Information Acquisition in Global Games of Regime Change (Online Appendix)

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1 Introduction

This appendix contains the proofs of all the intermediate results that have been omitted from the paper, as well as extensions of several results reported in the paper.

The appendix is divided into 6 sections. In Section 2 we show that for any distribution of precision choices Γ, there are no non-monotonic equilibria in the second stage of the game. We also establish properties of the ex-ante utility function that we invoked in the proof of Theorem 1. In particular, we show that the ex-ante utility function is increasing and that its derivative is uniformly bounded from above and converges to zero. Finally, we show that there exists a precision level $\overline{\tau} \in (\underline{\tau}, \infty)$ such that players never find it optimal to choose a precision higher than $\overline{\tau}$.

Section 3 is concerned with the results regarding the over-acquisition and under-acquisition of information. We first prove the technical result that we used to argue that the unique equilibrium of the game is generically inefficient. We then analyze the conditions under which agents globally over-acquire and under-acquire information and explain why a complete characterization of these global effects is not attainable.

In Section 4 we prove intermediate results that have been invoked in the paper when analyzing strategic complementarities in information choices and the welfare implications of an increase in transparency. We also discuss what happens when we relax the assumption that $T = \frac{1}{2}$. In particular, we provide an extension to Proposition 5 in the main text to the case when $T \neq \frac{1}{2}$. We also discuss why it is difficult to establish a full characterization of strategic complementarities and substitutabilities between private and public information when $T \neq \frac{1}{2}$.

In Section 5 we consider the effects of an increase in the precision of the prior on the probability of a successful investment and on welfare. In particular, we show that for any T the probability of a successful investment increases in τ_θ when μ_θ is sufficiently high and decreases when μ_θ is sufficiently low. We also include figures generated using numerical simulations that decompose the $\{T, \mu_\theta\}$ space into regions where the probability of successful investment is increasing or decreasing in τ_θ. In Section 5.2, we evaluate numerically the effect of an increase in τ_θ on the ex-ante utility of an investor for different parameter values. These numerical results provide support to the claim that welfare is increasing in the precision of public information when the mean of the prior is high and decreasing when the mean of the prior is low.

Finally, Section 6 contains the details of determining the appropriate lower bound for the possible precision choices, $\underline{\tau}$.

2 Uniqueness of Equilibrium and the Properties of the Ex-ante Utility

2.1 Uniqueness of equilibrium

Proposition 1 *For any* Γ, *suppose that* $\inf\left(supp(\Gamma)\right) > \frac{1}{2\pi}\tau_\theta^2$. *Then the coordination game has a unique equilibrium in which all investors use threshold strategies* $\{x_i^*\left(\tau_i, \Gamma\right), i \in [0,1]\}$ *and investment is successful if and only if* $\theta \geq \theta^*$.

Proof. In the appendix of the paper we show that for any Γ such that $\inf\left(supp(\Gamma)\right) > \frac{1}{2\pi}\tau_\theta^2$ we have a unique equilibrium in monotone strategies. We show here that there are no other types of equilibria.

Suppose that in the second stage of the game the distribution of precision choices among investors is given by some distribution function $\Gamma\left(\tau\right)$ with bounded support. The bounded support assumption follows from Assumptions $A1$ and $A2$ in the text. To show uniqueness of equilibrium for any $\Gamma\left(\tau\right)$ we use the procedure of iterative deletion of dominated strategies, a standard approach in the global games literature (see Carlsson and van Damme, 1993).

Suppose that investor i expects no one to invest, that is, he expects investment to be successful if and only if $\theta > \overline{\theta}_0 = 1$. Then, even though no one else invests, investor i will choose to invest if his signal is high enough, i.e. he will invest if

$$\Pr\left(\theta > 1 | x\left(\tau_i\right)\right) > T$$

Denote by $\overline{x}_1^i\left(\tau_i\right)$ the value of the signal that makes investor i indifferent between investing and not investing ($\overline{x}_1^i\left(\tau_i\right)$ solves the above equation with equality). Since the above equation is monotone in $x\left(\tau_i\right)$ it follows that the investor will invest if $x_i \geq \overline{x}_1^i\left(\tau_i\right)$ and not invest otherwise. This implies that each investor i who receives a signal with precision τ_i will find it optimal to invest if the signal he receives is higher than $\overline{x}_1^i\left(\tau_i\right)$, where

$$\overline{x}_1^i\left(\tau_i\right) = \frac{\tau_i + \tau_\theta}{\tau_i}\overline{\theta}_0 - \frac{\tau_\theta}{\tau_i}\mu_\theta + \frac{\left(\tau_i + \tau_\theta\right)^{1/2}}{\tau_i}\Phi^{-1}\left(T\right)$$

Note that all investors with the same precision level will have the same $\overline{x}_1^i\left(\tau_i\right)$. Therefore, in what follows we drop the superscript i. The fact that all investors follow a monotone strategy implies that investment will be successful whenever

$$\int \Pr\left(x_i \geq \overline{x}_1\left(\tau_i\right) | \theta\right) d\Gamma > 1 - \theta$$

Define $\overline{\theta}_1$ as the value of θ that solves the above equation with equality. For a normal distribution this corresponds to the following equation:

$$\int \Phi\left(\tau_i^{1/2}\left(\overline{x}_1^i\left(\tau_i\right)-\overline{\theta}_1\right)\right) d\Gamma=\overline{\theta}_1 \tag{1}$$

Note that $\int \Phi\left(\tau_i^{1/2}\left(\overline{x}_1\left(\tau_i\right)-\overline{\theta}_1\right)\right) d\Gamma > 0$ when $\overline{\theta}_1 = 0$ and $\int \Phi\left(\tau_i^{1/2}\left(\overline{x}_1\left(\tau_i\right)-\overline{\theta}_1\right)\right) d\Gamma < 1$ when $\overline{\theta}_1 = 1$. Moreover,

$$\begin{aligned} & \frac{\partial}{\partial \overline{\theta}_1} \int \Phi\left(\tau_i^{1/2}\left(\overline{x}_1\left(\tau_i\right)-\overline{\theta}_1\right)\right) d\Gamma = \\ & -\int \phi\left(\tau_i^{1/2}\left(\overline{x}_1\left(\tau_i\right)-\overline{\theta}_1\right)\right) \tau_i^{1/2} d\Gamma \\ < \;& 0 \ \forall \overline{\theta}_1 \in [0,1] \end{aligned}$$

Since the right hand side of (1) is increasing, it follows that there is a unique $\overline{\theta}_1 \in (0,1)$ that solves the above equation. Moreover, $\overline{\theta}_1 < \overline{\theta}_0 = 1$.

For the induction step, assume that there is a value $\overline{\theta}_{n-1}$ such that $\overline{\theta}_{n-1} < \overline{\theta}_{n-2} < ... < \overline{\theta}_1 < \overline{\theta}_0 = 1$. $\overline{\theta}_{n-1}$ is defined such that investors expect investment to be successful whenever $\theta > \overline{\theta}_{n-1}$. Hence, investor i who receives a signal with precision τ_i will invest whenever his signal is higher than $\overline{x}_n\left(\tau_i\right)$, where $\overline{x}_n\left(\tau_i\right)$ solves

$$\Pr\left(\theta \geq \overline{\theta}_{n-1} | \overline{x}_n\left(\tau_i\right)\right) = T$$

or

$$\overline{x}_n\left(\tau_i\right) = \frac{\tau_i+\tau_\theta}{\tau_i}\overline{\theta}_{n-1} - \frac{\tau_\theta}{\tau_i}\mu_\theta + \frac{\left(\tau_i+\tau_\theta\right)^{1/2}}{\tau_i}\Phi^{-1}\left(T\right)$$

Note that since $\overline{\theta}_{n-1} < \overline{\theta}_{n-2}$ it follows that $\forall \tau_i \in supp\left(\Gamma\right), \overline{x}_n\left(\tau_i\right) < \overline{x}_{n-1}\left(\tau_i\right)$. Consider again the CM condition. The fact that each investor i invests whenever his signal is higher than $\overline{x}_n\left(\tau_i\right)$ implies that investment will be successful for all $\theta \geq \overline{\theta}_n$ where $\overline{\theta}_n$ solves the following equation:

$$\int \Pr\left(x_i \geq \overline{x}_n\left(\tau_i\right) | \overline{\theta}_n\right) d\Gamma = 1 - \overline{\theta}_n$$

or

$$\int \Phi\left(\tau_i^{1/2}\left(\frac{\tau_i+\tau_\theta}{\tau_i}\overline{\theta}_{n-1} - \frac{\tau_\theta}{\tau_i}\mu_\theta + \frac{\left(\tau_i+\tau_\theta\right)^{1/2}}{\tau_i}\Phi^{-1}\left(T\right) - \overline{\theta}_n\right)\right) d\Gamma = \overline{\theta}_n$$

By the previous argument we know that there exists $\overline{\theta}_n$ that solves the above equation. Note that since $\forall \tau_i \in supp\left(\Gamma\right) \overline{x}_n\left(\tau_i\right) < \overline{x}_{n-1}\left(\tau_i\right)$, we have

$$\Pr\left(x_i \leq \overline{x}_n\left(\tau_i\right) | \overline{\theta}_{n-1}\right) < \Pr\left(x_i \leq \overline{x}_{n-1}\left(\tau_i\right) | \overline{\theta}_{n-1}\right)$$

Suppose now that $\overline{\theta}_n = \overline{\theta}_{n-1}$. Then the RHS of the CM condition above is now greater than the LHS. But note that the RHS is increasing in $\overline{\theta}_n$, while the LHS is decreasing in $\overline{\theta}_n$. Hence it must be the case, by continuity of the LHS and RHS expressions, that $\overline{\theta}_n < \overline{\theta}_{n-1}$ and $\overline{\theta}_n \in [0,1]$. This implies a new threshold signal for each investor, $\overline{x}_{n+1}\left(\tau_i\right)$ such that $\overline{x}_{n+1}\left(\tau_i\right) < \overline{x}_n\left(\tau_i\right) < \overline{x}_{n-1}\left(\tau_i\right)$.

By induction we obtain decreasing sequences $\left\{\overline{\theta}_n\right\}_{n=1}^{\infty}$ and $\{\overline{x}_n(\tau_i)\}_{n=1}^{\infty}$ for each τ_i. Since $\left\{\overline{\theta}_n\right\}_{n=1}^{\infty}$ is bounded from below by 0, it follows that this sequence is convergent. This implies that $\{\overline{x}_n(\tau_i)\}_{n=1}^{\infty}$ is also a convergent sequence. Define $\lim_{n\to\infty}\overline{\theta}_n = \overline{\theta}$ and $\lim_{n\to\infty}\overline{x}_n(\tau_i) = \overline{x}(\tau_i)$. This concludes the iterative deletion of strictly dominated strategies.

Consider now the case when the investor expects that everyone will invest regardless of their signal and again apply the procedure of iterative deletion of dominated strategies. Following exactly the same argument, we obtain increasing and bounded sequences $\{\underline{\theta}_n\}_{n=1}^{\infty}$ and $\{\underline{x}_n(\tau_i)\}_{n=1}^{\infty}$. Define $\lim_{n\to\infty}\underline{\theta}_n = \underline{\theta}$ and $\lim_{n\to\infty}\underline{x}_n(\tau_i) = \underline{x}(\tau_i)$.

By construction $\left\{\overline{\theta}, \{\overline{x}(\tau_i)\}_{\tau_i\in supp(\Gamma)}\right\}$ has to solve

$$\begin{aligned}
\Pr\left(\theta \geq \overline{\theta}|\overline{x}(\tau_i)\right) &= T\ ,\forall\tau_i \in supp(\Gamma)\\
\Pr\left(x_i \geq \overline{x}(\tau_i)|\overline{\theta}\right) &= 1-\overline{\theta}
\end{aligned}$$

and $\left\{\underline{\theta}, \{\underline{x}(\tau_i)\}_{\tau_i\in supp(\Gamma)}\right\}$ has to solve

$$\begin{aligned}
\Pr\left(\theta \geq \underline{\theta}|\underline{x}(\tau_i)\right) &= T\ ,\forall\tau_i \in supp(\Gamma)\\
\Pr\left(x_i \geq \underline{x}(\tau_i)|\underline{\theta}\right) &= 1-\underline{\theta}
\end{aligned}$$

However, in the paper we show that the above system of equations has a unique solution. Therefore $\underline{\theta} = \overline{\theta}$ and $\underline{x}(\tau_i) = \overline{x}(\tau_i)$ $\forall\tau_i \in supp(\Gamma)$ implying that there is a unique strategy profile that survives the iterative deletion of dominated strategies. This strategy profile constitutes the unique equilibrium in monotone strategies. $\blacksquare$

Lemma 2 *Consider the benefit function $B^i(\tau_i;\Gamma)$.*

1. *$B^i(\tau_i;\Gamma)$ is strictly increasing in τ_i,*
2. *$\frac{\partial B^i}{\partial\tau_i}$ is bounded from above,*
3. *$\lim_{\tau_i\to\infty}\frac{\partial B^i}{\partial\tau_i} = 0$,*
4. *For $\tau_i > \underline{\tau}$, $\frac{\partial^2 B^i}{\partial\tau_i^2} < 0$.*

Proof. (1) Condition (1) implies that the value of information in our setting is always positive.

Recall that

$$\begin{aligned}
B^i(\tau_i;\Gamma) &= -\int_{-\infty}^{\theta^*}\int_{x_i^*}^{\infty} TdF(x|\theta)\,dG(\theta) - \int_{\theta^*}^{\infty}\int_{-\infty}^{x_i^*}(1-T)\,dF(x|\theta)\,dG(\theta)\\
&\quad + \int_{\theta^*}^{\infty}(1-T)\,dG(\theta) - C(\tau_i)
\end{aligned}$$

We differentiate $B^i(\tau_i;\Gamma)$ with respect to τ_i and note that

$$\frac{\partial x_i^*}{\partial\tau_i}\int_{-\infty}^{\theta^*} f_{\tau_i}(x_i^*|\theta)\,g_{\tau_\theta}(\theta)\,d\theta - \frac{\partial x_i^*}{\partial\tau_i}\int_{\theta^*}^{\infty}(1-T)\,f_{\tau_i}(x_i^*|\theta)\,g_{\tau_\theta}(\theta)\,d\theta = 0$$

since x_i^* has to satisfy

$$\Pr\left(\theta \geq \theta^* | x_i^*\right) = T$$

Evaluating the remaining integrals and using the properties of the truncated normal distribution we obtain

$$\frac{\partial B^i}{\partial \tau_i} = \frac{1}{2}\frac{1}{\tau_i}\frac{1}{\tau_i + \tau_\theta}\tau_i^{1/2}\phi\left(\frac{x_i^* - \theta^*}{\tau_i^{-1/2}}\right)\tau_\theta^{1/2}\phi\left(\frac{\theta^* - \mu_\theta}{\tau_\theta^{-1/2}}\right) > 0$$

for all $\tau_i < \infty$. This proves the first part of the lemma.

(2) To see that the second part of the lemma is also true note that

$$\begin{aligned}
\frac{\partial B^i}{\partial \tau_i} &= \frac{1}{2}\frac{1}{\tau_i}\frac{1}{\tau_i + \tau_\theta}\tau_i^{1/2}\phi\left(\frac{x_i^* - \theta^*}{\tau_i^{-1/2}}\right)\tau_\theta^{1/2}\phi\left(\frac{\theta^* - \mu_\theta}{\tau_\theta^{-1/2}}\right) \\
&\leq \frac{1}{2}\frac{1}{\tau_i}\frac{1}{\tau_i + \tau_\theta}\tau_i^{1/2}\frac{1}{\sqrt{2\pi}}\tau_\theta^{1/2}\frac{1}{\sqrt{2\pi}} \\
&\leq \frac{1}{4\pi}\frac{1}{\underline{\tau}^{1/2}}\frac{\sqrt{\tau_\theta}}{\underline{\tau} + \tau_\theta} < \infty
\end{aligned}$$

(3) To prove the third part of the above lemma note that

$$0 \leq \frac{\partial B^i}{\partial \tau_i} \leq \frac{1}{2}\frac{1}{\tau_i}\frac{1}{\tau_i + \tau_\theta}\tau_i^{1/2}\frac{1}{\sqrt{2\pi}}\tau_\theta^{1/2}\frac{1}{\sqrt{2\pi}} \to 0 \text{ as } \tau_i \to \infty$$

(4) The last part of the lemma is less straightforward. The second derivative of the benefit function is given by

$$\begin{aligned}
\frac{\partial^2 B^i}{\partial \tau_i^2} &= -\frac{1}{2}\frac{1}{\tau_i}\frac{1}{\tau_i + \tau_\theta}\tau_i^{1/2}\phi\left(\frac{x_i^* - \theta^*}{\tau_i^{-1/2}}\right)\tau_\theta^{1/2}\phi\left(\frac{\theta^* - \mu_\theta}{\tau_\theta^{-1/2}}\right) \\
&\quad \times \left[\frac{1}{\tau_i} + \frac{1}{\tau_i + \tau_\theta} - \frac{1}{2\tau_i} + \frac{1}{2}\left(x_i^* - \theta^*\right)^2 + \tau_i\left(x_i^* - \theta^*\right)\frac{\partial\left(x_i^* - \theta^*\right)}{\partial \tau_i}\right]
\end{aligned}$$

Thus, if we show that the term in the square brackets is positive then the claim would hold.

Define

$$A\left(\tau_i, \tau_\theta\right) \equiv \left[\frac{1}{\tau_i} + \frac{1}{\tau_i + \tau_\theta} - \frac{1}{2\tau_i} + \frac{1}{2}\left(x_i^* - \theta^*\right)^2 + \tau_i\left(x_i^* - \theta^*\right)\frac{\partial\left(x_i^* - \theta^*\right)}{\partial \tau_i}\right]$$

Note that

$$\begin{aligned}
\left(x_i^* - \theta^*\right) &= \frac{\tau_\theta}{\tau_i}\left(\theta^* - \mu_\theta\right) + \frac{\sqrt{\tau_i + \tau_\theta}}{\tau_i}\Phi^{-1}(T) \\
\frac{\partial\left(x_i^* - \theta^*\right)}{\partial \tau_i} &= -\frac{\tau_\theta}{\tau_i^2}\left(\theta^* - \mu_\theta\right) - \frac{\frac{1}{2}\tau_i + \tau_\theta}{\tau_i^2\sqrt{\tau_i + \tau_\theta}}\Phi^{-1}(T)
\end{aligned}$$

and hence

$$
\begin{aligned}
& A\left(\tau_i, \tau_\theta\right) \\
= \; & \frac{1}{2\tau_i} + \frac{1}{\tau_i + \tau_\theta} + \frac{1}{2}\left(x_i^* - \theta^*\right)^2 \\
& + \tau_i \left(\frac{\tau_\theta}{\tau_i}\left(\theta^* - \mu_\theta\right) + \frac{\sqrt{\tau_i + \tau_\theta}}{\tau_i}\Phi^{-1}\left(T\right)\right)\left(-\frac{\tau_\theta}{\tau_i^2}\left(\theta^* - \mu_\theta\right) - \frac{\frac{1}{2}\tau_i + \tau_\theta}{\tau_i^2\sqrt{\tau_i + \tau_\theta}}\Phi^{-1}\left(T\right)\right) \\
= \; & \frac{3\tau_i + \tau_\theta}{2\tau_i\left(\tau_i + \tau_\theta\right)} + \frac{1}{2}\left(x_i^* - \theta^*\right)^2 \\
& - \left(\frac{\tau_\theta}{\tau_i}\left(\theta^* - \mu_\theta\right) + \frac{\sqrt{\tau_i + \tau_\theta}}{\tau_i}\Phi^{-1}\left(T\right)\right)\left(\frac{\tau_\theta}{\tau_i}\left(\theta^* - \mu_\theta\right) + \frac{\frac{1}{2}\tau_i + \tau_\theta}{\tau_i\sqrt{\tau_i + \tau_\theta}}\Phi^{-1}\left(T\right)\right)
\end{aligned}
$$

where the last term is in general negative.

Before proceeding, we further simplify the above expression further:

$$
\begin{aligned}
& \frac{1}{2}\left(x_i^* - \theta^*\right)^2 + \left(\frac{\tau_\theta}{\tau_i}\left(\theta^* - \mu_\theta\right) + \frac{\sqrt{\tau_i + \tau_\theta}}{\tau_i}\Phi^{-1}\left(T\right)\right)\left(\frac{\tau_\theta}{\tau_i}\left(\theta^* - \mu_\theta\right) + \frac{\frac{1}{2}\tau_i + \tau_\theta}{\tau_i\sqrt{\tau_i + \tau_\theta}}\Phi^{-1}\left(T\right)\right) \\
= \; & \left(\frac{\tau_\theta}{\tau_i}\left(\theta^* - \mu_\theta\right) + \frac{\sqrt{\tau_i + \tau_\theta}}{\tau_i}\Phi^{-1}\left(T\right)\right) \times \\
& \left[-\frac{1}{2}\frac{\tau_\theta}{\tau_i}\left(\theta^* - \mu_\theta\right) + \frac{\sqrt{\tau_i + \tau_\theta}}{\tau_i}\Phi^{-1}\left(T\right)\left(\frac{1}{2} - \frac{\frac{1}{2}\tau_i + \tau_\theta}{\left(\tau_i + \tau_\theta\right)}\right)\right] \\
= \; & -\frac{1}{2}\left(\frac{\tau_\theta}{\tau_i}\right)^2\left(\theta^* - \mu_\theta + \frac{\sqrt{\tau_i + \tau_\theta}}{\tau_\theta}\Phi^{-1}\left(T\right)\right)\left[\theta^* - \mu_\theta + \frac{1}{\sqrt{\tau_i + \tau_\theta}}\Phi^{-1}\left(T\right)\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial^2 B^i}{\partial \tau_i^2} = \; & -\frac{1}{4}\frac{1}{\tau_i^{3/2}}\frac{\tau_\theta^2}{\tau_i + \tau_\theta}\phi\left(\frac{x_i^* - \theta^*}{\tau_i^{-1/2}}\right)\tau_\theta^{1/2}\phi\left(\frac{\theta^* - \mu_\theta}{\tau_\theta^{-1/2}}\right) \\
& \times\left[\frac{3\tau_i + \tau_\theta}{\tau_\theta^2\left(\tau_i + \tau_\theta\right)} - \frac{1}{\tau_i}\left(\theta^* - \mu_\theta + \frac{\sqrt{\tau_i + \tau_\theta}}{\tau_\theta}\Phi^{-1}\left(T\right)\right)\left(\theta^* - \mu_\theta + \frac{1}{\sqrt{\tau_i + \tau_\theta}}\Phi^{-1}\left(T\right)\right)\right]
\end{aligned}
$$

where we took $\frac{1}{2}\frac{\tau_\theta^2}{\tau_i}$ in front of the square brackets. At this point we consider three cases: $T < \frac{1}{2}$, $T = \frac{1}{2}$, and $T > \frac{1}{2}$.

Let $T < \frac{1}{2}$. When $T < \frac{1}{2}$ then

$$
\begin{aligned}
& \frac{1}{\tau_i}\left(\theta^* - \mu_\theta + \frac{\sqrt{\tau_i + \tau_\theta}}{\tau_\theta}\Phi^{-1}\left(T\right)\right)\left(\theta^* - \mu_\theta + \frac{1}{\sqrt{\tau_i + \tau_\theta}}\Phi^{-1}\left(T\right)\right) \\
\leq \; & \frac{1}{\tau_i}\left(1 + \left|\underline{\mu}_\theta\right| + \frac{\sqrt{\tau_i + \tau_\theta}}{\tau_\theta}\Phi^{-1}\left(T\right)\right)\left(1 + \left|\underline{\mu}_\theta\right| + \frac{1}{\sqrt{\tau_i + \tau_\theta}}\Phi^{-1}\left(T\right)\right) \\
\rightarrow \; & 0 \text{ as } \tau_i \rightarrow \infty
\end{aligned}
$$

Note also that

$$\lim_{\tau_i \rightarrow \infty}\frac{3\tau_i + \tau_\theta}{\tau_\theta^2\left(\tau_i + \tau_\theta\right)} = \frac{3}{\tau_\theta^2}$$

Therefore, it follows that

$$\lim_{\tau_i \rightarrow \infty} A\left(\tau_i, \tau_\theta\right) = \frac{3}{\tau_\theta^2} > 0$$

We conclude that, for a fixed τ_θ, there exists a bound for τ_i such that above this bound the second derivative of investor i's benefit function is negative.

Next, recall that

$$\tau_\theta \in [\underline{\tau}_\theta, \overline{\tau}_\theta] \text{ where } 0 < \underline{\tau}_\theta < \overline{\tau}_\theta < \infty$$

Since $A(\tau_i, \tau_\theta)$ is continuous and converges to zero for any τ_θ as $\tau_i \to \infty$, and since $[\underline{\tau}_\theta, \overline{\tau}_\theta]$ is a compact subset of $\mathbb{R}$, it follows that there exists a uniform bound for τ_i that is finite, and is such that if τ_i is larger than this bound, then the second-order derivative of the benefit function with respect to τ_i is negative for all $\tau_\theta \in [\underline{\tau}_\theta, \overline{\tau}_\theta]$. We assume that $\underline{\tau}$ is greater or equal to this bound.

The argument for the case when $T \geq \frac{1}{2}$ is analogous, i.e., a similar argument establishes the existence of a bound on τ_i that guarantees that the second-order derivative of the benefit function is negative when $T \geq \frac{1}{2}$. We assume that $\underline{\tau}$ is greater or equal than both of these bounds. Since each of these bounds is finite, it follows that $\underline{\tau}$ is finite. This concludes the proof. ■

Lemma 3 *There exists $\overline{\tau} < \infty$ such that investor i will never find it optimal to choose $\tau_i^* > \overline{\tau}$.*

Proof. From Lemma 2 we know that $\frac{\partial B^i}{\partial \tau_i} \leq \frac{1}{4\pi} \frac{1}{\underline{\tau}^{1/2}} \frac{\sqrt{\overline{\tau}_\theta}}{\underline{\tau} + \underline{\tau}_\theta}$. Moreover, by Assumption $A2$ we have that $lim_{\tau_i \to \infty} C'(\tau_i) = \infty$. Hence, there exists $\overline{\tau} < \infty$ such that for all $\tau_i > \overline{\tau}$

$$\frac{\partial B_i}{\partial \tau_i} - C'(\tau_i) < 0$$

Since in the neighborhood of $\underline{\tau}$ the following is true

$$\frac{\partial B_i}{\partial \tau_i} - C'(\tau_i) > 0$$

then it follows that no investor will ever choose $\tau_i > \overline{\tau}$. ■

3 Efficiency of Information Choices

In this section we provide proofs for the results that we used in the efficiency section of the paper that were omitted in the main appendix. We also provide a partial characterization of global over and under acquisition of information and we explain why a complete characterization of the global results is not attainable.

We start with a technical result that we invoked when arguing that agents' equilibrium choices are generically inefficient.

Lemma 4 *Denote by $\tau^*(\mu_\theta)$ the equilibrium precision choice as a function of μ_θ. Then for each T there exists a unique μ_θ, call it $\mu_\theta^E(T)$, that solves*

$$\mu_\theta = \widehat{\mu}^\tau (T, \tau^*(\mu_\theta), \tau_\theta)$$

where

$$\widehat{\mu}^\tau (T, \tau^*(\mu_\theta), \tau_\theta) = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)}{\tau^*(\mu_\theta) + \tau_\theta}} \Phi^{-1}(T)\right) + \frac{1}{\sqrt{\tau^*(\mu_\theta) + \tau_\theta}} \Phi^{-1}(T)$$

Moreover, if $\mu_\theta > \mu_\theta^E(T)$ then $\mu_\theta > \widehat{\mu}^\tau (T, \tau^*(\mu_\theta), \tau_\theta)$ and if $\mu_\theta < \mu_\theta^E(T)$ then $\mu_\theta < \widehat{\mu}^\tau (T, \tau^*(\mu_\theta), \tau_\theta)$.

Proof. We are interested in solutions to

$$\mu_\theta = \widehat{\mu}^{\tau}\left(T, \tau^*\left(\mu_\theta\right), \tau_\theta\right)$$

Define

$$f\left(\mu_\theta\right) \equiv \mu_\theta - \widehat{\mu}^{\tau}\left(T, \tau^*\left(\mu_\theta\right), \tau_\theta\right)$$

To prove our claim we will show that $f\left(\mu_\theta\right)$ intersects zero only once and is positive as $\mu_\theta \to \infty$ and negative as $\mu_\theta \to -\infty$.

Fix T and τ_θ. Note that if $T > \frac{1}{2}$ then $\widehat{\mu}^{\tau}(\cdot)$ is bounded from below by 0 and from above by $1 + \frac{1}{\sqrt{\underline{\tau}+\tau_\theta}}\Phi^{-1}(T)$, and if $T < \frac{1}{2}$ then $\widehat{\mu}^{\tau}(\cdot)$ is bounded from below by $\frac{1}{\sqrt{\underline{\tau}+\tau_\theta}}\Phi^{-1}(T)$ and from above by 1. If follows that regardless of the value of T, the function $f\left(\mu_\theta\right)$ is positive for large enough μ_θ and negative for small enough μ_θ. Thus, it remains to show that $f\left(\mu_\theta\right)$ intersects zero only once.

Note that since $\widehat{\mu}^{\tau}\left(T, \tau^*\left(\mu_\theta\right), \tau_\theta\right)$ is continuous in μ_θ it follows that there exists μ_θ^E such that

$$f\left(\mu_\theta^E\right) = \mu_\theta^E - \widehat{\mu}^{\tau}\left(T, \tau^*\left(\mu_\theta^E\right), \tau_\theta\right) = 0$$

We now show that at any such μ_θ^E the derivative of $f\left(\mu_\theta\right)$ with respect to μ_θ is positive, implying that $f\left(\mu_\theta\right)$ crosses zero only once. Taking the derivative of $\widehat{\mu}^{\tau}\left(T, \tau^*\left(\mu_\theta^E\right), \tau_\theta\right)$ with respect to μ_θ we get:

$$\frac{\partial \widehat{\mu}^{\tau}\left(T, \tau^*\left(\mu_\theta\right), \tau_\theta\right)}{\partial \mu_\theta} = \frac{\partial \widehat{\mu}^{\tau}\left(T, \tau^*\left(\mu_\theta\right), \tau_\theta\right)}{\partial \tau^*}\frac{\partial \tau^*}{\partial \mu_\theta}$$

Consider first $\frac{\partial \tau^*}{\partial \mu_\theta}$ and recall that τ^* is a fixed point of the best response function, i.e. it satisfies:

$$\tau^* = \boldsymbol{\tau}^*\left(\tau^*; \mu_\theta\right)$$

where $\boldsymbol{\tau}^*$ denotes the investor's best response function, implicitly defined by the first-order condition. Hence,

$$\frac{\partial \tau^*}{\partial \mu_\theta} = -\frac{\frac{\partial \boldsymbol{\tau}^*\left(\tau^*; \mu_\theta\right)}{\partial \mu_\theta}}{1 - \frac{\partial \boldsymbol{\tau}^*\left(\tau^*; \mu_\theta\right)}{\partial \tau^*}} \propto -\frac{\frac{\partial^2 U^i}{\partial \tau \partial \mu_\theta}\left(\tau^*, \tau^*, \mu_\theta\right)}{\frac{\partial^2 U^i}{\partial \tau^2}\left(\tau^*, \tau^*, \mu_\theta\right) - C''\left(\tau^*\right)}$$

Since $\frac{\partial \boldsymbol{\tau}^*\left(\tau^*; \mu_\theta\right)}{\partial \tau^*} < 1$.[1]

Suppose that $\mu_\theta = \mu_\theta^E$ and consider CM condition:[2]

$$\frac{\tau_\theta}{\sqrt{\tau^*}}\left(\theta^* - \mu_\theta^E\right) + \sqrt{\frac{\tau^* + \tau_\theta}{\tau^*}}\Phi^{-1}(T) - \Phi^{-1}\left(\theta^*\right) = 0$$

Rearranging, and using the fact that $\mu_\theta^E = \widehat{\mu}^{\tau}\left(T, \tau^*\left(\mu_\theta^E\right), \tau_\theta\right)$ we obtain:

$$\frac{\tau_\theta}{\sqrt{\tau^*}}\theta^* - \Phi^{-1}\left(\theta^*\right) = \frac{\tau_\theta}{\sqrt{\tau^*}}\Phi\left(\sqrt{\frac{\tau^*}{\tau^* + \tau_\theta}}\Phi^{-1}(T)\right) - \sqrt{\frac{\tau^*}{\tau^* + \tau_\theta}}\Phi^{-1}(T)$$

[1] The fact that $\frac{\partial \boldsymbol{\tau}^*\left(\tau^*; \mu_\theta\right)}{\partial \tau^*} < 1$ is implied by the assumption that $\underline{\tau}$ is large enough. See Theorem 1 and its proof in the paper.

[2] To economize on notation we write below τ^* instead of $\tau^*\left(\mu_\theta^E\right)$.

implying that

$$\theta^*\left(\mu_\theta^E\right) = \Phi\left(\sqrt{\frac{\tau^*}{\tau^*+\tau_\theta}}\Phi^{-1}(T)\right) = \mu_\theta^E - \frac{1}{\sqrt{\tau^*+\tau_\theta}}\Phi^{-1}(T)$$

which in turn implies that $x^*\left(\tau^*\left(\mu_\theta^E\right);\tau^*\left(\mu_\theta^E\right),\mu_\theta^E\right) = \mu_\theta^E$. But then it follows that

$$\frac{\partial^2 U^i}{\partial \tau_i \partial \mu_\theta} = -\frac{1}{\tau_i}\frac{1}{\tau_i+\tau_\theta}\tau_i^{1/2}\phi\left(\frac{x_i^*-\theta^*}{\tau_i^{-1/2}}\right)\tau_\theta^{1/2}\phi\left(\frac{\theta^*-\mu_\theta}{\tau_\theta^{-1/2}}\right)\tau_\theta\left(x_i^*-\mu_\theta\right)\left(\frac{\partial\theta^*}{\partial\mu_\theta}-1\right)$$

is zero when $\mu_\theta = \mu_\theta^E$.[3] Therefore, $\frac{\partial^2 U^i}{\partial\tau\partial\mu_\theta}\left(\tau^*\left(\mu_\theta^E\right),\tau^*\left(\mu_\theta^E\right),\mu_\theta^E\right) = 0$ and so $\frac{\partial\tau^*}{\partial\mu_\theta}|_{\mu_\theta=\mu_\theta^E} = 0$. Since $\frac{\partial\widehat{\mu}^\tau(T,\tau^*(\mu_\theta),\tau_\theta)}{\partial\tau^*}$ is always finite, it follows that $\frac{\partial\widehat{\mu}^\tau(T,\tau^*(\mu_\theta),\tau_\theta)}{\partial\mu_\theta}|_{\mu_\theta=\mu_\theta^E} = 0$ and so at any μ_θ^E such that $f\left(\mu_\theta^E\right) = 0$, the derivative of $f(\cdot)$ with respect to μ_θ is positive. This implies that $f(\cdot)$ crosses zero only once, which completes the proof for the case when $T > \frac{1}{2}$. The case for $T < \frac{1}{2}$ follows by an analogous argument. ■

3.1 Global over-acquisition and under-acquisition of information

Now we consider the global over and under acquisition of information. Recall that we say that agents globally over-acquire information if the efficient choice of precision, τ^{**}, is less than the equilibrium precision choice, τ^*, where

$$\tau^{**} \in \arg\max_\tau U(\tau,\tau) - C(\tau)$$

while τ^* is the solution to

$$U_1(\tau,\tau) - C'(\tau) = 0$$

To study global results we need to make use of the following results, established by Szkup (2014):

Lemma 5 *Let τ be the precision choice of all other investors*

1. *If $\mu_\theta < \widehat{\mu}^\tau(T,\tau,\tau_\theta)$ then $\frac{\partial\theta^*}{\partial\tau} < 0$*

2. *If $\mu_\theta = \widehat{\mu}^\tau(T,\tau,\tau_\theta)$ then $\frac{\partial\theta^*}{\partial\tau} = 0$*

3. *If $\mu_\theta > \widehat{\mu}^\tau(T,\tau,\tau_\theta)$ then $\frac{\partial\theta^*}{\partial\tau} > 0$*

where

$$\widehat{\mu}^\tau(T,\tau,\tau_\theta) = \Phi\left(\sqrt{\frac{\tau}{\tau+\tau_\theta}}\Phi^{-1}(T)\right) + \frac{1}{\sqrt{\tau+\tau_\theta}}\Phi^{-1}(T)$$

Proof. This lemma then follows from Proposition 2 in Szkup (2014). ■

Lemma 6 *Let τ be the precision choice of all other investors:*

1. *Suppose that $T > \frac{1}{2}$, then*

   (a) *If $\mu_\theta \leq T$ then θ^* is decreasing for all $\tau > \underline{\tau}$;*

[3] In Lemma 11 we establish that $\frac{\partial\theta^*}{\partial\mu_\theta} < 0$.

(b) *If* $\mu_\theta \in (T, \widehat{\mu}^\tau (T, \underline{\tau}, \tau_\theta))$ *then* θ^* *is initially decreasing and then increasing in* τ;

(c) *If* $\mu_\theta \geq \widehat{\mu}^\tau (T, \underline{\tau}, \tau_\theta)$ *then* θ^* *is increasing for all* $\tau > \underline{\tau}$.

2. *Suppose that* $T = \frac{1}{2}$, *then*

(a) *if* $\mu_\theta < \frac{1}{2}$ *then* θ^* *is decreasing in* τ *for all* $\tau > \underline{\tau}$;

(b) *if* $\mu_\theta = \frac{1}{2}$ *then* θ^* *is constant in* τ;

(c) *if* $\mu_\theta > \frac{1}{2}$ *then* θ^* *is increasing in* τ *for all* $\tau > \underline{\tau}$.

3. *Suppose that* $T < \frac{1}{2}$, *then:*

(a) *If* $\mu_\theta < \widehat{\mu}^\tau (T, \underline{\tau}, \tau_\theta)$ *then* θ^* *is decreasing in* τ *for all* $\tau > \underline{\tau}$*;*

(b) *If* $\mu_\theta \in (\widehat{\mu}^\tau (T, \underline{\tau}, \tau_\theta), T)$ *then* θ^* *is initially increasing and then decreasing in* τ;

(c) $\mu_\theta \geq T$ *then* θ^* *is increasing in* τ *for all* $\tau > \underline{\tau}$.

where

$$\widehat{\mu}^\tau (T, \underline{\tau}, \tau_\theta) = \Phi\left(\sqrt{\frac{\underline{\tau}}{\underline{\tau} + \tau_\theta}}\Phi^{-1}(T)\right) + \frac{1}{\sqrt{\underline{\tau} + \tau_\theta}}\Phi^{-1}(T)$$

and

$$\begin{array}{lcll} \textit{if } T & > & \frac{1}{2} & \textit{then for all } \tau \geq \underline{\tau},\ \widehat{\mu}^\tau (T, \tau, \tau_\theta) > T \\ \textit{if } T & < & \frac{1}{2} & \textit{then for all } \tau \geq \underline{\tau},\ \widehat{\mu}^\tau (T, \tau, \tau_\theta) < T \end{array}$$

Proof. The result follows from Proposition 4 in Szkup (2014). ■

Next, we define our notion of global spillover effects and use Lemma 6 to establish conditions under which there are positive or negative spillover effects in information choices.

Definition 7 *Consider investor* i *and let* τ *be the precision choice of all other investors.*

1. *If* $\forall \tau > \underline{\tau}$ *we have* $\frac{\partial B^i}{\partial \tau} > 0$ *then we say that the game exhibits global positive spillovers*

2. *If* $\forall \tau > \underline{\tau}$ *we have* $\frac{\partial B^i}{\partial \tau} < 0$ *then we say that the game exhibits global negative spillovers*

Corollary 8 *Let* $\underline{\tau}$ *be the precision of private information that agents are initially endowed with.*

1. *Suppose that* $T > \frac{1}{2}$.

(a) *If* $\mu_\theta \leq T$ *then there are global positive spillover effects in information choices.*

(b) *If* $\mu_\theta \in (T, \widehat{\mu}^\tau (T, \underline{\tau}, \tau_\theta))$ *then there exists* $\widehat{\tau}(\mu_\theta)$ *such that for all* $\tau < \widehat{\tau}(\mu_\theta)$ *there are positive spillovers in information choices, while if* $\tau > \widehat{\tau}(\mu_\theta)$ *there are negative spillovers in information choices.*

(c) *If* $\mu_\theta \geq \widehat{\mu}^\tau (T, \underline{\tau}, \tau_\theta)$ *then there are global negative spillover effects in information choices.*

2. *Suppose that* $T = \frac{1}{2}$.

   (a) *if* $\mu_\theta < \frac{1}{2}$ *then there are global positive spillover effects in information choices.*

   (b) *if* $\mu_\theta = \frac{1}{2}$ *then there are no spillover effects in information choices.*

   (c) *if* $\mu_\theta > \frac{1}{2}$ *then there are global negative spillover effects in information choices.*

3. *Suppose that* $T < \frac{1}{2}$.

   (a) *If* $\mu_\theta < \widehat{\mu}^{\tau}(T, \underline{\tau}, \tau_\theta)$ *then there are global positive spillover effects in information choices.;*

   (b) *If* $\mu_\theta \in (\widehat{\mu}^{\tau}(T, \underline{\tau}, \tau_\theta), T)$ *then there exists* $\widehat{\tau}(\mu_\theta)$ *such that for all* $\tau < \widehat{\tau}(\mu_\theta)$ *there are negative spillovers in information choices, while if* $\tau > \widehat{\tau}(\mu_\theta)$ *there are positive spillovers in information choices.*

   (c) $\mu_\theta \geq T$ *then there are global negative spillover effects in information choices.*

Proof. This result follows from Lemma 6 and the observation that the sign of $\frac{\partial U^i}{\partial \tau}$ depends only on the effect that an increase in the precision of all investors has on the threshold θ^*, since

$$\frac{\partial U^i}{\partial \tau} = -\frac{\partial \theta^*}{\partial \tau}\left(1 - \Phi\left(\frac{\theta^* - x_i^*}{\tau_i^{-1/2}}\right)\right)\tau_\theta^{1/2}\phi\left(\frac{\theta^* - \mu_\theta}{\tau_\theta^{-1/2}}\right)$$

■

Lemma 6 and Corollary 8 are key for establishing the comparison between τ^*, the equilibrium choice of precision, and τ^{**}, the efficient precision. In the next proposition we utilize the above lemma to provide an almost complete characterization of global over and under acquisition of information (except for the case when $T < \frac{1}{2}$ and $\mu_\theta \in (\widehat{\mu}^{\tau}(T, \underline{\tau}, \tau_\theta), T)$).

Proposition 9

1. *Suppose that* $T > \frac{1}{2}$.

   (a) *If* $\mu_\theta \leq \mu_\theta^E(T)$ *then agents under-acquire information;*

   (b) *If* $\mu_\theta = \mu_\theta^E(T)$ *then agents acquire the efficient level of information;*

   (c) *If* $\mu_\theta \geq \mu_\theta^E(T)$ *then agents over-acquire information;*

   where $\mu_\theta^E(T)$ *is the unique solution to* $\mu_\theta = \widehat{\mu}^{\tau}(T, \tau^*(\mu_\theta), \tau_\theta)$.[4]

2. *Suppose that* $T = \frac{1}{2}$.

   (a) *If* $\mu_\theta > \frac{1}{2}$ *then agents under-acquire information;*

   (b) *If* $\mu_\theta = \frac{1}{2}$ *then agents acquire the efficient level of information;*

   (c) *If* $\mu_\theta < \frac{1}{2}$ *then agents over-acquire information.*

[4] See Lemma 4 for the precise definition of $\mu_\theta^E(T)$.

3. *Suppose that* $T < \frac{1}{2}$.

   (a) *If* $\mu_\theta \leq \widehat{\mu}^\tau(T, \underline{\tau}, \tau_\theta)$ *then agents under-acquire information;*

   (b) *If* $\mu_\theta \geq T$ *then agents over-acquire information.*

Proof. We divide the proof into three separate parts. First, we provide the proof for the case when $T = \frac{1}{2}$. Next, we consider the case when $T > \frac{1}{2}$. Finally, we consider the case when $T < \frac{1}{2}$.

Suppose first that $T = \frac{1}{2}$. In this case, the claim follows almost immediately from Corollary 8. To see this, consider first a situation where $\mu_\theta > \frac{1}{2}$ and note that Corollary 8 implies that in this case there are global negative externalities, i.e. for all $\tau_i > \underline{\tau}$ and all $\tau > \underline{\tau}$ we have $B_2^i(\tau_i; \tau) < 0$. Moreover, we know that $B_1^i(\tau_i; \tau)$ is decreasing in τ_i (part (iv) of Lemma 2). It follows that the planner's objective function, which is given by $B^i(\tau; \tau) - C(\tau)$, is decreasing for all $\tau \geq \tau^*$, i.e. for all $\tau \geq \tau^*$ we have $B_1^i(\tau; \tau) + B_2^i(\tau; \tau) - C(\tau) < 0$. It follows that the planner will choose precision level $\tau^{**} \leq \tau^*$. Moreover, at $\tau = \tau^{**}$ we have

$$B_1^i(\tau^*; \tau^*) + B_2^i(\tau^*; \tau^*) - C(\tau^*) = B_2^i(\tau^*; \tau^*) < 0$$

and, thus, the planner would choose $\tau^{**} < \tau^*$. Hence, when $T = \frac{1}{2}$ and $\mu_\theta > \frac{1}{2}$ then agents over-acquire information. Using a similar argument one can show that agents under-acquire information when $T = \frac{1}{2}$ and $\mu_\theta < \frac{1}{2}$ and that they acquire the efficient level of information when $T = \frac{1}{2}$ and $\mu_\theta = \frac{1}{2}$.

Next suppose that $T > \frac{1}{2}$. From Corollary 8 we know that when $\mu_\theta \leq T$ there are global positive externalities in information choices that are ignored by the agents at the time they choose the precision of their signals. Hence, in this case $B_1^i(\tau; \tau) - C'(\tau) \geq 0$ implies that $B_1^i(\tau; \tau) + B_2^i(\tau; \tau) - C'(\tau) > 0$. It follows that $\tau^{**} \geq \tau^*$. Moreover, since $B_2^i(\tau; \tau)$ is strictly positive, it follows that at τ^* we have $B_1^i(\tau^*; \tau^*) + B_2^i(\tau^*; \tau^*) - C'(\tau^*) > 0$, so that $\tau^{**} > \tau^*$. Using an analogous argument we can establish that if $T > \frac{1}{2}$ then for all $\mu_\theta \geq \widehat{\mu}^\tau(T, \underline{\tau}, \tau_\theta)$ we have $\tau^{**} < \tau^*$.

It remains to consider the case where $T > \frac{1}{2}$ and $\mu_\theta \in (T, \widehat{\mu}^\tau(T, \underline{\tau}, \tau_\theta))$. Since $T > \frac{1}{2}$, from Lemma 6 we know that in this case the threshold θ^* is first decreasing and then increasing in τ. Fix T, then for each $\mu_\theta \in (T, \widehat{\mu}^\tau(T, \underline{\tau}, \tau_\theta))$ there exists a unique precision level, call it $\widehat{\tau}(\mu_\theta)$, such that at this precision level $\frac{d\theta^*}{d\tau} = 0$. From the proof of Lemma 4 we know that $\tau^* = \widehat{\tau}(\mu_\theta)$ if and only if $\mu_\theta = \mu_\theta^E(T)$. From Lemma 4 we also know that $\mu_\theta > \mu_\theta^E(T)$ implies $\mu_\theta > \widehat{\mu}^\tau(T, \tau^*(\mu_\theta), \tau_\theta)$ and $\mu_\theta < \mu_\theta^E(T)$ implies $\mu_\theta < \widehat{\mu}^\tau(T, \tau^*(\mu_\theta), \tau_\theta)$. But then it follows that (i) if $\mu_\theta > \mu_\theta^E(T)$ then $\left.\frac{d\theta^*}{d\tau}\right|_{\tau=\tau^*} < 0$, (ii) if $\mu_\theta = \mu_\theta^E(T)$ then $\left.\frac{d\theta^*}{d\tau}\right|_{\tau=\tau^*} = 0$, and (iii) if $\mu_\theta < \mu_\theta^E(T)$ then $\left.\frac{d\theta^*}{d\tau}\right|_{\tau=\tau^*} > 0$.

From the above discussion we conclude that if $\mu_\theta \in \left(\mu_\theta^E(T), \widehat{\mu}^\tau(T, \underline{\tau}, \tau_\theta)\right)$ then the equilibrium choice of τ^* is such that θ^* is increasing in τ at τ^*, so τ^* lies at the increasing portion of $\theta^*(\tau)$. This means that for all $\tau \in [\tau^*, \infty)$ we have negative spillover effects (Corollary 8). Thus, the optimal choice of precision τ^{**} must satisfy $\tau^{**} < \tau^*$ implying that investors over-acquire information. Similarly, if $\mu_\theta \in \left(T, \mu_\theta^E(T)\right)$ then the choice of τ^* is such that θ^* is decreasing in τ^*, so τ^* lies at the decreasing portion of $\theta^*(\tau)$. This means that for all $\tau \in [\underline{\tau}, \tau^*]$ we have positive spillover effects and, thus, it has to be the case that $\tau^{**} > \tau^*$. Finally, consider the case when $\mu_\theta = \mu_\theta^E(T)$. We know that for all $\tau < \tau^*$, θ^* is decreasing in τ (there are positive spillover effects in information choices), so that $\tau^{**} \geq \tau^*$. Similarly, we know that for all $\tau > \tau^*$, θ^* is increasing in τ (there are negative spillover effects in information choices), implying that $\tau^{**} \leq \tau^*$. Therefore, it has to be the case that $\tau^{**} = \tau^*$. We conclude that when $\mu_\theta = \mu_\theta^E(T)$ investors choose the efficient level of information.

Finally consider the case when $T < \frac{1}{2}$. From Corollary 8 we know that if $\mu_\theta < \widehat{\mu}^\tau(T, \underline{\tau}, \tau_\theta)$ then there are global positive externalities, hence agents under-acquire information. On the other hand, if $\mu_\theta > T$ then there are global negative externalities and agents over-acquire information. ■

3.2 Discussion

Proposition 9 allows us to determine when agents globally over-acquire or under-acquire information. The logic of the proof might be unclear and the reader might wonder why we have not used a more direct approach based on the analysis of first-order and second-order necessary conditions for local and global maxima. Therefore, in this subsection we explain why such an approach is not feasible. We also provide detailed intuition behind our approach and explain why it does not work when $T < \frac{1}{2}$ and $\mu_\theta \in (\widehat{\mu}^\tau(T, \underline{\tau}, \tau_\theta), T)$.

The optimal level of precision solves

$$\max_{\tau \geq \underline{\tau}} B^i(\tau, \tau) - C(\tau)$$

i.e. it has to satisfy the following first order condition:

$$B_1^i(\tau, \tau) - C'(\tau) + B_2^i(\tau, \tau) \leq 0$$

with equality if $\tau^{**} > \underline{\tau}$. The usual brute force approach would be to verify that the objective function is quasiconcave, so that the first-order condition is sufficient and then find the value of τ that satisfies this condition. Finally, we would compare the solution we found this way to the equilibrium precision choice.

This standard approach, however, is not applicable in the model. The first difficulty stems from the fact that the first-order and the second-order conditions are complex objects. In particular, we were not able to verify that the objective function is quasiconcave or even if the solution to the first-order condition, if it exists, is a local maximum. Indeed, our intuition together with numerical examples tend to suggest that both need not be true, i.e. that the utility function, depending on parameters, can take many different shapes and may have a local interior minimum. Moreover, it is possible that the efficient precision choice is a corner solution, but given the lack of a closed-form solution to the model we do not have the tools to determine when this is the case.

In order to circumvent these problems we used an indirect approach. This approach is based on investigating the local and global behavior of θ^* as a function of τ, and builds on the comparative statics results for global games established in Szkup (2014). This approach follows two steps. First, we establish whether a small increase or decrease in the precision from its equilibrium level would increase or decrease welfare. We refer to this as a local over-acquisition or local under-acquisition of information. This is, of course, a much simpler task since we know that at the equilibrium precision we have $B_1^i(\tau,\tau) - C'(\tau) = 0$ and, thus, the local results are fully determined by the sign of $\left|\frac{\partial\theta^*}{\partial\tau}\right|_{\tau=\tau^*}$. In the second step we use the global properties of $\frac{\partial\theta^*}{\partial\tau}$ in order to extend this local result to the planner's precision choice. For example, if θ^* is monotone we know that this local result will automatically imply a global result.

Unfortunately, this approach fails when $T < \frac{1}{2}$ and μ_θ takes on values between $\Phi\left(\sqrt{\frac{\underline{\tau}}{\underline{\tau}+\tau_\theta}}\Phi^{-1}T\right) + \frac{1}{\sqrt{\underline{\tau}+\tau_\theta}}\Phi^{-1}(T)$ and T. The issue is that when $T < \frac{1}{2}$ then the threshold θ^* is first increasing and then decreasing in τ (Lemma 6). While we can establish whether the equilibrium precision choice lies on the increasing or decreasing portion of θ^* (when we consider θ^* as a function of τ) and we still can obtain our local results, these local results do not necessarily describe the efficient choice of information. In particular, when τ^* lies on the decreasing portion of θ^* the local behavior of the objective function indicates that a small increase in precision would improve welfare compared to the equilibrium. However, it is possible that the actual efficient choice of precision is smaller than equilibrium. θ^* might be initially increasing very fast, and then decreasing very gradually, in which case the planner would choose to acquire no information. Without closed-form solutions we are unable to determine when such cases arise. Note that this is not a problem when $T > \frac{1}{2}$ because in this case θ^*, if non-monotonic, is first decreasing and then increasing.

3.3 Numerical Results

In this subsection we provide a concrete numerical example which shows why the case $T < \frac{1}{2}$ is so problematic. Our goal is to show that the difficulties described above not only are a theoretical possibility that cannot be excluded, but do actually arise for some parameter values.

The social planner's objective function can be expressed as

$$U^i(\tau;\tau) = -\int_{-\infty}^{\theta^*}\int_{x^*}^{\infty} T dF_\tau(x|\theta)\, dG_{\tau_\theta}(\theta) - \int_{\theta^*}^{\infty}\int_{-\infty}^{x^*}(1-T)\, dF_\tau(x|\theta)\, dG_{\tau_\theta}(\theta) + \int_{\theta^*}^{\infty}(1-T)\, dG_{\tau_\theta}(\theta) - C(\tau),$$

i.e., social planner's objective function is the utility of investor i given that all investors choose the same precision level $\tau_i = \tau$. Thus, Planner's objective function can be decomposed into three terms: (1) the expected cost of mistakes (the first two terms), (2) the expected gain from investment (the third term), and (3) the cost of precision (the last term).

We set $T = 0.25$ and $\mu_\theta = 0.0755$. The remaining parameters are $\underline{\tau} = 3$, $C(\tau) = \frac{1}{200}(\tau - \underline{\tau})^{1.01}$, and $\tau_\theta = 3$.[5] Figure 1 depicts, as a function of precision choice τ, the welfare (the top left panel), the expected gain from investment (the top right panel), the expected cost of mistakes (the bottom left panel), and the cost of precision (the bottom right panel). From Figure 1 we can see that for these parameters the welfare function is not only not concave, but also not even quasiconcave (the top left

[5] The welfare function is non-quasiconcave also for values of parameters in the neighborhood of the parameters considered in this example. The reason for this result is that the cost function is sufficiently flat and the expected gain from investment initially decreases sharply as τ increases.

panel). This implies that the standard approach based on first-order and second-order conditions is not applicable.

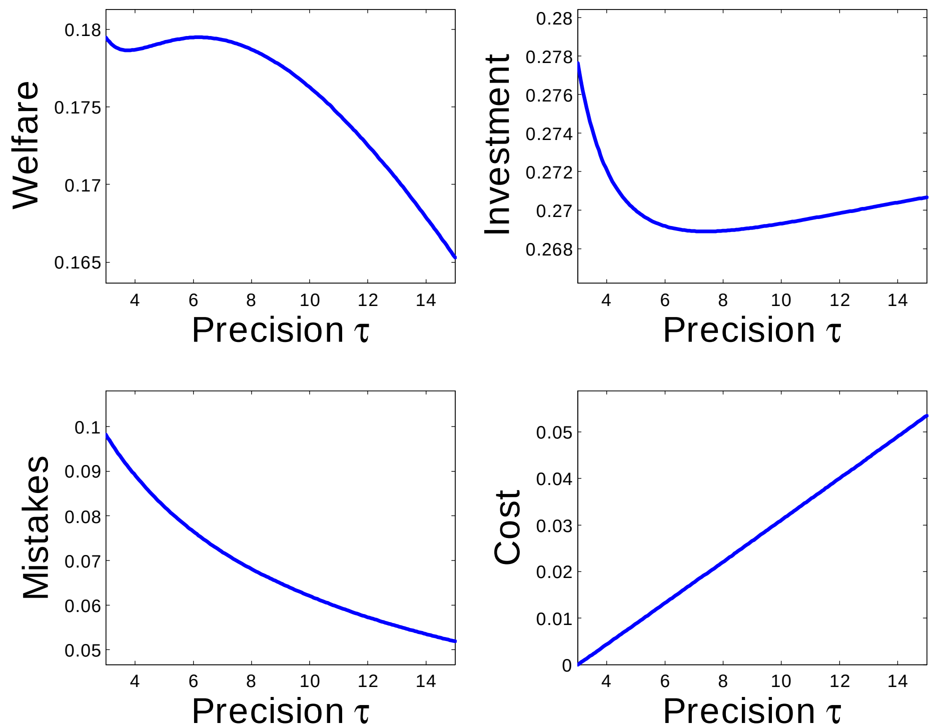

Figure 1: Welfare as a function of τ

The remaining panels provide explanation why the welfare function is not quasiconcave.[6] In particular, we see that an initial increase in τ leads to a sharp decrease in the expected gain from investment (which is driven by a sharp increase in θ^*). This implies that welfare is initially decreasing as τ increases from $\underline{\tau}$. However, as τ increases further, the negative effect of a higher τ on investment decreases sharply as θ^* approaches its global maximum. On the other hand, higher precision helps agent avoid costly mistakes (the bottom left panel). As τ increases, this benefit due to lower expected cost of mistakes dominates and the welfare function becomes increasing in τ. As τ increases further, the reduction in the expected cost of mistake becomes smaller and smaller. As such, the welfare function becomes again decreasing in τ, driven by the increasing cost of higher precision (the bottom right panel).

It is important to note that for the welfare function to be non-quasiconcave it is crucial that θ^* achieves a global maximum at a finite τ. Only then the rate at which θ^* increases falls sufficiently fast that the positive effect of a decrease in the expected cost of mistake becomes a dominant force for some values of τ.

[6] Note that, for each τ, by subtracting from the value of investment the value of mistakes and the information cost, we obtain the value of welfare corresponding to this precision level.

4 Strategic Complementarities and Welfare

Lemma 10 *As $\tau \to \infty$ the threshold $\theta^* \to T$.*

Proof. Recall that θ^* is the unique solution to the critical mass condition

$$\frac{\tau_\theta}{\tau^{1/2}}\left(\theta^* - \mu_\theta\right) + \sqrt{\frac{\tau + \tau_\theta}{\tau}}\Phi^{-1}\left(T\right) - \Phi^{-1}\left(\theta^*\right) = 0$$

or

$$\Phi^{-1}\left(\theta^*\right) - \frac{\tau_\theta}{\tau^{1/2}}\left(\theta^* - \mu_\theta\right) = \sqrt{\frac{\tau + \tau_\theta}{\tau}}\Phi^{-1}\left(T\right)$$

Taking the limit as $\tau \to \infty$ on both sides of this equation, we obtain

$$\lim_{\tau \to \infty} \Phi^{-1}\left(\theta^*\right) = \Phi^{-1}\left(T\right)$$

or,

$$\lim_{\tau \to \infty} \theta^* = T$$

■

Lemma 11 *θ^* is a decreasing function of μ_θ.*

Proof. Consider the CM condition:

$$\frac{\tau_\theta}{\tau^{1/2}}\left(\theta^* - \mu_\theta\right) + \sqrt{\frac{\tau + \tau_\theta}{\tau}}\Phi^{-1}\left(T\right) - \Phi^{-1}\left(\theta^*\right) = 0$$

Recall also that

$$\frac{\tau^{1/2}}{\tau_\theta} > \frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)}$$

for all $\tau \in [\underline{\tau}, \infty)$. Applying the Implicit Function Theorem to the CM condition we obtain

$$\frac{\partial \theta^*}{\partial \mu_\theta} = \frac{\frac{\tau_\theta}{\tau^{1/2}}}{\frac{\tau_\theta}{\tau^{1/2}} - \frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)}} < 0$$

since $\frac{\tau_\theta}{\tau^{1/2}} > \sqrt{2\pi}$. ■

Lemma 12 *Let $T = \frac{1}{2}$ and consider θ^*.*

1. *If $\mu_\theta < \frac{1}{2}$ then $\theta^* > \frac{1}{2}$, $\frac{\partial \theta^*}{\partial \tau_\theta} > 0$ and $\frac{\partial \theta^*}{\partial \tau} < 0$;*
2. *If $\mu_\theta = \frac{1}{2}$ then $\theta^* = \frac{1}{2}$, $\frac{\partial \theta^*}{\partial \tau_\theta} = 0$ and $\frac{\partial \theta^*}{\partial \tau} = 0$;*
3. *If $\mu_\theta > \frac{1}{2}$ then $\theta^* < \frac{1}{2}$, $\frac{\partial \theta^*}{\partial \tau_\theta} < 0$ and $\frac{\partial \theta^*}{\partial \tau} > 0$.*

Proof. When $T=\frac{1}{2}$ the CM condition becomes

$$\frac{\tau_\theta}{\tau^{1/2}}\left(\theta^*-\mu_\theta\right)-\Phi^{-1}\left(\theta^*\right)=0$$

Since $\frac{\tau^{1/2}}{\tau_\theta}>\frac{1}{\sqrt{2\pi}}$, the left hand side of the above equation is decreasing in θ^*. Moreover, it is easy to see that when $\mu_\theta=\frac{1}{2}$ the above equation implies that $\theta^*=\frac{1}{2}$. It then follows that when $\mu_\theta>\frac{1}{2}$ it has to be the case that $\theta^*<\frac{1}{2}$ and when $\mu_\theta<\frac{1}{2}$ then $\theta^*>\frac{1}{2}$. Using the above observations and applying the Implicit Function Theorem to the CM condition we obtain the remaining results regarding $\frac{\partial\theta^*}{\partial\tau_\theta}$ and $\frac{\partial\theta^*}{\partial\tau}$. ■

Lemma 13 *Let $T=\frac{1}{2}$ and consider $\frac{\partial\tau^*}{\partial\mu_\theta}$.*

1. *If $\mu_\theta<\frac{1}{2}$ then $\frac{\partial\tau^*}{\partial\mu_\theta}>0$;*
2. *If $\mu_\theta=\frac{1}{2}$ then $\frac{\partial\tau^*}{\partial\mu_\theta}=0$;*
3. *If $\mu_\theta>\frac{1}{2}$ then $\frac{\partial\tau^*}{\partial\mu_\theta}<0$;*

Proof. Note that

$$\frac{d\tau^*}{d\mu_\theta}=\frac{1}{1-\left.\frac{\partial\tau_i^*}{\partial\tau}\right|_{\tau=\tau^*}}\left.\frac{\partial\tau_i^*}{\partial\mu_\theta}\right|_{\tau_i=\tau^*}$$

where $\tau_i^*\left(\tau,\mu_\theta\right)$ is the optimal precision choice of investor i when the mean of the prior is equal to μ_θ and all other investors choose precision τ. The above expression tells us that the derivative of the equilibrium precision choice with respect to μ_θ is equal to the change in investor i's equilibrium precision choice due to change in μ_θ, captured by $\frac{\partial\tau_i^*}{\partial\mu_\theta}$, times the "multiplier effect" that captures the fact that μ_θ affects equilibrium precision choices of the remaining investors, leading to a further adjustment in τ_i^*. Finally, since we are interested in studying a change from an initial symmetric equilibrium precision, these effects have to be evaluated at $\tau_i=\tau^*$ and $\tau=\tau^*$. As shown in the proof of Theorem 1 in the appendix of the paper, the equilibrium multiplier effect is always positive. Thus, to establish whether an increase in μ_θ leads to an increase or decrease in τ^* we focus on the partial effect of a change in μ_θ has on τ_i^*, i.e. $\left.\frac{\partial\tau_i^*}{\partial\mu_\theta}\right|_{\tau_i=\tau^*}$.

Applying the Implicit Function Theorem to investor i's first order condition we obtain

$$\frac{\partial\tau_i^*}{\partial\mu_\theta}=-\frac{\frac{\partial^2U^i}{\partial\tau_i\partial\mu_\theta}}{\frac{\partial^2U^i}{\partial\tau_i^2}}$$

where the denominator is always negative (see Lemma 2 above).

$$\frac{\partial^2U^i}{\partial\tau_i\partial\mu_\theta}=-\frac{1}{\tau_i}\frac{1}{\tau_i+\tau_\theta}\tau_i^{1/2}\phi\left(\frac{x_i^*-\theta^*}{\tau_i^{-1/2}}\right)\tau_\theta^{1/2}\phi\left(\frac{\theta^*-\mu_\theta}{\tau_\theta^{-1/2}}\right)\tau_\theta\left(x_i^*-\mu_\theta\right)\left(\frac{\partial\theta^*}{\partial\mu_\theta}-1\right).$$

Since $\frac{\partial\theta^*}{\partial\mu_\theta}<0$, it follows that:

$$sgn\left(\frac{\partial^2U^i}{\partial\tau_i\partial\mu_\theta}\right)=sgn\left(x_i^*-\mu_\theta\right)$$

Next, note that when $T = \frac{1}{2}$ then

$$x_i^* - \mu_\theta = \frac{\tau + \tau_\theta}{\tau} \left(\theta^* - \mu_\theta\right)$$

It follows then from Lemma 12 that if $\mu_\theta < \frac{1}{2}$ then $x_i^* - \mu_\theta > 0$, if $\mu_\theta = \frac{1}{2}$ then $x_i^* - \mu_\theta = 0$ and if $\mu_\theta > \frac{1}{2}$ then $x_i^* - \mu_\theta < 0$. The results follows immediately from this observation. ■

Lemma 14 *Let $T = \frac{1}{2}$ and consider $\frac{\partial^2 \theta^*}{\partial \tau_\theta \partial \mu_\theta}$. For all $\mu_\theta \in (\mu^-, \mu^+)$ we have $\frac{\partial^2 \theta^*}{\partial \tau_\theta \partial \mu_\theta} < 0$ where $\mu^- < \frac{1}{2} < \mu^+$ are the two solutions to the equation*

$$\left(\theta^* - \mu_\theta\right)^2 = \frac{\tau - \tau_\theta}{\tau_\theta \left(\tau_\theta + \tau\right)}.$$

Proof. The derivative of θ^* with respect to μ_θ is equal to

$$\frac{\partial \theta^*}{\partial \mu_\theta} = \frac{\frac{\tau_\theta}{\tau^{1/2}}}{\frac{\tau_\theta}{\tau^{1/2}} - \frac{1}{\phi(\Phi^{-1}(\theta^*))}}$$

Differentiating again, this time with respect to τ_θ, we get

$$\begin{aligned} \frac{\partial^2 \theta^*}{\partial \tau_\theta \partial \mu_\theta} &= \frac{\frac{1}{\tau^{1/2}} \left[\frac{\tau_\theta}{\tau^{1/2}} - \frac{1}{\phi(\Phi^{-1}(\theta^*))}\right] - \frac{\tau_\theta}{\tau^{1/2}} \left[\frac{1}{\tau^{1/2}} - \frac{\Phi^{-1}(\theta^*)}{(\phi(\Phi^{-1}(\theta^*)))^2} \frac{\partial \theta^*}{\partial \tau_\theta}\right]}{\left[\frac{\tau_\theta}{\tau^{1/2}} - \frac{1}{\phi(\Phi^{-1}(\theta^*))}\right]^2} \\ &= \frac{-\frac{1}{\tau^{1/2}} \frac{1}{\phi(\Phi^{-1}(\theta^*))} + \frac{\tau_\theta}{\tau^{1/2}} \frac{\Phi^{-1}(\theta^*)}{(\phi(\Phi^{-1}(\theta^*)))^2} \frac{\partial \theta^*}{\partial \tau_\theta}}{\left[\frac{\tau_\theta}{\tau^{1/2}} - \frac{1}{\phi(\Phi^{-1}(\theta^*))}\right]^2} \end{aligned}$$

Since the denominator is always positive, we focus on the numerator of the above expression.

$$\begin{aligned} & -\frac{1}{\tau^{1/2}} \frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)} + \frac{\tau_\theta}{\tau^{1/2}} \frac{\Phi^{-1}\left(\theta^*\right)}{\left(\phi\left(\Phi^{-1}\left(\theta^*\right)\right)\right)^2} \frac{\partial \theta^*}{\partial \tau_\theta} \\ = \; & \frac{1}{\tau^{1/2}} \frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)} \left[-1 + \tau_\theta \frac{\Phi^{-1}\left(\theta^*\right)}{\left(\phi\left(\Phi^{-1}\left(\theta^*\right)\right)\right)} \frac{\partial \theta^*}{\partial \tau_\theta}\right] \end{aligned}$$

When $T = \frac{1}{2}$, the CM condition implies that $\Phi^{-1}\left(\theta^*\right) = \frac{\tau_\theta}{\tau^{1/2}} \left(\theta^* - \mu_\theta\right)$. Moreover,

$$\frac{\partial \theta^*}{\partial \tau_\theta} = \frac{-\frac{1}{\tau^{1/2}} \left(\theta^* - \mu_\theta\right)}{\frac{\tau_\theta}{\tau^{1/2}} - \frac{1}{\phi(\Phi^{-1}(\theta^*))}}$$

Therefore, the numerator can be written as

$$\frac{1}{\tau^{1/2}} \frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)} \left[-1 - \frac{\tau_\theta^2}{\tau^{1/2}} \frac{\left(\theta^* - \mu_\theta\right)^2}{\left(\phi\left(\Phi^{-1}\left(\theta^*\right)\right)\right)} \frac{1}{\tau^{1/2}} \left(\frac{\tau_\theta}{\tau^{1/2}} - \frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)}\right)^{-1}\right]$$

We argue below that the term in the square brackets is always negative.

Note that $\theta^* = \mu_\theta$ if and only if $\mu_\theta = \frac{1}{2}$ and θ^* is decreasing in μ_θ, it follows that for all $\mu_\theta \in (\mu^-, \mu^+)$

$$\left(\theta^* - \mu_\theta\right)^2 \leq \left(\theta^*\left(\mu^-\right) - \mu^-\right)^2 = \left(\theta^*\left(\mu^+\right) - \mu^+\right)^2 = \frac{\tau - \tau_\theta}{\tau_\theta \left(\tau + \tau_\theta\right)}$$

Thus,

$$
\begin{aligned}
& -1-\frac{\tau_\theta^2}{\tau}\frac{\left(\theta^*-\mu_\theta\right)^2}{\left(\phi\left(\Phi^{-1}\left(\theta^*\right)\right)\right)}\left(\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)}\right)^{-1} \\
< \quad & -\left(\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)}\right)^{-1}\left[\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)}+\frac{\tau_\theta^2}{\tau}\frac{\frac{\tau-\tau_\theta}{\tau_\theta(\tau+\tau_\theta)}}{\left(\phi\left(\Phi^{-1}\left(\theta^*\right)\right)\right)}\right] \\
= \quad & -\frac{\tau_\theta}{\tau^{1/2}}\frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)}\left(\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)}\right)^{-1}\left[\phi\left(\Phi^{-1}\left(\theta^*\right)\right)-\frac{\tau^{1/2}}{\tau_\theta}+\frac{1}{\tau^{1/2}}\frac{\tau-\tau_\theta}{(\tau+\tau_\theta)}\right] \\
< \quad & -\frac{\tau_\theta}{\tau^{1/2}}\frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)}\left(\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)}\right)^{-1}\left[\frac{1}{\sqrt{2\pi}}-\frac{\tau^{1/2}}{\tau_\theta}\left(1-\frac{\tau_\theta}{\tau}\frac{\tau-\tau_\theta}{(\tau+\tau_\theta)}\right)\right]
\end{aligned}
$$

where the first inequality followed from the fact that $\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi(\Phi^{-1}(\theta^*))}<0$ and observation that $\left(\theta^*-\mu_\theta\right)^2<\frac{\tau-\tau_\theta}{\tau_\theta(\tau+\tau_\theta)}$. Therefore, as long as $\underline{\tau}$ satisfies

$$
\frac{\underline{\tau}^{1/2}}{\overline{\tau}_\theta}\left(1-\frac{\overline{\tau}_\theta}{\underline{\tau}}\frac{\underline{\tau}-\underline{\tau}_\theta}{(\underline{\tau}+\underline{\tau}_\theta)}\right)>\frac{1}{\sqrt{2\pi}}
$$

then for all $\mu_\theta\in(\mu^-,\mu^+)$ we have $\frac{\partial^2\theta^*}{\partial\tau_\theta\partial\mu_\theta}<0$. ■

Lemma 15 *Consider* $\frac{\partial^2\theta^*}{\partial\tau_\theta\partial\tau^*}$.

1. *If* $\mu_\theta<\frac{1}{2}$ *then* $\frac{\partial^2\theta^*}{\partial\tau_\theta\partial\tau^*}<0$
2. *If* $\mu_\theta=\frac{1}{2}$ *then* $\frac{\partial^2\theta^*}{\partial\tau_\theta\partial\tau^*}=0$
3. *If* $\mu_\theta>\frac{1}{2}$ *then* $\frac{\partial^2\theta^*}{\partial\tau_\theta\partial\tau^*}>0$

Proof. The derivative of θ^* with respect to τ is given by:

$$
\frac{\partial\theta^*}{\partial\tau}=\frac{\frac{1}{2}\frac{\tau_\theta}{\tau^{3/2}}\left(\theta^*-\mu_\theta\right)}{\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi(\Phi^{-1}(\theta^*))}}
$$

Differentiating the above equation with respect to τ_θ we get:

$$
\begin{aligned}
\frac{\partial^2\theta^*}{\partial\tau\partial\tau_\theta} \quad = \quad & \frac{\left[\frac{1}{2}\frac{1}{\tau^{3/2}}\left(\theta^*-\mu_\theta\right)+\frac{1}{2}\frac{\tau_\theta}{\tau^{3/2}}\frac{\partial\theta^*}{\partial\tau_\theta}\right]\left[\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi(\Phi^{-1}(\theta^*))}\right]}{\left[\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi(\Phi^{-1}(\theta^*))}\right]^2} \\
& -\frac{\left[\frac{1}{2}\frac{\tau_\theta}{\tau^{3/2}}\left(\theta^*-\mu_\theta\right)\right]\left[\frac{1}{\tau^{1/2}}+\frac{\Phi^{-1}(\theta^*)}{(\phi(\Phi^{-1}(\theta^*)))^2}\frac{\partial\theta^*}{\partial\tau_\theta}\right]}{\left[\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi(\Phi^{-1}(\theta^*))}\right]^2}
\end{aligned}
$$

Note that when $\mu_\theta=\frac{1}{2}$ then $\theta^*=\mu_\theta=\frac{1}{2}$, $\frac{\partial\theta^*}{\partial\tau_\theta}=0$. It follows that $\frac{\partial^2\theta^*}{\partial\tau\partial\tau_\theta}=0$.

Next, consider the case when $\mu_\theta<\frac{1}{2}$. In that case, $\theta^*>\frac{1}{2}>\mu_\theta$ and $\frac{\partial\theta^*}{\partial\tau_\theta}>0$. Since $\frac{\tau_\theta}{\tau^{1/2}}-\frac{1}{\phi(\Phi^{-1}(\theta^*))}<0$ it follows that $\frac{\partial^2\theta^*}{\partial\tau\partial\tau_\theta}<0$. Similarly, when $\mu_\theta>\frac{1}{2}$ then $\theta^*<\frac{1}{2}<\mu_\theta$ and $\frac{\partial\theta^*}{\partial\tau_\theta}<0$, and, thus $\frac{\partial^2\theta^*}{\partial\tau\partial\tau_\theta}>0$. ■

4.1 The trade-off between public and private information when $T \neq \frac{1}{2}$

When analyzing the effect of an increase in the precision of public information on private information acquisition we assumed that $T = \frac{1}{2}$. This assumption simplified the analysis and allowed us to characterize fully when an increase in the precision of public information encourages and when it discourages private information acquisition. Here we consider the trade-off between public and private information when $T \neq 1/2$. In particular, we provide sufficient conditions for the private and public information to be complements and substitutes. We complement our analytical results with results obtained from numerical simulations. Numerical simulations confirm our intuition that the passive information effect is the key for determining whether private and public information are complements or substitutes.

Proposition 16 *There exists T_L and T_H, $0 < T_L < \frac{1}{2} < T_H < 1$ such that for each $T \in (T_L, T_H)$ there exists a non-empty interval of μ_θ, call it $I(T)$, such that for all $\mu_\theta \in I(T)$ public and private information are complements.*

This result extends Proposition 5 in the paper that characterizes complementarity between public and private information for the special case when $T = \frac{1}{2}$. The proof consists of three steps and is similar to the proof of Proposition 5. First, we first fix T and show that whether private and public information are complements depends on the relative strengths of three effects: the passive information effect, the active information effect and the coordination effect. Next, we investigate separately when each of these effects is positive (encouraging information acquisition) and when they are negative (discouraging information acquisition). Finally, we show that there exists an interval (T_L, T_H) such that for each $T \in (T_L, T_H)$ there is an interval of values for μ_θ where all these effects are positive.

Proof. As shown in the proof of Proposition 5 in the paper, whether private and public information are complements depends on the sign of $\left.\frac{\partial^2 U_i}{\partial \tau_i \delta \tau_\theta}\right|_{\tau_i=\tau^*}$ which is given by

$$\begin{aligned}\left.\frac{\partial^2 U_i}{\partial \tau_i \delta \tau_\theta}\right|_{\tau_i=\tau^*} = & -\frac{1}{2}\frac{1}{\tau^*}\frac{1}{\tau^*+\tau_\theta}{\tau^*}^{1/2}\phi\left(\frac{x^*-\theta^*}{{\tau^*}^{-1/2}}\right)\tau_\theta^{1/2}\phi\left(\frac{\theta^*-\mu_\theta}{\tau_\theta^{-1/2}}\right) \\ & \left[\underbrace{\frac{1}{\tau^*+\tau_\theta}-\frac{1}{2\tau_\theta}+\frac{1}{2}(\theta-\mu_\theta)^2}_{\text{the passive information effect}} + \underbrace{\tau^*(x^*-\theta^*)\left.\frac{\partial x_i^*}{\partial \tau_\theta}\right|_{\tau_i=\tau^*}}_{\text{the active information effect}} + \underbrace{\tau_\theta(x^*-\mu_\theta)\left.\frac{\partial \theta^*}{\partial \tau_\theta}\right|_{\tau=\tau^*}}_{\text{the coordination effect}}\right]\end{aligned}$$

We consider first the passive information effect. Following an analogous approach as in the proof of Proposition 5 one can show that for each $T \in (0,1)$ there exists $\widehat{\mu}_\theta^-(T)$ and $\widehat{\mu}_\theta^+(T)$, with $\widehat{\mu}_\theta^-(T) < \widehat{\mu}_\theta^+(T)$ such that for all $\mu_\theta \in \left(\widehat{\mu}_\theta^-(T), \widehat{\mu}_\theta^+(T)\right)$ the passive information effect is positive.[7] The thresholds $\widehat{\mu}_\theta^-(T)$

[7] The difference with the case of $T = \frac{1}{2}$ is that now we cannot claim that for all $\mu_\theta \notin \left(\widehat{\mu}_\theta^-(T), \widehat{\mu}_\theta^+(T)\right)$ the passive information effect is negative. Instead, we can only show that there exist $\widehat{\widehat{\mu}}_\theta^-(T)$ and $\widehat{\widehat{\mu}}_\theta^+(T)$, with $\widehat{\widehat{\mu}}_\theta^-(T) > \widehat{\mu}_\theta^-(T)$ and $\widehat{\mu}_\theta^+(T) < \widehat{\widehat{\mu}}_\theta^+(T)$, such that for all $\mu_\theta \in \left(-\infty, \widehat{\widehat{\mu}}_\theta^-(T)\right] \cup \left[\widehat{\widehat{\mu}}_\theta^+(T), \infty\right)$ the passive information effect is negative. See Proposition 17.

is defined implicitly as the largest solution to the equation[8]

$$\mu_\theta = \Phi\left(\frac{\tau_\theta}{\tau(\mu_\theta)^{1/2}}\sqrt{\frac{\tau(\mu_\theta)-\tau_\theta}{\tau_\theta(\tau(\mu_\theta)+\tau_\theta)}} + \sqrt{\frac{\tau(\mu_\theta)+\tau_\theta}{\tau(\mu_\theta)}}\phi^{-1}(T)\right) - \sqrt{\frac{\tau(\mu_\theta)-\tau_\theta}{\tau_\theta(\tau(\mu_\theta)+\tau_\theta)}}$$

while $\widehat{\mu}_\theta^+(T)$ is defined as the smallest solution to the equation

$$\mu_\theta = \Phi\left(-\frac{\tau_\theta}{\tau(\mu_\theta)^{1/2}}\sqrt{\frac{\tau(\mu_\theta)-\tau_\theta}{\tau_\theta(\tau(\mu_\theta)+\tau_\theta)}} + \sqrt{\frac{\tau(\mu_\theta)+\tau_\theta}{\tau(\mu_\theta)}}\phi^{-1}(T)\right) + \sqrt{\frac{\tau(\mu_\theta)-\tau_\theta}{\tau_\theta(\tau(\mu_\theta)+\tau_\theta)}}$$

For each $T \in (0,1)$ let $I_P(T) = \left(\widehat{\mu}_\theta^-(T), \widehat{\mu}_\theta^+(T)\right)$. Thus, $I_P(T)$ is the set of values for μ_θ for a fixed T such that the passive information effect is positive (i.e., it encourages information acquisition).

Next, consider the active information effect and note that the active information effect encourages private information acquisition if and only if

$$(x^* - \theta^*)\left.\frac{\partial x_i^*}{\partial \tau_\theta}\right|_{\tau_i=\tau^*} < 0$$

where[9]

$$\left.\frac{\partial x_i^*}{\partial \tau_\theta}\right|_{\tau_i=\tau^*} = \frac{1}{\tau^*}(\theta^* - \mu_\theta) + \frac{1}{2\tau^*\sqrt{\tau^*+\tau_\theta}}\Phi^{-1}(T),$$

so that

$$\left.\frac{\partial x_i^*}{\partial \tau_\theta}\right|_{\tau_i=\tau^*} < 0 \text{ iff } \theta^* < \mu_\theta - \frac{1}{2}\frac{1}{\sqrt{\tau^*+\tau_\theta}}\Phi^{-1}(T).$$

Then, from the critical mass condition it follows that $\partial x_i^*/\partial\tau_\theta|_{\tau_i=\tau^*} < 0$ iff

$$\mu_\theta > \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta + \tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T).$$

From inspecting the left hand side and the right hand side of the above inequality we conclude that the above inequality is satisfied for sufficiently high μ_θ while for sufficiently low μ_θ the above inequality is reversed. Moreover, since both sides are continuous in μ_θ and the right-hand side is bounded as a function of μ_θ, it follows that there exists at least one value of μ_θ such that

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta + \tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

Let $\widetilde{\mu}_\theta(T)$ denote the largest value of μ_θ that satisfies the above equality when $T > \frac{1}{2}$ and the smallest solution when $T < \frac{1}{2}$. Then, for $T > \frac{1}{2}$ whenever $\mu_\theta > \widetilde{\mu}_\theta(T)$ we have $\partial x_i^*/\partial\tau_\theta|_{\tau_i=\tau^*} < 0$ while for $T < \frac{1}{2}$ we know that whenever $\mu_\theta < \widetilde{\mu}_\theta(T)$ then $\partial x_i^*/\partial\tau_\theta|_{\tau_i=\tau^*} > 0$

[8] To see this solve

$$\frac{1}{\tau^*+\tau_\theta} - \frac{1}{2\tau_\theta} + \frac{1}{2}(\theta - \mu_\theta)^2 = 0$$

for θ^*. This delivers two solutions for θ^*, one for the case when $\theta^* > \mu_\theta$ and the other one for the case when the opposite is true, i.e., $\theta^* < \mu_\theta$. Substituting the first solution into the critical mass condition and rearranging yields the equation that defines $\widehat{\mu}_\theta^-$. Following the same steps for the case when $\theta^* < \mu_\theta$ yields the equations that defines $\widehat{\mu}_\theta^+$.

[9] Here, we ignore the effect of a change in τ_θ on θ^*. This effect is captured by the coordination effect.

Next, consider equation

$$\mu_\theta = \frac{1}{2} + \frac{\sqrt{\tau^*(\mu_\theta) + \tau_\theta}}{\tau_\theta} \Phi^{-1}(T)$$

Again, it is immediate that the above equation has at least one solution. Moreover, it can be shown that the solution to the above equation is unique. Call the unique solution of the above equation $\overline{\mu}_\theta(T)$. Using the critical mass and payoff indifference conditions one can establish that $x^* - \theta^* > 0$ if $\mu_\theta > \overline{\mu}_\theta(T)$, $x^* - \theta^* = 0$ if $\mu_\theta = \overline{\mu}_\theta(T)$ and $x^* - \theta^* < 0$ if $\mu_\theta < \overline{\mu}_\theta(T)$.

We claim next that for all $T > \frac{1}{2}$ we have $\overline{\mu}_\theta(T) > \widetilde{\mu}_\theta(T)$ while for all $T < \frac{1}{2}$ we have $\overline{\mu}_\theta(T) < \widetilde{\mu}_\theta(T)$. To see this fix τ and note that at $T = \frac{1}{2}$ we have

$$\left[\frac{1}{2} + \frac{\sqrt{\tau + \tau_\theta}}{\tau_\theta}\Phi^{-1}(T)\right] - \left[\Phi\left(\sqrt{\frac{\tau + \tau_\theta}{\tau}} \frac{\frac{1}{2}\tau_\theta + \tau}{\tau + \tau_\theta}\Phi^{-1}(T)\right) - \frac{1}{2}\frac{1}{\sqrt{\tau + \tau_\theta}}\Phi^{-1}(T)\right] = 0$$

Moreover,

$$\begin{aligned}
&\frac{\partial}{\partial T}\left[\frac{1}{2} + \frac{\sqrt{\tau + \tau_\theta}}{\tau_\theta}\Phi^{-1}(T) - \Phi\left(\sqrt{\frac{\tau + \tau_\theta}{\tau}} \frac{\frac{1}{2}\tau_\theta + \tau}{\tau + \tau_\theta}\Phi^{-1}(T)\right) - \frac{1}{2}\frac{1}{\sqrt{\tau + \tau_\theta}}\Phi^{-1}(T)\right] \\
= \;& \frac{\sqrt{\tau + \tau_\theta}}{\tau_\theta}\frac{1}{\phi(\Phi^{-1}(T))} - \phi\left(\sqrt{\frac{\tau + \tau_\theta}{\tau}} \frac{\frac{1}{2}\tau_\theta + \tau}{\tau + \tau_\theta}\Phi^{-1}(T)\right)\sqrt{\frac{\tau + \tau_\theta}{\tau}} \frac{\frac{1}{2}\tau_\theta + \tau}{\tau + \tau_\theta}\frac{1}{\phi(\Phi^{-1}(T))} \\
& -\frac{1}{2}\frac{1}{\sqrt{\tau + \tau_\theta}}\frac{1}{\phi(\Phi^{-1}(T))} \\
\geq \;& \frac{1}{\sqrt{\tau + \tau_\theta}}\frac{1}{\phi(\Phi^{-1}(T))}\left[\frac{\tau + \tau_\theta}{\tau_\theta} - \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\tau + \tau_\theta}{\tau}}\frac{\frac{1}{2}\tau_\theta + \tau}{\sqrt{\tau + \tau_\theta}} - \frac{1}{2}\right] \\
= \;& \frac{\left(\frac{1}{2}\tau_\theta + \tau\right)}{\sqrt{\tau}}\frac{1}{\sqrt{\tau + \tau_\theta}}\frac{1}{\phi(\Phi^{-1}(T))}\left[\frac{\sqrt{\tau}}{\tau_\theta} - \frac{1}{\sqrt{2\pi}}\right] > 0
\end{aligned}$$

since we assumed that $\sqrt{\tau}/\tau_\theta > 1/\sqrt{2\pi}$ (see Assumption 1 in the paper). It follows that, for an exogenously given τ, if $T > \frac{1}{2}$ then

$$\frac{1}{2} + \frac{\sqrt{\tau + \tau_\theta}}{\tau_\theta}\Phi^{-1}(T) > \Phi\left(\sqrt{\frac{\tau + \tau_\theta}{\tau}} \frac{\frac{1}{2}\tau_\theta + \tau}{\tau + \tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau + \tau_\theta}}\Phi^{-1}(T)$$

while the opposite inequality holding when $T < \frac{1}{2}$. Thus, if τ was exogenous then $\overline{\mu}_\theta(T) > \widetilde{\mu}_\theta(T)$ for $T > \frac{1}{2}$ and $\overline{\mu}_\theta(T) < \widetilde{\mu}_\theta(T)$ if $T < \frac{1}{2}$. Next, recall that

$$\mu_\theta = \frac{1}{2} + \frac{\sqrt{\tau^*(\mu_\theta) + \tau_\theta}}{\tau_\theta} \Phi^{-1}(T)$$

has a unique solution and note that $\mu_\theta - \frac{1}{2} + \frac{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}{\tau_\theta}\Phi^{-1}(T)$ tends to $-\infty$ as $T \to 0$ and tends to $+\infty$ as $T \to 1$. Putting together the above observations we conclude that $\overline{\mu}_\theta(T)$ is larger than $\widetilde{\mu}_\theta(T)$ for $T > \frac{1}{2}$ and smaller than $\widetilde{\mu}_\theta(T)$ for $T < \frac{1}{2}$.[10]

[10] More precisely, define

$$F(\mu_\theta) \equiv \frac{1}{2} + \frac{\sqrt{\tau^*(\mu_\theta) + \tau_\theta}}{\tau_\theta} \Phi^{-1}(T)$$

Above we have established that if $T > \frac{1}{2}$ then: (1) $\overline{\mu}_\theta(T) > \widetilde{\mu}_\theta(T)$, (2) for all $\mu_\theta < \overline{\mu}_\theta(T)$ we have $x^* - \theta^* < 0$, and (3) for all $\mu_\theta > \widetilde{\mu}_\theta(T)$ we have $\partial x_i^*/\partial\tau_\theta|_{\tau_i=\tau^*} > 0$. Thus, if $T > \frac{1}{2}$ then for all $\mu_\theta \in (\widetilde{\mu}_\theta(T), \overline{\mu}_\theta(T))$ we have $(x^* - \theta^*)\,\partial x_i^*/\partial\tau_\theta|_{\tau_i=\tau^*} < 0$ and, hence, the active information effect is positive. Similarly, if $T < \frac{1}{2}$ then for all $\mu_\theta \in (\overline{\mu}_\theta(T), \widetilde{\mu}_\theta(T))$ we have $(x^* - \theta^*)\,\partial x_i^*/\partial\tau_\theta|_{\tau_i=\tau^*} < 0$ and, hence, the active information effect is positive. Finally, if $T = \frac{1}{2}$ then $\overline{\mu}_\theta(T) = \widetilde{\mu}_\theta(T)$ and the active information effect is always non-positive (as argued in the proof of Proposition 5 in the paper). For each $T \neq \frac{1}{2}$, let $I_A(T)$ denote the interval of μ_θ such that the active information effect is positive.

Finally consider the coordination effect. The sign of the coordination effect is determined by the sign of $(x^* - \mu_\theta)\frac{\partial\theta^*}{\partial\tau_\theta}$. In particular, the coordination effect encourages information if and only if

$$(x^* - \mu_\theta)\frac{\partial\theta^*}{\partial\tau_\theta} < 0.$$

First consider $\partial\theta^*/\partial\tau_\theta$. From Szkup (2014) we know that $\partial\theta^*/\partial\tau_\theta > 0$ whenever

$$\mu_\theta > \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta + \tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

As above, let $\widetilde{\mu}_\theta(T)$ be the largest solution to

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta + \tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

when $T > \frac{1}{2}$, and the smallest solution when $T < \frac{1}{2}$.[11] Given this definition of $\widetilde{\mu}_\theta(T)$ it follows that if $T > 1/2$ and $\mu_\theta > \widetilde{\mu}_\theta(T)$ then $\partial\theta^*/\partial\tau_\theta > 0$ while if $T < 1/2$ and $\mu_\theta < \widetilde{\mu}_\theta(T)$ then $\partial\theta^*/\partial\tau_\theta < 0$.

Next, consider $x^* - \mu_\theta$. Using the Payoff Indifference condition we have

$$\begin{aligned}
x^* - \mu_\theta &> 0 \quad \text{iff} \quad \theta^* > \mu_\theta - \sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\Phi^{-1}(T) \\
x^* - \mu_\theta &= 0 \quad \text{iff} \quad \theta^* = \mu_\theta - \sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\Phi^{-1}(T) \\
x^* - \mu_\theta &< 0 \quad \text{iff} \quad \theta^* < \mu_\theta - \sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\Phi^{-1}(T)
\end{aligned}$$

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and

$$G(\mu_\theta) \equiv \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta + \tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

and suppose that $T > \frac{1}{2}$. Then, for any τ we have:

$$\mu_\theta - F(\mu_\theta) < \mu_\theta - G(\mu_\theta).$$

Thus, at $\widetilde{\mu}_\theta(T)$ we have

$$\widetilde{\mu}_\theta(T) - F(\widetilde{\mu}_\theta(T)) < \widetilde{\mu}_\theta(T) - G(\widetilde{\mu}_\theta(T)) = 0.$$

Since $\mu_\theta = F(\mu_\theta)$ has a unique fixed point and since $\mu_\theta - F(\mu_\theta) < 0$ for sufficiently small μ_θ and $\mu_\theta - F(\mu_\theta) > 0$ for sufficiently large μ_θ, it follows that the unique fixed point of equation $\mu_\theta = F(\mu_\theta)$, that we denote by $\overline{\mu}_\theta(T)$, is larger then $\widetilde{\mu}_\theta(T)$. An analogous argument establishes that for $T < \frac{1}{2}$ we have $\overline{\mu}_\theta(T) < \widetilde{\mu}_\theta(T)$.

[11] Note that $\widetilde{\mu}_\theta(T)$ is continuous in T since at $T = \frac{1}{2}$ the above equation has unique solution, $\widetilde{\mu}_\theta(T) = \frac{1}{2}$.

Using the Critical Mass condition it can be shown that $\theta^* > \mu_\theta - \sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\Phi^{-1}(T)$ if and only if

$$\mu_\theta < \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)\right) + \frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

and $\theta^* = \mu_\theta - \sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\Phi^{-1}(T)$ if and only if

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)\right) + \frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

In Lemma A.2 in the paper we showed that the above equation has a unique solution, which we call $\mu_\theta^E(T)$, such that if $\mu_\theta > \mu_\theta^E(T)$ then

$$\mu_\theta > \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)\right) + \frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

while the opposite is true if $\mu_\theta < \mu_\theta^E(T)$. Therefore, it follows that if $\mu_\theta < \mu_\theta^E(T)$ then $x^* - \mu_\theta > 0$, if $\mu_\theta = \mu_\theta^E(T)$ then $x^* - \mu_\theta = 0$, and if $\mu_\theta > \mu_\theta^E(T)$ then $x^* - \mu_\theta < 0$.

Next, following analogous steps as in the case of the active information effect one can show that if $T > \frac{1}{2}$ then $\mu_\theta^E(T) > \widetilde{\mu}_\theta(T)$, if $T = \frac{1}{2}$ then $\mu_\theta^E(T) = \widetilde{\mu}_\theta(T)$ and if $T < \frac{1}{2}$ then $\mu_\theta^E(T) < \widetilde{\mu}_\theta(T)$. Thus, we have established that if $T > \frac{1}{2}$ then: (i) $\mu_\theta^E(T) > \widetilde{\mu}_\theta(T)$, (ii) for all $\mu_\theta > \widetilde{\mu}_\theta(T)$ we have $\partial\theta^*/\partial\tau_\theta|_{\tau=\tau^*} < 0$, and (iii) for all $\mu_\theta < \mu_\theta^E(T)$ we have $x^* - \mu_\theta > 0$. Thus, if $T > \frac{1}{2}$ then for all $\mu_\theta \in \left(\widetilde{\mu}_\theta(T), \mu_\theta^E(T)\right)$ we have $(x^* - \mu_\theta)\,\partial\theta^*/\partial\tau_\theta|_{\tau=\tau^*} < 0$ and, hence, the coordination effect if positive. Similarly, if $T < \frac{1}{2}$ then: (i) $\mu_\theta^E(T) < \widetilde{\mu}_\theta(T)$, (ii) for all $\mu_\theta < \widetilde{\mu}_\theta(T)$ we have $\partial\theta^*/\partial\tau_\theta|_{\tau=\tau^*} > 0$, and (iii) for all $\mu_\theta > \mu_\theta^E(T)$ we have $x^* - \mu_\theta < 0$. It follows that if $T < \frac{1}{2}$ then for all $\mu_\theta \in \left(\widetilde{\mu}_\theta(T), \mu_\theta^E(T)\right)$ we have $(x^* - \mu_\theta)\,\partial\theta^*/\partial\tau_\theta|_{\tau=\tau^*} < 0$ and, hence, the coordination effect is positive. Finally, if $T = \frac{1}{2}$ then $\mu_\theta^E(T) = \widetilde{\mu}_\theta(T)$ and the coordination effect is always non-positive (as argued in the proof of Proposition 5 in the paper). For each $T \neq \frac{1}{2}$ denote by $I_C(T)$ the interval of μ_θ such that the coordination effect is positive.

The final step of the proof requires us to show that there exists an interval (T_L, T_H) such that for each $T \in (T_L, T_H)$, the set $I(T) \equiv I_P(T) \cap I_A(T) \cap I_C(T)$ has a non-empty interior. First note that for $T > \frac{1}{2}$ we have $\left(\widetilde{\mu}_\theta(T), \mu_\theta^E(T)\right) \subset I_A(T) \cap I_C(T)$ and for $T < \frac{1}{2}$ we have $\left(\mu_\theta^E(T), \widetilde{\mu}_\theta(T)\right) \subset I_A(T) \cap I_C(T)$.[12] Thus, it remains to consider the intersection of $I_P(T)$ with $\left(\widetilde{\mu}_\theta(T), \mu_\theta^E(T)\right)$. More precisely, we need to compare $\widehat{\mu}_\theta^-(T)$ and $\widehat{\mu}_\theta^+(T)$ with $\widetilde{\mu}_\theta(T)$ and $\mu_\theta^E(T)$.

First, consider the case when $T = 1/2$. When $T = 1/2$ then $\widetilde{\mu}_\theta(T) = \mu_\theta^E(T) = 1/2$ while $\widehat{\mu}_\theta^-(T) < 1/2 < \widehat{\mu}_\theta^+(T)$. Thus, at $T = 1/2$ we have $\widehat{\mu}_\theta^-(T) < \widetilde{\mu}_\theta(T) = \mu_\theta^E(T) < \widehat{\mu}_\theta^+(T)$. Consider now the case when $T < 1/2$. Let T_L^- be the largest $T < 1/2$ such that $\widehat{\mu}_\theta^-(T) = \widetilde{\mu}_\theta(T)$. Such T_L^- exists since at $T = 1/2$ we have $\widehat{\mu}_\theta^-(T) < \widetilde{\mu}_\theta(T)$ and $\lim_{T\to 0}\widehat{\mu}_\theta^-(T) > \lim_{T\to 0}\widetilde{\mu}_\theta(T) = -\infty$. Thus, we know that for all $T \in \left(T_L^-, 1/2\right]$ we have $\widehat{\mu}_\theta^-(T) < \widetilde{\mu}_\theta(T)$. Next, define T_L^+ as the largest $T < 1/2$ such

[12] This requires establishing that for each $T > \frac{1}{2}$ we have $\mu_\theta^E(T) < \overline{\mu}(T)$ while for each $T < \frac{1}{2}$ we have $\mu_\theta^E(T) > \overline{\mu}(T)$. This can be established in exactly the same way as previous inequalities in this proof.

that $\widehat{\mu}_\theta^+(T) = \mu_\theta^E(T)$. If there is no such T, i.e., if for all $T < 1/2$ we have $\widehat{\mu}_\theta^+(T) > \mu_\theta^E(T)$ (note that at $T = 1/2$ we have $\mu_\theta^E(T) < \widehat{\mu}_\theta^+(T)$) then set $T_L^+ = 0$. Thus, for all $T \in \left(T_L^+, 1/2\right]$ we have $\widehat{\mu}_\theta^+(T) > \mu_\theta^E(T)$. Finally, define $T_L \equiv \max\left\{T_L^-, T_L^+\right\}$. We claim that for all $T \in (T_L, 1/2)$ we have $I_P(T) \cap \left(\mu_\theta^E(T), \widetilde{\mu}_\theta(T)\right) = \varnothing$.

To see this suppose that $T_L = T_L^-$, so that $T_L^- > T_L^+$, and consider any $T > T_L^-$. By definition of T_L^- it has to be the case that at $T > T_L^-$ we have $\widetilde{\mu}_\theta(T) > \widehat{\mu}_\theta^-(T)$. Since $\widehat{\mu}_\theta^+(T) > \widehat{\mu}_\theta^-(T)$ it follows that $\left(\mu_\theta^E(T), \widetilde{\mu}_\theta(T)\right) \cap \left(\widehat{\mu}_\theta^-(T), \widehat{\mu}_\theta^+(T)\right) = \varnothing$. It follows that implying that for all $T > T_L^-$ we have $I_P(T) \cap \left(\mu_\theta^E(T), \widetilde{\mu}_\theta(T)\right) = \varnothing$ and by definition of these intervals we know that for all $\mu_\theta \in I_P(T) \cap \left(\mu_\theta^E(T), \widetilde{\mu}_\theta(T)\right)$ private and public information are complements. Next, consider the case when $T_L = T_L^+$. This implies that $T_L^+ > T_L^-$ and hence there exists $0 < T < 1/2$ such that $\mu_\theta^E(T) = \widehat{\mu}_\theta^+(T)$. Recall that by definition of T_L^+ we know that for all $T > T_L^+$, $\widehat{\mu}_\theta^+(T) > \mu_\theta^E(T)$. Since $T_L^+ > T_L^-$, we know that for all $T > T_L^+$ we have $\widetilde{\mu}_\theta(T) > \widehat{\mu}_\theta^-(T)$. Thus, $\left(\widehat{\mu}_\theta^-(T), \widehat{\mu}_\theta^+(T)\right) \cap \left(\mu_\theta^E(T), \widetilde{\mu}_\theta(T)\right) \neq \varnothing$. This established the above claim.

In a similar fashion we define $T_H > 1/2$. This completes the proof as we establish existence of T_L and T_H, where $T_L < 1/2 < T_H$ such that for all $T \in (T_L, T_H)$ we have $I(T) \neq \varnothing$. ■

Above we showed that for all $T \in (T_L, T_H) \subset (0, 1)$ there are values of μ_θ such that private and public information are complements. One may wonder whether we can extend the above result to show that for each $T \in (0, 1)$ there is a set of values for μ_θ such that private and public information are complements. Unfortunately, the argument used above does not allow us to prove this more general result. In particular, our argument relies on the fact that for all $T \in (T_L, T_H)$ we have $I_P(T) \cap I_A(T) \cap I_C(T) \neq \varnothing$. However, for T close enough to 1 (or close enough to 0) we necessarily have $I_P(T) \cap I_A(T) \cap I_C(T) = \varnothing$. In other words, for extreme values of T there are no values of μ_θ such that all the effects of an increase in the precision in public information are positive. To see this, note that

$$\begin{aligned}
\lim_{T \to 1} \widetilde{\mu}_\theta(T) &= \infty \\
\lim_{T \to 1} \mu_\theta^E(T) &= \infty
\end{aligned}$$

while

$$\begin{aligned}
\lim_{T \to 1} \widehat{\mu}_\theta^+(T) &= 1 + \sqrt{\frac{\tau^* - \tau_\theta}{\tau_\theta(\tau^* + \tau_\theta)}} \\
\lim_{T \to 1} \widehat{\mu}_\theta^-(T) &= 1 - \sqrt{\frac{\tau^* - \tau_\theta}{\tau_\theta(\tau^* + \tau_\theta)}}
\end{aligned}$$

where for notational convenience we wrote τ^* instead of $\lim_{T \to 1} \tau^*\left(\widehat{\mu}_\theta^+(T), T\right)$. It follows that for T close enough to 1 we have $\mu_\theta^E(T) > \widetilde{\mu}_\theta(T) > \widehat{\mu}_\theta^+(T) > \widehat{\mu}_\theta^-(T)$. Therefore, when the cost of investment, T, is very high or very low one would need to compare the magnitude of each of the effects. Given the complexity of the model, this, however, is infeasible.

Above we investigated conditions under which public and private information are complements for the case $T \neq \frac{1}{2}$. In the next proposition, we provide sufficient conditions under which private and public information are substitute. In particular, below we characterize regions in which passive, active and coordination effects are negative. As such, the next proof will utilize several results established in the proof of Proposition 16.

Before we state our result we need to define several important thresholds for μ_θ. These thresholds also played a crucial role in the proof of Proposition 16 where we also addressed the issue of their existence. Let $\widehat{\widehat{\mu}}^{-}(T)$ be the smallest solution to equation:

$$\mu_\theta = \Phi\left(\frac{\tau_\theta}{\tau(\mu_\theta)^{1/2}}\sqrt{\frac{\tau(\mu_\theta)-\tau_\theta}{\tau_\theta(\tau(\mu_\theta)+\tau_\theta)}} + \sqrt{\frac{\tau(\mu_\theta)+\tau_\theta}{\tau(\mu_\theta)}}\phi^{-1}(T)\right) - \sqrt{\frac{\tau(\mu_\theta)-\tau_\theta}{\tau_\theta(\tau(\mu_\theta)+\tau_\theta)}},$$

while $\widehat{\widehat{\mu}}^{+}(T)$ be the largest solution to equation:

$$\mu_\theta = \Phi\left(-\frac{\tau_\theta}{\tau(\mu_\theta)^{1/2}}\sqrt{\frac{\tau(\mu_\theta)-\tau_\theta}{\tau_\theta(\tau(\mu_\theta)+\tau_\theta)}} + \sqrt{\frac{\tau(\mu_\theta)+\tau_\theta}{\tau(\mu_\theta)}}\phi^{-1}(T)\right) + \sqrt{\frac{\tau(\mu_\theta)-\tau_\theta}{\tau_\theta(\tau(\mu_\theta)+\tau_\theta)}}.$$

Similarly, let $\widetilde{\widetilde{\mu}}^{\tau_\theta}$ be the smallest, if $T > \frac{1}{2}$, or the largest, if $T < \frac{1}{2}$, solution to equation

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

and $\overline{\mu}(T)$ be the unique solution to

$$\mu_\theta = \frac{1}{2} + \frac{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}{\tau_\theta}\Phi^{-1}(T)$$

With the above definitions, we can state our most general result that provides sufficient conditions for the substitutability between private and public information.

Proposition 17 *Consider an investor's incentives to acquire information.*

1. *Suppose that $T > \frac{1}{2}$. If $\mu_\theta \notin \left(\min\left\{\widetilde{\widetilde{\mu}}^{\tau_\theta}, \widehat{\widehat{\mu}}^{-}(T)\right\}, \max\left\{\overline{\mu}(T), \widehat{\widehat{\mu}}^{+}(T)\right\}\right)$ then private and public information are substitutes.*

2. *Suppose that $T = \frac{1}{2}$. If $\mu_\theta \notin \left(\widehat{\widehat{\mu}}^{-}(T), \widehat{\widehat{\mu}}^{+}(T)\right)$ then private and public information are substitutes.*

3. *Suppose that $T < \frac{1}{2}$. If $\mu_\theta \notin \left(\min\left\{\overline{\mu}(T), \widehat{\widehat{\mu}}^{-}(T)\right\}, \max\left\{\widetilde{\widetilde{\mu}}^{\tau_\theta}(T), \widehat{\widehat{\mu}}^{+}(T)\right\}\right)$ then private and public information are substitutes.*

Proof. To prove this result we use the same approach as we used to establish Proposition 16 with the difference that now we focus on the regions in which the passive information effect, the active information effect and the coordination effect are negative. Note, also that part (2) of the above proposition follows immediately from Proposition 5 in the paper. Therefore, we only need to prove part (1) and part (3). Below, we consider the case when $T > \frac{1}{2}$. The proof for the case when $T < \frac{1}{2}$ is analogous.

Fix $T > \frac{1}{2}$ and consider first the passive information effect. The passive information effect discourages information acquisition if and only if

$$\frac{1}{\tau^* + \tau_\theta} - \frac{1}{2\tau_\theta} + \frac{1}{2}(\theta - \mu_\theta)^2 > 0$$

Note that $\frac{1}{\tau^*+\tau_\theta} > \frac{1}{2\tau_\theta}$ and θ^* is increasing in μ_θ. Moreover, $\lim_{\mu_\theta \to -\infty} \theta^* = 0$ while $\lim_{\mu_\theta \to +\infty} \theta^* = 1$. It follows that there exists $\widehat{\widehat{\mu}}^-(T)$ and $\widehat{\widehat{\mu}}^+(T)$, with $\widehat{\widehat{\mu}}^-(T) < \widehat{\widehat{\mu}}^+(T)$, such that for all $\mu_\theta \notin \left(\widehat{\widehat{\mu}}^-(T), \widehat{\widehat{\mu}}^+(T)\right)$ we have $\frac{1}{\tau^*+\tau_\theta} - \frac{1}{2\tau_\theta} + \frac{1}{2}(\theta - \mu_\theta)^2 > 0$, and if $\mu_\theta = \widehat{\widehat{\mu}}^-(T)$ or $\mu_\theta = \widehat{\widehat{\mu}}^+(T)$ then

$$\frac{1}{\tau^* + \tau_\theta} - \frac{1}{2\tau_\theta} + \frac{1}{2}(\theta - \mu_\theta)^2 = 0$$

Following an analogous argument as in Proposition 5 in the paper it can be shown, using the Critical Mass condition and the above observations, that $\widehat{\widehat{\mu}}^-(T)$ is the smallest solution to the equation

$$\mu_\theta = \Phi\left(\frac{\tau_\theta}{\tau(\mu_\theta)^{1/2}}\sqrt{\frac{\tau(\mu_\theta) - \tau_\theta}{\tau_\theta(\tau(\mu_\theta) + \tau_\theta)}} + \sqrt{\frac{\tau(\mu_\theta) + \tau_\theta}{\tau(\mu_\theta)}}\phi^{-1}(T)\right) - \sqrt{\frac{\tau(\mu_\theta) - \tau_\theta}{\tau_\theta(\tau(\mu_\theta) + \tau_\theta)}}$$

while $\widehat{\widehat{\mu}}^+(T)$ is the largest solution to the equation

$$\mu_\theta = \Phi\left(-\frac{\tau_\theta}{\tau(\mu_\theta)^{1/2}}\sqrt{\frac{\tau(\mu_\theta) - \tau_\theta}{\tau_\theta(\tau(\mu_\theta) + \tau_\theta)}} + \sqrt{\frac{\tau(\mu_\theta) + \tau_\theta}{\tau(\mu_\theta)}}\phi^{-1}(T)\right) + \sqrt{\frac{\tau(\mu_\theta) - \tau_\theta}{\tau_\theta(\tau(\mu_\theta) + \tau_\theta)}}$$

Thus, we conclude that whenever $\mu_\theta \notin \left(\widehat{\widehat{\mu}}^-(T), \widehat{\widehat{\mu}}^+(T)\right)$ then the passive information effect is negative.[13]

Consider now the active information effect and note that the active information effect discourages information acquisition if and only if

$$\tau^*(x^* - \theta^*)\left.\frac{\partial x_i^*}{\partial \tau_\theta}\right|_{\tau_i = \tau_\theta} > 0.$$

As we showed in the proof of Proposition 16, for a given T, we have $x^* > \theta^*$ if and only if $\mu_\theta < \overline{\mu}_\theta(T)$, $x^* = \theta^*$ if and only if $\mu_\theta = \overline{\mu}_\theta(T)$ and $x^* < \theta^*$ if and only if $\mu_\theta > \overline{\mu}_\theta(T)$ where $\overline{\mu}_\theta(T)$ is the unique solution to

$$\mu_\theta = \frac{1}{2} + \frac{\sqrt{\tau^*(\mu) + \tau_\theta}}{\tau_\theta},$$

where $\overline{\mu}_\theta(T)$ depends on T through the impact the cost of investment has on equilibrium precision choice, τ^*. Next, note that a change in x^* holding θ^* constant is given by

$$\frac{\partial x^*}{\partial \tau_\theta} = \frac{1}{\tau^*}(\theta^* - \mu_\theta) + \frac{1}{2}\frac{1}{\tau^*\sqrt{\tau^* + \tau_\theta}}.$$

[13] In Proposition 16 we found bounds $\widehat{\mu}^-(T)$ and $\widehat{\mu}^+(T)$ such that for all $\mu_\theta \in \left(\widehat{\mu}^-(T), \widehat{\mu}^+(T)\right)$ the passive information effect encourages private information acquisition. Unless the equation defining $\widehat{\mu}^-(T)$ and $\widehat{\widehat{\mu}}^-(T)$ has a unique solution, we have $\widehat{\widehat{\mu}}^-(T) < \widehat{\mu}^-(T)$. Similarly, unless the equation defining $\widehat{\mu}^+(T)$ and $\widehat{\widehat{\mu}}^+(T)$ has a unique solution, we have $\widehat{\mu}^+(T) < \widehat{\widehat{\mu}}^+(T)$. In general, it is unclear whether these conditions have a unique solution or not and, hence, it is not necessarily true that the passive information effect is positive for $\mu_\theta \in \left(\widehat{\widehat{\mu}}^-(T), \widehat{\widehat{\mu}}^+(T)\right)$.

As argued in the above Proposition, $\partial x^*/\partial\tau_\theta < 0$ if and only

$$\mu_\theta < \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right)+\frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T),$$

and $\partial x^*/\partial\tau_\theta > 0$ if and only if

$$\mu_\theta > \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right)+\frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T).$$

Finally, $\partial x^*/\partial\tau_\theta = 0$ if and only if μ_θ is a solution to the following equation:

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right)+\frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T),$$

Let $\tilde{\tilde{\mu}}_\theta(T)$ be the smallest solution to the above equation. Since for a fixed T the right hand side of the above equation is bounded and $\mu_\theta\in(-\infty,\infty)$ it follows that such $\tilde{\tilde{\mu}}_\theta(T)$ exists.

Recall that in the proof of Proposition 16 we showed that any solution to the equation

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right)+\frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

has to be smaller than $\overline{\mu}_\theta(T)$ (see Footnote 9, page 19 of the Online Appendix). Therefore, $\tilde{\tilde{\mu}}_\theta(T) < \overline{\mu}_\theta(T)$.

From the above discussion it follows that if $\mu_\theta > \overline{\mu}_\theta(T)$ then $x^* < \theta^*$ and $\partial x^*/\partial\tau_\theta < 0$, while if $\mu_\theta < \tilde{\tilde{\mu}}_\theta(T)$ then $x^* > \theta^*$ and $\partial x^*/\partial\tau_\theta > 0$. Hence, we conclude that for all $\mu_\theta \in \left(-\infty, \tilde{\tilde{\mu}}_\theta(T)\right)\cup(\overline{\mu}_\theta(T),\infty)$ the active information effect discourages private information acquisition.

Finally, we consider the coordination effect. The coordination effect discourages information acquisition if and only if

$$\tau_\theta(x^*-\mu_\theta)\left.\frac{\partial\theta^*}{\partial\tau_\theta}\right|_{\tau=\tau^*} > 0$$

In the proof of the Proposition 16 we showed that $x^*-\mu_\theta > 0$ if $\mu_\theta < \mu_\theta^E(T)$, $x^*-\mu_\theta = 0$ if $\mu_\theta = \mu_\theta^E(T)$ and $x^*-\mu_\theta < 0$ if $\mu_\theta > \mu_\theta^E(T)$. Next, consider $\partial\theta^*/\partial\tau_\theta|_{\tau=\tau^*}$ and recall that while analyzing the active information effect we defined $\tilde{\tilde{\mu}}_\theta(T)$ as the smallest solution to

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right)+\frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

As shown in Szkup (2014), whenever

$$\mu_\theta < \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right)+\frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

then $\partial\theta^*/\partial\tau_\theta|_{\tau=\tau^*} > 0$ while if the direction of the above inequality is reversed then $\partial\theta^*/\partial\tau_\theta|_{\tau=\tau^*} < 0$. Therefore, from definition of $\tilde{\tilde{\mu}}_\theta(T)$ it follows that for all $\mu_\theta < \tilde{\tilde{\mu}}_\theta(T)$ we have $\partial\theta^*/\partial\tau_\theta|_{\tau=\tau^*} > 0$.

Moreover, in the proof of Proposition 16, we argued that, for $T > 1/2$, $\mu_\theta^E(T)$ is larger than any solution to

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right)+\frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T).$$

It follows that: (1) $\mu_\theta^E(T) > \widetilde{\widetilde{\mu}}_\theta(T)$, and (2) for all $\mu_\theta > \mu_\theta^E(T)$ we have $\partial\theta^*/\partial\tau_\theta|_{\tau=\tau^*} < 0$. Thus, we have established that for all $\mu_\theta \notin \left(\widetilde{\widetilde{\mu}}_\theta(T), \mu_\theta^E(T)\right)$ the coordination effect is negative.

Above we have established that for each $T > \frac{1}{2}$ the passive information effect is negative if $\mu_\theta \notin \left(\widehat{\widehat{\mu}}^-(T), \widehat{\widehat{\mu}}^+(T)\right)$, the active information effect is negative if $\mu_\theta \in \left(-\infty, \widetilde{\widetilde{\mu}}_\theta(T)\right) \cup (\overline{\mu}_\theta(T), \infty)$ and the coordination effect is negative if $\mu_\theta \in \left(-\infty, \widetilde{\widetilde{\mu}}_\theta(T)\right) \cup \left(\mu_\theta^E(T), +\infty\right)$. Since for $T > \frac{1}{2}$ we have $\overline{\mu}_\theta(T) > \mu_\theta^E(T)$ (see the proof of Proposition 16) part (1) of the Proposition follows immediately.

The proof of part (3), i.e., the case when $T < \frac{1}{2}$ is analogous. The main differences, compared to the case when $T > \frac{1}{2}$, is that when $T < \frac{1}{2}$ then $\mu_\theta^E(T) > \overline{\mu}_\theta(T)$. Moreover, for $T < \frac{1}{2}$ we need to define $\widetilde{\widetilde{\mu}}_\theta(T)$ as the largest solution to

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right)+\frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T),$$

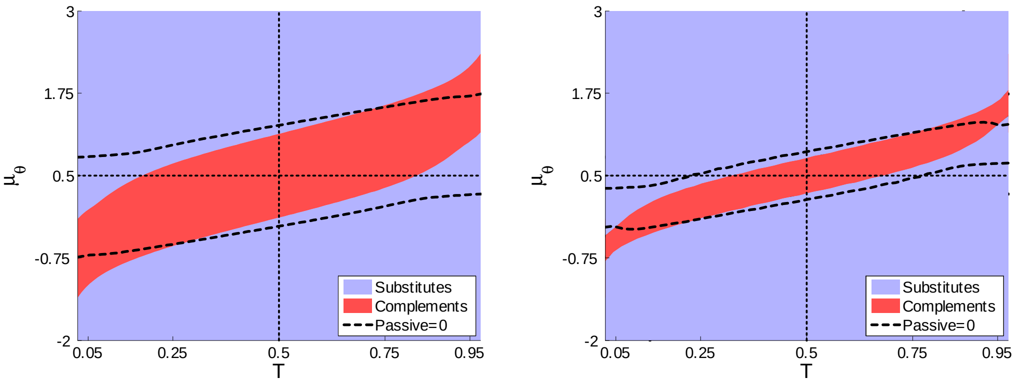
where it can be shown that for $T < \frac{1}{2}$ we have $\widetilde{\widetilde{\mu}}_\theta(T) > \mu_\theta^E(T)$. ■

4.2 The trade-off between public and private information when $T \neq \frac{1}{2}$: Numerical Results

In Propositions 16 and 17 we established the sufficient conditions under which private and public information are complements (Proposition 16) and substitutes (Proposition 17). We complement the above analytical results with numerical simulations. In particular, Figure 2 provide a complete characterization of the regions where a marginal increase in τ_θ leads to an increase in investors' incentives to acquire information (the dark shaded region in each panel) when $\tau_\theta = 1$ (left panel) and when $\tau_\theta = 2$ (the right panel).

Figure 2 shows that complementarity between public and private information is a pervasive phenomenon not limited only to the case when $T = \frac{1}{2}$. Moreover, unless T takes extreme values, the region where private and public information are complements falls inside the region where the passive information effect is positive which suggests that even when $T \neq 1/2$ the passive information effect plays an important role in driving this result.[14] Note that as τ_θ increases the region in which the two types of information are complements shrinks. The main reason is that as τ_θ increases, the region where passive information effect is positive shrinks (this can be deduced from the expression for the passive information effect, see proof of Proposition 17). This further underscores the importance of the passive information effect.

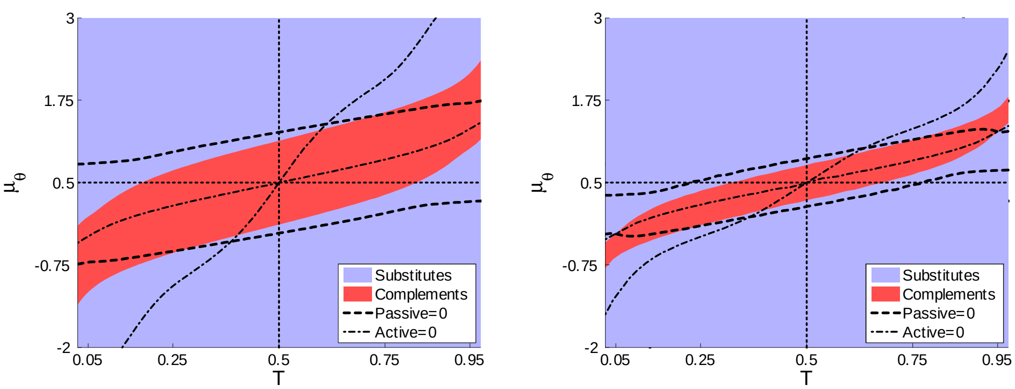
[14] In Figure 2 the region between the two dashed lines corresponds to the region where the passive information effect is positive.



Panel A: $\tau_\theta = 1$ Panel B: $\tau_\theta = 2.5$

Figure 2: Relation between private and public information: passive information effect

Recall that when $T \neq 1/2$, it is possible that the active information effect is positive. In order to investigate the role played by the active information effect, in Figure 3 we plot the region where the active information effect is positive (the region between the dash-dot lines) against the region where the passive information effect is positive (the region between the dashed lines).



Panel A: $\tau_\theta = 1$ Panel B: $\tau_\theta = 2.5$

Figure 3: Relation between private and public information: passive and active informations effects

We see that the region where the active information effect is positive, with exception for the case when T is extreme, differs substantially from the region where the public and private information are complements. Thus, Figure 3 provides additional support for the importance of the passive information effect in explaining the complementarity between public and private information.

5 Change in Investment and Welfare: Robustness

In this section we consider the effect of an increase in τ_θ on the probability of successful investment and welfare when $T \neq 1/2$.

5.1 Change in the Probability of Successful Investment

In section 6.2 of the paper we analyzed how a change in the precision of the prior affects the probability of successful investment when $T = 1/2$. Here, we investigate the case when $T \neq 1/2$. We state first intermediate technical results and then address the above question.

5.1.1 Intermediate Results

Lemma 18 *Suppose that τ is exogenously given.*

1. *If $\mu_\theta > \Phi\left(\sqrt{\frac{\tau+\tau_\theta}{\tau}}\Phi^{-1}(T)\right)$ then $\theta^* < \mu_\theta$.*
2. *If $\mu_\theta = \Phi\left(\sqrt{\frac{\tau+\tau_\theta}{\tau}}\Phi^{-1}(T)\right)$ then $\theta^* = \mu_\theta$.*
3. *If $\mu_\theta < \Phi\left(\sqrt{\frac{\tau+\tau_\theta}{\tau}}\Phi^{-1}(T)\right)$ then $\theta^* > \mu_\theta$.*

Proof. Recall that θ^* is defined as the solution to the Critical Mass condition:

$$\frac{\tau_\theta}{\tau^{1/2}}(\theta^* - \mu_\theta) + \sqrt{\frac{\tau+\tau_\theta}{\tau}}\Phi^{-1}(T) - \Phi^{-1}(\theta^*) = 0$$

The above lemma follows from the observation that the left hand side of the above equation is decreasing in θ^*. ■

Next, consider the following equation

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\Phi^{-1}(T)\right).$$

Define $\mu_\theta^M(T)$ as the largest solution to the above equation when $T > 1/2$ and the smallest solution when $T < 1/2$. Note that $\mu_\theta^M(T)$ is well defined since the right hand side can takes only values between 0 and 1. Next, recall from Section 4.1 that we defined $\tilde{\tilde{\mu}}^{\tau_\theta}(T)$ as the smallest, if $T > \frac{1}{2}$, or the largest, if $T < \frac{1}{2}$, solution to equation

$$\mu_\theta = \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right)+\frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

Lemma 19 *Consider $\mu_\theta^M(T)$.*

1. *If $T > 1/2$ then $\mu_\theta^M(T) < \tilde{\tilde{\mu}}^{\tau_\theta}(T)$;*
2. *If $T = 1/2$ then $\mu_\theta^M(T) = \tilde{\tilde{\mu}}^{\tau_\theta}(T)$;*
3. *If $T < 1/2$ then $\mu_\theta^M(T) > \tilde{\tilde{\mu}}^{\tau_\theta}(T)$.*

Proof. First, note that if $T = 1/2$ then $\mu_\theta^M(T) = \widetilde{\widetilde{\mu}}^{\tau_\theta}(T) = 1/2$. Next, holding τ constant, take the derivative of $\mu_\theta^M(T) - \widetilde{\widetilde{\mu}}^{\tau_\theta}(T)$ with respect to T to obtain:

$$\begin{aligned}
\frac{\partial}{\partial T}\left[\mu_\theta^M(T) - \widetilde{\widetilde{\mu}}^{\tau_\theta}(T)\right] &= \sqrt{\frac{\tau+\tau_\theta}{\tau}}\frac{1}{\phi\left(\Phi^{-1}(T)\right)}\phi\left(\sqrt{\frac{\tau+\tau_\theta}{\tau}}\Phi^{-1}(T)\right) \\
&\quad -\frac{1}{\phi\left(\Phi^{-1}(T)\right)}\phi\left(\sqrt{\frac{\tau+\tau_\theta}{\tau}}\frac{\tau+\frac{1}{2}\tau_\theta}{\tau+\tau_\theta}\Phi^{-1}(T)\right)\sqrt{\frac{\tau+\tau_\theta}{\tau}}\frac{\frac{1}{2}\tau+\tau_\theta}{\tau+\tau_\theta} \\
&\quad -\frac{1}{2}\frac{1}{\sqrt{\tau+\tau_\theta}}\frac{1}{\phi\left(\Phi^{-1}(T)\right)} \\
&\geq \left[\frac{1}{\sqrt{2\pi}}\sqrt{\frac{\tau+\tau_\theta}{\tau}}\frac{\frac{1}{2}\tau_\theta}{\tau+\tau_\theta} - \frac{1}{2}\frac{1}{\sqrt{\tau+\tau_\theta}}\right]\frac{1}{\phi\left(\Phi^{-1}(T)\right)} \\
&= \frac{1}{2}\frac{1}{\sqrt{\tau\left(\tau+\tau_\theta\right)}}\left[\frac{1}{\sqrt{2\pi}}\tau_\theta - \sqrt{\tau}\right]\frac{1}{\phi\left(\Phi^{-1}(T)\right)} < 0
\end{aligned}$$

Hence, it follows that if $T > 1/2$ then, for a given τ, we have

$$\Phi\left(\sqrt{\frac{\tau+\tau_\theta}{\tau}}\Phi^{-1}(T)\right) < \Phi\left(\sqrt{\frac{\tau+\tau_\theta}{\tau}}\frac{\frac{1}{2}\tau_\theta+\tau}{\tau+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau+\tau_\theta}}\Phi^{-1}(T)$$

In particular, it at $\mu_\theta^M(T)$ we have

$$\begin{aligned}
\mu_\theta^M(T) &= \Phi\left(\sqrt{\frac{\tau^*\left(\mu_\theta^M(T)\right)+\tau_\theta}{\tau^*\left(\mu_\theta^M(T)\right)}}\Phi^{-1}(T)\right) \\
&< \Phi\left(\sqrt{\frac{\tau^*\left(\mu_\theta^M(T)\right)+\tau_\theta}{\tau^*\left(\mu_\theta^M(T)\right)}}\frac{\frac{1}{2}\tau_\theta+\tau^*\left(\mu_\theta^M(T)\right)}{\tau^*\left(\mu_\theta^M(T)\right)+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau^*\left(\mu_\theta^M(T)\right)+\tau_\theta}}\Phi^{-1}(T)
\end{aligned}$$

Recall from Section 4.1 that, for $T > 1/2$, $\widetilde{\widetilde{\mu}}^{\tau_\theta}(T)$ is defined as the value of μ_θ such that for all $\mu_\theta \geq \widetilde{\widetilde{\mu}}_\theta(T)$ we have

$$\mu_\theta > \Phi\left(\sqrt{\frac{\tau^*\left(\mu_\theta^M(T)\right)+\tau_\theta}{\tau^*\left(\mu_\theta^M(T)\right)}}\frac{\frac{1}{2}\tau_\theta+\tau^*\left(\mu_\theta^M(T)\right)}{\tau^*\left(\mu_\theta^M(T)\right)+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau^*\left(\mu_\theta^M(T)\right)+\tau_\theta}}\Phi^{-1}(T)$$

Thus, if $T > 1/2$ then, $\mu_\theta^M(T) < \widetilde{\widetilde{\mu}}^{\tau_\theta}(T)$. The proof for the case $T < 1/2$ is analogous. ■

Finally, define $\mu^{Inv+}(T)$ as

$$\begin{aligned}
\mu^{Inv+}(T) &\equiv \max\left\{\overline{\mu}_\theta(T), \widehat{\widehat{\mu}}_\theta^{+}(T)\right\} \text{ if } T > 1/2 \\
\mu^{Inv+}(T) &\equiv \max\left\{\mu_\theta^M(T), \widehat{\widehat{\mu}}_\theta^{-}(T)\right\} \text{ if } T < 1/2
\end{aligned}$$

and $\mu^{Inv-}(T)$ as

$$\begin{aligned}
\mu^{Inv-}(T) &\equiv \min\left\{\mu_\theta^M(T), \widehat{\widehat{\mu}}_\theta^{+}(T)\right\} \text{ if } T > 1/2 \\
\mu^{Inv-}(T) &\equiv \min\left\{\overline{\mu}_\theta(T), \widehat{\widehat{\mu}}_\theta^{-}(T)\right\} \text{ if } T < 1/2
\end{aligned}$$

It follows from the definitions of $\widehat{\widehat{\mu}}_\theta^{+}(T)$, , $\widehat{\widehat{\mu}}_\theta^{+}(T)$, $\mu_\theta^M(T)$ and $\overline{\mu}_\theta(T)$ that both $\mu^{Inv+}(T)$ and $\mu^{Inv-}(T)$ are continuous and $\mu^{Inv+}(T) = \mu^{Inv-}(T) = 1/2$ when $T = 1/2$. Below, we use $\mu^{Inv+}(T)$ and $\mu^{Inv-}(T)$ to state our main result of this section.

5.1.2 Change in the Probability of Successful Investment: Main Results

In Section 6.2 we identified three channels through which a change in τ_θ affects the probability of successful investment. First, it affects the ex-ante distribution of θ. Second, it affects directly the value of the threshold θ^* through an adjustment in investors' investment strategies (holding their precision choices constant). Third, it leads indirectly to a change in θ^* by affecting investors' precision choices, and through this channel it further affects their investment behavior. We then proved that when $T = 1/2$ then a change in τ_θ increases probability of successful investment when $\mu_\theta > 1/2$, has no effect if $\mu_\theta = 1/2$, and decreases it when $\mu_\theta < 1/2$.

When $T \neq 1/2$ such characterization is not feasible. The main issue here is that it is difficult to establish both the sign and the magnitude of the second and third channels once $T \neq 1/2$. Intuitively, however, we should expect that the ex-ante probability of successful investment increases when μ_θ is high and decreases when μ_θ is low. This is because, the direct effect of a change in τ_θ on θ^* (the second channel identified above) tends to dominate the other two channels and it is positive for high values of the prior mean and is negative otherwise. Proposition 20 shows that this intuition is correct when μ_θ is sufficiently high or low.

Define $\mu^{Inv+}(T)$ as

$$\begin{aligned}\mu^{Inv+}(T) &\equiv \max\left\{\overline{\mu}_\theta(T), \widehat{\widehat{\mu}}_\theta^{+}(T)\right\} \text{ if } T > 1/2 \\ \mu^{Inv+}(T) &\equiv \max\left\{\mu_\theta^M(T), \widehat{\widehat{\mu}}_\theta^{-}(T)\right\} \text{ if } T < 1/2\end{aligned}$$

and $\mu^{Inv-}(T)$ as

$$\begin{aligned}\mu^{Inv-}(T) &\equiv \min\left\{\mu_\theta^M(T), \widehat{\widehat{\mu}}_\theta^{+}(T)\right\} \text{ if } T > 1/2 \\ \mu^{Inv-}(T) &\equiv \min\left\{\overline{\mu}_\theta(T), \widehat{\widehat{\mu}}_\theta^{-}(T)\right\} \text{ if } T < 1/2\end{aligned}$$

Proposition 20 *Suppose that the precision of the prior, τ_θ, increases.*

1. *Let $T > 1/2$:*
   (a) *If $\mu_\theta > \mu^{Inv+}(T)$ then the ex-ante probability of successful investment increases;*
   (b) *if $\mu_\theta < \mu^{Inv-}(T)$ then the ex-ante probability of successful investment decreases;*
2. *Let $T < 1/2$*
   (a) *If $\mu_\theta > \mu^{Inv+}(T)$ then the ex-ante probability of successful investment increases;*
   (b) *If $\mu_\theta < \mu^{Inv-}(T)$ then the ex-ante probability of successful investment decreases.*

Proof. Recall from the proof of Proposition 6 in the paper that the total effect of a change in τ_θ on the probability of successful investment is given by

$$\begin{aligned}\frac{d\Pr(\theta > \theta^*)}{d\tau_\theta} &= \frac{d}{d\tau_\theta}\left[1 - \Phi\left(\frac{\theta^* - \mu_\theta}{\tau_\theta^{-1/2}}\right)\right] \\ &= -\tau_\theta^{1/2}\phi\left(\frac{\theta^* - \mu_\theta}{\tau_\theta^{-1/2}}\right)\left[\frac{d\theta^*}{d\tau_\theta} + \frac{1}{2\tau_\theta}(\theta^* - \mu_\theta)\right]\end{aligned}$$

where

$$\frac{d\theta^*}{d\tau_\theta} = \frac{\partial\theta^*}{\partial\tau_\theta} + \frac{\partial\theta^*}{\partial\tau^*} \times \frac{d\tau^*}{d\tau_\theta}.$$

We consider case when $T > 1/2$ since the case of $T < 1/2$ is analogous. From Proposition 17 stated above, we know that if $T > 1/2$ and $\mu_\theta \notin \left(\min\left\{\widetilde{\widetilde{\mu}}^{\tau_\theta}, \widehat{\widehat{\mu}}^{-}(T)\right\}, \max\left\{\overline{\mu}(T), \widehat{\widehat{\mu}}^{+}(T)\right\}\right)$ then

$$\frac{d\tau^*}{d\tau_\theta} < 0.$$

Moreover, from Szkup (2014) we know that if precision choices are exogenous then for all μ_θ such that

$$\mu_\theta < \Phi\left(\sqrt{\frac{\tau+\tau_\theta}{\tau}}\frac{\frac{1}{2}\tau_\theta+\tau}{\tau+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau+\tau_\theta}}\Phi^{-1}(T)$$

we have

$$\frac{\partial\theta^*}{\partial\tau_\theta} > 0 \text{ and } \frac{\partial\theta^*}{\partial\tau^*} < 0.$$

Recall that by definition of $\widetilde{\widetilde{\mu}}^{\tau_\theta}(T)$, we know that

$$\mu_\theta < \widetilde{\widetilde{\mu}}^{\tau_\theta} \Longrightarrow \mu_\theta < \Phi\left(\sqrt{\frac{\tau^*(\mu_\theta)+\tau_\theta}{\tau^*(\mu_\theta)}}\frac{\frac{1}{2}\tau_\theta+\tau^*(\mu_\theta)}{\tau^*(\mu_\theta)+\tau_\theta}\Phi^{-1}(T)\right) + \frac{1}{2}\frac{1}{\sqrt{\tau^*(\mu_\theta)+\tau_\theta}}\Phi^{-1}(T)$$

and, thus, it follows that for all $\mu_\theta < \widetilde{\widetilde{\mu}}^{\tau_\theta}(T)$ we have

$$\frac{d\theta^*}{d\tau_\theta} > 0.$$

Moreover, if $\mu_\theta < \mu_\theta^M(T)$ then

$$\theta^* < \mu_\theta$$

By Lemma 19, $\mu_\theta^M(T) < \widetilde{\widetilde{\mu}}^{\tau_\theta}(T)$ for all $T > 1/2$. It follows that if $\mu_\theta < \mu^{Inv-}(T) \equiv \min\left\{\mu_\theta^M(T), \widehat{\widehat{\mu}}^{-}(T)\right\}$ we have

$$\frac{d\Pr(\theta > \theta^*)}{d\tau_\theta} < 0.$$

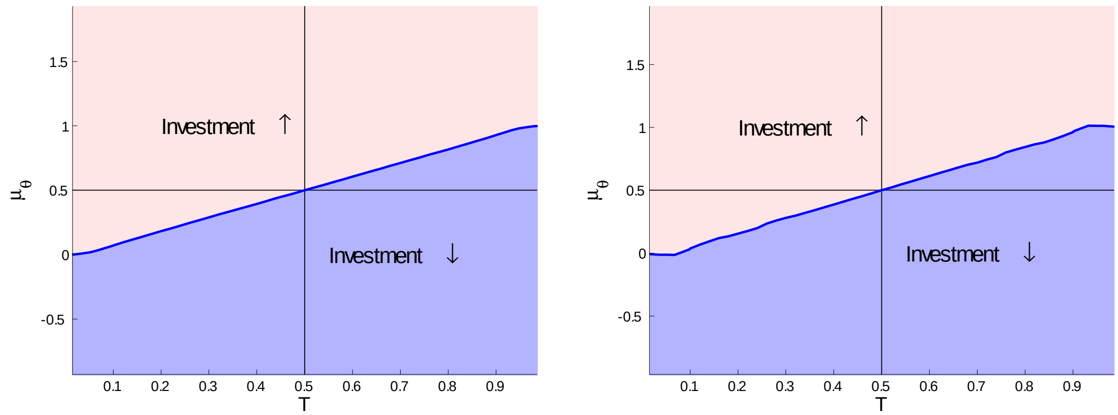
Similarly, when $\mu_\theta > \mu^{Inv+}(T) \equiv \max\left\{\overline{\mu}(T), \widehat{\widehat{\mu}}^{+}(T)\right\}$ then $d\theta^*/d\tau_\theta < 0$, $\theta < \mu_\theta$ and so

$$\frac{d\Pr(\theta > \theta^*)}{d\tau_\theta} < 0.$$

■

The above proposition establishes conditions under which the ex-ante probability of investment increases or decreases when $T \neq 1/2$. The proof of this result is substantially simpler than the proof of Proposition 6 in the main text. This simplification comes from the fact that here we dealt only with the case where an increase in the precision of prior crowds out private information acquisition. In contrast, in Proposition 6 we fixed $T = 1/2$ but consider all possible values of μ_θ. As explained above, however, such result is not feasible in the case when $T \neq 1/2$.

To check how the ex-ante probability of successful investment vary with a change in τ_θ for any pair $\{T, \mu_\theta\}$ we resorted to numerical simulations. Figure 4 presents two numerical examples that suggest that the finding of Proposition 6 extends to the case when $T \neq 1/2$.



Panel A: Initial $\tau_\theta = 0.5$ Panel B: Initial $\tau_\theta = 1.5$

Figure 4: Change in the probability of successful investment

Each panel of Figure 4 depicts the regions where a change in τ_θ leads to an increase and decrease in the probability of investment. The left panel corresponds to the situation where we consider a small increase in the precision of the prior from $\tau_\theta = 0.5$ while the right panel corresponds to the case where the initial precision of the prior is $\tau_\theta = 1.5$. We see that, as suggested above, for a fixed T, the probability of successful investment is positive for high enough μ_θ and negative otherwise.

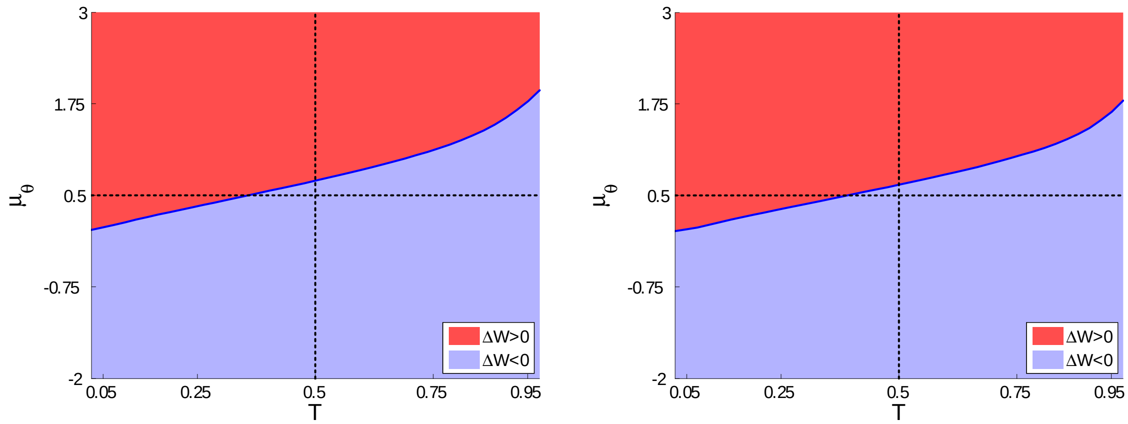
The above figures indicate that the threshold for μ_θ above which probability of successful investment increases with a change in τ_θ is itself increasing in T. To understand this, recall from Section 6.2 of the paper, that an increase in τ_θ leads to an increase in the probability of successful investment only if $x^* < \mu_\theta$, and leads to a decrease in the probability of successful investment if $x^* > \mu_\theta$. An increase in T decreases the benefit from successful investment and increases the cost of unsuccessful investment. Therefore, investors become less willing to invest and set higher threshold x^*. It follows that the region where the probability of successful investment is increasing in the precision of the prior shifts up as T increases.

5.2 Welfare consequences of higher τ_θ

In this section we investigate numerically how an increase in the precision of public information (i.e., precision of the prior), τ_θ, affects investors' ex-ante utility.

5.2.1 Change in Welfare:

We first investigate for which values of T and μ_θ welfare is increasing or decreasing in τ_θ and how does this depend on the initial precision of the prior. In Section 6.3 of the paper, we argued that one should expect welfare is increasing in τ_θ when μ_θ is sufficiently high and decreasing when μ_θ is low. Figure 5 support this intuition. In particular, we see that, for a fixed T, welfare is increasing in the precision of public information when μ_θ is sufficiently high and is decreasing in τ_θ when μ_θ is sufficiently low.



Panel A: Initial $\tau_\theta = 0.5$ Panel B: Initial $\tau_\theta = 1.5$

Figure 5: Welfare consequences of a higher τ_θ

Figure 5 suggests that the values of μ_θ for which welfare in increasing in τ_θ vary with T. This result is driven by the behavior of the probability of successful investment. In particular, as we explained in Section 5.1 the threshold for μ_θ above which probability of successful investment increases with a change in τ_θ is increasing in T. Since, as argued below, the change in the probability of successful investment is typically the main driver of the change in welfare a similar pattern in observed when we consider investors' welfare.

5.2.2 Change in Welfare: Decomposition

Next, we investigate what the main drivers of changes in welfare are. In particular, we check how much of the change in welfare is due to a change in the expected gain from successful investment[15] (labelled below as $Investment$), the change in the expected cost of mistakes (labeled as $Mistake$) and the change in the cost of acquiring information (labeled as $Cost$).[16] That is, we are interested in the following decomposition

$$\frac{\Delta U}{U} = \frac{\Delta Investment}{U} + \frac{\Delta Mistakes}{U} + \frac{\Delta Cost}{U}$$

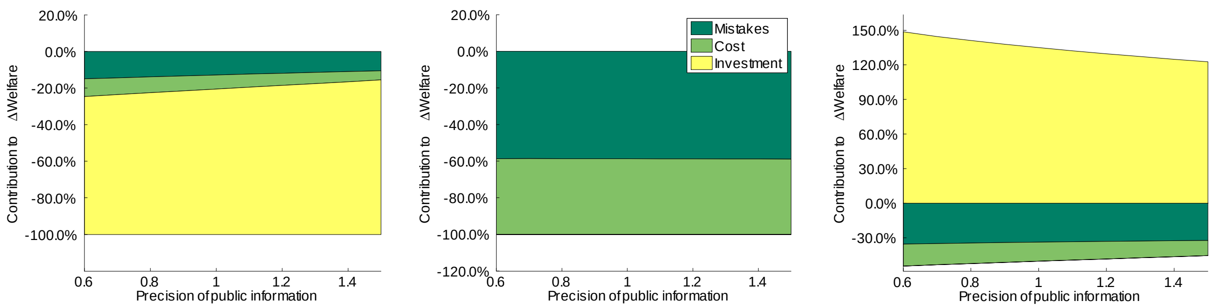
where U denotes investors' ex-ante utility. In order to compute the above decomposition we fix $\{T, \mu_\theta\}$ and compute the change in welfare implied by a gradual increase in τ_θ gradually from 0.5 to 1.5. We then decompose the implied change into the three effects mentioned above. We expect to find that, unless a small change in precision has little effect on the investment threshold (which happens only when $\{T, \mu_\theta\}$ lies near the line dividing region $Investment \uparrow$ and $Investment \downarrow$ in Figure 4), the investment effect will be the driving force behind the changes in welfare.

[15] The expected gain from investment is defined as the payoff from a successful investment times the probability of successful investment. It corresponds to the third term in investors' ex-ante utility, equation (4) in the main text.

[16] While a marginal change in the precision of public information has no effect on investor i's expected utility (by the Envelope Theorem), this is not the case when we consider discrete changes in the informativeness of the prior.

To test the above hypothesis we pick three values of T, $T \in \{0.25, 0.5, 0.75\}$, and three values of μ_θ, $\mu_\theta \in \{0, 0.5, 1\}$. The point $\{T = 0.5, \mu_\theta = 0.5\}$ plays a special role since we know that for these parameter values a change in the precision of prior has no effect on θ^*.

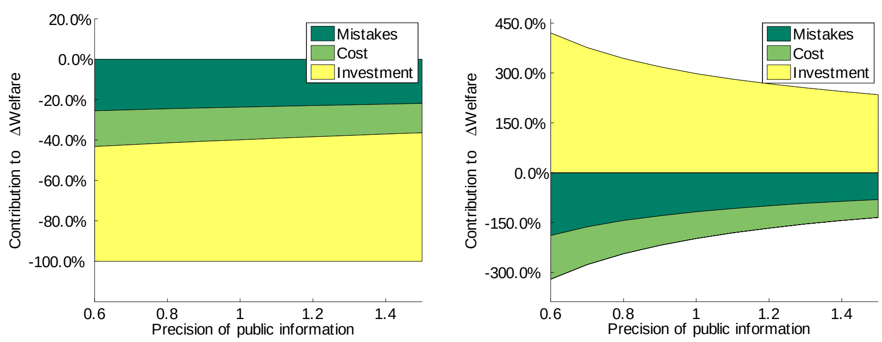
We first consider the case where $T = 0.5$ (Figure 6). We see that both in the case when the mean of the prior is low ($\mu_\theta = 0$, Panel A) or high ($\mu_\theta = 1$, Panel C) the investment effect is the dominant force driving the change in welfare. Panel B depicts the case where the change in welfare is driven by the change in the expected cost of mistakes and the change in the cost of precision. When $T = 0.5$ and $\mu_\theta = 0.5$ the investment effect is zero since a change in precision of the prior has no effect on the probability of successful investment (Proposition 6 in the paper).



Panel A: $T = 0.5$ and $\mu_\theta = 0$ Panel B: $T = 5$ and $\mu_\theta = 0.5$ Panel C: $T = 0.5$ and $\mu_\theta = 1$

Figure 6: Welfare Decomposition when $T = 0.5$

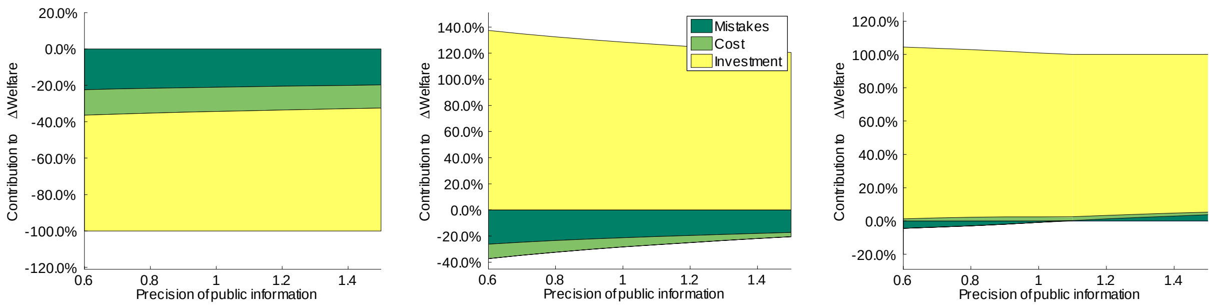
These results are consistent with our hypothesis stated above. To provide further evidence that support our hypothesis, for the case of $T = 0.5$ we also computed this decomposition for less extreme values of $\mu_\theta = 0.25$ and $\mu_\theta = 0.75$. Since these values of the mean of the prior are close to the critical point of 0.5 the effect of a change in τ_θ has a smaller effect on the probability of successful investment. However, if our hypothesis is true, the "investment effect" should continue to be dominant force also at these values. Figure 7 shows that indeed, for these values of parameters, the investment effect contributes more than 50% to the change in welfare.



Panel A: $T = 0.5$ and $\mu_\theta = 0.25$ Panel A: $T = 0.5$ and $\mu_\theta = 0.75$

Figure 7: Additional Results when $T = 0.5$

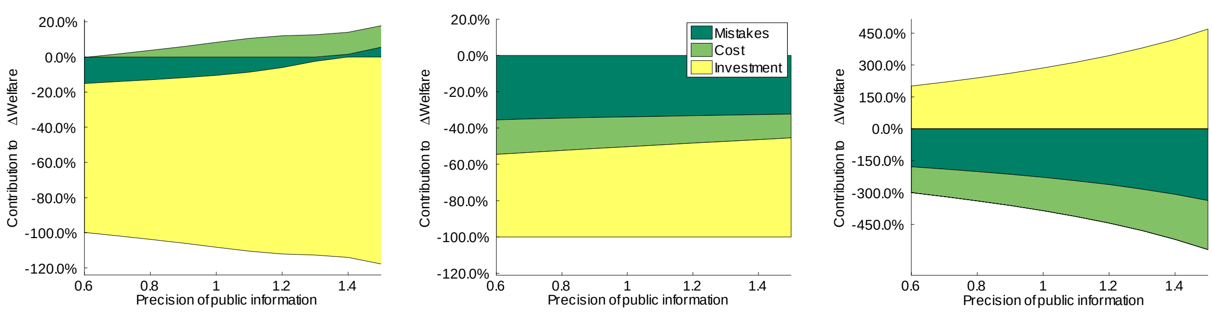
We next consider the case where $T = 0.25$. Figure 8 shows that for $\mu_\theta \in \{0, 0.5, 1\}$ the investment effect is the dominant force. Note that compared to the case when $T = 0.5$, now the change in the expected gain from successful investment contributes even more to the changes in welfare. This is because when T is low then the payoff from successful investment is high and hence a small change in the probability of successful investment has a large impact of the expected gain from successful investment and, thus, on investors' welfare.



Panel A: $T = 0.25$ and $\mu_\theta = 0$ Panel B: $T = 0.25$ and $\mu_\theta = 0.5$ Panel C: $T = 0.25$ and $\mu_\theta = 1$

Figure 8: Welfare Decomposition when $T = 0.25$

Finally, we consider the case when $T = 0.75$. Note that when T is high then payoff from successful investment is low and hence changes in τ_θ lead to relatively small adjustments in the expected gain from investment (even though they can result in large changes in the probability of successful investment). This is reflected in Figure 9 which shows that when T is high the contribution of the investment effect is smaller than at in the cases considered above. Nevertheless, the investment effect still remains a significant factor driving changes in welfare. Finally, as Panel A of Figure 9 suggest, when μ_θ takes extreme values (i.e., values far from the critical value of μ_θ at which $\partial W/\partial \tau_\theta = 0$) then investment effect is the main force driving changes in welfare.



Panel A: $T = 0.75$ and $\mu_\theta = 0$ Panel A: $T = 0.75$ and $\mu_\theta = 0.5$ Panel A: $T = 0.75$ and $\mu_\theta = 1$

Figure 9: Welfare Decomposition when $T = 0.75$

Given the above numerical results, we conclude that investment effect tends to be dominant force driving changes in welfare though its importance varies with T. As T gets higher changes in the expected gain from successful investment become smaller. As a consequence, the relative importance of the other two channels grows.

6 The Lower Bound for Precision Choices $\underline{\tau}$

In this section we discuss in detail the choice of the lower bound for precision choices, $\underline{\tau}$. There are several technical results where the choice of $\underline{\tau}$ matters. First, as is standard in global games with public information, in order to ensure uniqueness of equilibrium in the coordination game we need to have $\underline{\tau} > \frac{1}{2\pi}\overline{\tau}_\theta$ (see e.g. Hellwig, 2002, or Morris and Shin, 2004). That is, the lowest precision that can be chosen by agents needs to be sufficiently large compared to the highest precision of public signal that we consider. If this assumption is not satisfied, then it is possible that there are multiple equilibria in the second stage of the game. While interesting, the occurrence of multiple equilibria would substantially complicate the analysis of the information acquisition stage, which is why we do not consider this possibility in the paper.

The next restriction of $\underline{\tau}$ comes from requiring that the ex-ante utility function is concave. It is well known that the value of information may be non-concave, especially when the informativeness of signals available to the decision-maker is low (see Radner and Stiglitz, 1984, and Chade and Schlee, 2002). In order to ensure that the ex-ante utility is concave we have to require that $\underline{\tau}$ is high enough. We do not provide a closed-form lower bound on $\underline{\tau}$. Instead, we simply assume that $\underline{\tau}$ satisfies this condition.

Another restriction on $\underline{\tau}$ comes from the technical results regarding the behavior of $\frac{\partial^2\theta^*}{\partial\tau_\theta\partial\mu_\theta}$. In order for $\frac{\partial^2\theta^*}{\partial\tau_\theta\partial\mu_\theta} < 0$ in the required range of μ_θ we need to impose the assumption that

$$\frac{\underline{\tau}^{1/2}}{\overline{\tau}_\theta}\left(1 - \frac{\overline{\tau}_\theta}{\underline{\tau}}\frac{\underline{\tau} - \underline{\tau}_\theta}{(\underline{\tau} + \underline{\tau}_\theta)}\right) > \frac{1}{\sqrt{2\pi}}$$

where $\underline{\tau}_\theta$ and $\overline{\tau}_\theta$ are the lower and upper bounds, respectively, for the precision of the prior that we consider in the paper. This is the condition reported in Assumption $A1$.

The final restriction comes from the analysis of a change in the probability of investment resulting from a change in τ_θ. In order to determine the change of this probability we have to ensure that the best-response function is not too steep. The proof suggests that the slope of the best response function, $\tau_i^*(\tau)$, at the equilibrium precision level has to be less than 5/6 in order for our result to go through. In the proof of Theorem 1 we show that the slope of the best response function converges to zero as $\tau \to \infty$. It follows that choosing a sufficiently high $\underline{\tau}$ will ensure that the above restriction on the slope of the best response function is satisfied. Again, rather than providing an implicit bound for the value of $\underline{\tau}$ we simply assume that $\underline{\tau}$ satisfies this condition.

To sum up, one should think of $\underline{\tau}$ as being high enough so that the ex-ante utility function is concave, the slope of the best response function is less than 5/6 and that

$$\frac{\underline{\tau}^{1/2}}{\overline{\tau}_\theta}\left(1 - \frac{\overline{\tau}_\theta}{\underline{\tau}}\frac{\underline{\tau} - \underline{\tau}_\theta}{(\underline{\tau} + \underline{\tau}_\theta)}\right) > \frac{1}{\sqrt{2\pi}}$$

References

[1] Carlsson, H. and E. van Damme (1993), "Global Games and Equilibrium Selection", *Econometrica*, 61(5), pp. 989-1018.

[2] Chade, H. and E. Schlee (2002), "Another Look at the Radner–Stiglitz Nonconcavity in the Value of Information", *Journal of Economic Theory* 107(2), pp. 421–452

[3] Hellwig, C. (2002), "Public Information, Private Information, and the Multiplicity of Equilibria in Coordination Games", *Journal of Economic Theory* 107(2), pp. 191–222.

[4] Morris, S. and H. Shin (2004), "Coordination Risk and Price of Debt", *European Economic Review*, 48, pp. 133–153.

[5] Radner, R. and J. Stiglitz (1984), "A Nonconcavity in the Value of Information", in Boyer and Kihlstorm ed. *Bayesian Models in Economic Theory*, Elsevier Science Publisher, pp 33-52

[6] Szkup, M. (2014), "Information Structure and Comparative Statics in Simple Global Games", mimeo University of British Columbia