A Note on the Flexibility of the Barnett and Hahm Functional Form

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Abstract

It is very desirable to find flexible functional forms for unit cost (or profit) functions that are globally concave (or convex). It is easy to find flexible functional forms that are locally well behaved but do not satisfy the required regularity conditions globally. This note examines the global flexibility properties of a unit profit function that was originally suggested by Barnett and Hahm (1994). It is found that this functional form is an improvement over other locally flexible functional forms, in that Barnett and Hahm (BH) functional form for a unit profit function can maintain global convexity while at the same time, it can allow for an arbitrary pattern of substitutes and complements for pairs of outputs and inputs. However, an example shows that the BH functional form is not fully flexible.

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Key Words

Flexible functional forms, unit profit functions, duality, Barnett and Hahm functional form, substitutes and complements in production, Barnett regularity, Diewert regularity, Miniflex Laurent functional form.

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1. Introduction

Diewert and Wales (1987; 55-58) considered various generalizations and modifications of Barnett's Miniflex Laurent functional form in the cost function context[2] and Barnett and Hahm (1994) and Isakin and Serletis (2014) further adapted these functional forms to the unit profit function context.[3] In this note, we will consider the flexibility properties of the specific functional form for the Barnett and Hahm unit profit function that was suggested by Isakin and Serletis (2014).

The flexibility properties of the Barnett and Hahm (BH) functional form are not clear. It is suspected that the BH functional form is not flexible but a formal proof of its inflexibility is not available in the literature. In this note, we will show that the BH functional form is definitely not flexible if the number of net outputs is greater than 2. On the other hand, we will also show that it does have certain desirable properties.

2. The Case of Three Net Outputs and One Fixed Input

We consider the case of 3 net outputs and one fixed input. Let the positive prices of these net outputs[4] be denoted by p_i for i = 1,2,3 and define $p \equiv [p_1,p_2,p_3]$. The producer's unit profit function, $\pi(p)$, is defined as the maximum net revenue that the producer can achieve (taking the output prices p as fixed) given that only one unit of the fixed input is available.[5] For the case of 3 net outputs, the BH unit profit function is defined as follows:

$$(1)\ \pi(p) \equiv a_{11}p_1 + a_{22}p_2 + a_{33}p_3 - 2a_{12}p_1^{1/2}p_2^{1/2} - 2a_{13}p_1^{1/2}p_3^{1/2} - 2a_{23}p_2^{1/2}p_3^{1/2} + b_1p_1^2p_2^{-1/2}p_3^{-1/2} + b_2p_2^2p_1^{-1/2}p_3^{-1/2} + b_3p_3^2p_1^{-1/2}p_2^{-1/2}$$

where the parameters a_{11}, a_{22} and a_{33} are not restricted in sign and the parameters a_{12}, a_{13}, a_{23}, b_1, b_2 and b_3 are restricted to be nonnegative.[6] With these sign restrictions, it is

[2] See Barnett (1983) and Barnett and Lee (1985). For a discussion of the properties of the Miniflex Laurent functional form in the consumer context, see Diewert and Wales (1993).

[3] For a discussion of the role of unit profit functions in modeling producer behavior, see Diewert and Wales (1992). The Barnett and Hahm (1994; 39) unit profit function was defined by their equation (27). Diewert and Wales (1987; 57) defined the analogue of this function in the unit cost function context and called it the Symmetric Generalized Barnett functional form. Diewert and Wales (1987; 57-58) also showed that this functional form was *quasiflexible*; i.e., it can approximate the second derivatives of an arbitrary twice continuously differentiable linearly homogeneous function to the second order around an arbitrary point except for one row (and the corresponding column) and the main diagonals of the matrix of second order partial derivatives. Thus the BH functional form is quasiflexible.

[4] If commodity i is an output, then the quantity of commodity i, y_i, is a positive number. If commodity i is a variable input, then its quantity y_i is set equal to the negative of the (positive) input demand.

[5] This is a special case of the duality theorems developed in section 5 of Diewert (1973).

[6] If we set $b_1 = b_2 = b_3 = 0$, then $\pi(p)$ collapses to the Generalized Leontief unit profit function that is similar to the Generalized Leontief unit cost function defined by Diewert (1971). If we allow the a_{ij} to be unrestricted in sign, then this functional form is flexible; simply apply the same proof of flexibility that is in Diewert (1971). However, unless the a_{ij} for $i \neq j$ are nonnegative, the resulting functional form will not be globally convex. If the a_{ij} for $i \neq j$ are all nonnegative, then complementarity is ruled out and hence the sign restricted Generalized Leontief unit profit function will not be flexible.

straightforward to show that each term on the right hand side of (1) is a convex, linearly homogeneous function over the positive orthant and hence $\pi(p)$ is a globally convex, linearly homogeneous function as well. Let $\pi_i(p) \equiv \partial\pi(p)/\partial p_i$ denote the first order partial derivative of $\pi(p)$ with respect to p_i for i = 1,2,3. Differentiating the right hand side of (1) with respect to p_1, we find that:[7]

$$(2)\ \pi_1(p) = a_{11} - a_{12}p_1^{-1/2}p_2^{1/2} - a_{13}p_1^{-1/2}p_3^{1/2} + 2b_1p_1p_2^{-1/2}p_3^{-1/2} - (1/2)b_2p_2^2p_1^{-3/2}p_3^{-1/2} - (1/2)b_3p_3^2p_1^{-3/2}p_2^{-1/2}.$$

Differentiate $\pi_1(p)$ with respect to p_1, p_2 and p_3 and evaluate the resulting second order partial derivatives at $p = [1,1,1] \equiv 1_3$. We obtain the following derivatives:

$$(3)\ \pi_{11}(1_3) = (1/2)a_{12} + (1/2)a_{13} + 2b_1 + (3/4)b_2 + (3/4)b_3;$$
$$(4)\ \pi_{12}(1_3) = -(1/2)a_{12} - b_1 - b_2 + (1/4)b_3;$$
$$(5)\ \pi_{13}(1_3) = -(1/2)a_{13} - b_1 + (1/4)b_2 - b_3;$$
$$(6)\ \pi_{23}(1_3) = -(1/2)a_{23} + (1/4)b_1 - b_2 - b_3.$$

The net outputs i and j (where $i \neq j$) are *substitutes* in production if $\pi_{ij} < 0$, *complements* in production if $\pi_{ij} > 0$ and *unrelated* if $\pi_{ij} = 0$. It is straightforward to show that when N = 3, there can be at most one complementary pair of net outputs.[8]

We now study the flexibility properties of the unit profit function defined by (1) at the point $p = 1_3$. If net outputs 1 and 2, 1 and 3 and 2 and 3 are all substitutes or unrelated at the point $p = 1_3$, then it can be seen that we can set $b_1 = b_2 = b_3 = 0$, and choose nonnegative a_{12}, a_{13} and a_{23} so that equations (4)-(6) will be satisfied. This special case of (1) collapses to the Generalized Leontief unit profit function which will be flexible at the point $p = 1_3$ and globally convex over the positive orthant.[9] Now suppose $\pi_{12}(1_3) > 0$ so that net outputs 1 and 2 are complements at $p = 1_3$. Then using equation (4) above, it can

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[7] Let y_i denote the net revenue maximizing supply of net output i for i = 1,2,3 and suppose the producer has the amount $k > 0$ of fixed input available for use. Then Hotelling's Lemma implies that $y_i = \pi_i(p)k$ for i = 1,2,3.

[8] See Diewert (1983) for a proof.

[9] However, following the terminology introduced by Barnett (2002), the resulting functional form need not be globally *regular*; i.e., if all of the net outputs are outputs (and not variable inputs), then the revenue maximizing output supply functions need not have the correct curvature and monotonicity properties. It should be noted that Barnett's definition of the monotonicity requirement in regularity for an estimated functional form differs from that used by Diewert in his research. Consider a situation where a single output, many input cost function is econometrically estimated. Then Barnett monotonicity regularity involves taking the estimated cost function and using duality theory to trace out the production function that is dual to the estimated cost function. He then takes each observed input vector and checks whether the dual production function satisfies the required monotonicity conditions at that input point. Thus Barnett regularity checks monotonicity of the production function, conditioning on each input vector. On the other hand, Diewert regularity uses the estimated cost function, conditions on each observed output point and input price vector and checks whether the vector of first order partial derivatives of the estimated cost function is nonnegative. This will be the case if the fitted demand is nonnegative at each observed output quantity and input price point using Shephard's Lemma. Thus Barnett regularity conditions on observed input *quantities* while Diewert regularity conditions on observed input *prices*. The differences in the two concepts of regularity carry over to the estimation of profit functions.

be seen that this restriction can be satisfied if we choose $b_3 > 0$ to be large enough relative to the other parameters. If 1 and 3 are complements so that $\pi_{13}(1_3) > 0$, then choose $b_2 > 0$ large enough relative to the other parameters. Finally, if 2 and 3 are complements so that $\pi_{23}(1_3) > 0$, then choose $b_1 > 0$ large enough relative to the other parameters. Thus the BH functional form has a big advantage over the Generalized Leontief special case of (1) in that the BH functional form *can accommodate any pattern of complementarity* while maintaining global convexity (and linear homogeneity) whereas the GL special case can maintain global convexity only if all net outputs are substitutes.[10]

However, the Barnett and Hahm functional form is not completely flexible; i.e., it cannot always approximate an arbitrary twice continuously differentiable unit profit function to the second order at a given point as we show below. Suppose the unit profit function that we are attempting to approximate by the BH functional form at $p = 1_3$ has the following matrix of second order partial derivatives:

$$(7)\ \nabla^2\pi(1_3) = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}.$$

It can be seen that $\nabla^2\pi(1_3)$ is a positive semidefinite symmetric matrix (of rank 1) and so it is an admissible substitution matrix. Recall equations (3) and (4) above. From (7), we have $\pi_{11}(1_3) = 1 = \pi_{12}(1_3)$. Insert these values for $\pi_{11}(1_3)$ and $\pi_{12}(1_3)$ into equations (3) and (4) and we obtain the following system of equations:

(8) $1 = (1/2)a_{12} + (1/2)a_{13} + 2b_1 + (3/4)b_2 + (3/4)b_3$;
(9) $1 = -(1/2)a_{12} - b_1 - b_2 + (1/4)b_3$.

A necessary condition for the BH functional form to be flexible is that there exist nonnegative a_{12}, a_{13}, b_1, b_2 and b_3 that solve equations (8) and (9). Use equation (9) to solve for b_3 in terms of the other parameters:

(10) $b_3 = 4 + 2a_{12} + 4b_1 + 4b_2$.

Now substitute (10) into (9) and after simplification, we obtain the following equation:

(11) $2a_{12} + (1/2)a_{13} + 5b_1 + (15/4)b_2 = -2$.

It can be seen that there is no nonnegative solution for a_{12}, a_{13}, b_1 and b_2 that solves (11).[11] This establishes the inflexibility of the BH functional form.

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[10] This advantage of the BH functional form carries over to the case of many net outputs. However, for the case where N (the number of net outputs) is arbitrary, the BH functional form has $N(N^2 - 2N + 3)/2$ unknown parameters; see Isakin and Serletis (2014). Thus when N = 10, the number of parameters in the BH functional form is 415. Hence the functional form is not suitable for time series applications but Isakin and Serletis apply it successfully in a cross sectional context.

[11] A nonnegative solution for (8) and (9) could be obtained if we increased $\pi_{11}(1_3)$ from 1 to 3.

Thus while the BH functional form is a big improvement over the sign restricted Generalized Leontief unit profit function, it suffers from two problems:

- It is not flexible and
- It is not parsimonious.

The second problem is not a problem in the context of working with a large cross sectional data set as is the case with the application by Isakin and Serletis, but it will be a problem in a time series context. For a parsimonious flexible functional form that is globally convex, see the normalized quadratic unit profit function that is described in detail in Diewert and Wales (1992).

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