilibrium characterization, I consider the three subsets of types separately. For every $X = x$ let $Q^j(x)$ be the set of choices made by consumers with type $\theta \in S^j_\theta$ for $j = 0, 1, 2$, respectively. Since $q(\cdot)$ is continuous (Corollary 1), these sets are well defined. In what follows, I will use data from $Q^j_k(x)$ to identify the model parameters restricted to $S^j_\theta$, beginning with the subset $Q^2_k(x)$. The allocation rule $\rho(\cdot)$ is one-to-one when restricted to $S^2_\theta$, and hence its inverse $\rho^{-1}(\cdot)$ exists on $Q^2_k(x)$, but not when restricted to $S^1_\theta$ because of bunching, and as a consequence the identification strategies are different.

In what follows, I suppress the dependence on $X$, until it is relevant. Let $M(\cdot)$ and $m(\cdot)$ be the distribution and density of $q$, respectively. Since the equilibrium indirect utility function $U^*$ is unique, it implies that there is a unique distribution $M(\cdot)$ that corresponds to the model structure $[F_\theta(\cdot), C(\cdot)]$. Thus the structure is said to be identified if for given $m(\cdot)$ there exists a (unique) pair $[F_\theta(\cdot), C(\cdot)]$ that satisfies Equations (8). Let $\tilde{\theta}(\cdot) : Q^2 \rightarrow S^2_\theta$ be the inverse of $\rho(\cdot)$ when restricted on $Q^2$, i.e. $\forall q \in Q^2, \tilde{\theta}(q) = \rho^{-1}(q)$. Similarly, let $M^*(q)$ and $m^*(q)$ be the truncated distribution and density of $q \in Q^2_k(x)$ defined, respectively, as:

$$
M^*(q) := \Pr(\tilde{q} < q | q \in Q_2) = \Pr(\theta < \tilde{\theta}(q) | \theta \in S^2_\theta) = \int_{S^2_\theta} \mathbb{1}\{\theta < \tilde{\theta}(q)\} f_\theta(\theta | \theta \in S^2_\theta) d\theta;
$$

$$
m^*(q) := \frac{m(q)}{\int_{Q_2} m(q) dq} = \frac{f(\tilde{\theta}(q))}{\int_{Q_2} f(\tilde{\theta}(q)) dq} |\det(D\tilde{\theta})(q)|, 
$$

where $det(\cdot)$ is the determinant function. Then the bijection between high-type and high-qualities gives

$$
M^*(q) = \Pr(\rho(\theta, F_\theta, C) \leq q | q \in Q^2) = F_\theta \circ \rho^{-1}(q | q \in Q^2), 
$$

which will be a key relationship for identification. Before moving on, I introduce new short-hand notations. Let $F_\theta(\cdot | j)$ be the CDF $F_\theta(\cdot)$ restricted to be in the set $S^j_\theta$ and let $N_j$ be the set of consumers who buy $q \in Q_j$, for $j = 0, 1, 2$.

---

6 See Rothenberg (1971) and Chen, Chernozhukov, Lee, and Newey (2014) for local identification in parametric and (semi)nonparametric setup, respectively.

7 Since $U^*(\theta)$ is convex on $S^2_\theta$, the inverse $\rho^{-1}(\cdot)$ that solves $q = \rho(\theta)$ exists, Kachurovskii (1960). Also see Parthasarathy (1983) and Fujimoto and Herrero (2000) for more details.
4.1. **Linear Utility.** I begin by showing that without any further restrictions \( F_\theta(\cdot | 2) \) can be identified and \( C(\cdot) \) can be identified on \( Q^2 \). When the utility function is linear, consumer optimality for the high-types imply that the marginal utility, which is \( \theta \), is equal to the marginal prices \( \nabla P(\cdot) \) – the gradient of price function. Therefore the type that chooses \( q \in Q^2 \) must satisfy

\[
\nabla P(q) = \theta = \tilde{\theta}(q),
\]

which identifies the (pseudo) type \( \theta_i = \tilde{\theta}(q_i) \) for all \( i \in \mathbb{N}_2 \).

This identification arguments uses the demand side optimality and the price gradients. We lose this identification when either the utility function is nonlinear (Subsection 4.3) or when there are discrete options and we cannot calculate the price gradients (Subsection 6.4).

As \( \tilde{\theta}(\cdot) \) restricted to \( S^2_\theta \) is bijective we can identify the truncated joint distribution of types, or simply the joint distribution of high-types as

\[
F_\theta(\xi|2) = \Pr(q \leq (\nabla P)^{-1}(\xi)|Q \in Q^2) = M^*((\nabla P)^{-1}(\xi)).
\]

Next, I consider identification of the cost function. The equilibrium allocation condition (4) is

\[
\alpha(\theta) = 0, \text{ or } \nabla \{ f_\theta(\theta) (\theta - \nabla C(\nabla U^*)) \} = -f_\theta(\theta).
\]

If we divide both sides by \( \int_{S^2_\theta} f_\theta(t) dt \) we get

\[
\begin{align*}
\text{div} \left\{ \frac{f_\theta(\theta)}{\int_{S^2_\theta} f_\theta(t) dt} (\theta - \nabla C(\nabla U^*)) \right\} &= -\frac{f_\theta(\theta)}{\int_{S^2_\theta} f_\theta(t) dt} \\
\text{div} \left\{ \frac{m^*(q)}{|\det(D\tilde{\theta}(q))|} (\tilde{\theta}(q) - \nabla C(q)) \right\} &= -\frac{m^*(q)}{|\det(D\tilde{\theta}(q))|} \\
\text{div} \left\{ \frac{m^*(q)}{|\det(D\nabla P(q))|} (\nabla P(q) - \nabla C(q)) \right\} &= -\frac{m^*(q)}{|\det(D\nabla P(q))|},
\end{align*}
\]

where the second equality follows from Equation (9) and the last equality follows from the definition of the curvature of the pricing function, i.e. \( D\nabla P(q) = D\tilde{\theta}(q). \) This means the cost function \( C(\cdot) \) is the solution to the partial differential equation (PDE) with the following boundary condition, \( \beta(\theta) = 0 \) on \( \partial Q^2 \), i.e.,

\[
\frac{m^*(q)}{|\det(D\nabla P(q))|} (\nabla C(q) - \nabla P(q)) \cdot \vec{n}(\nabla P(q)) = 0.
\]

The cost function \( C(\cdot) \) is on the convex set \( Q^2 \) identified as the unique solution of this PDE, Evans (2010). To extend it to the entire domain we need:

**Assumption 2.** Cost function \( C : Q \to \mathbb{R} \) is a real analytic function at \( q \in Q \).
This assumption about $C(\cdot)$ being analytic is a technical assumption that assumes $C(\cdot)$ is infinitely differentiable and can be expressed (uniquely) as a Taylor series. Hence, it allows for any convex polynomial, trigonometric and exponential functions. Once the cost function is identified on an open convex set $Q^2$, analytic extension theorem implies that the function has a unique extension to the entire domain $Q$. Since the cost function is completely unspecified, besides convexity, we need to restrict the space of functions to be able to extend it everywhere. Assuming that the cost function is analytic is sufficient. Undoubtedly this is stronger than anything I have assumed thus far, but this assumption has also been used in the literature on nonparametric identification. Newey and Powell (2003) assume that the density belongs to the exponential family, which is analytic, (see Liese and Miescke, 2008, Lemma 1.16), and therefore has a unique extension; Fox, Il Kim, Ryan, and Bajari (2012) also assume real analyticity to identify random coefficient logit model; and Fox and Gandhi (2013) assume the utility function is real analytic to identify random utility model. This result is formalized below.

**Theorem 4.1.** Under the Assumptions 1- (i)-(iv-a), (v), and 2, the model structure $[F_\theta(\cdot|2), C(\cdot)]$ is nonparametrically identified.

It is clear that the monotonicity of $\rho(\cdot)$ is the key to identification, and since we lose monotonicity on $S^1_{\theta}$ we lose identification, as shown in the example below which is taken from Rochet-Choné.

**Example 4.1.** Let $J = 2$ and the cost function be $C(q) = c/2(q_1^2 + q_2^2)$ and types are independent and uniformly distributed on $S_\theta = [0, 1]^2$ and $q_0 = 0$ and $P_0 = 0$. Then, the optimal indirect utility function $U^*$ has different shapes in the three regions: (i) in the non participation region $S^0_{\theta}, U^*(\theta) = 0$; (ii) in the bunching region $S^1_{\theta}, U^*$ depends only on $\theta_1 + \theta_2$; and (iii) in the perfect screening region $S^2_{\theta}, U^*$ is strictly convex.

On $S^0_{\theta}, \rho(\theta) = 0$, which means $\alpha(\theta) = \text{div}(\theta f(\theta)) + f(\theta) = 3$ and $\beta(\theta) = a$ on $\partial S^0_{\theta}$. The boundary that separates $S^0_{\theta}$ and $S^1_{\theta}$ is a linear line $\tau_0 = \theta_1 + \theta_2$, where $\tau_0 = \sqrt{\frac{3}{c}}$. On $S^1_{\theta}, \rho(\theta) = (\rho_1(\theta), \rho_2(\theta)) = (\rho_0(\tau), \rho_0(\tau)), \text{ with } \theta_1 + \theta_2 = \tau$. In other words, all consumers with type $\theta_1 + \theta_2 = \tau$ are treated the same and they get the same $\rho_1(\tau) = \rho_2(\tau) = \rho_0(\tau)$. So $\alpha(\theta) = 3 - 2cq_0(\tau)$ and on $\partial S^1_{\theta}, \beta(\theta) = (cp(\theta) - \theta) \cdot \hat{n}(\theta) = -c \rho_0(\tau)$. Sweeping conditions are satisfied if $\alpha(\theta) \geq 0$ and $\beta(\theta) \geq 0$ and on each bunch

$$\int_{\theta_1}^{\tau} \alpha(\theta_1, \tau - \theta_1)d\theta_1 + \beta(0, \tau) + \beta(\tau, 0) = 0,$$
which can be used to solve for \( q_b \) as \( \rho_b(\tau) = \frac{3\tau}{4c} - \frac{1}{2c} \). Then \( S^1_\theta = \{ \theta : \tau_0 \leq \theta_1 + \theta_2 \leq \tau_1 \} \) where \( \tau_1 \) is determined by the continuity condition on \( S_\theta \) of \( \rho(\cdot) \), i.e. \( \rho_b(\tau_1) = 0 \). Now, define \( \tau = \rho_b^{-1}(q) \) as the inverse of the optimal (bunching) mechanism. Then identification is to determine the joint cdf of \( (\theta_1, \theta_2) \) from that of \( \tau = \theta_1 + \theta_2 \), which is not possible.

To summarize: the seller divides the agents into three categories and perfectly screens only the top ones. We can then use the distribution of their choices to determine their types and the cost function. To understand the welfare consequence of asymmetric information we might also want to understand the heterogeneity in preference of those in the medium categories that are not perfectly screened but they are not excluded from the market either. The example above shows that if we restrict the utility function to be linear and independent of the consumer characteristics then because the bunching is also linear we cannot identify the types.

This brings me to the next question. If the utility is also a function of observed characteristics \( X \), then can we use the variation in those observed characteristics to identify the medium-types, the types that are bunched? In the following subsection I show that the answer is positive. Under the Assumption 1-(iv-b) that the utility is bilinear, if the observed characteristics \( X \) are (statistically) independent of the type \( \theta \) and if the dimension of \( X \) is the same as the dimension of \( \theta \) then we can identify \( f_\theta(\cdot|1) \).

4.2. Bilinear Utility. In this subsection I assume that the base utility function satisfies Assumption 1-(iv-b) and \( X_1 \) is independent of \( \theta \). Recall from assumption 1-(iv) that \( X_1 \) denotes those characteristics that interact with product characteristics, while \( X_2 \) do not.

**Assumption 3.** Characteristics \( X = (X_1, X_2) \) and \( \theta \) are mutually independent.
In particular, suppose that the net utility of choosing $q$ by an agent with characteristics $X$ and unobserved $\theta$ is

$$V(q; \theta, X) = \sum_{j=1}^{J} \theta_j X_{1j} q_j - P(q).$$

Now, $X_1$ affects the utility, and hence it will also affect the product line and price functions because now they will depend on $X$. However, once we fix the value of $X_1 = x_1$ (which is observed by the seller) and change the unit of measurement of product quality from $q$ to $\tilde{q} := x_1 \cdot q$, we can apply Theorem 4.1 to identify $f(\theta|2)$. Next, I show that we can also use exogenous variation in $X_1$ to identify the density $f(\theta|1)$ over the bunching region $S_1^\theta$. So, independent variation in $X_1$ is an important assumption for identification. As it will be clear, we will use the notation $f_\theta(\cdot|1)$ to make it clear that it is the density of $\theta \in S_1^\theta$.

In the example above we saw that all agents with type such that $\tau = \sum_{j \in [J]} \theta$ selected the same $q(\tau)$. Now, that the agents vary in $X$, agents are bunched according to $W = \sum_{j \in [J]} \theta_j X_{1j}$, in other words, all agents with the same $W$ select $\rho(W)$, i.e. $\rho(\theta) = (\rho_1(\theta), \ldots, \rho_J(\theta)) = (\rho_1(W), \ldots, \rho_J(W))$ for all $\theta \in S_1^\theta$. In other words, $W$ acts as a sufficient statistics, and incentive compatibility requires that $q(W)$ be monotonic in $W$ and hence invertible. So from the observed $q$ we can determine the index $W := \rho^{-1}(q)$. Then, the identification problem is to recover $f_\theta(\cdot)$ from the joint density $f_{W,X_1}(\cdot, \cdot)$ of $(W, X_1)$ when

$$W = \theta_1 X_{11} + \cdots + \theta_J X_{1J}.$$ 

I begin by normalizing the equation above by multiplying both sides by $||X||^{-1}$. Let $D := ||X_1||^{-1} X_1 \in S_{J-1}$, and $B := ||X_1||^{-1} W \in \mathbb{R}$ where $S_{J-1} = \{\omega \in \mathbb{R}^J : ||\omega|| = 1\}$ is a $J-$ dimensional unit sphere, so that $B = \theta \cdot D$. Then the conditional density of $B$ given $D$ is

$$f_{B|D}(b|d) = \int_{S_B^\theta} f_{B,D,\theta}(b|d, \theta) f_\theta(\theta|1) d\theta = \int_{\{\theta \cdot d = b\}} f_\theta(\theta|1) d\sigma(\theta) := Rf_\theta(b, d),$$

where $Rf_\theta(b, d)$ stands for the Radon transform, see Helgason (1999), of $f_\theta(\cdot|1)$. So to identify $f_\theta(\cdot|1)$ we must show that $Rf_\theta(\cdot, \cdot)$ is invertible, for which we need sufficient variation in $X$. Suppose not, and suppose $X$ is a vector of constants $(a_1, \ldots, a_J)$. Then we cannot identify $f_\theta(\cdot|1)$ from $B = a_1 \theta_1 + \cdots + a_J \theta_J$. Let

$$Ch_{Rf}(\xi) := \int_{-\infty}^{\infty} e^{-2\pi i \xi} Rf(d, b) db,$$

be the Fourier transform of $Rf(d, b)$ that can be identified from $f_{B,D}(\cdot, \cdot)$, and
be the Fourier transform of \(f_\theta(\cdot|1)\) evaluated at \(\xi d\), which we do not know. However, the Projection slice theorem implies that these two functions are the same for a fixed \(d\), i.e., \(Ch_f(\xi d) = Ch_Rf(\xi)\), and hence \(f_\theta(\cdot)\) can be identified as the Fourier inverse:

\[
f_\theta(\theta|1) = \int_{-\infty}^{\infty} e^{2\pi i \xi \cdot \psi} Rf(\xi) \, d\xi.
\]

**Theorem 4.2.** Under Assumptions 1-(i)-(iv-b), (v), and 2 and 3 the densities \(f_\theta(\cdot|1), f_\theta(\cdot|2)\) and the cost function \(C(\cdot)\) are nonparametrically identified.

Intuitively, the identification exploits the fact that two consumers with same \(\theta\) but different \(X_1\) will face different menus and different choices. So if we consider the population with fixed \(X_1\), the variation in the choices must be due to the variation in \(\theta\). But as we change \(X_1\) from \(x_1\) to \(x_1'\), the choices change but variation in \(\theta\) remains the same because \(X_1 \parallel \theta\). So with continuous variation in \(X_1\), we have infinitely many moment conditions for \(\theta\), which allows us to express the conditional choice density given \(X_1\) as a (mixture) Radon transform of the \(f_\theta(\cdot|1)\) with mixing density being the marginal density of \(X_1\). This shows that even when the equilibrium fails to be bijective, we might be able to use variation in consumer socioeconomic and demographic characteristics \(X_1\) for identification. Since the joint density of types in \(S^2_\theta\) (who were perfectly screened) was identified even without \(X_1\) this result suggests that the model is over identified, which can then be used for specification testing. Even though this intuition is correct, we will postpone the discussion of over identification until the next subsection when I consider nonlinear utility function. I will show that when utility is nonlinear and if we have access to discrete cost shifter then to identify the model it is sufficient that the cost shifter causes the gradient of pricing functions to intersect.

**Note:** So far I have implicitly assumed that we can divide the observed choices \(\{q_i\}\) into three subsets. We know the outside option \(Q^0 = \{q_0\}\), so the only thing left is to determine the bunching set \(Q^1\). As seen in the Figure 2, the product line \(Q^1\) is congruent to one dimensional \(\mathbb{R}_+\), which is the main characteristic of bunching. In higher dimension, the set \(Q^1\) will consist of all products that are congruent with the positive real of dimension lower than \(J\).
4.3. Nonlinear Utility. In this section I consider the model with nonlinear utility (Assumption 1-(iv-c)) that is the (gross) utility function is equal to $X_1 \cdot v(q; X_2)$. To keep the arguments clear, I will ignore $X$, which is tantamount to assuming that $d_X = 0$, and focus on the identification of $v(q)$ on $S_2^q$. Once we have understood the what variation in the data drives identification, we can introduce $X$ and consider the possibility of over-identification.

I begin by first showing that the model $[F_{\theta}(\cdot|2), C(\cdot), v(\cdot)]$ cannot be identified because the two optimality conditions Equations (2) and (10) are insufficient. Identification fails, because of the substitutability between the type $\theta$ and the curvature of the utility function $v(\cdot)$ as shown below.

**Lemma 2.** Under Assumptions 1-(i)–(iv-c) and (v) the model $\{F_{\theta}(\cdot|2), C(\cdot), v(\cdot)\}$, where the domain of the cost and utility functions are restricted to be $Q^2$ and $S_2^q$, respectively, are not identified.

**Proof.** Since the optimality condition (4) is used to determine the cost function, we can treat the cost function as known. I will suppress the dependence on $v$, and let $\theta_1(\cdot) = \theta_1(\cdot|2)$, so $\partial P(\cdot|2) = p_j, \ j = 1, 2$.

Using the change of variable, the joint (truncated) density of $(q_1, q_2)$ is

$$m_{q}^*(q_1, q_2) = f_{\theta} \left( \frac{P_1}{\omega_1 q_1^{\omega_1 - 1}}, \frac{P_2}{\omega_2 q_2^{\omega_2 - 1}} \right) \frac{P_1 P_2 (1 - \omega_1)(1 - \omega_2)}{\omega_1 \omega_2 q_1^{\omega_1 - 1} q_2^{\omega_2 - 1}}.$$ 

Let $\tilde{\theta}_j \equiv \theta_j \times \omega_j \sim F_{\theta}(\cdot|2)$, where $F_{\theta}(\cdot|2) = F_{\theta}(\cdot/\omega|2)$ with $\omega \equiv (\omega_1, \omega_2)$ and $v(q_j) = v(q_j)/\omega = q_j^{\omega_j}/\omega_j$, be a new model. It is easy to check that $\{q_j, p_j\}$ solves the first-order condition implied by $[\tilde{v}(\cdot), F_{\tilde{\theta}}(\cdot)]$, and the joint (truncated) density of $(q_1, q_2)$ is

$$\tilde{m}_{q}^*(q_1, q_2) = f_{\tilde{\theta}} \left( \frac{P_1}{\omega_1 q_1^{\omega_1 - 1}}, \frac{P_2}{\omega_2 q_2^{\omega_2 - 1}} \right) \frac{P_1 P_2 (1 - \omega_1)(1 - \omega_2)}{\omega_1 \omega_2 q_1^{\omega_1 - 1} q_2^{\omega_2 - 1}}.$$ 

As we have seen here, we can increase the type to $\theta \cdot \omega$ and decrease the curvature of utility to $v(q)/\omega$ without changing the observable choices. Therefore, data from only one market is not enough for identification. To that end, let
$Z \in S_Z = \{z_1, z_2\}$ be an exogenous and binary cost shifter, that only affects the cost function $C(\cdot; Z)$ and is independent of the consumer type and the utility function $Z \parallel (\theta, \nu(\cdot))$. For such a shifter to have identification power it must not only change cost but also change the relative prices, however it is not necessary for us to observe $Z$. Such a cost shifter could be either in the form of some exogenous changes in law that affects prices over two period, or different tax or marketing expenses across two independent markets. Either way, as I will show later, it is sufficient for $Z$ to be binary. This exclusion restriction implies that at different values of the cost shifter: a), the ratio of the types will be equal to the ratio of the slope of the prices at different values of the cost shifter; and b) the (multivariate) quantiles of choices by the high-types are the same.

To see how $Z$ can help in the identification, lets go back to the non-identification example in Lemma 2. I will use the subscript $\ell \in \{1, 2\}$ in $\{P_\ell(\cdot), \rho_\ell(\cdot)\}$ to denote the price function and allocation rule when $Z = z_\ell$. As in Lemma 2, the utility function is $v(q_1, q_2) = \begin{pmatrix} v_1(q_1) \\ v_2(q_2) \end{pmatrix} = \begin{pmatrix} (q_1)^{\omega_1} \\ (q_2)^{\omega_2} \end{pmatrix}$. As before let us focus only on the high-types $S_2^\ell$ and let’s further assume that $Q^2$ is also invariant to $Z$. Then the demand side optimality (marginal utility equals the marginal price) can be written as

$$\begin{pmatrix} \nabla_1 P_\ell(q) \\ \nabla_2 P_\ell(q) \end{pmatrix} = \begin{pmatrix} \hat{\theta}_{11}(q) \cdot v_1'(q_1) \\ \hat{\theta}_{21}(q) \cdot v_2'(q_2) \end{pmatrix} = \begin{pmatrix} \omega_1(q_1)^{\omega_1-1} \\ \omega_2(q_2)^{\omega_2-1} \end{pmatrix}, \quad \ell = 1, 2.$$  

Solving for $\nabla v(q)$ for $\ell = 1, 2$ and equating the two gives

$$\begin{pmatrix} \hat{\theta}_{11}(q)/\hat{\theta}_{21}(q) \\ \hat{\theta}_{21}(q)/\hat{\theta}_{22}(q) \end{pmatrix} = \begin{pmatrix} \nabla_1 P_1(q)/\nabla_1 P_2(q) \\ \nabla_2 P_1(q)/\nabla_2 P_2(q) \end{pmatrix},$$  

i.e., the ratio of types should equal the ratio of marginal prices, or equivalently

$$\begin{pmatrix} \hat{\theta}_{11}(q) \\ \hat{\theta}_{21}(q) \end{pmatrix} = \begin{pmatrix} \nabla_1 P_1(q)/\nabla_1 P_2(q) \cdot \hat{\theta}_{21}(q) \\ \nabla_2 P_1(q)/\nabla_2 P_2(q) \cdot \hat{\theta}_{22}(q) \end{pmatrix}. \quad (12)$$

Equation (12) captures the fact that a consumer who pays higher marginal price for a $q$ when $Z = z_1$ than when $Z = z_2$ must have higher type $\hat{\theta}_1(q)$ than $\hat{\theta}_2(q)$. So, if we know $\theta$’s choice $q = q_1(\theta)$ when $Z = z_1$ then we can use the curvature of the pricing functions to determine $\theta$ that chooses the same $q$ when $Z = z_2$.  

Now, consider the supply side. The allocation rule for the high-types is monotonic (IC constraint) so we know:

\[ F_\theta(t|z_2) = F_\theta(t_1, t_2|z_2) = \Pr(\theta_1 \leq t_1, \theta_2 \leq t_2|S_\theta^2) = \Pr(\rho(\theta, z_\ell) \leq \rho(t, z_\ell)|Q^2) \]

\[ = \Pr(q \leq \rho(t, z_\ell)) = \Pr(q_1 \leq \rho_1(t, z_\ell), q_2 \leq \rho_2(t, z_\ell)|Q^2) \]

\[ = M^*_2(\rho(t)), \ell = 1, 2, \]

where the third equality follows from monotonicity of \( \rho(\cdot, Z) \) and exogeneity of \( Z \). This relationship is independent of \( Z \), which gives the following equality

\[ M^*_1(\rho_1(t)) = M^*_2(\rho_2(t)). \]

Hence, the (multivariate) quantiles of the choice distribution when \( Z = z_1 \) are equal to those when \( Z = z_2 \).

\[ \rho_1(t) = (M^*_1)^{-1}[M^*_2(\rho_2(t))], \quad (13) \]

and since \((M^*_1)^{-1} \circ M^*_2(\cdot)\) is identified, we can identify \( \rho_1(\theta) \) if we know \( \rho_2(\theta) \). Therefore, the difference, \(( (M^*_1)^{-1} \circ M^*_2(\rho(\tau)) - \rho(\tau) ))\), measures the change in \( q \) when \( Z \) moves from \( z_2 \) to \( z_1 \), while fixing the quantile of \( q \) at \( \tau \). This variation (13) together with (12) can be used to first identify \( \hat{\theta}(\cdot) \) and then \( \nabla v(q) \) as a (vector valued) function that solves \( \nabla P(q) = \hat{\theta}(q) \circ \nabla v(q) \).

The intuition behind identification is as follows: Start with a normalization \( \theta^0 \equiv \hat{\theta}_2(q^0) \) for some bundle \( q^0 = (q^0_1, q^0_2) \in Q^2 \), and determine \( \nabla P_1(q^0), \nabla P_2(q^0) \), the quantile \( \tau = M^*_2(q^0) \), and \( \theta^1 \equiv \hat{\theta}_1(q^0) \) from (12). Using (13) determine \( q^1 \) with the same quantile \( \tau \) under \( Z = z_1 \). Then, for \( q^1 \) determine \( \nabla P_1(q^1) \) and \( \nabla P_2(q^1) \), which can determine \( \theta^2 = \hat{\theta}_2(q^1) = \nabla P_2(q^1) \circ (\nabla P_1(q^1))^{-1} \circ \theta^1 \) (inverse of (12)). Then iterating these steps we can identify a sequence \{\( \theta^0, \theta^1, \ldots, \theta^L, \ldots \)\} and the corresponding quantile. If these sequence form a dense subset of \( Q^2 \) then the function \( \hat{\theta}(\cdot) : Q^2 \times S_Z \rightarrow S^2_\theta \) is identified everywhere. I formalize this intuition for \( J \geq 2 \) below, starting with the assumption about exclusion restriction.

**Assumption 4.** Let \( Z \in S_Z = \{z_1, z_2\} \) be independent of \( \theta \) and \( v(q) \).

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\(^8\) Here, the superscript is an index of the sequence of bundles, and should not be confused with the utility function \( v_j(q_j) = (q_j)^{\omega_j} \), similarly for the superscript on \( \theta \).
As before, consumer optimality implies \( \nabla P_t(q) = \hat{\theta}_t(q) \circ \nabla v(q) \), and the general version of Equation (12) can be written as

\[
\hat{\theta}_t(q) = \nabla P_t(q) \circ \hat{\theta}_v(q) \circ \left( \nabla P_v(q) \right)^{-1}
\]

\[
\equiv r_{\ell,\ell}(\hat{\theta}_v(q), q) = \begin{pmatrix}
    r_{\ell,\ell}(\hat{\theta}_v(q), q) \\
    \vdots \\
    r_{\ell,\ell}(\hat{\theta}_v(q), q)
\end{pmatrix}.
\]  \( (14) \)

Next, Assumption 4 and the incentive compatibility condition for high types imply \( F_\ell(t|2) = M^*(q(t; z_\ell); z_\ell), \ell = 1, 2 \) and hence

\[
M^*_\ell(\rho_\ell(t)) := M^*(\rho(t; z_\ell); z_\ell) = M^*(\rho(t; z_\ell^*); z_\ell^*) := M^*_\ell(\rho_\ell(t)).
\]  \( (15) \)

Once we determine multivariate quantiles, (15) generalizes (13). Quantiles are the proper inverse of a distribution function, but defining multivariate quantiles is not straightforward because of the lack of a natural order in \( \mathbb{R}^J, J \geq 2 \). One way around this problem is to choose an order (or a rank) function, and define the quantiles with respect to that order. I follow the definition of a multivariate quantiles proposed by Koltchinskii (1997); see Section 2. He shows that if we choose a continuously differentiable convex function \( g_M(\cdot) \), then we can define the quantile function as the inverse of some transformation of \( g_M(\cdot) \), denoted as \( (\partial g_M)^{-1}(\tau) \in \mathbb{R}^J \) for quantile \( \tau \in [0, 1] \). For this procedure to make sense, it must be the case that, conditional on the choice of \( g_M(\cdot) \), there is a one to one mapping between the quantile function and the joint distribution. In fact Koltchinskii (1997) shows that for any two distributions \( M_1(\cdot) \) and \( M_2(\cdot) \), the corresponding quantile functions are equal, \( (\partial g_{M_1})^{-1}(\cdot) = (\partial g_{M_2})^{-1}(\cdot) \), if and only if \( M_1(\cdot) = M_2(\cdot) \). Henceforth, I assume that such a function \( g_M(\cdot) \) is chosen and fixed, then (15) and (1) imply

\[
\rho_1(\tau) = (\partial g_{M_1})^{-1}(M_2^*(\rho_2(\tau))) := s_{2,1}(\rho_2(\tau)), \quad \tau \in (0, 1).
\]  \( (16) \)

This means we can then use

\[
\hat{\theta}_t(q) = r_{\ell,\ell}(\hat{\theta}_v(q), q);
\]

\[
\rho_\ell(\tau) = s_{\ell,\ell}(\rho_\ell(\tau))
\]

to identify \( \hat{\theta}_\ell(\cdot) \), for either \( \ell = 1 \) or \( \ell = 2 \). Since, for a \( q \) the probability that \( \{\theta \leq t|Z = z_\ell\} \) is equal to the probability that \( \{\theta \leq r_{\ell,\ell}(t, q)|Z = z_\ell\} \), i.e., \( \Pr(\theta \leq t|Z = z_\ell) = \Pr(\theta \leq r_{\ell,\ell}(t, q)|Z = z_\ell) \), it means

\[
\rho_\ell(r_{\ell,\ell}(\theta, q)) = s_{\ell,\ell}(\rho_\ell(\theta));
\]  \( (17) \)
so if we know $\rho(\cdot)$ at some $\theta$ then we can identify $\rho(\cdot)$ at $r_{\ell,\ell}(\theta, q)$. As mentioned earlier, let us normalize $v(q^0) = q^0$ for some $q^0 \in Q^2$ so that we know $\{q^0, \theta^0 = \hat{\theta}_1(q^0)\}$. Then this will allow us to identify $\{q^1, \hat{\theta}_1(q^1)\}$ where $q^1 = s_{1,2}(q^0)$ and $\hat{\theta}_1(q^1) = r_{2,1}(\theta^0, q^1)$, which further identifies $\{q^2, \hat{\theta}_1(q^2)\}$ with $q^2 = s_{1,2}(q^1)$ and $\hat{\theta}_1(q^2) = r_{2,1}(\hat{\theta}_1(q^1), q^2)$ and so on. To complete the identification it must be the case that we can begin with any quantile $\rho(\tau) \in Q^2$ and identify $\hat{\theta}(\rho(\tau))$, possibly by constructing a sequence as above.

To do that we can exploit the Assumption 4, which implies that for some $\theta$ the difference $(\theta - r_{2,1}(\theta, q))$ measures the resulting change in $\theta$ if we switch from $z_2$ to $z_1$ for a fixed $q$ so that we can trace $\hat{\theta}(\cdot)$ as we move back and forth between $z_2$ and $z_1$. But for identification it is important that this “tracing” steps come to a halt or equivalently for some (fixed point) $\hat{q} \in Q^2$ the mapping $(\theta(\cdot) - r_{2,1}(\theta, \cdot)) = 0$. For this it is sufficient that the marginal prices at $\hat{q}$ are equal ($\nabla P_1(\hat{q}) = \nabla P_2(\hat{q})$). Since this is multidimensional problem, it is also important that the fixed point is attractive (stable), for which it is sufficient that the slope of of all $\ell$ components of $r_{\ell,\ell}(\cdot)$ (see (14)) depend only on whether $q_j > \hat{q}_j$ or not irrespective of what $j$ is.

**Assumption 5.** There exist a $q \in Q^2$ such that $r_{\ell,\ell}(\theta(q), \hat{q}) = \theta(q)$ and $\text{sgn}[(r_{\ell,\ell}^j(q_j) - q_j)(q_j - \hat{q}_j)]$ is independent of $j \in \{1, \ldots, J\}$.

Both the components of Assumption 5 are technical assumption but they are testable and hence, verifiable from the data. Similar assumption has been used by D’Haultfoeuille and Février (2015) to identify a nonseparable model with discrete instrument and multivariate errors. Without loss of generality I assume the initial normalization be the fixed point $\hat{q}$, so that $\theta^0 = \hat{\theta}_1(\hat{q})$ is known. In other words $\theta^0$ is such that $q(\theta^0, z_1) = \hat{q}$. And from Assumption 4 suppose

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9 We can also normalize some quantile of $F_\theta(\cdot)$. 
\[ \nabla P_1(q) \circ \nabla P_2(q)^{-1} \ll 1, \text{ whenever } q \ll q. \] Then, for \( \tau^{th} \) quantile \( q(\tau) < q \):

\[
\hat{\theta}_1(\tau) := (q)^{-1}(q(\tau); z_1) = \hat{\theta}_1(q^0) = r_{1,2}(\hat{\theta}_1(q^1), q^1)
\]

\[ = [\nabla P_2(q^1) \circ \nabla P_1(q^1)^{-1}] \circ \hat{\theta}_1(q^1)
\]

\[ = [\nabla P_2(q^1) \circ \nabla P_1(q^1)^{-1}] \circ [r_{1,2}(\hat{\theta}_1(q^2), q^2)]
\]

\[ = [\nabla P_2(q^1) \circ \nabla P_1(q^1)^{-1}] \circ [\nabla P_2(q^2) \circ \nabla P_1(q^2)^{-1}] \circ [r_{1,2}(\hat{\theta}_1(q^3), q^3)]
\]

\[ = \ldots = r^L[\hat{\theta}_1(s_{1,2}^L(q(\tau))), s_{1,2}^L(q(\tau))]
\]

\[ = \lim_{L \to \infty} [\nabla P_2(q^1) \circ \nabla P_1(q^1)^{-1}] \circ \ldots \circ [\nabla P_2(q^L) \circ \nabla P_1(q^L)^{-1}] \circ [r_{1,2}(\hat{\theta}_1(q^{L+1}), q^{L+1})]
\]

\[ = \left\{ \prod_{L=1}^{\infty} \nabla P_2(q^L) \circ \nabla P_1(q^L)^{-1} \right\} \lim_{L \to \infty} \hat{\theta}_1(s_{1,2}(q^{L+1}))
\]

\[ = \left\{ \prod_{L=1}^{\infty} \nabla P_2(q^L) \circ \nabla P_1(q^L)^{-1} \right\} \lim_{L \to \infty} \theta_0.
\]

where the first equality is simply the definition, the second equality is the normalization, the third equality follows from (17) with \( q^1 := s_{1,2}(q^0 = \rho(\tau)) \) so that \( \hat{\theta}_1(q^0) = r_{1,2}(\hat{\theta}_1(q^1), q^1) \) and the fourth equality follows from (14). Repeating this procedure \( L \) times leads to the seventh equality. The last equality uses the following facts: a) \( q^L = s_{1,2}(q^{L-1}) \); b) \( q(\tau) < q \); c) \( s_{1,2}(\cdot) \) is an increasing continuous function so \( \lim_{L \to \infty} s_{1,2}(q^L) = s_{1,2}(q^{\infty}) = s_{1,2}(q) \); and d) \( \hat{\theta}_1(q) = \theta_0. \)

Since the quantile \( \tau \) was arbitrary, we identify \( \hat{\theta}_1(\cdot) \).

Once the quantile function of \( \theta \) is identified we can identify \( C(\cdot, Z) \) as before.

The optimality condition \( \alpha(\theta) = 0 \) (Equation (7)) and Equation (9) give

\[
\text{div} \left\{ \frac{m^*_s(q)}{|\det(D\theta_k(q))|} (\hat{\theta}_k(q) \nabla v(q) - \nabla C(q; z_k)) \right\} = -\frac{m^*(q)}{|\det(D\theta(q))|}.
\]

Differentiating \( \theta_k \circ \nabla v(q) = \nabla P_k(q) \) with respect to \( q \) gives

\[
D\nabla P_k(q) = D\hat{\theta}_k(q) \circ \nabla v(q) + \hat{\theta}_k(q) \circ D\nabla v(q)
\]

\[
D\hat{\theta}_k(q) \circ \nabla v(q) = D\nabla P_k(q) - \hat{\theta}_k(q) \circ (\nabla v(q)) \circ (\nabla v(q))^{-1} \circ D\nabla v(q)
\]

\[
D\hat{\theta}_k(q) = D\nabla P_k(q) \circ (\nabla v(q))^{-1} - \nabla P_k(q) \circ (\nabla v(q))^{-2} \circ D\nabla v(q),
\]

which identifies \( |\det(D\theta(q))| \). Then substituting \( |\det(D\theta(q))| \) in above gives

\[
\text{div} \left\{ \frac{m^*(q)}{|\det(D\theta(q))|} (\nabla P(q) - \nabla C(q)) \right\} = -\frac{m^*(q)}{|\det(D\theta(q))|}.
\]
(a partial differential equation for $C(\cdot, z_k)$), with boundary condition
\[
\frac{m_k^*(q)}{|\det(D\theta_k)(q)|} \left( \nabla C(q; z_k) - \nabla P_k(q) \right) \cdot n \left( \nabla P_k(q) \right) = 0, \forall q \in \partial Q^2.
\]
This PDE has a unique solution $C(q)$, and hence, we have the following result:

**Theorem 4.3.** Under Assumptions 1-(i)–(iv-c) and (v) and Assumptions 2–5, $[F_\theta(\cdot|2), v(\cdot), C(\cdot; Z)]$ are identified.

To identify the density $f_\theta(\cdot|1)$ we can use Theorem 4.2, except now the gross utility function is $\sum_{j \in [J]} \theta_j X_j v_j(q_j, X_2)$. Therefore to account for $v(\cdot, X_2)$ we need to to be able to extend the utility function from $Q^2$ to $Q^2 \cup Q^1$. For the identification strategy then if $v(\cdot)$ is a real-analytic, like the cost function, then we can extend the domain of $v(\cdot)$ to include $Q^1$.

**Assumption 6.** Let the utility function $v(\cdot, X_2)$ be a real analytic function.

Then under Assumption 6, we can change the unit of measurement from $q$ to $\tilde{q} \equiv v(q, X_2)$, then apply Theorem 4.2 with gross utility as $\sum_{j \in [J]} \theta_j X_j \tilde{q}_j$.  

### 4.4. Overidentification

Now that we know identification depends on how many cost shifters we have and whether or not the gradient of the pricing function cross, the next step is analyze the effect of observed characteristics $X$ on identification. Before we begin, let us assume that the nonlinear utility model is identified. Then I ask the following question: if the utility function depends on $X$, and $X$ is independent of $\theta$ is the model over identified?

**Lemma 3.** Consider the optimal allocation rule restricted for high types $S^2_\theta$, where $q = \rho(\theta, X, z_t) := \rho(\theta, X)$. Suppose $F_\theta(\cdot|2)$ and $M_{q|X,Z}(\cdot|\cdot, \cdot)$ have finite second moments. Then the CDF $F_\theta(\cdot|2)$ is over identified.

**Proof.** From the previous results $F_\theta(\cdot|2)$ and $M_{q|X,Z}(\cdot|X)$ are nonparametrically identified. Since $Z$ is observed, we can suppress the notation. We want to use the data $\{q, X\}$ and the knowledge of $F_\theta(\cdot|2)$ and the truncated distribution $M^*_{q|X}(\cdot|X)$ to identify $\rho(\cdot, X)$. Let $\mathcal{L}(S^2_\theta, Q^2)$ be the set of joint distribution defined as
\[
\mathcal{L}(S^2_\theta, Q^2) = \{ L(q, \theta) : \int_{S^2_\theta} L(q, \theta)d\theta = M^*_{q|X}(q|\cdot); \int_{Q^2_\theta} L(q, \theta)dq = F_\theta(\theta|2) \}. \tag{19}
\]

To that end consider the following optimization problem:
\[
\min_{L(q, \theta) \in \mathcal{L}} \mathbb{E}(|q - \theta|^2|X).
\]
In other words, given two sets $S^2_\theta$ and $Q^2_X$ of equal volume we want to find the optimal volume-preserving map between them, where optimality is measured against cost function $|\theta - q|^2$. If the observed $q \in Q^2$ were generated under equilibrium then the solution will map $q$ to the right $\theta$ such that $q = \rho(\theta; X)$, for a fixed $X$. The minimization problem is equivalent to

$$\max_{L(q,\theta) \in \mathcal{L}} \mathbb{E}(\theta \cdot q|X),$$

such that the solution maximizes the (conditional) covariance between $\theta$ and $q$. So either we minimize the quadratic distance or the covariance, our objective is to find an optimal way to “transport” $q$ to $\theta$. Let $\delta[\cdot]$ be a Dirac measure or a degenerate distribution. Brenier (1991); McCann (1995) show that that there exists a unique convex function $\Gamma(q, X)$ such that $dL(q, \theta) = dM^*_{q|X}(q)\delta[\theta = \nabla_q\Gamma(q, X)]$ is the solution. Therefore for all $q \in Q^2_X$ we can determine its inverse $\theta = \nabla_q\Gamma(q, X)$ which identifies $F^{\theta}(\cdot|2)$. □

This means, we can use $\Gamma(q, X)$ to test the validity of the supply side equilibrium. There are many ways to think of a “specification test.” One way is by verifying that using $\nabla_q\Gamma(q, X)$ (instead of $\theta$) in Equation (4) leads to the same equilibrium $\rho(\theta; X)$.

5. Model Restrictions

In this section I derive the restrictions imposed by the model on observables under the Assumption 1-(iv) –a, b and c, respectively. These restrictions can be used to test the model validity. For every agent we observe $[p_i, q_i, X_i]$ and for the seller we observe $\{z_1, z_2\}$. From the model $p_i$ and $q_i$ are given by $p = P_\ell(q, z_\ell)$ and $q = \rho_\ell(\theta, z_\ell)$. Specifically, suppose a researcher observes a sequence of price and quantity data, and some agents and cost characteristics. Does there exist any possibility to rationalize the data such that the underlying screening model is optimal when the utility function satisfies Assumption 1-(iv-a) (Model 1) or Assumption 1-(iv-b) (Model 2) or Assumption 1-(iv-c) (Model 3)? In all three models we ask, in the presence of multidimensional asymmetric information, what are the restrictions on the sequence of data $(Z, X_i, \{q_i, p_i\})$ we can test if and only if it is generated by an optimal screening model, without knowing the cost function, the type distribution and for Model 3 the utility function. We say that a distribution of the observables is rationalized by a model if and only if it satisfies all the restrictions of the model. In other words, a distribution of
the observables is rationalized if and only if there is a structure (not necessarily unique) in the model that generates such a distribution.

Let $D_1 = (q, p), D_2 = (q, p, X_1), D_3 = (q, p, X, Z)$ distributed, respectively, as $\Psi_{D_i}(\cdot), i = 1, 2, 3$, and let

$$\mathcal{M}_1 = \{(F_\theta(\cdot), C(\cdot)) \in \mathcal{F} \times \mathcal{C} : \text{satisfy Assumption} \ 1 - (i - (iv - a), (v))\}$$
$$\mathcal{M}_2 = \{(F_\theta(\cdot), C(\cdot)) \in \mathcal{F} \times \mathcal{C} : \text{satisfy Assumption} \ 1 - (i - (iv - b), (v))\}$$
$$\mathcal{M}_3 = \{(F_\theta(\cdot), C(\cdot, Z)) \in \mathcal{F} \times \mathcal{C}_Z : \text{satisfy Assumptions} \ 1 - (i - (iv - c), (v), 3 \ \text{and} \ 4)\}$$

Define the following conditions:

C1. $\Psi_{D_1}(\cdot) = \delta[p = P(q)] \times M(q)$, with density $m(q) > 0$ for all $q \in Q^1 \cup Q^2$.

C2. There is a subset $Q^1 \subsetneq Q$ which is a $J - 1$ dimensional flat (hyperplane) in $\mathbb{R}^d$.

C3. $p = P(q)$ has non vanishing gradient and Hessian for all $q \in Q^2$.

C4. Let $\{W\} := \{\nabla P(q) : q \in Q^2\}$. Then $F_W(w) = \text{Pr}(W \leq w) = M^*(q)$ and let $m^*(\cdot) > 0$ be the density of $M^*(\cdot)$.

C5. Let $C(\cdot)$ be the solution of the differential equation

$$\text{div} \left\{ \frac{m^*(q)}{|\text{det}(D\nabla P(q))|} (\nabla P(q) - \nabla C(q)) \right\} = -\frac{m^*(q)}{|\text{det}(D\nabla P(q))|}, \quad (20)$$

with boundary conditions

$$\frac{m^*(q)}{|\text{det}(D\nabla P(q))|} (\nabla C(q) - \nabla P(q)) \cdot \vec{n} (\nabla P(q)) = 0.$$

5.1. Linear Utility. For every consumer we observe $D_1$ and the objective is to determine the necessary and sufficient conditions on the joint distribution $\Psi_{D_1}(\cdot, \cdot)$ for it to be rationalized by model $\mathcal{M}_1$.

Lemma 5.1. If $\mathcal{M}_1$ rationalizes $\Psi_{D_1}(\cdot)$ then $\Psi_{D_1}(\cdot)$ satisfies conditions C1. – C5. Conversely, if $F_\theta(\cdot|0)$ and $F_\theta(\cdot|1)$ are known and $\Psi_{D_1}(\cdot)$ satisfies the C1. – C5 then there is a model $\mathcal{M}_1$ that generates $D_1$.

Proof. If. Since $F_\theta(\cdot)$ is such that the density $f_\theta(\cdot) > 0$ everywhere on $\mathcal{S}_\theta$ and the equilibrium allocation rule $\rho : \mathcal{S}_\theta \to Q$ is onto, and continuous, the CDF $M(q)$ is well defined and the density $m(q) > 0$. Moreover, since the equilibrium allocation rule is deterministic, for every $q$ there is only one price $P(q)$, hence the Dirac measure, which completes C1. Rochet-Choné shows that in equilibrium the bunching set $Q^1$ is nonempty, and hence $m(q) > 0$ for all $q \in Q^1$. Moreover the allocation rule $\rho : \mathcal{S}_\theta \to Q^1$ is not bijective, and as
a result $Q^1$ as a subset of $\mathbb{R}_+^J$ is flat, which completes C2. The optimality condition for the types that are perfectly screened is $\theta = \nabla P(q) := \hat{\theta}(q)$, and incentive compatibility implies the indirect utility function is convex and hence $P(q)$ has non vanishing gradient and Hessian, which completes C3. Then, $M^*(q) = \Pr(q \leq q) = \Pr(\nabla P(q) \leq \nabla P(q)) = \Pr(W \leq w) = F_W(w)$, hence C4. Finally, if we use (9) to replace $f_\theta(\cdot)$ in $\alpha(\theta) = 0, \forall \theta \in S_\theta^2$ with the boundary condition $\beta(\theta) = 0, \forall \theta \in \partial S_\theta^2 \cap \partial S_\theta$ we get C5.

*Only if.* Now, we show that if $\Psi_{D_1}(\cdot)$ satisfies all C1. – C5. conditions listed above then we can determine a model $\mathcal{M}_1$ that rationalizes $\Psi_{D_1}(\cdot)$. Let $C(\cdot)$ satisfy C5, then we can determine the cost function $C(\cdot)$. Moreover it is real analytic so it can be extend uniquely to all $Q$. From C4. we can determine the vector $W$ which is also the type $\theta$ and it satisfies the first order optimality condition. Thus the indirect utility of the type $\theta$ that corresponds to the choices $q \in Q^2$ is convex and hence satisfies the incentive compatibility constraint. Moreover, since $m^*(q) > 0$ the density $f_\theta(\cdot|2) > 0$ and $F_\theta(\cdot|2) = \int_{\theta \in \{W = \nabla P(q); q \in Q^2\}} f_\theta(\theta|2)d\theta$. As far as $F_\theta(\cdot|1)$ is concerned we can simply ignore bunching and define $F_\theta(\theta) = M(q|q \in Q^1)$ where $\theta = \nabla P(q)$. □

5.2. Bi-Linear Utility. Now, I consider the case of bi-linear utility function. Since $X_2$ is redundant information, we can ignore it. The only difference between this and the previous model is now there is $X_2$ but everything else is the same. So to save more notations, I slightly abuse notations and use the same conditions C1. – C5. except now they are understood with respect to $D_2$. For instance C1. becomes $\Psi_{D_2}(\cdot) = \delta[p = P(q; X_1)] \times M(q) \times \Psi_{X_1}$.

**Lemma 5.2.** If $\mathcal{M}_2$ rationalizes $\Psi_{D_2}(\cdot)$ then $\Psi_{D_2}(\cdot)$ satisfies conditions C1. – C5. Conversely, if $F_\theta(\cdot|0)$ is known, $dim(X_1) = dim(q) = J$, and $\Psi_{D_2}(\cdot)$ satisfies C1. – C5 then there is a model $\mathcal{M}_2$ that generates $D_2$.

The proof of this lemma is very similar to that of Lemma 5.2, except in here the menu (allocation and prices) depend on $X_1$ but the cost function and the type CDF do not depend on, and the conditional density $f_\theta(\cdot|1)$ can be determined from the data. In view of the space I omit the proof.

5.3. Nonlinear Utility. Finally, I consider the case of nonlinear utility. Before I proceed, I introduce two more conditions.

C4'. If $\rho_\tau(X_2, Z)$ is the $\tau \in [0, 1]$ quantile of $q \in Q^2_{X_2, Z}$ then $\rho_\tau(\cdot, z_1) = \rho_\tau(\cdot, z_2)$.

C6. The truncated distribution of choices $M^*_q|X, Z(\cdot|\cdot, \cdot)$ has finite second moment,
and for a given \( Z = z \) (henceforth suppressed) the solution of

\[
\max_{\theta \in \mathcal{L}(Q^2, S^2)} \mathbb{E}(\theta \cdot q | X),
\]

where \( \mathcal{L}(Q^2, S^2) \) is defined in (19) is given by a mapping \( \theta = \nabla q \Gamma(q, X) \) for some convex function \( \Gamma(q, X) \) such that it solves the optimality condition (4).

So with nonlinear utility, condition C4' replaces condition C4, and as with the bi-linear utility the conditions should be interpreted as being conditioned on both \( X \) and \( Z \), wherever appropriate.

**Lemma 5.3.** Let \( F_0(\cdot|2) \) have finite second moment. If \( \mathcal{M}_3 \) rationalizes \( \Psi_{D_3}(\cdot) \) then \( \Psi_{D_3}(\cdot) \) satisfies C1 - C3, C4' - C7. Conversely, if \( F_0(\cdot|0) \), and a quantile \( \tilde{\theta}(q) \) is known, \( \dim(X_1) = \dim(q) = J, Q_{X,z_k}^2 = Q_{X,z_{k'}}^2 \) (common support) and \( \Psi_{D_3}(\cdot) \) satisfies C1 - C3, C4' - C6 then there exists a model \( \mathcal{M}_3 \) that rationalizes \( \Psi_{D_3}(\cdot) \).

**Proof.** If \( \mathcal{M}_3 \) is \( F_0(\cdot|2) \) with non vanishing density \( f_0(\cdot) \) everywhere on the support \( S_\theta \). Moreover, the equilibrium allocation rule \( \rho : S_\theta \times X \times Z \rightarrow Q \) is onto, and continuous for given \( (X, Z) \). Therefore the CDF \( M_{q|X,Z}(\cdot|\cdot, \cdot) \) is a push forward of \( F_0(\cdot|2) \) given \( (X, Z) \). Since \( Q = Q_{X,Z}^2 \cup Q_{X,Z}^1 \cup \{q_0\} \) the (truncated) density \( m_{q|X,Z}(q|\cdot, \cdot) > 0 \) for all \( q \in Q_{X,Z}^2 \cup Q_{X,Z}^1 \). In equilibrium, for a given \( (q, X, Z) \) the pricing function is deterministic, therefore the distribution is degenerate at \( p = P(q; X, Z) \). Hence the Dirac measure. This completes C1. For C2 note that the allocation rule is not bijective, and as a result \( \rho(S_{\theta}^2; X, Z) = Q^1 \subset \mathbb{R}^J_+ \) is a hyperplane. For the high-types, optimality requires the marginal utility \( \theta \cdot v(q; X_2) \) is equal to the marginal price \( P(q; X, Z) \), and since \( v(\cdot; X_2) \) has non vanishing Hessian, \( P(\cdot; X, Z) \) also has non vanishing gradient \( \nabla P(\cdot; X, Z) \) and Hessian, which completes C3. Since \( Z \parallel \theta \), using Equation (15) gives \( F_\theta(\xi|2) = \int_M \rho_1(\xi) |X, z_1) = \int_M \rho_2(\xi) |X, z_2) \), as desired for C4'. The condition C5 follows once we replace \( m(\cdot) \) and \( P(\cdot) \) in (20) with \( m_{q|X,Z}(\cdot|\cdot, \cdot) \) and \( P(q; X, Z) \), respectively and observe that for any pair \( (X, Z) \) the equilibrium for high-type is given by \( \alpha(\theta) = 0 \). Since \( F_0(\cdot|2) \) is known and \( M_{q|X,Z}(\cdot|\cdot) \) is determined, condition C6 follows from Lemma 3.

Only if. We want to show that if \( \Psi_{D_3}(\cdot) \) satisfies all conditions in the statement, then we can construct a model \( \mathcal{M}_3 \) that rationalizes \( \Psi_{D_3}(\cdot) \). For \( Z = z_k \), using condition C6, we can determine two cost functions \( C(\cdot; z_1) \) and \( C(\cdot; z_2) \). Since (20) is applicable only to \( Q_{X}^2 \), we need to extend the domain of the cost function. Of many ways to extend the domain, the simplest is to assume that the cost is quadratic, i.e. \( C(q; X, Z) = 1/2 \sum_{j=1}^J q_{j}^2 \) for all \( q \in Q_{X,Z}^1 \cup \{q_0\} \).
Using the exclusion restriction and (18) for all \( q \in Q_{X,Z}^2 \) we can determine the function \( \hat{\theta}(q; Z = z_k) \) along a set \( \tilde{Q}_{X,Z}^2 \subseteq Q_{X,Z}^2 \) for \( k = 1, 2 \). If the set \( \tilde{Q}_{X,Z}^2 \) is a dense subset then there is a unique extension of \( \hat{\theta}(\cdot; \cdot) \) over all \( Q_{X,Z}^2 \). If not, then, let us linearly extend the function to the entire domain of \( Q_{X,Z}^2 \). Then define \( v(q; X_2) = \nabla P(q; X, Z) \circ (\hat{\theta}_k(q))^{-1} \). Finally, to extend the function to \( Q \) we can assume that each function \( v_j(q; X) = q_j^{1/2}, j = 1, \ldots, J \) for all \( q \in Q_{X,Z}^1 \cup \{q_0\} \). As far as \( F_\theta(\cdot|1) \) is concerned we can simply ignore bunching and define \( F_\theta(\theta) = M(q|q \in Q^1) \) where \( q \in Q^1 \) is such that \( \theta = \nabla P(q; X, Z) \circ (v(q; X_2))^{-1} \). Since the probability of \( q = \{q_0\}, q \in Q_{X,Z}^1 \) and \( q \in Q_{X,Z}^2 \) is equal to the probability of \( \theta \in S_\theta^0, \theta \in S_\theta^1 \) and \( \theta \in S_\theta^2 \), respectively we can determine \( F_\theta(\cdot) \). It is then straightforward to verify that the triplet thus constructed belongs to \( \mathcal{M}_3 \). □

6. Discussion

6.1. Unobserved Taste Shifter. So far I have assumed that consumer’s tastes are completely characterized by a vector \( \theta \). But suppose there is an unobserved market level taste shifter \( Y \) that scales the taste for all consumers, and as such it is observed by all consumers and the seller but not the econometrician.

Assumption 7. Let

1. The random variables \( (\theta, Y) \) are distributed on \( S_\theta \times \mathbb{R}_{++} \) according to the CDF \( F_{\theta,Y}(\cdot, \cdot) \) such that \( \Pr(\theta \leq \theta_0, Y \leq y_0) = F_{\theta,Y}(\theta_0, y_0) \).
2. Let \( \theta^* := Y \times \theta \) be such that \( \theta^* \sim F_{\theta^*Y}(|y|) = F_{\theta^*}(\cdot) \) and \( \mathbb{E}(\log Y) = 0 \).

Let \( S_{\theta^*Y}^2 \) denote the types that are perfectly screened. Then under assumption 7 optimality of these types means \( \theta^*_i = \nabla P(q_i) \) and since \( \theta^*_i = \theta_i y, i \in [N_2] \), we want to identify \( F_{\theta^*}(\cdot) \) and \( F_{Y}(\cdot) \) from above. Dividing \([N_2]\) into two parts and reindexing \( \{1, \ldots, N_21\} \) and \( \{1, \ldots, N_22\} \) and taking the log of the above we get

\[
\log \theta^*_i = \log \theta_{ij} + \log Y, \quad i_j = 1, \ldots N_2j, j = 1, 2.
\]

Let \( Ch(\cdot, \cdot) \) be the joint characteristic function of \( (\log \theta_{i1}, \log \theta_{i2}) \) and \( Ch_1(\cdot, \cdot) \) be the partial derivative of this characteristic function with respect to the first component. Similarly, let \( Ch_{\log Y}(\cdot) \) and \( Ch_{\log \theta_j}(\cdot) \) denote characteristic functions of \( \log Y \) and \( \log \theta_j \), which is the short hand for \( \theta_{i_j}, i_j \in [N_2] \). Then from Kotlarski (1966):

\[
Ch_{\log Y}(\xi) = \exp \left( \int_0^\xi \frac{Ch_1(0, t)}{Ch(0, t)} \, dt \right) - i\xi \mathbb{E}[\log \theta_1].
\]
Then the characteristic function of $Ch_{\log \theta_1}(\xi) = \frac{Ch(\xi,0)}{Ch_{\log Y}(\xi)}$, which identifies $F_\theta(\cdot)$.

**Lemma 6.1.** Under Assumption 7, the model $[F_\theta(\cdot), F_Y(\cdot), C(\cdot), v(\cdot)]$ with unobserved heterogeneity is identified.

### 6.2. Measurement Errors

So far we have assumed that the econometrician observes both the transfers and the contract characteristics without an error. Such an assumption could be strong in some environment. Sometimes it is hard to measure the transfers (wages, prices etc) and sometimes it is hard to measure different attributes of contracts. For instance a monopoly who sells differentiated products it is possible that some if not all of the attributes of the product are measured with error. In this subsection we allow data to be measured with error.

#### 6.2.1. Measurement Error in Prices

I begin by considering the case when only the transfers are measured with error, and subsequently consider the case when only the contract choices are measured with error. If only the prices are measured with additive error, and if the error is independent of the true price then the model is still identified. The intuition behind this is simple. When choices $\{q\}$ are observed without error, but only prices are observed with error, and if this error is additively separable and independent of the true prices, i.e.,

$$P^\varepsilon(q) = P(q) + \varepsilon, \quad P(q) \parallel \varepsilon,$$

then the observed marginal prices $\nabla P^\varepsilon(\cdot)$ and the true marginal prices $\nabla P(\cdot)$ are the same, which means the previous identification arguments are still applicable.

**Lemma 6.2.** If $\{P^\varepsilon = P + \varepsilon\}$ is observed, where $P$ is the price and $P \parallel \varepsilon$ is the measurement error, then the model parameters $[F_\theta(\cdot), C(\cdot)]$ are identified.

#### 6.2.2. Measurement Error in Choices

Now consider a case where the choices $q$’s are observed with error. Furthermore, lets assume that there is one (dimensional) error $\eta \in \mathbb{R}^+$ that affects all $J$ characteristics. In other words, we envision a situation where there is one $\eta$ for each consumer, and instead of the choice $q$, we only measure $q^\eta = q + \eta \cdot 1$, where $1$ is $J$-dimensional vector of ones. Since there is no reason why each component $q$ should have unique measurement error associated with it, having one error unique to each consumer choice seems more natural in this environment. We also assume that $\eta \parallel q$ and $\eta \sim F_\eta(\cdot)$. The data is then $\{P, q^\eta\}$ pair for every consumer with type $\theta \in S_\theta^2$. Then $P = P(q) = P(q^\eta - \eta \cdot 1)$ implies $\nabla P(q) \neq \nabla P(q^\eta)$, which means without correcting for $\eta$ the taste parameter $\theta$ cannot be identified. Following the
same logic as Lemma 6.1 we can identify $F_\eta(\cdot)$ but that still does not mean we can identify $\theta$ because we have $J + 1$ unknowns and only $J$ equation for each consumer.

**Lemma 6.3.** If $\{q^n = q + \eta \cdot 1\}$ is observed, where $\eta \parallel q$ is the measurement error then the model $[F_\theta(\cdot), C(\cdot)]$ cannot be identified.

### 6.3. Unobserved Product Characteristic

In this section I extend the linear utility model to allow for unobserved product characteristic. Suppose we observe two bundles $q$ and $q'$ where the former dominates the later $q > q'$ and $P(q) \leq P(q')$, with positive demand for both. This suggests either that the model is wrong, and hence we can reject the Rochet-Choné model as a good description of the data generating process, or that some product characteristic is missing in the data. Let $Y \in \mathbb{R}_+$ denote such characteristic that is observed only by the consumers and the seller such that the net utility when a type-$\theta := (\theta_1, \ldots, \theta_J, \theta_y)$ chooses $\{q\} \cup \{Y\}$ bundle and pays $P$ is

$$V(q,Y; \theta) = \theta \cdot q + \theta_y Y - P,$$

where $\theta_y \in \mathcal{S}_y$ is the consumer’s taste for $Y$. The two bundles mentioned above are in fact $(q,Y)$ and $(q',Y')$ with $P(q,Y) \leq P(q',Y')$. This means our econometrics model can be written as

$$P = P(q,Y)$$

$$\begin{pmatrix} q \\ Y \end{pmatrix} = \rho(\theta_1, \ldots, \theta_J, \theta_y, F_\theta, C),$$

where $(Y, P(\cdot, \cdot), F_{\theta, \theta_y}(\cdot, \cdot), C(\cdot))$ are unknown. Since the product characteristics are endogenous, the observed product characteristics are correlated with the unobserved characteristics $Y$ and hence the model cannot be identified.\(^{10}\)

### 6.4. Discrete Options

The identification arguments rely on observing continuous (or continuum) of many options for the consumers. This is not an assumption I made, but an outcome of the seller’s maximization problem: the more the seller can discriminate the better the revenue as long as it respects incentive compatibility constraints. There can be instances where we observe only discrete (very few) options, Wilson (1993), but the theory is silent about why a seller would offer fewer (than optimal number of) options, without imposing arbitrary size restrictions on the consumer type space $\Theta$. If the set $Q$ is

\(^{10}\)Bajari and Benkard (2005) assume $\theta_y \equiv 1$ and $Y \parallel q$, which are very strong assumptions.
discrete but Θ is continuous, then the identification results do not apply. Without a theory model to rationalize the supply side, I would then have to resort to using some form of exclusion restriction for identification. For instance, it seems safe to conjecture that we can use variation in consumer characteristics to extend Theorem 4.2 to identify \( f(\theta|1) \) and \( f(\theta|2) \), but not the cost function.

In view of this lacuna in the theory, researchers have started to estimate the loss of payoff from using simpler contracts and find that in some cases the loss is small. For example, Chu, Leslie, and Sorensen (2011) find that the revenue loss of using two-part tariff instead of a multi-part tariff is small. If the loss of offering simple (or few options) is not too much, but if there is a “menu-cost” (for the lack of better term) in offering multiple options, then it stands to reason that offering fewer options could be optimal for the seller. Such menu-cost could be a proxy for cost of marketing – it is easier and hence cheaper to sell ten options than fifty options. Although this argument is intuitive, it is not clear how one can make this menu-cost operable in a multidimensional adverse selection environment, and achieve point identification. There is so little research in this area that it is not obvious what sort of menu-cost will generate a sharp prediction about the exact number of options observed in the data.

In the discussion above, finite or discrete options is meant to denote cases where there are indeed very few options. If the seller offers numerous discrete options then we might be able to fit in a continuous function to ‘fill-the-gap’ and treat the fitted function as if it were a continuous equilibrium outcome and apply the previous results. For example Aryal (2013) observes more than two hundred ad choices and he fits a nonlinear tariff function and use that for identification. Similarly, in the telecommunication market of Figure 1, if we define a product as a bundle of ex-post usage of voice data and SMS then we can make use of heterogeneity in ex-post usage to treat the options as if it were continuous. The right approach depends on the nature of the market and the data, and whether or not we want to abandon point-identification in favor of partial identification. In multi-unit auctions, for example, bidders often submit very few steps, and Kastl (2011), and more recently Cassola, Hortaçsu, and Kastl (2013), posit that it could be because bidders incur an additional cost when submitting a bid having more steps, which leads to partial identification. Exploring partial identification of Rochet-Choné model with this menu-cost is an important problem, but it is beyond the scope of this paper.
7. Conclusion

In this paper I study the identification of a screening model studied by Rochet and Choné (1998) where consumers have multidimensional private information. I show that if the utility is linear or bi-linear, as is often used in empirical industrial organization literature, then we can use the optimality of both supply side and the demand side to nonparametrically identify the multidimensional unobserved consumer taste distribution and the cost function of the seller. The key to identification is to exploit equilibrium bijection between the unobserved types and observed choices and the fact that in equilibrium consumer will choose a bundle that equates marginal utility to marginal prices. When private information is multidimensional, however, the allocation rule need not be bijective for all types. For those medium-types who are bunched, I show that if we have information about consumers’ socioeconomic and demographic characteristics that are independent of the types and if there are as many such characteristics as products, the joint density of types can be identified.

When utility is nonlinear, having a binary and exogenous cost shifter is sufficient for identification. I also show that with nonlinear utility if we have independent consumer characteristics then the model is over identified, which can be used to test the validity of supply side optimality. To the best of my knowledge, this is a first study that provides a way to test optimality of equilibrium in a principal-agent model. Furthermore, I characterize all testable restrictions of the model on the data, and extend the identification to consider measurement error and unobserved heterogeneity.

One of the other areas for future extension would be to allow multiple sellers, like say the cable television market of Crawford and Yurukoglu (2012), where consumers’ preferences for multiple channels is private information, and as mentioned above to study the effect of observing discrete options on identification. Much like the problems with discrete options, there are difficulties associated with competition, even with one dimensional private information; see Epstein and Peters (1999); Martimort and Stole (2002); and Yang and Ye (2008). Laf- font, Maskin, and Rochet (1987) propose an aggregation technique, and it has been used in empirical setting by Ivaldi and Martimort (1994) and Aryal (2013) but can be used only with bi-dimensional types. Another method is to use generalized Hotelling model (Rochet and Stole, 2002; Bonatti, 2011), which is similar to Bresnahan (1987). Much needs to be done in this field, and my hope is that this paper will be useful.
References


