INDEX NUMBER THEORY AND MEASUREMENT ECONOMICS

By W.E. Diewert. March 15, 2015.

University of British Columbia and the University of New South Wales
Email: erwin.diewert@ubc.ca
Website: http://www.economics.ubc.ca/faculty-and-staff/w-erwin-diewert/

Chapter 10: Multilateral Index Number Theory: Economic and Axiomatic Approaches

1. Introduction

The World Bank conducts periodic comparisons of real GDP across countries in its International Comparison Program (ICP). The results for the 2005 and 2011 ICP rounds have been released; see the World Bank (2008) (2013) (2014). The 2005 ICP program compared the level of prices and the quantities or volumes of GDP (and its components) for 146 countries for the year 2005 and the 2011 ICP compared prices and volumes for 179 countries. International price statisticians developed Structured Product Descriptions (SPDs) for over 1000 products and the individual countries collected price information on these products for the comparison years. The products were grouped into 155 Basic Heading (BH) categories. The price information collected in each country was then compared across countries, leading to a matrix of 155 Basic Heading prices by 146 countries for the 2005 comparisons. The precise way in which the individual product prices in each BH category were aggregated into a single country price for each BH heading will not be discussed in this Chapter;[1] our focus here is on multilateral methods which are used to form overall estimates of the price level in each country (these indexes are called Purchasing Power Parities or PPPs) and estimates of relative real GDP across countries.[2] It should be noted that the ICP PPPs and relative GDP volumes are important inputs into the Penn World Tables,[3] which in turn have been used in a huge number of economic studies. Note also that multilateral index number methods can be used in order to construct estimates of *regional* real GDP or consumption levels within a country that are comparable across time and space.[4]

There are a large number of methods that can be used to construct these aggregate Purchasing Power Parities and relative country volumes. Hill (2007a) (2007b) surveyed the main methods that have been used in previous rounds of the ICP as well as other

[1] This topic is discussed in detail in Diewert (2004) and Rao (2009) (2013).
[2] See Deaton and Heston (2010), Diewert (2010b) and Vogel (2013) for discussions of the complications for the ICP 2005 that were caused by the need to respect regional parities in the overall world comparisons.
[3] See Feenstra, Inklaar and Timmer (2013) on the Penn World Tables. See McCarthy (2013) on the reasons why cross sectional estimates of real GDP based on PPPs are in general not consistent with country estimates of real GDP over time. See Diewert (2014) on the importance of PPPs in forming accurate measures of inflation and growth for a group of countries.
[4] In this case, all exchange rates are one.

methods that could be used.[5] Basically, only two multilateral methods have been used in ICP rounds prior to 2005:

- The Gini Eltetö Köves Szulc (GEKS) method based on Fisher (1922) bilateral indexes and
- The Geary (1958) Khamis (1972) (GK) method, which is an additive method.

In the 2005 ICP round, aggregate PPPs and relative country volumes for countries within each region were constructed for four of the five regions using the Gini-EKS (GEKS) method.[6] However, the African region wanted to use an additive method and so this region used a relatively new additive method, the Iklé Dikhanov Balk (IDB) method, for constructing PPPs and relative volumes within the region.[7] We will describe the properties of these three methods (GEKS, GK and IDB) for making multilateral comparisons between countries in a region.[8] These three methods will be discussed in sections 2, 3 and 4 below.

A brief comment on the relative merits of the GEKS, GK and IDB methods is warranted. The GK and IDB methods are *additive methods*; i.e., the real final demand of each country can be expressed as a *sum* of the country's individual Basic Heading final demand components where each real final demand component is weighted by an *international price* which is constant across countries. This feature of an additive method is tremendously convenient for users since components of final demand can be aggregated consistently across both countries and commodity groups and so for many purposes, it is useful to have available a set of additive international comparisons. However, additive methods are not consistent with the *economic approach* to index number theory (which allows for substitution effects), whereas the GEKS method is consistent. Section 6 will explain the economic approach and explain why additive methods are not fully consistent with the economic approach.

In order to discriminate between the various multilateral index number methods that have been suggested for the ICP, it is useful to look at the *axiomatic properties* of the various methods. Thus in Section 5, we will list various axioms or properties or tests that have been suggested for multilateral indexes and see which tests are satisfied by GEKS, GK and IDB.

[5] For additional methods, see Rao (1990), Balk (1996) (2009; 232-260), R.J. Hill (1997) (1999a) (1999b) (2001) (2004) (2009) and Diewert (1999).
[6] In the 2011 round of comparisons, each region used the GEKS method to calculate relative country GDP levels within each region.
[7] Iklé (1972; 203) proposed the equations for the method in a rather difficult to interpret manner and provided a proof for the existence of a solution for the case of two countries. Dikhanov (1994; 6-9) used the much more transparent equations (13) and (14) below, explained the advantages of the method over the GK method and illustrated the method with an extensive set of computations. Balk (1996; 207-208) used the Dikhanov equations and provided a proof of the existence of a solution to the system for an arbitrary number of countries. Van Ijzeren (1983; 42) also used Iklé's equations and provided an existence proof for the case of two countries.
[8] Most of the material in this Chapter is drawn from Diewert (2013a) (2013b).

The GEKS multilateral method is consistent with the economic approach to making multilateral comparisons. The GEKS approach also has the property that each country in the comparison is treated in a fully symmetric manner; i.e., the method is a democratic one. This aspect of GEKS can be considered as an advantage of the method. However, from a technical point of view, there are some disadvantages to the method in that countries that are at very different stages of development and which face very different relative prices are given the same weight in the method as countries which are at a very similar stage of development and face the same structure of relative prices. Bilateral comparisons between similar in structure countries are likely to be much more accurate than comparisons between countries which are very dissimilar. Thus in Section 7, an economic approach is introduced that builds up a complete multilateral set of comparisons that rests on making bilateral comparisons between very *similar* in structure countries. This method is called the *Minimum Spanning Tree* (MST) *method* by Robert Hill (1999a) (1999b) (2001) (2004) (2009), who introduced the method.[9] This method has some advantages over GEKS and thus it could be considered for use in the next ICP round.

Section 8 uses the artificial data example in Diewert (1999) to illustrate how the four methods (GEKS, GK, IDB and MST) differ in a rather extreme numerical example. Another numerical example based on ICP data for 1985 will be presented in section 9.

Section 10 concludes.

2. The GEKS Method

The GEKS method is due to Gini (1924) (1931) and it was rediscovered by Eltetö and Köves (1964) and Szulc (1964).

In order to explain the method, it will be useful to introduce some notation at this point. Suppose that we have collected information on the expenditures of K countries (or other economic units that can be compared) and these expenditures have been grouped into N categories.[10] Denote the Basic Heading PPP for commodity category n and for country k in the region by $p_n^k > 0$ and the corresponding expenditure (in local currency units) on commodity class n by country k in the reference year by e_n^k for n = 1,...,N and k = 1,...,K.[11] Basically, the Basic Heading PPPs p_n^k are elementary indexes for each category of expenditure n for each country k; i.e., if there were only one commodity in each category of expenditure, p_n^k would represent the price of a unit (using the same unit of

---

[9] Fisher (1922; 272-274) in his discussion on comparing the price levels of Norway, Egypt and Georgia, came close to introducing this method. Kravis, Heston and Summers (1982;104-111) used similarity measures to cluster countries into groups and also came close to introducing *Hill's spatial linking method*.

[10] In the case of the international comparisons that are made by the World Bank, these N groups are called Basic Heading expenditure categories as noted above. In the 2005 round of international comparisons, N was equal to 155 and K was equal to 146 so that there were 146 countries in the comparisons of real GDP across countries in 2005.

[11] In the World Bank comparisons for 2005 and 2011, the expenditures e_n^k were drawn from the national accounts of country k in the reference year and they referred to total expenditures on commodity category n.

measurement across countries) of commodity n in country k in terms of the local currency.[12] Given this price information, we can define *volumes*[13] or *implicit quantity levels* q_n^k for each Basic Heading (BH) category n and for each country k as the category expenditure deflated by the corresponding Basic Heading commodity PPP for that country:

$$(1)\ q_n^k \equiv e_n^k/p_n^k\ ; \qquad n = 1,...,N\ ;\ k = 1,...,K.$$

It will be useful to define *country commodity expenditure shares* s_n^k for BH class n and country k as follows:

$$(2)\ s_n^k \equiv e_n^k/\sum_{i=1}^N e_i^k\ ; \qquad n = 1,...,N\ ;\ k = 1,...,K.$$

Now define *country vectors of BH PPPs* as $p^k \equiv [p_1^k,...,p_N^k]$, *country vectors of BH volumes* as $q^k \equiv [q_1^k,...,q_N^k]$, *country expenditure vectors* as $e^k \equiv [e_1^k,...,e_N^k]$ and *country expenditure share vectors* as $s^k \equiv [s_1^k,...,s_N^k]$ for k = 1,...,K.

In order to define the GEKS parities $P^1,P^2,...,P^K$ between the K countries in the comparison, we first need to define the *Fisher* (1922) *ideal bilateral price index* P_F between country j relative to k:

$$(3)\ P_F(p^k,p^j,q^k,q^j) \equiv [p^j\cdot q^j\ p^j\cdot q^k/p^k\cdot q^j\ p^k\cdot q^k]^{1/2}\ ; \qquad j = 1,...,K\ ;\ k = 1,...,K.$$

Note that the Fisher ideal price index is the geometric mean of the Laspeyres price index between countries j and k, $P_L(p^k,p^j,q^k,q^j) \equiv p^j\cdot q^k/p^k\cdot q^k$,[14] and the Paasche price index, $P_P(p^k,p^j,q^k,q^j) \equiv p^j\cdot q^j/p^k\cdot q^j$.[15] Various justifications for the use of the Fisher ideal index in the bilateral context have been made by Diewert (1976) (1992) (2002; 569) and others.[16] As we have seen in Chapters 1 and 2, the Fisher index can be justified from the point of

[12] Since we have not formally introduced elementary indexes up to this point, we can think of the p_n^k as being unit value prices for homogeneous commodities that are measured in the same units of measurement across countries. A bilateral elementary index is a price index that does not make use of quantity information; i.e., it has the form $P(p^0,p^1)$ instead of $P(p^0,p^1,q^0,q^1)$. Examples of elementary indexes are the Carli and Jevons indexes that were defined in previous chapters.

[13] National income accountants distinguish between a "quantity" and a "volume". A *volume* is an aggregate of a group of actual quantities. Since country expenditures in each of the Basic Heading categories are aggregates over many commodities, it is appropriate to refer to the q_n^k as volumes rather than quantities. The price levels p_n^k that correspond to the q_n^k are called Basic Heading (BH) PPPs.

[14] Define the country k expenditure share on commodity group n as $s_n^k \equiv p_n^k q_n^k/p^k\cdot q^k$ for n = 1,...,N. Then the Laspeyres price index between countries j and k can be written in the following expenditure share form: $P_L(p^k,p^j,q^k,q^j) \equiv p^j\cdot q^k/p^k\cdot q^k = \sum_{n=1}^N p_n^j q_n^k/p^k\cdot q^k = \sum_{n=1}^N (p_n^j/p_n^k)p_n^k q_n^k/p^k\cdot q^k = \sum_{n=1}^N (p_n^j/p_n^k)s_n^k$, which is a country k share weighted *arithmetic* mean of the price relatives p_n^j/p_n^k.

[15] Define the country j expenditure share on commodity group n as $s_n^j \equiv p_n^j q_n^j/p^j\cdot q^j$ for n = 1,...,N. Then the Paasche price index between countries j and k can be written in the following expenditure share form: $P_P(p^k,p^j,q^k,q^j) \equiv p^j\cdot q^j/p^k\cdot q^j = [\sum_{n=1}^N p_n^k q_n^j/p^j\cdot q^j]^{-1} = [\sum_{n=1}^N (p_n^j/p_n^k)^{-1} p_n^j q_n^j/p^j\cdot q^j]^{-1} = [\sum_{n=1}^N (p_n^j/p_n^k)^{-1} s_n^j]^{-1}$, which is a country j share weighted *harmonic* mean of the price relatives p_n^j/p_n^k. Using these formulae for the Laspeyres and Paasche price indexes, it can be seen that the Fisher price index can also be written in terms of expenditure shares and price relatives.

[16] See Balk (2008; 91-97) for a review of the literature on axiomatic justifications for the Fisher index.

view of finding the "best" symmetric average of the Laspeyres and Paasche indexes, or from the point of view of the axiomatic or test approach to index number theory, or from the viewpoint of the economic approach to index number theory; see also Chapters 15, 16 and 17 in the *Consumer Price Index Manual*, ILO/IMF/OECD/UNECE/Eurostat/World Bank (2004).

The *aggregate PPP for country* j, P^j, is defined as follows:

$$(4)\ P^j \equiv \prod_{k=1}^{K} [P_F(p^k,p^j,q^k,q^j)]^{1/K}\ ; \qquad j = 1,...,K.$$

What is the rational for the system of country aggregate price indexes defined by (4)? The index $P_F(p^1,p^j,q^1,q^j)$ represents the Fisher index level of prices in country j relative to country 1 and so if we fix country 1 as the base country and we allow j to equal 1,2,...,K, the resulting sequence of price indexes, $P_F(p^1,p^1,q^1,q^1)$, $P_F(p^1,p^2,q^1,q^2)$, ... , $P_F(p^1,p^K,q^1,q^K)$ represents the level of prices in countries 1, 2, ..., K relative to country 1 as the numeraire or comparison country. But we could allow any country k to be the base country and the resulting sequence of price indexes, $P_F(p^k,p^1,q^k,q^1)$, $P_F(p^k,p^2,q^k,q^2)$, ... , $P_F(p^k,p^K,q^k,q^K)$ represents the level of prices in countries 1, 2, ..., K relative to country k as the comparison country. Thus there are K sets of country parities that we could compute, using each country in turn as the base country. The parities represented by (4) simply take the equally weighted geometric mean of all of these base country specific comparisons as the final set of parities.

Once the GEKS P^j's have been defined by (4), the corresponding GEKS *country real expenditures or volumes* Q^j can be defined as the country expenditures $p^j \cdot q^j$ in the reference year divided by the corresponding GEKS purchasing power parity P^j:

$$(5)\ Q^j \equiv p^j \cdot q^j / P^j\ ; \qquad j = 1,...,K.$$

If all of the P^j defined by (4) are divided by a positive number, α say, then all of the Q^j defined by (5) can be multiplied by this same α without materially changing the GEKS multilateral method. If country 1 is chosen as the numeraire country in the region, then set α equal to P^1 defined by (4) for j = 1 and the resulting price level P^j is interpreted as the number of units of country j's currency it takes to purchase 1 unit of country 1's currency and get an equivalent amount of utility. The rescaled Q^j is interpreted as the volume of final demand of country j in the currency units of country 1.

It is also possible to normalize the aggregate real expenditure of each country in common units (the Q^k) by dividing each Q^k by the sum $\sum_{j=1}^{K} Q^j$ in order to express each country's real expenditure or real final demand as a fraction or share of total regional real expenditure; i.e., define the country k's *share of regional real expenditures*, S^k, as follows:[17]

[17] There are several additional ways of expressing the GEKS PPP's and relative volumes; see Balk (1996), Diewert (1999; 34-37) and section 6 below.

(6) $S^k \equiv Q^k/\sum_{j=1}^K Q^j$; k = 1,...,K.

Of course, the country shares of regional real final demand, the S^k, remain unchanged after rescaling the PPPs by the scalar α.

This completes a brief description of the GEKS method for making multilateral comparisons.[18]

3. The Geary Khamis Method

The method was suggested by Geary (1958) and Khamis (1972) showed that the equations that define the method have a positive solution under certain conditions.

The GK system of equations involves K *country price levels* or PPPs, $P^1,...,P^K$, and N *international Basic Heading commodity reference prices*, $\pi_1,...,\pi_N$. The equations which determine these unknowns (up to a scalar multiple) are the following ones:

(7) $\pi_n = \sum_{k=1}^K [q_n^k/\sum_{j=1}^K q_n^j][p_n^k/P^k]$; n = 1,...,N ;
(8) $P^k = p^k \cdot q^k/\pi \cdot q^k$; k = 1,...,K

where $\pi \equiv [\pi_1,...,\pi_N]$ is the vector of GK regional average reference prices. It can be seen that if a solution to equations (7) and (8) exists, then if all of the country parities P^k are multiplied by a positive scalar λ say and all of the reference prices π_n are divided by the same λ, then another solution to (7) and (8) is obtained. Hence, the π_n and P^k are only determined up to a scalar multiple and an additional normalization is required such as

(9) $P^1 = 1$

in order to uniquely determine the parities. It can also be shown that only N + K − 1 of the N equations in (7) and (8) are independent. Once the parities P^k have been determined, the real expenditure or volume for country k, Q^k, can be defined as country k's *nominal value of final demand in domestic currency units*, $p^k \cdot q^k$, divided by its PPP, P^k:

(10) $Q^k = p^k \cdot q^k/P^k$; k = 1,...,K
$\quad = \pi \cdot q^k$ using (8).

The second set of equations in (10) are the equations which characterize an *additive method*;[19] i.e., the real final demand of each country can be expressed as a *sum* of the country's individual Basic Heading final demand volume components where each real final demand component is weighted by an *international price* which is constant across countries.

[18] For additional material on the GEKS index, see the problems at the end of the chapter. In particular, see Problem 3 which provides an economic interpretation for the GEKS price and quantity indexes.
[19] An additive multilateral system is sometimes said to have the property of *matrix consistency*.

Finally, if equations (10) are substituted into the regional share equations (6), then country k's share of regional real expenditures is

$$(11)\ S^k = \pi\cdot q^k/\pi\cdot q \qquad k = 1,\dots,K$$

where the *region's total volume vector* q is defined as the sum of the country volume vectors:

$$(12)\ q \equiv \textstyle\sum_{j=1}^K q^j .$$

Equations (10) show how convenient it is to have an additive multilateral comparison method: when country outputs are valued at the international reference prices, values are additive across both countries and commodities. However, additive multilateral methods are not consistent with economic comparisons of utility across countries if the number of countries in the comparison is greater than two; see section 6 below. In addition, looking at equations (7), it can be seen that large countries will have a larger contribution to the determination of the international prices π_n and thus these international prices will be much more representative for the largest countries in the comparison as compared to the smaller ones.[20] This leads to the next method for making multilateral comparisons: an additive method that does not suffer from this problem of big countries having an undue influence in the comparison.

4. The Iklé Dikhanov Balk Method

Iklé (1972; 202-204) suggested this method in a very indirect way, Dikhanov (1994) (1997) suggested the much clearer system (13)-(14) below and Balk (1996; 207-208) provided the first existence proof. Dikhanov's (1994; 9-12) equations that are the counterparts to the GK equations (7) and (8) are the following ones:

$$(13)\ \pi_n = [\textstyle\sum_{k=1}^K s_n^k\, [p_n^k/P^k]^{-1}/\sum_{j=1}^K s_n^j]^{-1} ; \qquad n = 1,\dots,N$$
$$(14)\ P^k = [\textstyle\sum_{n=1}^N s_n^k\, [p_n^k/\pi_n]^{-1}]^{-1} \qquad k = 1,\dots,K$$

where the country expenditure shares s_n^k are defined by (2) above.

As in the GK method, equations (13) and (14) involve the K country price levels or PPPs, $P^1,\dots,P^K$, and N international commodity reference prices, $\pi_1,\dots,\pi_N$. Equations (13) indicate that the *nth international price*, π_n, is a *share weighted harmonic mean of the country k Basic Heading PPPs for commodity n*, p_n^k, deflated by country k's overall PPP, P^k. The country k share weights for commodity n, s_n^k, do not sum (over countries k) to unity but when s_n^k is divided by $\sum_{j=1}^K s_n^j$, the resulting normalized shares do sum (over countries k) to unity. Thus equations (13) are similar to the GK equations (7), except that now a harmonic mean of the deflated BH commodity n "prices", p_n^k/P^k, is used in place of the old arithmetic mean and in the GK equations, country k's *volume* share of

[20] Hill (1997) and Dikhanov (1994; 5) made this point.

commodity group n in the region, $q_n^k/\sum_{j=1}^K q_n^j$, was used as a weighting factor (and hence large countries had a large influence in forming these weights) but now the weights involve ***country*** *expenditure* shares and so each country in the region has a more equal influence in forming the weighted average. Equations (14) indicate that P^k, the *PPP for country* k, P^k, is equal to a *weighted harmonic mean of the country k BH PPPs*, p_n^k, deflated by the international price for commodity group n, π_n, where the summation is over commodities n instead of over countries k as in equations (13). The share weights in the harmonic means defined by (14), the s_n^k, of course sum to one when the summation is over n, so there is no need to normalize these weights as was the case for equations (13).

It can be seen that if a solution to equations (13) and (14) exists, then multiplication of all of the country parities P^k by a positive scalar λ and division all of the reference prices π_n by the same λ will lead to another solution to (13) and (14). Hence, the π_n and P^k are only determined up to a scalar multiple and an additional normalization is required such as (9), $P^1 = 1$.

Although the IDB equations (14) do not appear to be related very closely to the corresponding GK equations (8), it can be shown that these two sets of equation are actually the same system. To see this, note that the country k expenditure share for commodity group n, s_n^k, has the following representation:

$$(15)\ s_n^k = p_n^k q_n^k / p^k \cdot q^k \ ; \qquad n = 1,\ldots,N\ ;\ k = 1,\ldots,K.$$

Now substitute equations (15) into equations (14) to obtain the following equations:

$$\begin{aligned}(16)\ P^k &= 1/\textstyle\sum_{n=1}^N s_n^k\,[p_n^k/\pi_n]^{-1} \qquad k = 1,\ldots,K\\ &= 1/\textstyle\sum_{n=1}^N [p_n^k q_n^k/p^k\cdot q^k][\pi_n/p_n^k]\\ &= p^k\cdot q^k/\textstyle\sum_{n=1}^N \pi_n q_n^k\\ &= p^k\cdot q^k/\pi\cdot q^k.\end{aligned}$$

Thus equations (14) are equivalent to equations (8) and the IDB system is an *additive system*; i.e., equations (10)-(12) can be applied to the present method just as they were applied to the GK method for making international comparisons.[21]

The IDB method was used by the African region in order to construct comparable regional aggregates for countries within Africa in the World Bank's 2005 round of international comparisons. Basically, this method appears to be an "improvement" over the GK method in that large countries no longer have a dominant influence on the determination of the international reference prices π_n and so if an additive method is required with more democratic reference prices, IDB appears to be "better" than GK. In addition, Deaton and Heston (2010) have shown empirically that the IDB method

[21] What makes the IDB system special is the fact that equations (16) are equivalent to equations (14). Instead of using harmonic means in equations (13) and (14), we could use more general means, such as means of order r; i.e., we could replace equations (13) by $\pi_n = [\sum_{k=1}^K s_n^k\,[p_n^k/P^k]^r/\sum_{j=1}^K s_n^j]^{1/r}$ and equations (14) by $P^k = [\sum_{n=1}^N s_n^k\,[p_n^k/\pi_n]^r]^{1/r}$ where $r \neq 0$. But it is only when $r = -1$ that the second set of equations simplifies to equations (16), which implies additivity of the method.

generates aggregate PPPs that are much closer to the GEKS PPPs than are GK PPPs, using ICP 2005 data. However, in section 6 below, it will be shown that if one takes the economic approach to index number comparisons, then any additive multilateral method will be subject to some substitution bias.

However, for many users, the issue of possible substitution bias in the multilateral method is not an important one: these users want an additive multilateral method so that they can aggregate in a consistent fashion across countries and commodity groups. For these users, it may be useful to look at the axiomatic properties of the GK and IDB multilateral methods in order to determine a preference for one or the other of these additive methods. Thus in the next section, various multilateral axioms or tests are listed and the consistency of GK, IDB and GEKS with these axioms will be determined.

5. The Test or Axiomatic Approach to Making Multilateral Comparisons

Balk (1996) proposed a system of nine axioms for multilateral methods based on the earlier work of Diewert (1988).[22] Diewert (1999; 16-20) further refined his set of axioms and in this section, eleven of his thirteen "reasonable" axioms he proposed for a multilateral system will be listed. Some new notation will be used in the present section: $P \equiv [p^1,...,p^K]$ will signify an N by K matrix which has the domestic Basic Heading parities (or "price" vectors) $p^1,...,p^K$ as its K columns and $Q \equiv [q^1,...,q^K]$ will signify an N by K matrix which has the country Basic Heading volumes (or "quantity" vectors) $q^1,...,q^K$ as its K columns.

Any multilateral method applied to K countries in the comparison determines the country aggregate volumes, $Q^1,...,Q^K$, along with the corresponding country PPPs, $P^1,...,P^K$, The country volumes Q^k can be regarded as functions of the data matrices P and Q, so that the country volumes can be written as functions of the two data matrices, P and Q; i.e., the multilateral method the functions, $Q^k(P,Q)$ for k = 1,...,K. Once these functions $Q^k(P,Q)$ have been determined by the multilateral method, then *country k's share of total regional real expenditures*, $S^k(P,Q)$, can be defined as follows:

$$(17)\quad S^k(P,Q) \equiv Q^k(P,Q)/[Q^1(P,Q) + ... + Q^K(P,Q)] ; \qquad k = 1,...,K.$$

Both Balk (1996) (2008) and Diewert (1988) (1999) used the system of regional share equations $S^k(P,Q)$ as the basis for their axioms.

Eleven of Diewert's (1999; 16-20) 13 tests or axioms for a multilateral share system, $S^1(P,Q),...,S^K(P,Q)$, are listed below.[23] Before listing these tests, it will be assumed that

[22] Balk's axioms were somewhat different from those proposed by Diewert since Balk also introduced an extra set of country weights into Diewert's axioms. Balk's example will not be followed here since it is difficult to determine precisely what these country weights should be. Rao (2009) also considered adding an extra set of weights to multilateral methods. For the most up to date review of the axiomatic approach to multilateral indexes, see Balk (2008; 232-260).

[23] Diewert's (1999; 18) bilateral consistency in aggregation test is omitted, since this test depends on choosing a "best" bilateral quantity index and there may be no consensus on what this "best" functional

the two data matrices, P and Q, satisfy some mild regularity conditions, R1-R3, which we will now list.

R1: For each country k and each commodity n in country k, either p_n^k, q_n^k and s_n^k are all zero or p_n^k, q_n^k and s_n^k are all positive.

This assumption allows for the possibility that some countries do not consume all of the basic heading commodities.

R2: For every basic heading commodity n, there exists a country k such that p_n^k, q_n^k and s_n^k are all positive so that each commodity is demanded by some country.

R3: For every country k, there exists a commodity n such that p_n^k, q_n^k and s_n^k are all positive so that each country demands at least one basic heading commodity.

In keeping with the literature on test approaches to index number theory, the components of the data matrix Q will be referred to as "quantities" (when in practice, they are usually BH volumes by commodity group and country) and the components of the data matrix P will be referred to as "prices" (when in practice they are usually BH PPPs by commodity group and by country).

T1: *Share Test*: There exist K continuous, positive functions, $S^k(P,Q)$, k = 1,...,K, such that $\sum_{k=1}^K S^k(P,Q) = 1$ for all P,Q in the appropriate domain of definition.

This is a very mild test of consistency for the multilateral system.

T2: *Proportional Quantities Test*: Suppose that $q^k = \beta_k q$ for some $q >> 0_N$ and $\beta_k > 0$ for k = 1,...,K with $\sum_{k=1}^K \beta_k = 1$. Then $S^k(P,Q) = \beta_k$ for k = 1,...,K.

This test says that if the quantity vector for country k, q^k, is equal to the positive fraction β_k times the total regional quantity vector q, then that country's share of regional real expenditures, $S^k(P,Q)$, should equal that same fraction β_k. Note that this condition is to hold no matter what P is.

T3: *Proportional Prices Test*: Suppose that $p^k = \alpha_k p$ for $p >> 0_N$ and $\alpha_k > 0$ for k = 1,...,K. Then $S^k(P,Q) = p\cdot q^k/[p\cdot\sum_{i=1}^K q^i]$ for k = 1,...,K.

This test says the following: suppose that the all of the country price vectors p^k are proportional to a common "price" vector p. Then the country k share of regional real expenditure, $S^k(P,Q)$, is equal to the value of its quantity vector, valued at the common prices p, $p\cdot q^k \equiv \sum_{n=1}^N p_n q_n^k$, divided by the regional value of real expenditures, also valued at the common prices p, $p\cdot\sum_{i=1}^K q^i$. Thus if prices are proportional to a common set of prices p across all countries, then these prices p can act as a set of *reference*

form is. His final axiom involving the consistency of the multilateral system with the economic approach to index number theory will be discussed in section 6 below.

international prices and the real expenditure volume of country k, Q^k, should equal $p \cdot q^k$ up to a normalizing factor.

T4: *Commensurability Test*: Let $\delta_n > 0$ for n = 1,...,N and let Δ denote the N by N diagonal matrix with the δ_n on the main diagonal. Then $S^k(\Delta P, \Delta^{-1} Q) = S^k(P,Q)$ for k = 1,...,K.

This test implies that the country shares $S^k(P,Q)$ are invariant to changes in the units of measurement. This is a standard (but important) test in the axiomatic approach to index number theory that dates back to Fisher (1922; 420).

T5: *Commodity Reversal Test*: Let Π denote an N by N permutation matrix. Then $S^k(\Pi P, \Pi Q) = S^k(P,Q)$ for k = 1,...,K.

This test says that the ordering of the N commodity groups should not affect each country's share of regional real expenditure. This test also dates back to Fisher (1922; 63) in the context of bilateral index number formulae.

T6: *Multilateral Country Reversal Test*: Let S(P,Q) denote a K dimensional column vector that has the country shares $S^1(P,Q)$,.., $S^K(P,Q)$ as components and let Π^* be a K by K permutation matrix. Then $S(P\Pi^*, Q\Pi^*) = S(P,Q)\Pi^*$.

This test implies that countries are treated in a symmetric manner; i.e., the country shares of world output are not affected by a reordering of the countries. The next two tests are homogeneity tests.

T7: *Monetary Units Test*: Let $\alpha_k > 0$ for k = 1,...,K. Then $S^k(\alpha_1 p^1,...,\alpha_K p^K, Q) = S^k(p^1,...,p^K,Q) = S^k(P,Q)$ for k = 1,...,K.

This test implies that the absolute scale of domestic prices in each country does not affect each country's share of world output; i.e., only relative prices within each country affect the multilateral volume parities.

T8: *Homogeneity in Quantities Test*: For i = 1,...,K, let $\lambda_i > 0$ and let j denote another country not equal to country i. Then $S^i(P,q^1,...,\lambda_i q^i,...,q^K)/S^j(P,q^1,...,\lambda_i q^i,...,q^K) = \lambda_i S^i(P, q^1,..., q^i,...,q^K)/S^j(P,q^1,..., q^i,...,q^K) = \lambda_i S^i(P,Q)/S^j(P,Q)$.

This test is equivalent to saying that the volume share of country i relative to country j is linearly homogeneous in the components of the country i quantity vector q^i.

T9: *Monotonicity Test in Quantities Test*: For each k, $S^k(P,q^1,...,q^{k-1},q^k,q^{k+1},...,q^K) = S^k(P,Q)$ is increasing in the components of q^k.

This test says that country k's share of world output increases as any component of the country k quantity vector q^k increases.

T10: *Country Partitioning Test*: Let A be a strict subset of the indexes (1,2,...,K) with at least two members. Suppose that for each $i \in A$, $p^i = \alpha_i p^a$ for $\alpha_i > 0$, $p^a >> 0_N$ and $q^i = \beta_i q^a$ for $\beta_i > 0$, $q^a >> 0_N$ with $\sum_{i \in A} \beta_i = 1$. Denote the subset of {1,2,...,K} that does not belong to A by B and denote the matrices of country price and quantity vectors that belong to B by P^b and Q^b respectively. Then: (i) for $i \in A$, $j \in A$, $S^i(P,Q)/S^j(P,Q) = \beta_i/\beta_j$ and (ii) for $i \in B$, $S^i(P,Q) = S^{i*}(p^a,P^b,q^a,Q^b)$ where $S^{k*}(p^a,P^b,q^a,Q^b)$ is the system of share functions that is obtained by adding the group A aggregate price and quantity vectors, p^a and q^a respectively, to the group B price and quantity data, P^b,Q^b.

Thus if the aggregate quantity vector for the countries in group A were distributed proportionally among its members (using the weights β_i) and each group A country faced prices that were proportional to p^a, then part (i) of T10 requires that the group A share functions reflect this proportional allocation. Part (ii) of T10 requires that the group B share functions are equal to the same values no matter whether we use the original share system or a new share system where all of the group A countries have been aggregated up into the single country which has the price vector p^a and the group A aggregate quantity vector q^a. Conversely, this test can be viewed as a *consistency in aggregation test* if a single group A country is partitioned into a group of smaller countries.

T11: *Additivity Test*: For each set of price and quantity data, P,Q, belonging to the appropriate domain of definition, there exists a set of positive world reference prices $\pi >> 0_N$ such that $S^k(P,Q) = \pi \cdot q^k/[\pi \cdot \sum_{i=1}^K q^i]$ for k = 1,...,K.

Thus if the multilateral system satisfies test T11, then it is an *additive method* since the real expenditure Q^k of each country k is proportional to the inner product of the vector of international prices π with the country k vector of commodity volumes (or "quantities"), q^k.

It is useful to contrast the axiomatic properties of the IDB method with the other additive method that has been used in ICP, the GK system. Using the results in Diewert (1999) on the GK system and the results on the IDB system in Rao and Vogel (2012), it can be seen that both methods satisfy tests T1-T7 and T11 and both methods fail the monotonicity in quantities test T9. Thus the tests that discriminate between the two methods are T8 and T10: the IDB multilateral system passes the homogeneity test T8 and fails the country partitioning test T10 and vice versa for the GK system.[24] There has been more discussion about test T10 than test T8. Proponents of the GK system like the fact that it has good aggregation (across countries) properties and the fact that big countries have more influence on the determination of the world reference price vector π is regarded as a reasonable price to pay for these "good" aggregation properties.[25] On the other hand,

---

[24] Balk (1996; 212) also compares the performance of the two methods (along with other multilateral methods) using his axiomatic system.

[25] Note that the fact that big countries play a more important role in the determination of the international prices when test T10 is satisfied is analogous to a property that national prices have to regional prices when a country's national accounts by product are constructed: the national price for a commodity is taken to be the unit value price for that commodity over the regions within the country. Thus large regions with large

proponents of the IDB method like the fact that the world reference prices are more democratically determined (large countries play a smaller role in the determination of the vector of international prices π) and they place less weight on having good aggregation properties. Also, from evidence presented by Deaton and Heston (2010) using the ICP 2005 data base, it appears that the IDB parities are closer to the GEKS parities than the GK parities. Thus the IDB method has the advantage that it is an additive method that does not depart too far from the parities that are generated by the GEKS method.

Diewert (1999; 18) showed that the GEKS system (using the Fisher ideal index as the basic building block) passed Tests 1-9 but failed the country partitioning test T10 and the additivity test T11. Thus all three of the multilateral methods considered thus far fail two out of the eleven tests.

At this point, it is difficult to unambiguously recommend any one of the three multilateral methods over the other two. In the following section, an economic approach to making multilateral comparisons will be considered which may help in evaluating the three methods.

6. Additive Multilateral Methods and the Economic Approach to Making Index Number Comparisons

It is useful to begin this section by reviewing what are the essential assumptions for the *economic approach to index number theory*:

- Purchasers have preferences over alternative bundles of goods and services that they purchase.
- As a result, they buy more of things that have gone down in relative price and less of things which have gone up in relative price.

The above type of substitution behaviour is well documented and hence, it is useful to attempt to take it into account when doing international comparisons.

The economic approach to index number theory does take substitution behavior into account. This approach was developed by Diewert (1976) in the bilateral context[26] and by Diewert (1999) in the multilateral context. Basically this theory works as follows:

- Assume that all purchasers have the same preferences over commodities and that these preferences can be represented by a homogeneous utility function.
- Find a functional form that can approximate preferences to the second order[27] and has an exact index number formula associated with it. The resulting index number formula is called a *superlative index number formula*.[28]

---

final demands will have a more important role in the determination of the national price vector than the smaller regions.

[26] The pioneers in this approach were Konüs and Byushgens (1926).

[27] Diewert (1974;113) termed such functional forms *flexible*.

[28] Diewert (1976; 117) introduced this concept and terminology.

- Use the superlative index number formula in a bilateral context so that the real output of every country in the region can be compared to the real output of a numeraire country using this formula. The resulting relative volumes are dependent on the choice of the numeraire country.
- Take the geometric average of all K sets of relative volumes using each country in the region as the numeraire country. This set of average relative volumes can then be converted into regional shares as in section 2 above. The resulting method is called a *superlative multilateral method*.[29]

It turns out that the GEKS method discussed in section 2 above is a superlative multilateral method; see Diewert (1999; 36). The GEKS method also has quite good axiomatic properties as was seen in section 5 above.

Given the importance of the GEKS multilateral method, it is worth explaining that the GEKS volume parities can be obtained by alternative methods.

The first alternative method is explained by Deaton and Heston (2010).[30] In this method, the GEKS parities can be obtained by using a least squares minimization problem, due originally to Gini (1924), that will essentially make an K by K matrix of bilateral Fisher volume parities that are not transitive into a best fitting set of transitive parities. The second method for deriving the GEKS parities was explained above. Pick any country as the base country and use the Fisher bilateral quantity index to form the real final demand volume of every country relative to the chosen base country. This gives estimated volumes for all countries in the comparison relative to the chosen base country. Now repeat this process, choosing each country in turn as the base country, which leads to K sets of relative volume estimates. The final step for obtaining the GEKS relative volumes is to take the geometric mean of all of the K base country dependent sets of parities.

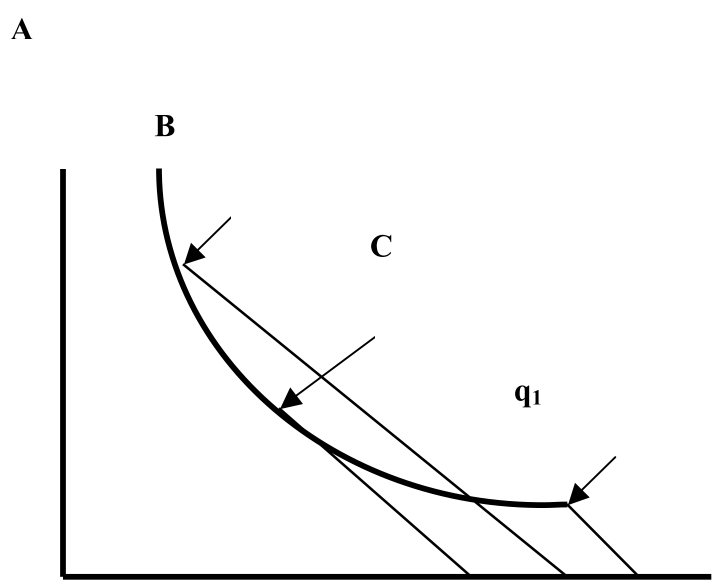
The problem with an additive multilateral method (from the perspective of the economic approach) if the number of countries in the region is greater than two can now be explained with the help of a diagram.[31]

Figure 1: Problems with Making Additive Comparisons when there are Three Countries

[29] See Diewert (1999; 22).

[30] The method is also explained in Problem 1 at the end of the chapter.

[31] This diagram is basically due to Marris (1984; 52) and Diewert (1999; 48-50).



The solid curved line in the above Figure represents an indifference curve for purchasers of the two goods under consideration. The consumption vectors of Countries A, B and C are all on the same indifference curve and hence, the multilateral method should show the same volume for the three countries. If we use the relative prices that country B faces as "world" reference prices in an additive method, then country B has the lowest volume or real consumption, followed by country A and finally, C has the highest volume. But they all have equal volumes! It can be seen that we can devise an additive method that will make the volumes of any two countries equal but we cannot devise an additive method that will equalize the volumes for all three countries. On the other hand, the common indifference curve in Figure 1 can be approximated reasonably well by a flexible functional form that has a corresponding exact index number formula (such as the Fisher index) and thus a GEKS method that used the Fisher bilateral index as a basic building block would give the right answer to a reasonable degree of approximation. The bottom line is that an additive multilateral method is not really consistent with economic comparisons of utility across countries if the number of countries in the comparison is greater than two.[32]

Although additive multilateral methods have their problems in that they are not consistent with substitution in the face of changing relative prices, the economic approach as

---

[32] "Figure 1.1 also illustrates the Gerschenkron effect: in the consumer theory context, countries whose price vectors are far from the 'international' or world average prices used in an additive method will have quantity shares that are biased upward. ... It can be seen that these biases are simply quantity index counterparts to the usual substitution biases encountered in the theory of the consumer price index. However, the biases will usually be much larger in the multilateral context than in the intertemporal context since relative prices and quantities will be much more variable in the former context. ... The bottom line on the discussion presented above is that the quest for an additive multilateral method with good economic properties (i.e., a lack of substitution bias) is a doomed venture: nonlinear preferences and production functions cannot be adequately approximated by linear functions. Put another way, if technology and preferences were always linear, there would be no index number problem and hundreds of papers and monographs on the subject would be superfluous!" W. Erwin Diewert (1999; 50).

explained above is not without its problems. Two important criticisms of the economic approach are:

- The assumption that all final purchasers have the same preferences over different baskets of final demand purchases is suspect and
- The assumption that preferences are homothetic (i.e., can be represented by a linearly homogeneous utility function) is also suspect.

The second criticism of the economic approach to multilateral comparisons based on superlative bilateral index number formulae has been discussed in the recent literature on international comparisons and some brief comments on this literature are in order here.

An important recent development is Neary's (2004) GAIA multilateral system, which can be described as a consumer theory consistent version of the GK system, which allows for nonhomothetic preferences on the part of final demanders. Deaton and Heston (2010) point out that a weakness of the Neary multilateral system is that it uses a single set of relative prices to value consumption or GDP in all countries, no matter how different are the actual relative prices in each country. This problem was also noticed by Feenstra, Ma and Rao (2009) and these authors generalized Neary's framework to work with two sets of cross sectional data in order to estimate preferences and they also experimented with alternative sets of reference prices. Barnett, Diewert and Zellner (2009) in their discussion of Feenstra, Mao and Rao, noted that a natural generalization of their model would be to use a set of reference prices which would be representative for each country in the comparison. Using representative prices for each country would lead to K sets of relative volumes and in the end, these country specific parities could be averaged just as the GEKS method averages country specific parities. Barnett, Diewert and Zellner conjectured that this geometric average of the country estimates will probably be close to GEKS estimates based on traditional multilateral index number theory, which of course, does not use econometrics. It remains to be seen if econometric approaches to the multilateral index number problem can be reconciled with superlative multilateral methods.[33]

In the following section, another economic approach to constructing multilateral comparisons will be described: a method that is based on linking countries that have *similar economic structures*.

7. The Minimum Spanning Tree Method for Making Multilateral Comparisons

Recall that the Fisher ideal quantity index can be used to construct real volumes for all K countries in the comparison, using one country as the base country. Thus as each country is used as the base country, K sets of relative volumes can be obtained. The GEKS multilateral method treats each country's set of relative volumes as being equally valid and hence an averaging of the parities is appropriate under this hypothesis. Thus the

[33] One limitation of econometric approaches is that it will be impossible to estimate flexible functional forms for preferences when the number of commodity groups is as large as 155 since approximately 12,000 parameters would have to be estimated in this case.

method is "democratic" in that each bilateral index number comparison between any two countries gets the same weight in the overall method. However, it is not the case that all bilateral comparisons of volume between two countries are equally accurate: if the relative prices in countries A and B are very similar, then the Laspeyres and Paasche volume or quantity indexes will be very close to each other and hence it is likely that the "true" volume comparison between these two countries (using the economic approach to index number theory) will be very close to the Fisher volume comparison. On the other hand, if the structure of relative prices in the two countries is very different, then it is likely that the structure of relative quantities in the two countries will also be different and hence the Laspeyres and Paasche quantity indexes will likely differ considerably and it is no longer so certain that the Fisher quantity index will be close to the "true" volume comparison. The above considerations suggest that a more accurate set of world product shares could be constructed if initially a bilateral comparison was made between the two countries which had the most *similar relative price structures*.[34] At the next stage of the comparison, look for a third country which had the most similar relative price structure to the first two countries and link in this third country to the comparisons of volume between the first two countries and so on. At the end of this procedure, a *minimum spanning tree* would be constructed, which is a path between all countries that minimized the sum of the relative price dissimilarity measures. This linking methodology has been developed by Robert Hill (1999a) (1999b) (2004) (2009). The conclusion is that similarity linking[35] using Fisher ideal quantity indexes as the bilateral links is an alternative to GEKS which has some advantages over it.[36] Both methods are consistent with the economic approach to index number theory.

A key aspect of this methodology is the choice of the measure of similarity (or dissimilarity) of the relative price structures of two countries. Various measures of the similarity or dissimilarity of relative price structures have been proposed by Allen and Diewert (1981), Kravis, Heston and Summers (1982; 104-106), Aten and Heston (2009),

[34] Note that if all countries in the multilateral comparison have proportional "price" vectors, then the GEKS relative volume for any two countries j relative to i, S^j/S^i, is simply the Fisher ideal quantity index between the two countries, which in turn is equal to $p^i{\cdot}q^j/p^i{\cdot}q^i$ and to $p^j{\cdot}q^j/p^j{\cdot}q^i$, the Laspeyres and Paasche quantity indexes between the two countries. It can be seen that if we choose a vector of international prices π to be any one of the country price vectors, then $S^j/S^i = \pi{\cdot}q^j/\pi{\cdot}q^i = Q^j/Q^i$. Thus under the hypothesis of price proportionality across countries, the country real expenditure levels, Q^k, are proportional to $\pi{\cdot}q^k$ and the GEKS multilateral method can be regarded as an additive method.

[35] Perhaps more descriptive labels for the MST method for making international comparisons is the *similarity linking method* or the *spatial chaining method.*

[36] Deaton (2010; 33-34) noticed the following problem with the GEKS method: suppose we have two countries where the expenditure share on commodity 1 is tiny for country A and very big for country B. Suppose also that the price of commodity 1 in country A is very large relative to the price in country B. Then looking at the Törnqvist price index between A and B, it can be seen that the overall price level for country A will be blown up by the relatively high price for good 1 in A relative to B and by the big expenditure share in B on commodity 1. Since the Törnqvist will generally approximate the corresponding Fisher index closely, it can be seen that we have ended up exaggerating the price level of country A relative to B. This problem can be mitigated by spatial linking of countries that have similar price and quantity structures.

Diewert (2009), Hill (1997) (2009) and Sergeev (2001) (2009). A few of these suggested measures of dissimilarity will now be reviewed.[37]

Suppose that we wish to compare how similar the structure of relative prices is for two countries, 1 and 2, which have the strictly positive Basic Heading PPP vectors p^k and the Basic Heading volume vectors q^k for $k = 1,2$. For convenience of exposition, in remainder of this section, we will refer to the PPP vector p^k as a "price" vector and the volume vector q^k as a "quantity" vector. A *dissimilarity index*, $\Delta(p^1,p^2,q^1,q^2)$, is a function defined over the "price" and "quantity" data pertaining to the two countries, p^1,p^2,q^1,q^2, which indicates how similar or dissimilar the structure of relative prices is in the two countries being considered. If the two price vectors are proportional, so that relative prices in the two countries are equal, then we want the dissimilarity index to equal its minimum possible value, 0; i.e., we want $\Delta(p^1,p^2,q^1,q^2)$ to equal 0 if $p^2 = \lambda p^1$ for any positive scalar λ. If the price vectors are not proportional, then we want the dissimilarity measure to be positive. Thus the larger is $\Delta(p^1,p^2,q^1,q^2)$, the more dissimilar is the structure of relative prices between the two countries.

The first measure of dissimilarity in relative price structures was suggested by Kravis, Heston and Summers (1982; 105)[38] and Robert Hill (1999a) (1999b) (2001) (2004) and it is essentially a normalization of the relative spread between the Paasche and Laspeyres price indexes, so it is known as the *Paasche and Laspeyres Spread relative price dissimilarity measure*, $\Delta_{PLS}(p^1,p^2,q^1,q^2)$:

$$(18)\ \Delta_{PLS}(p^1,p^2,q^1,q^2) \equiv \max\{P_L/P_P,\ P_P/P_L\} - 1 \geq 0$$

where $P_L \equiv p^2{\cdot}q^1/p^1{\cdot}q^1$, and $\equiv p^2{\cdot}q^2/p^1{\cdot}q^2$. Thus if P_L equals P_P, the dissimilarity measure defined by (18) takes on its minimum value of 0; as P_L differs more markedly from P_P, the dissimilarity measure increases and the relative price structures are regarded as being increasingly dissimilar. Diewert (2009; 184) pointed out a major problem with this measure of relative price dissimilarity; namely that it is possible for P_L to equal P_P but yet p^2 could be very far from being proportional to p^1. The following two measures of dissimilarity do not suffer from this problem.

Diewert (2009; 207) suggested the following measure of relative price similarity, the *weighted log quadratic measure of relative price dissimilarity*, $\Delta_{WLQ}(p^1,p^2,q^1,q^2)$:

$$(19)\ \Delta_{WLQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} (1/2)(s_n^{1} + s_n^{2})[\ln(p_n^{2}/p_n^{1}P_F(p^1,p^2,q^1,q^2))]^2$$

where $P_F(p^1,p^2,q^1,q^2) \equiv [p^2{\cdot}q^1\ p^2{\cdot}q^2/p^1{\cdot}q^1\ p^1{\cdot}q^2]^{1/2}$ is the Fisher ideal price index between countries 2 and 1 and $s_n^{c} \equiv p_n^{c}q_n^{c}/p^c{\cdot}q^c$ is the country c expenditure share on commodity n for $c = 1,2$ and $n = 1,...,N$.

---

[37] Recall our more extensive discussion of these measures in Chapter 8, which drew on Diewert (2009).

[38] Kravis, Heston and Summers (1982; 105) proposed another similarity measure that is related to a weighted correlation coefficient between two country price or PPP vectors. However, their measure is not a "pure" bilateral similarity measure since their weights depend on the data of all countries in the comparison.

There is a problem with the dissimilarity measure defined by (19) if for some commodity group n, either p_n^1 or p_n^2 equal 0 (or both prices equal 0), because in these cases, the measure can become infinite.[39] If both prices are 0, then commodity group n is irrelevant for both countries and the nth term in the summation in (19) can be dropped. In the case where one of the prices, say p_n^1 equals 0, but the other price p_n^2 is positive, then it would be useful to have an imputed PPP or "price" for commodity group n in country 1 which will make the final demand volume for that commodity group equal to 0. This reservation price, p_n^{1*} say, could be approximated by simply setting p_n^{1*} equal to $p_n^2/P_F(p^1,p^2,q^1,q^2)$. If p_n^1 equal to 0 in (19) is replaced by this imputed price p_n^{1*}, then it can be seen that $p_n^2/p_n^{1*}P_F(p^1,p^2,q^1,q^2)$ is equal to 1 and the nth term on the right hand side of (19) vanishes. Similarly, in the case where p_n^2 equals 0, but the other price p_n^1 is positive, then set the reservation price for the nth commodity group in country 2, p_n^{2*} say, equal to $p_n^1P_F(p^1,p^2,q^1,q^2)$. If the 0 price p_n^2 in (19) is replaced by the imputed price p_n^{2*}, then it can be seen that $p_n^{2*}/p_n^1P_F(p^1,p^2,q^1,q^2)$ is equal to 1 and the nth term on the right hand side of (19) also vanishes in this case. Thus if there is a zero "price" for either country for commodity group n, then the above convention for constructing an imputed price for the zero price leads to the dropping of nth term on the right hand side of (19).[40]

It can be seen that if prices are proportional for the two countries so that $p^2 = \lambda p^1$ for some positive scalar λ, then $P_F(p^1,p^2,q^1,q^2) = \lambda$ and the measure of relative price dissimilarity $\Delta_{PLQ}(p^1,p^2,q^1,q^2)$ defined by (19) will equal its minimum of 0. Thus the smaller is $\Delta_{PLQ}(p^1,p^2,q^1,q^2)$, the more similar is the structure of relative prices in the two countries.

The method of spatial linking using the relative price dissimilarity measure defined by (19) will be illustrated in the next section.[41] Basically, instead of using the GEKS country shares defined by (6) in section 2, the shares generated by the minimum spanning tree are used to link all of the countries in the comparison.

Diewert (2009; 208) also suggested the following measure of relative price similarity, the *weighted asymptotically quadratic measure of relative price dissimilarity*, $\Delta_{WAQ}(p^1,p^2,q^1,q^2)$:

$$(20)\ \Delta_{WAQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} (1/2)(s_n^1 + s_n^2)\{[(p_n^2/p_n^1P_F(p^1,p^2,q^1,q^2)) - 1]^2 + [(P_F(p^1,p^2,q^1,q^2)p_n^1/p_n^2) - 1]^2\}.$$

As was the case with the dissimilarity index defined by (19), the index defined by (20) will equal plus infinity if one of the prices for commodity group n, p_n^1 or p_n^2, equals

[39] If a price p_n^k equals 0, then we will assume that the corresponding quantity is also 0.
[40] Diewert (2009) did not deal with the zero price problem but it is a real problem that needs to be addressed in order to implement his suggested dissimilarity measures for relative price structures using real data. For additional discussion on the difficulties associated with making comparisons across countries where different commodities are being consumed, see Deaton and Heston (2010) and Diewert (2010a).
[41] Some additional examples will be presented in Chapter 8.

zero.[42] Again, it is useful to define an imputed price for the zero price to insert into the formula and a reasonable convention is to use the same imputed prices that were suggested for (19); i.e., if $p_n^1 = 0$, then define $p_n^{1*} \equiv p_n^2/P_F(p^1,p^2,q^1,q^2)$ and if if $p_n^2 = 0$, then define $p_n^{2*} \equiv p_n^1 P_F(p^1,p^2,q^1,q^2)$. These choices for the imputed prices will cause the nth term in the summation of N terms on the right hand side of (20) to vanish.

Rao, Shankar and Hajarghasht (2010) have used the MST method for constructing PPPs across OECD countries using data for 1996. They used the PLS and WAQ dissimilarity measures defined by (18) and (20) and compared the resulting spatial chains with the standard GEKS method. They found some fairly significant differences between the three sets of parities for the 24 countries in the comparison, with differences in the PPP for a single country of up to 10%. Thus the choice of method does matter, even if the methods of comparison are restricted to multilateral methods that allow for substitution effects. An interesting aspect of their study is that they found when WAQ was used as the dissimilarity measure as opposed to PLS, the linking of the countries was much more intuitive:

"As is generally the case with MSTs, there are a number of counter intuitive paths. For example, Spain and Greece are connected through Portugal, Denmark, USA, UK, Germany, Switzerland, Austria, Sweden, Italy. Similarly Australia and New Zealand are connected through the UK, Germany, Switzerland and Austria. Now we turn to Figure 2 where MST based on relative price distance measure is provided. The links in WRPD based MST are a lot more intuitive and are consistent with the notion of price similarity of the countries. For example, Spain, Italy, Portugal, Greece and Turkey are all connected directly, USA-Canada has a direct link so is the pair Ireland-United Kingdom. Countries like Sweden, Finland, Iceland, Norway and Denmark are all connected together. The main conclusion emerging from Figure 2 is that the WRPD [WAQ] is a better measure of price similarity than the PLS used in the standard MST applications." D.S. Prasada Rao, Sriram Shankar and Gholamreza Hajarghasht (2010).

Thus it appears that the pattern of bilateral links that emerges when using the MST method is much more "sensible" when a more discriminating measure of dissimilarity is used in the linking algorithm, as compared to the use of the Paasche and Laspeyres Spread measure defined by (18). Hence in future applications of the MST method, it is recommended that (18) *not* be used as the dissimilarity measure that is a key input into the MST method.

The narrowing of Paasche and Laspeyres spreads by the use of a spatial chaining method is not the only advantage of this method of linking countries. There are advantages at *lower levels of aggregation* in that if similar in structure countries are compared, generally, it will be found that product overlaps are maximized and hence the BH PPPs will be more accurately determined for similar in structure countries:

"Many differences in quality and proportion of high tech items discussed above are likely to be more pronounced between countries with very different economic structures. If criteria can be developed to identify countries with similar economic structure and they are compared only with each other, then it may overcome many of the issues of quality and lowest common denominator item comparisons. Economically similar countries are likely to have outlet types in similar proportions carrying the same types of goods and

---

[42] If both prices are 0, then simply drop the nth term in the summation on the right hand side of (20).

services. So direct comparisons between such countries will do a better job of holding constant the quality of the items than comparisons across more diverse countries." Bettina Aten and Alan Heston (2009; 251).

"Using the same spanning tree for a number of years would dramatically simplify multilateral international comparisons. Each country would only have to compare itself with its immediate neighbors in the spanning tree, thus reducing the cost and increasing the timeliness of international comparisons. Furthermore, by construction, each country's immediate neighbors in the minimum spanning tree will tend to have similar consumption patterns. This may substantially increase the characteristicity of the comparisons. Geary-Khamis by contrast, compares all countries using a single average price vector. In a comparison over rich and poor countries the average price vector may bear little resemblance to the actual price vectors of many of the countries in the comparison. Conversely, EKS uses all possible combinations of bilateral comparisons. This also requires all countries to provide price and expenditure data on the same set of basic headings, thus reducing the characteristicity of each comparison." Robert Hill (2009; 236-237).

Thus the method of spatial linking, if adopted, would involve some changes to country commodity lists. Each country in the Minimum Spanning Tree would be linked to at least one other country and so for each bilateral link, a list of representative commodities pertaining to that link would have to be priced by the two countries in the link. If a country was a local "star" country and linked to several other countries, then the local star country would have to price out a commodity list that pertained to each pair of bilateral links.

Hill (2009; 237) also pointed out that the basic MST methodology could be adapted to impose a priori restrictions on possible links between certain countries:

"Suppose for example ... we do not want India to be linked directly with Hong Kong. This *exclusion* restriction can be imposed by replacing the PLS between India and Hong Kong, in the K×K PLS matrix, by a large dummy value ... Similarly, suppose we want Korea to be linked directly with Japan. This *inclusion* restriction can be imposed by replacing the PLS measure between Korea and Japan with a small dummy value ... This ensures that the corresponding edge is selected." Robert Hill (2009; 237).

Finally, Hill noted that not all statistical agencies produce data of the same quality and the MST method can be adapted to take this fact into account:

"In particular, some countries have better resourced national statistical offices than others. It would make little sense to put a country with an under resourced national statistical office at the center of a regional star even if so specified by the minimum spanning tree." Robert Hill (2009; 237).

The MST algorithm can be modified to ensure that countries with under resourced statistical offices enter the spanning tree with only one bilateral link to the other countries in the comparison.

To sum up, the *advantages* of the MST method for making multilateral comparisons are as follows:

- The MST method, using a superlative index number formula for forming bilateral links, like GEKS is consistent with the economic approach to making multilateral comparisons; i.e., it takes into account substitution effects.
- The MST method is likely to lead to a more accurate set of parities than those generated by the GEKS method, since the bilateral links between pairs of

countries are based on comparisons between countries with the most similar structures of relative prices; i.e., the MST method is the spatial counterpart to chained annual indexes in the time series context.

- The influence of countries with under resourced statistical agencies can be minimized in a simple modification of the basic MST method.

There are also some *disadvantages* to the spatial linking method:

- The method is not as familiar as GEKS and GK and hence, it will be more difficult to build up a constituency for the use of this method.
- There are some arbitrary aspects to the method as compared to GEKS in that: (i) different measures of dissimilarity could be used and there is no universal agreement at this stage as to which measure is the most appropriate one to use; (ii) the treatment of zero "prices" and "quantities" in the measures of dissimilarity is not completely straightforward and (iii) the treatment of countries with under resourced statistical agencies is also not completely straightforward and moreover, it may prove to be difficult to decide exactly which countries are under resourced.
- The path of bilateral links between countries generated by the method could be unstable; i.e., the Minimum Spanning Tree linking the countries could change when we move from one cross sectional comparison between countries to another cross sectional comparison.[43]

Spatial linking was not used in ICP 2011. Before the MST method is more widely adopted, it will be necessary to do more experimentation and trial runs using the method.

8. An Artificial Data Set Numerical Example

Diewert (1999; 79-84) illustrated the differences between various multilateral methods by constructing country PPPs and shares of "world" final demand volumes for a three country, two commodity example. The GEKS, GK, IDB and MST parities will be calculated in this section using his numerical example.

The price and quantity vectors for the three countries are as follows:

(21) $p^1 \equiv [1,1]$; $p^2 \equiv [10, 1/10]$; $p^3 \equiv [1/10,10]$; $q^1 \equiv [1,2]$; $q^2 \equiv [1,100]$; $q^3 \equiv [1000,10]$.

Note that the geometric average of the prices in each country is 1, so that average price levels are roughly comparable across countries, except that the price of commodity 1 is very high and the price of commodity 2 is very low in country 2 and vice versa for country 3. As a result of these price differences, consumption of commodity 1 is relatively low and consumption of commodity 2 is relatively high in country 2 and vice

[43] However, this evidence of unstable links comes from the results of the MST method using the Paasche and Laspeyres Spread measure of dissimilarity. Drawing on the recent research of Rao, Shankar and Hajarghasht (2010), it is likely that this instability will be reduced if a better measure of dissimilarity is used in the MST algorithm, like those defined by (19) and (20), as opposed to the use of the PLS measure defined by (18).

versa in country 3. Country 1 can be regarded as a tiny country, with total expenditure (in national currency units) equal to 3, country 2 is a medium country with total expenditure equal to 20 and country 3 is a large country with expenditure equal to 200.

The Fisher (1922) quantity index Q_F can be used to calculate the volume Q^k of each country k relative to country 1; i.e., calculate Q^k/Q^1 as $Q_F(p^1,p^k,q^1,q^k) \equiv [p^1 \cdot q^k p^k \cdot q^k / p^1 \cdot q^1 p^k \cdot q^1]^{1/2}$ for k = 2,3. Set $Q^1 = 1$ and then Q^2 and Q^3 are determined and these volumes using country 1 as the base or star country are reported in the Fisher 1 column of Table 1. In a similar manner, use country 2 as the base and use the Fisher formula to calculate Q^1, $Q^2 = 1$ and Q^3. Then divide these numbers by Q^1 and the numbers listed in the Fisher 2 column of Table 1 are obtained. Finally, use country 3 as the base and use the Fisher formula to calculate Q^1, Q^2 and $Q^3 =1$. Then divide these numbers by Q^1 and obtain the numbers listed in the Fisher 3 column of Table 1. Ideally, these Fisher star parities would all coincide but since they do not, take the geometric mean of them and obtain the GEKS parities which are listed in the fourth column of Table 1. Thus for this example, the GEKS economic approach to forming multilateral quantity indexes leads to the volumes of countries 2 and 3 to be equal to 7.26 and 64.81 times the volume of country 1.[44]

Table 1: Fisher Star, GEKS, GK and IDB Relative Volumes for Three Countries

	Fisher 1	**Fisher 2**	**Fisher 3**	**GEKS**	**GK**	**IDB**
Q^1	1.00	1.00	1.00	1.00	1.00	1.00
Q^2	8.12	8.12	5.79	7.26	47.42	33.67
Q^3	57.88	81.25	57.88	64.81	57.35	336.67

Turning to the spatial linking method, it can be seen that country 1 has the most similar structure of prices with both countries 2 and 3; i.e., countries 2 and 3 have the most dissimilar structure of relative prices.[45] Thus in this case, the spatial linking method leads to the Fisher star parities for country 1; i.e., the spatial linking relative outputs are given by the Fisher 1 column in Table 1. Note that these parities are reasonably close to the GEKS parities.

The GK parities for P^k and π_n can be obtained by iterating between equations (7) and (8) until convergence has been achieved.[46] Once these parities have been determined, the Q^k can be determined using equations (10). These country volumes were then normalized so that $Q^1 = 1$. The resulting parities are listed in the GK column in Table 1. It can be seen that the GK parity for Q^3/Q^1, 57.35, is reasonable but the parity for Q^2/Q^1, 47.42, is much too large to be reasonable from an economic perspective. The cause of this unreasonable

[44] Since the Fisher star parities are not all equal, it needs to be recognized that the GEKS parities are only an approximation to the "truth". Thus it could be expected that an economic approach would lead to a Q^2/Q^1 parity in the 5 to 9 range and to a Q^3/Q^1 parity in the 50 to 90 range. Note that the IDB parities are well outside these ranges and the GK parity for Q^2/QY^1 is well outside this suggested range.

[45] This MTS result is obtained for all three measures of dissimilarity (18), (19) and (20) considered in the previous section.

[46] Only 5 iterations were required for convergence.

estimate for Q^2 is the fact that the GK international price vector, $[\pi_1,\pi_2]$, is equal to [1, 9.00] so that these relative prices are closest to the structure of relative prices in country 3, the large country. Thus the relatively large consumption of commodity 2 in country 2 gets an unduly high price weight using the GK vector of international reference prices, leading to an exaggerated estimate for its volume, Q^2. This illustrates a frequent criticism of the GK method: the structure of international prices that it gives rise to is "biased" towards the price structure of the biggest countries.

The IDB parities for the above numerical example are now calculated in order to see if the method can avoid the unreasonable results generated by the GK method. The parities for P^k and π_n can be obtained by iterating between equations (13) and (14) until convergence has been achieved.[47] Once these parities have been determined, the Q^k can be determined using equations (10). These country volumes were then normalized so that $Q^1 = 1$. The resulting parities are listed in the IBD column in Table 1. It can be seen that the GK parity for Q^2/Q^1 is 33.67 which is well outside the suggested reasonable range (from the viewpoint of the economic approach) of 5 to 9 and the GK parity for Q^3/Q^1 is 336.7 which is well outside the suggested reasonable range of 50 to 90. What is the cause of these problematic parities?

The problematic IDB volume estimates are not caused by an unrepresentative vector of international prices since the IBD international price vector, $[\pi_1,\pi_2]$, is equal to [1, 1], which in turn is equal to the vector of (equally weighted) geometric mean commodity prices across countries. The problem is due to the fact that *any* additive method cannot take into account the problem of declining marginal utility as consumption increases if there are 3 or more countries in the comparison. Thus the IBD vector of international prices $\pi = [1,1]$ is exactly equal to the country 1 price vector $p^1 = [1,1]$ and so the use of these international prices leads to an accurate volume measure for country 1. But the structure of the IBD international prices is far different from the prices facing consumers in country 2, where the price vector is $p^2 \equiv [10, 1/10]$. The very low relative price for commodity 2 leads consumers to demand a relatively large amount of this commodity (100 units) and the relatively high price for commodity 1 leads to a relatively low demand for this commodity (1 unit). Thus at international prices, the output of country 2 is $\pi \cdot q^2$ which is equal to 101 as compared to its nominal output $p^2 \cdot q^2$ which is equal to 20. Thus the use of international prices overvalues the output of country 2 relative to country 1 because the international price of commodity 2 is equal to 1 which is very much larger than the actual price of commodity 2 in country 2 (which is 1/10). Note that Q^2/Q^1 is equal to $\pi \cdot q^2/\pi \cdot q^1 = 101/3 = 33.67$, an estimate which fails to take into account the declining marginal utility of the relatively large consumption of commodity 2 in country 1. A similar problem occurs when the outputs of countries 1 and 3 are compared using international prices except in this case, the use of international prices tremendously overvalues country 3's consumption of commodity 1. The problem of finding international reference prices that are "fair" for two country comparisons can be solved[48]

[47] Since all of the prices and quantities are positive in this example, equations (13) and (14) in the main text can be used. Eighteen iterations were required for convergence.

[48] See Diewert (1996; 246) for examples of superlative indexes that are additive if there are only two countries or observations.

but the problem cannot be solved in general if there are three or more countries in the comparison as was seen in section 6 above.

The tentative conclusion at this point is that additive methods for making international price and quantity comparisons where there are tremendous differences in the structure of prices and quantities across countries are likely to give rather different answers than methods that are based on economic approaches. This is why it is important for the International Comparison Program to provide two sets of results: one set based on a multilateral method like GEKS or MST that allows for substitution effects and another set that is based on an additive method like GK or IDB. Thus users can decide which set of estimates to use in their empirical work based on whether they need an additive method (with all of its accompanying desirable consistency in aggregation properties) or whether they need a method that allows for substitution effects.

9. A Numerical Example Based on ICP 1985 Data

Yuri Dikhanov at the World Bank generated some highly aggregated data (across Basic Heading groups) from ICP 1985 on 5 consumption components for 8 countries.
The 8 countries are:

- 1 = Hong Kong;
- 2 = Bangladesh;
- 3 = India;
- 4 = Indonesia;
- 5 = Brazil;
- 6 = Japan;
- 7 = Canada and
- 8 = U.S.

Note that Hong Kong, Japan, Canada and the U.S. can be considered to be "rich" countries while Bangladesh, India, Indonesia and Brazil can be considered to be "less rich".

The 5 commodity groups were:

- 1 = durables;
- 2 = food, alcohol and tobacco;
- 3 = other nondurables excluding food, alcohol, tobacco and energy;
- 4 = energy and
- 5 = services.

The expenditure data (converted to US dollars) and the volume or "quantity" data for the 8 countries are listed in the following Tables 2 and 3.

Table 2: Expenditures in US Dollars for Eight Countries and Five Consumption Categories

n	Hong Kong	Bangladesh	India	Indonesia	Brazil	Japan	Canada	U.S.
1	14320	1963	23207	8234	52722	307547	94121	967374
2	10562	24835	176782	83882	105527	448995	82056	778665
3	14951	5100	60748	15158	60798	272875	69461	992761
4	2619	3094	42126	17573	39933	125835	43342	524288
5	62124	11627	166826	61248	273669	1736977	379629	5559458

Table 3: Quantities (Volumes) in Comparable Units for Eight Countries and Five Consumption Categories

	Hong Kong	Bangladesh	India	Indonesia	Brazil	Japan	Canada	U.S.
1	15523	2312	30189	9781	46146	280001	81021	967374
2	9164	47509	356756	138273	163868	251846	63689	778665
3	17564	10588	180964	29879	65274	200614	58261	992761
4	1095	3033	38377	22084	23963	59439	35714	524288
5	81148	47611	786182	223588	541236	1695136	417210	5559458

If the entries in Table 2 (expenditures converted to US dollars at market exchange rates) are divided by the entries in Table 3 (quantities in comparable units), the Basic Heading prices (converted into US dollars at market exchange rates) for each commodity class for each country are obtained. These prices are listed in Table 4 below.

Table 4: Prices or PPPs of Consumption Components in US Dollars for Eight Countries and Five Consumption Categories

	Hong Kong	Bangladesh	India	Indonesia	Brazil	Japan	Canada	U.S.
1	0.92250	0.84905	0.76872	0.84184	1.14250	1.09838	1.16169	1.0
2	1.15255	0.52274	0.49553	0.60664	0.64398	1.78282	1.28839	1.0
3	0.85123	0.48168	0.33569	0.50731	0.93143	1.36020	1.19224	1.0
4	2.39178	1.02011	1.09769	0.79573	1.66644	2.11704	1.21359	1.0
5	0.76556	0.24421	0.21220	0.27393	0.50564	1.02468	0.90992	1.0

Thus the US price level for each commodity group is set equal to 1 and the other prices are the average domestic prices for the commodity group converted into US dollars at the 2005 market exchange rates. Note that for durables, India has the lowest price level at 0.77 and Canada has the highest level at 1.16; for food, India has the lowest prices at 0.495 while Japan has the highest at 1.78; for other nondurables, India has the lowest price level at 0.33 while Japan has the highest level at 1.36; for energy, Indonesia has the lowest price level at 0.79 while Japan has the highest at 2.12 and for services, India has the lowest price level at 0.21 while Japan has the highest level at 1.02. Thus the amount of price level variation across countries ranges from 38% for durables to 500% for services.

The above data are used to compute relative consumption volumes for the 8 countries using various multilateral methods. Instead of normalizing the relative volumes into shares of "world" consumption, the consumption of each country relative to the consumption of the US will be calculated. This is simply an alternative normalization of the country relative volumes.

The star method for constructing relative volumes for the 8 countries was explained in the previous section. Basically, one country is chosen as the "star" country and the Fisher quantity index of all other countries is calculated relative to the star country, which gives the volume of all 8 countries relative to the star country. In Table 5 below, these star relative volumes are listed except that they have been normalized so that country 8's volume (for the U.S.) has been set equal to unity. This will give the reader some indication of the variability in the data.

Table 5: Fisher Star Volumes Relative to the U.S.

	Hong Kong	Bangladesh	India	Indonesia	Brazil	Japan	Canada	U.S.
Star 1	0.01346	0.01367	0.16021	0.05158	0.09192	0.27530	0.07444	1.00000
Star 2	0.01257	0.01277	0.14351	0.04660	0.08984	0.24629	0.07129	1.00000
Star 3	0.01275	0.01350	0.15178	0.04984	0.09040	0.25596	0.07328	1.00000
Star 4	0.01277	0.01341	0.14902	0.04894	0.09141	0.25496	0.07262	1.00000
Star 5	0.01323	0.01284	0.15169	0.04837	0.09035	0.26357	0.07372	1.00000
Star 6	0.01355	0.01437	0.16439	0.05322	0.09504	0.27724	0.07464	1.00000
Star 7	0.01343	0.01331	0.15387	0.05007	0.09105	0.27596	0.07429	1.00000
Star 8	0.01346	0.01277	0.15178	0.04894	0.09035	0.27724	0.07429	1.00000

The Fisher star parities for the 7 countries relative to the US had the following relative volume ranges: Hong Kong: 0.01257 to 0.01355 (7.7% variation); Bangladesh: 0.01277 to 0.01437 (12.5%); India: 0.14351 to 0.16439 (14.5%); Indonesia: 0.04660 to 0.05322 (14.2%); Brazil: 0.08984 to 0.09504 (5.8%); Japan: 0.24629 to 0.27724 (12.6%) and Canada: 0.07129 to 0.07464 (4.7%). Thus the variation in relative volumes is quite large, depending on which country is used as the base country in a comparison based on the use of Fisher star parities.

The GEKS, GK and IDB methods for comparing relative volumes were explained in previous sections. Consumption volumes (relative to the U.S.) for the 8 countries were computed using these methods and they are listed in Table 9 below.

The Hill MST spatial linking method relative volumes were also computed using the 3 dissimilarity measures (26)-(28) above. The first measure of relative price dissimilarity is the Weighted Log Quadratic measure of relative price dissimilarity defined by (19) above. The 8 by 8 matrix of relative price dissimilarity measures is listed in Table 11 below.

Table 6: Weighted Log Quadratic Relative Price Dissimilarities between Eight Countries

	HK	**BGD**	**IND**	**INDO**	**BRA**	**JPN**	**CAN**	**US**
HK	0.00000	0.10056	0.11017	0.09067	0.07011	0.01381	0.03660	0.06143

BGD	0.10056	0.00000	0.01188	0.01165	0.05632	0.10506	0.13237	0.23223
IND	0.11017	0.01188	0.00000	0.03133	0.08980	0.13429	0.18955	0.29841
INDO	0.09067	0.01165	0.03133	0.00000	0.07084	0.07726	0.09610	0.19600
BRA	0.07011	0.05632	0.08980	0.07084	0.00000	0.09146	0.08770	0.14328
JPN	0.01381	0.10506	0.13429	0.07726	0.09146	0.00000	0.01904	0.05322
CAN	0.03660	0.13237	0.18955	0.09610	0.08770	0.01904	0.00000	0.02020
US	0.06143	0.23223	0.29841	0.19600	0.14328	0.05322	0.02020	0.00000

Looking at the above Table, it can be seen that the 8 countries group themselves into two groups that have similar price structures: the rich countries HK, JPN, CAN and US (countries 1, 6, 7 and 8) form one group and the less rich countries BGD, IND, INDO and BRA (2, 3, 4 and 5) form the other group. The linking between the two groups took place via Hong Kong and Brazil.[49] The details of the spatial linking process are as follows. Country 7 (CAN) is linked to 8 (US) (the WLQ dissimilarity measure Δ_{WLQ} equals 0.0202) and 7 (CAN) is linked to 6 (JPN) as well (Δ = 0.019). Then country 6 (JPN) is linked to 1 (HK) (Δ = 0.0138) and this completes the linking of the rich countries. Country 2 acts as a star country for the poorer countries: 2 (BGD) is linked to 4 (INDO) (Δ = 0.0116); 2 (BGD) is linked to 3 (INDIA) (Δ = 0.0118) and 2 (BGD) is linked to 5 (BRAZIL) (Δ = 0.056). Finally, the two groups of countries are linked via countries 1 (HK) and 5 (BRAZIL) (Δ = 0.070). The resulting MST volumes relative to the US are listed in Table 14 below.

The second measure of relative price dissimilarity is the Weighted Asymptotic Quadratic measure of relative price dissimilarity defined by (20) above. The 8 by 8 matrix of relative price dissimilarity measures is listed in Table 12 below.

Table 7: Weighted Asymptotic Quadratic Relative Price Dissimilarities between Eight Countries

	HK	**BGD**	**IND**	**INDO**	**BRA**	**JPN**	**CAN**	**US**
HK	0.00000	0.24097	0.27260	0.20885	0.15436	0.02914	0.10005	0.18062
BGD	0.24097	0.00000	0.02444	0.02455	0.11694	0.28243	0.32119	0.58729
IND	0.27260	0.02444	0.00000	0.07050	0.19169	0.36590	0.50994	0.85586
INDO	0.20885	0.02455	0.07050	0.00000	0.15038	0.20144	0.21328	0.45346
BRA	0.15436	0.11694	0.19169	0.15038	0.00000	0.21447	0.20876	0.36462
JPN	0.02914	0.28243	0.36590	0.20144	0.21447	0.00000	0.04023	0.11820
CAN	0.10005	0.32119	0.50994	0.21328	0.20876	0.04023	0.00000	0.04106
US	0.18062	0.58729	0.85586	0.45346	0.36462	0.11820	0.04106	0.00000

Note that the WAQ dissimilarity measures listed in Table 7 are roughly two to three times the size of the WLQ measures listed in Table 6. The lowest measure of dissimilarity is between Bangladesh and India (Δ_{WAQ} = 0.02444) and then between

[49] Note that another possible bilateral link between the two regions would be via Indonesia and Japan which have a dissimilarity measure equal to 0.07726 which is a bit higher than the Hong Kong and Brazil dissimilarity measure which was equal to 0.07011.

Bangladesh and Indonesia (Δ = 0.02444). Then there is a shift to the rich countries where the next lowest measure of dissimilarity is between Hong Kong and Japan (Δ = 0.02914). The next lowest measure is between Japan and Canada (Δ = 0.04023) and then between Canada and the U.S. (Δ = 0.04106). Thus the rich countries are linked: Hong Kong to Japan, then Japan to Canada and then Canada to the U.S. The next lowest measure of dissimilarity is between India and Indonesia but we have already linked Bangladesh to both India and Indonesia so we move to the next lowest measure of dissimilarity which is between Bangladesh and Brazil (Δ = 0.11694). Thus now all of the poor countries are linked: Bangladesh is a poor country star, directly linked to India, Indonesia and Brazil. Now we need to link the rich and poor countries and the lowest dissimilarity measure between these two groups is again between Hong Kong and Brazil. Thus the MST generated by the Weighted Asymptotic Quadratic measure of relative price dissimilarity is exactly the same as the tree generated by the Weighted Log Quadratic measure. Thus the MST (WLQ) relative volume parities will be exactly the same as the MST (WAQ) parities; see Table 14 below.

The third measure of relative price dissimilarity is the Paasche and Laspeyres Spread measure of relative price dissimilarity defined by (18) above. The 8 by 8 matrix of relative price dissimilarity measures is listed in Table 8 below.

Table 8: Paasche and Laspeyres Spread Relative Price Dissimilarities between Eight Countries

	HK	**BGD**	**IND**	**INDO**	**BRA**	**JPN**	**CAN**	**US**
HK	0.00000	0.06486	0.06860	0.08716	0.12845	0.01907	0.08539	0.09904
BGD	0.06486	0.00000	0.02484	0.02354	0.09876	0.09420	0.06389	0.02905
IND	0.06860	0.02484	0.00000	0.04596	0.00007	0.10421	0.13058	0.08134
INDO	0.08716	0.02354	0.04596	0.00000	0.05308	0.06806	0.02150	0.03897
BRA	0.12845	0.09876	0.00007	0.05308	0.00000	0.09387	0.11206	0.11177
JPN	0.01907	0.09420	0.10421	0.06806	0.09387	0.00000	0.01615	0.03390
CAN	0.08539	0.06389	0.13058	0.02150	0.11206	0.01615	0.00000	0.00133
US	0.09904	0.02905	0.08134	0.03897	0.11177	0.03390	0.00133	0.00000

The lowest measure of dissimilarity is between countries India and Brazil (Δ_{PLS} = 0.00007) and then between Canada and the U.S. (Δ = 0.00133). The next lowest measure of dissimilarity is between Japan and Canada (Δ = 0.01615) and then the next lowest measure is between Japan and Hong Kong (Δ = 0.01907). Thus the rich countries are linked: Hong Kong to Japan, then Japan to Canada and then Canada to the U.S, which is exactly the same set of linkages generated by the WLQ and WAQ measures of dissimilarity. The next lowest measure of dissimilarity is between Indonesia and Canada (Δ = 0.02150) so the rich and poor countries are now linked by Indonesia and Canada! Recall that in the previous two spanning trees, the rich and poor countries were linked by Hong Kong and Brazil. The next lowest measure of dissimilarity is between Bangladesh and Indonesia (Δ = 0.02354) and then between Bangladesh and India (Δ = 0.02484) Thus now all of the poor countries are linked: Indonesia to Bangladesh; Bangladesh to India and India to Brazil. As mentioned above, rich and poor countries are linked via Indonesia and Canada. Thus the MST generated by the Paasche and Laspeyres Spread measure of relative price dissimilarity is quite different from the trees generated by the WLQ and WAQ measures. The MST (PLS) relative volume parities are reported in Table 9 below.

The country consumption volumes relative to the US for the 6 multilateral methods are listed in Table 9 below.

Table 9: Country Consumption Volumes Relative to the US Using Six Multilateral Methods

Method	HK	BGD	INDIA	INDO	BRA	JPN	CAN	US
GEKS	0.01315	0.01332	0.15317	0.04966	0.09128	0.26556	0.07357	1.0
MTS (WLQ)	0.01349	0.01310	0.14720	0.04779	0.09214	0.27596	0.07429	1.0
MTS (WAQ)	0.01349	0.01310	0.14720	0.04779	0.09214	0.27596	0.07429	1.0
MTS (PLS)	0.01349	0.01372	0.15420	0.05007	0.09184	0.27596	0.07429	1.0
GK	0.01386	0.01357	0.16258	0.05057	0.09613	0.27814	0.07431	1.0
IDB	0.01346	0.01392	0.16187	0.05143	0.09441	0.27076	0.07417	1.0

The above volume parities for the 7 countries relative to the US had the following ranges: Hong Kong: 0.01315 to 0.01386 (5.4% variation); Bangladesh: 0.01310 to 0.01392 (6.3%); India: 0.14720 to 0.16258 (10.4%); Indonesia: 0.04779 to 0.05143 (7.6%); Brazil: 0.09128 to 0.09613 (5.3%); Japan: 0.26556 to 0.27814 (4.7%) and Canada: 0.07357 to 0.07439 (1.1%). Thus the variation in relative volumes is quite large, depending on which multilateral method is used.

It can be seen that the relative consumption volumes generated by the four methods based on the use of a bilateral superlative index (the GEKS and the MTS or similarity linking methods) are fairly close to each other and the relative consumption volumes generated by the two additive methods (GK and IDB) are also fairly close to each other but the additive methods tend to overstate the consumption levels of the poorer countries (Bangladesh, India, Indonesia and Brazil) relative to the US.[50]

From the above Table 9, it is difficult to choose between GK and IDB if an additive method is required: both methods tend to overstate the relative volume of poor countries relative to rich countries but the degree of overstatement seems to vary between poor countries.

Turning to methods based on the economic approach to multilateral comparisons, the MST method based on the Paasche and Laspeyres spread is not recommended since this measure of dissimilarity does not adequately distinguish dissimilar price vectors. In the

[50] The GK volumes relative to the GEKS volumes (with U.S. volumes normalized to equal 1) were all higher for the seven non U.S. countries by the following percentages: 5.4% for Hong Hong, 1.9% for Bangladesh, 6.1% for India, 1.8% for Indonesia, 5.3% for Brazil, 4.7% for Japan and 1.0% for Canada. The IDB volumes relative to the GEKS volumes (with U.S. volumes normalized to equal 1) were also all higher for the seven non U.S. countries by the following percentages: 2.3% for Hong Hong, 4.5% for Bangladesh, 5.7% for India, 3.6% for Indonesia, 3.4% for Brazil, 2.0% for Japan and 0.8% for Canada.

above empirical example, the WLQ and WAQ measures of dissimilarity gave rise to the same set of comparisons and so for this example, these two variants of the MST method cannot be distinguished from one another. The differences between the GEKS volume estimates and the MST(WLQ) estimates are smaller than the differences between the GEKS estimates and the two additive methods but there are some significant differences.[51]

What are the advantages and disadvantages of using GEKS versus MST (WLQ) or MTS (WAQ)? The GEKS method has the advantage that it makes use of all possible bilateral comparisons between each pair of countries in the comparison and thus it is more robust to data problems in any one country. On the other hand, the MST method is very dependent on each set of bilateral comparisons in the final tree of comparisons and so poor quality data for any single country could adversely affect the overall quality of the comparison. But if the quality of data is roughly the same across countries, *the MST method is the spatial counterpart to the use of the chain principle in annual intertemporal comparisons*; i.e., using the MST method, the countries which have the most similar structure of relative prices are compared and bilateral comparisons are generally regarded as being more accurate if the structure of relative prices is similar. Thus in the example above, the U.S. and Canada (which have very similar structures of relative prices) are linked directly via the Fisher index between these two countries using the MST method, whereas using GEKS, links involving all other countries enter the comparison. Thus if data quality were uniformly high across countries, the MST method would seem to be preferred over GEKS.[52]

10. Conclusion

This chapter discussed four multilateral methods for constructing PPPs and relative volumes for countries in a region.

Two of the methods were additive methods: the Geary Khamis (GK) method and the Iklé Dikhanov Balk (IDB) method. Additive methods are preferred by many users due to the fact that components of real GDP add up across countries and across commodities when an additive multilateral method is used.

Which additive method is "best"? The axiomatic properties of the IDB and GK systems are very similar and so it is difficult to discriminate between the two methods based on their axiomatic properties. The main *advantages of the IDB method* are as follows:

- The IDB international prices are not as influenced by the structure of relative prices in the biggest countries in the region as compared to the GK international

[51] The MTS(WLQ) volumes relative to the GEKS volumes (with U.S. volumes normalized to equal 1) differed by the by the following percentages: 2.6% for Hong Hong, −1.7% for Bangladesh, −3.9% for India, −3.8% for Indonesia, 0.9% for Brazil, 3.9% for Japan and 1.0% for Canada.
[52] However, data quality is not uniformly high across countries so this argument for the use of the MTS method is not decisive.

prices ; i.e., the IDB method is more "democratic" than the GK method in its choice of international prices.
- From evidence presented by Deaton and Heston (2010) using the ICP 2005 data base, it appears that the IDB parities are closer to the GEKS parities than the GK parities. Thus the IDB method may have the advantage that it is an additive method that does not depart too far from the parities that are generated by the GEKS method.[53]

The main *advantages of the GK system* are as follows:

- The GK system has been widely used in previous ICP rounds and so users are familiar with the method and may want to continue to use the results of this method.
- The GK system has some similarity with the construction of national accounts data when quantities are aggregated over regions and thus GK estimates may be regarded as a reasonable extension of country wide national accounts to the world.

The other two methods discussed in this chapter were the Gini-Eltetö-Köves-Szulc (GEKS) method and the Minimum Spanning Tree (MST) method of similarity or spatial linking developed by Robert Hill using Fisher ideal indexes as basic bilateral building blocks. These two methods can be regarded as being consistent with an economic approach to a multilateral method; i.e., these methods deal adequately with substitution behavior on the part of purchasers of a country's outputs. The spatial linking method was not used in ICP 2005 or ICP 2011 but it has some attractive features which were discussed in sections 7-9 above.

Problems

(1). (Alternative derivations of Generalized GEKS for an arbitrary bilateral price index number formula). Recall the notation around equations (1) and (2) in section 2 above that defined the country k PPP (prices at constant units of measurement across countries) and volume (quantities in constant units) vectors, p^k and q^k for $k = 1,...,K$. Suppose that the bilateral price index formula $P(p^0,p^1,q^0,q^1)$ satisfies the *identity* and *time reversal tests*, (i) and (ii):

(i) $P(p,p,q,q) = 1$;
(ii) $P(p^1,p^0,q^1,q^0) = 1/P(p^0,p^1,q^0,q^1)$.

When making international comparisons of prices across countries for the same group of N commodities during a time period, it would be ideal if there were K positive *country price levels*, $P^1,P^2,...,P^K$ such that the bilateral indexes $P(p^i,p^j,q^i,q^j)$ that compared the prices in country j relative to those of country i satisfied the following K^2 equations:

[53] However, the second example in Chapter 8 below indicates that the IDB parities may not always be closer to the GEKS parities than the GK parities.

(iii) $P(p^i,p^j,q^i,q^j) = P^j/P^i$; $i = 1,...,K; j = 1,...,K.$

Note that if there exist $P^{1*} > 0$, $P^{2*} > 0,..., P^{K*} > 0$ that satisfies equations (iii), then λP^{1*}, $\lambda P^{2*},..., \lambda P^{K*}$ for any $\lambda > 0$ will also satisfy equations (iii). But equations (iii) will only hold exactly if the bilateral index number formula satisfies the circularity test and from Chapter 3, we saw that this essentially implies that the bilateral formula must be either a Lowe index or a Cobb-Douglas index and since neither of these indexes are superlative, they will not deal adequately with substitution bias. Thus we will look for K price levels which satisfy the transitivity equations (iii) approximately. Following Eltetö and Köves (1964) and Szulc (1964),[54] we essentially add error terms to equations (iii) and treat the $P^1,P^2,...,P^K$ (or their logarithms) as parameters to be estimated in an econometric model. Thus define y_{ij} and α_j as follows:

(iv) $y_{ij} \equiv \ln P(p^i,p^j,q^i,q^j)$; $i = 1,...,K; j = 1,...,K;$
(v) $\alpha_i \equiv \ln P^i$; $i = 1,...,K.$

Now take logarithms of both sides of equations (iii) and add error terms ε_{ij} to equation i,j and we obtain the following linear regression model:

(vi) $y_{ij} = \alpha_j - \alpha_i + \varepsilon_{ij}$; $i = 1,...,K; j = 1,...,K.$

Suppose that $\alpha_1^*,\alpha_2^*,...,\alpha_K^*$ is a solution to the following least squares minimization problem:

(vii) $\min_{\alpha\text{'s}} \Sigma_{i=1}^K \Sigma_{j=1}^K (y_{ij} - \alpha_j + \alpha_i)^2$.

It can be seen that if $\alpha_1^*,\alpha_2^*,...,\alpha_K^*$ is a solution to (vii), then $\alpha_1^*+\beta,\alpha_2^*+\beta,...,\alpha_K^*+\beta$ is also a solution to the minimization problem (vii), where β is any constant. Thus the solution to the least squares minimization problem (vii) will not be unique and we will need a normalization on the α_j (like $\alpha_1 = 0$ which corresponds to $P_1 \equiv \exp(\alpha_1) = 1$) in order to obtain a unique solution.

(a). Differentiate the objective function in (vii) with respect to α_j and set the partial derivative equal to 0. Show that the resulting first order necessary condition is equivalent to the following system of equations:

(viii) $\alpha_j - (1/K)\Sigma_{i=1}^K \alpha_i = (1/K)\Sigma_{i=1}^K y_{ij}$; $j = 1,...,K.$

(b) Use definitions (iv) and (v) to show that equations (viii) for the α_j imply the following equations for the P^j:

[54] Gini (1924) (1931), Eltetö and Köves (1964) and Szulc (1964) used the Fisher ideal index as their bilateral index number formula; here we develop their algebraic method using a general bilateral price index satisfying properties (i) and (ii). Van Ijzeren (1956) (1983) (1987) considered a number of alternative choices (including the Paasche, Laspeyres and Fisher indexes) for the bilateral index using the methodology explained in this problem.

(ix) $P^j/[\Pi_{i=1}{}^K P^i]^{1/K} = [\Pi_{i=1}{}^K P(p^i,p^j,q^i,q^j)]^{1/K}$; $j = 1,...,K.$

Thus it can be seen that the set of generalized GEKS price levels (where the logarithms of these price levels solve the least squares minimization problem (vii)) is equal to $\lambda P^{1*},\lambda P^{2*},...,\lambda P^{K*}$ where $\lambda > 0$ and $P^{j*} \equiv [\Pi_{i=1}{}^K P(p^i,p^j,q^i,q^j)]^{1/K}$ for $j = 1,...,K$. If the bilateral index number formula $P(p^i,p^j,q^i,q^j)$ is the Fisher ideal formula, then these PPPs for $\lambda = 1$ (the P^{j*}) are the same as the PPPs defined by equations (4) in section 2 above.

(c) Use equations (ix) and the properties of $P(p^i,p^j,q^i,q^j)$ to show that the ratio of P^{j*} to P^{k*} for any j and k is equal to the following expression:

(x) $P^{j*}/P^{k*} = [\Pi_{i=1}{}^K P(p^k,p^i,q^k,q^i)P(p^i,p^j,q^i,q^j)]^{1/K}$; $i = 1,...,K; j = 1,...,K.$

The above formula appears in the GEKS literature fairly often.

2. (Alternative derivation of GEKS using bilateral *quantity* indexes).[55] This problem is a continuation of Problem 1 above. Let $Q(p^0,p^1.q^0,q^1)$ be the implicit bilateral quantity index that corresponds to the bilateral price index, $P(p^0,p^1.q^0,q^1)$; i.e., define Q as follows:

(xi) $Q(p^0,p^1.q^0,q^1) \equiv (p^1\cdot q^1/p^0\cdot q^0)/P(p^0,p^1.q^0,q^1)$.

(a) If $P(p^0,p^1.q^0,q^1)$ satisfies tests (i) and (ii) in Problem 1 above, show that Q satisfies the following *identity and time reversal tests* for quantity indexes:

(xii) $Q(p,p.q,q) = 1$;
(xiii) $Q(p^1,p^0,q^1,q^0) = 1/Q(p^0,p^1,q^0,q^1)$.

Since Q is the implicit quantity index corresponding to the bilateral price index P, using (xi) repeatedly shows that the bilateral indexes P and Q satisfy the following K^2 equations:

(xiv) $P(p^i,p^j,q^i,q^j) = p^j\cdot q^j/[p^i\cdot q^i Q(p^i,p^j,q^i,q^j)]$; $i = 1,...,K; j = 1,...,K.$

Recall that from Problem 1, the K positive *country price levels* or PPPs were defined as the scalars $P^1,P^2,...,P^K$. Once these price levels have been determined, the corresponding country aggregate *quantity levels* (in comparable units) or *volume levels*, $Q^1,Q^2,...,Q^K$, are defined as follows:

(xv) $Q^k \equiv p^k\cdot q^k/P^k$; $k = 1,...,K.$

(Note that the PPPs are similar to country exchange rates: if all goods services were internationally traded and there were no transport costs, then the PPPs should be equal to corresponding country exchange rates if the law of one price held).

[55] This kind of derivation is also due to van Ijzeren.

(b) Show that the following equations hold:

(xvi) $\ln[P(p^i,p^j,q^i,q^j)P^i/P^j] = y_{ij} + \alpha_i - \alpha_j$; $\quad i = 1,...,K; j = 1,...,K$

where the y_{ij} and α_i are defined by (v) and (vi) in Problem 1. [This is very easy!]

(c) Show that the following equations hold using equations (xi), (xiv) and (xv):

(xvii) $P(p^i,p^j,q^i,q^j)P^i/P^j = [Q(p^i,p^j,q^i,q^j)Q^i/Q^j]^{-1}$; $\quad i = 1,...,K; j = 1,...,K.$

Now we look at the problem of making comparisons across countries in terms of country volume or quantity levels $Q^1,...,Q^K$. It would be ideal if these quantity levels were such the bilateral implicit quantity indexes $Q(p^i,p^j,q^i,q^j)$ that compared the quantities in country j relative to those of country i satisfied the following K^2 equations:

(xviii) $Q(p^i,p^j,q^i,q^j) = Q^j/Q^i$; $\quad i = 1,...,K; j = 1,...,K.$

Again note that if there exist $Q^{1*} > 0, Q^{2*} > 0,..., Q^{K*} > 0$ that satisfies equations (xviii), then λQ^{1*}, $\lambda Q^{2*},..., \lambda Q^{K*}$ for any $\lambda > 0$ will also satisfy equations (xviii). We will look for K quantity levels which satisfy the transitivity equations (xviii) approximately. Again we add error terms to equations (xviii) and treat the $Q^1,Q^2,...,Q^K$ (or their logarithms) as parameters to be estimated in an econometric model. Thus define z_{ij} and β_j as follows:

(xix) $z_{ij} \equiv \ln Q(p^i,p^j,q^i,q^j)$; $\quad i = 1,...,K; j = 1,...,K;$
(xx) $\beta_i \equiv \ln Q^i$; $\quad i = 1,...,K.$

Now take logarithms of both sides of equations (xviii) and add error terms η_{ij} to equation i,j and we obtain the following linear regression model:

(xxi) $z_{ij} = \beta_j - \beta_i + \eta_{ij}$; $\quad i = 1,...,K; j = 1,...,K.$

Suppose that $\beta_1^*,\beta_2^*,...,\beta_K^*$ is a solution to the following least squares minimization problem:

(xxii) $\min_{\beta\text{'s}} \Sigma_{i=1}^K \Sigma_{j=1}^K (z_{ij} - \beta_j + \beta_i)^2$.

We can repeat the analysis we did for the PPPs and it can be seen that the set of generalized GEKS volume levels (where the logarithms of these quantity levels solve the least squares minimization problem (vii)) is equal to $\lambda Q^{1*},\lambda Q^{2*},...,\lambda Q^{K*}$ where $\lambda > 0$ and $Q^{j*} \equiv [\Pi_{i=1}^K Q(p^i,p^j,q^i,q^j)]^{1/K}$ for $j = 1,...,K$. Using equations (xvii), it can be seen the minimum in (xxii) is equal to the minimum in (vii). Moreover, it can be shown that if we generate the PPPs $P^{j*} \equiv [\Pi_{i=1}^K P(p^i,p^j,q^i,q^j)]^{1/K}$ for $j = 1,...,K$ whose logs solve the PPP least squares problem (vii) and define the companion country volumes as $Q^{j*} \equiv p^j \cdot q^j/P^{j*}$ for $k = 1,...,K$, then the logarithms of these Q^{j*} will be a solution to the volumes least squares problem (xxii). Finally in order for the solution $P^{1*},...,P^{K*}$ and $Q^{1*},...,Q^{K*}$ to satisfy equations (xv), we can only impose one normalization on this solution, like $P^{1*} = 1$ or $\Sigma_{k=1}^{K} Q^{k*} = 1$ (the volumes Q^{k*} can be interpreted as shares of world "output" if we impose the last normalization).

3. (Economic properties of the GEKS indexes). Assume that country k has N(k) households and each household has the same preferences f(q) that are dual to the square root quadratic unit cost function c(p) defined by:

(i) $c(p) \equiv (p^TBp)^{1/2}$; $B = B^T$

where the b_{nm} parameters in the matrix B are such that c(p) is positive, increasing and concave in the components of p over a convex region of prices that includes the country price vectors $p^1,p^2,...,p^K$. Suppose that each household in country k faces the same price vector p^k for k = 1,...,K and the observed quantity vector for household i in country k, q^{ki}, generates the utility level u^{ki} defined as:

(ii) $u^{ki} \equiv f(q^{ki})$; k = 1,...,K; i = 1,...,N(k).

Note that we are assuming that each household in country k faces the same price vector p^k but of course, these price vectors can differ across countries. Assume that q^{ki} is a solution to the following cost (or expenditure) minimization problem:

(iii) $\min_q \{p^k\cdot q : f(q) \geq u^{ki}\} = c(p^k)u^{ki} = c(p^k)f(q^{ki})$; k = 1,...,K; i = 1,...,N(k).

The aggregate quantity vector for each country k, q^k, is defined as the sum of the individual household quantity vectors for country k:

(iv) $q^k \equiv \Sigma_{i=1}^{N(k)} q^{ki}$; k = 1,...,K.

(a) Show that

(v) $q^k = \Sigma_{i=1}^{N(k)} u^{ki} Bp^k/c(p^k)$; k = 1,...,K.

Hint: Apply Shephard's Lemma and use (i)-(iv).

(b) Show that the following expressions for the Laspeyres and Paasche price indexes between countries j relative to k are valid:

(vi) $P_L(p^k,p^j,q^k,q^j) \equiv p^j\cdot q^k/p^k\cdot q^k = p^{jT}Bp^k/c(p^k)^2$; j = 1,...,K ; k = 1,...,K;
(vii) $P_P(p^k,p^j,q^k,q^j) \equiv p^j\cdot q^j/p^k\cdot q^j = c(p^j)^2/p^{kT}Bp^j$; j = 1,...,K ; k = 1,...,K.

The Fisher price indexes $P_F(p^k,p^j,q^k,q^j)$ between countries j relative to k are defined as the geometric means of the corresponding Laspeyres and Paasche price indexes:

(viii) $P_F(p^k,p^j,q^k,q^j) \equiv [p^j\cdot q^j\ p^j\cdot q^k/p^k\cdot q^j\ p^k\cdot q^k]^{1/2}$; j = 1,...,K ; k = 1,...,K.

(c) Show that $P_F(p^k,p^j,q^k,q^j)$ is equal to the following expression:

(ix) $P_F(p^k,p^j,q^k,q^j) = c(p^j)/c(p^k)$; $j = 1,...,K$; $k = 1,...,K$.

From section 2, recall that the (unnormalized) GEKS PPP for country j, P^j, was defined by equation (4) as follows:

(x) $P^j \equiv \prod_{k=1}^K [P_F(p^k,p^j,q^k,q^j)]^{1/K}$; $j = 1,...,K$.

(d) Show that P^j is equal to the following expression under our assumptions on preferences:

(xi) $P^j \equiv c(p^j)/[\prod_{k=1}^K c(p^k)]^{1/K}$; $j = 1,...,K$.

Thus the GEKS country PPPs are proportional to the unit cost function that defines country preferences evaluated at the country price vector divided by a common factor (the geometric mean of the country unit cost functions evaluated at their country price vectors).

(e) Show that the geometric mean of the P^j defined by (xi) is numerically equal to unity; i.e., show that

(xii) $[\Pi_{j=1}^K P^j]^{1/K} \equiv [\Pi_{j=1}^K \{\prod_{k=1}^K P_F(p^k,p^j,q^k,q^j)\}^{1/K}]^{1/K} = 1$.

Hint: P_F satisfies the time reversal test and the identity test.

Recall from section 2 that once the GEKS P^j's have been defined by (x), the corresponding GEKS *country real expenditures or volumes* Q^j can be defined as the country expenditures $p^j \cdot q^j$ in the reference year divided by the corresponding GEKS purchasing power parity P^j:

(xii) $Q^j \equiv p^j \cdot q^j/P^j$; $j = 1,...,K$.

(f) Show that under our assumptions, the following equalities hold:

(xiii) $p^j \cdot q^j = \Sigma_{i=1}^{N(j)} u^{ji}\, c(p^j)$; $j = 1,...,K$.

(g) Show that the following equations hold:

(xiv) $Q^j = [\Sigma_{i=1}^{N(j)} u^{ji}]\, [\prod_{k=1}^K c(p^k)]^{1/K}$; $j = 1,...,K$.

Give an economic interpretation for equations (xiv).

4. (GK Computational Method 1). The GK system of N+K equations is the system defined by (7) and (8) in section 3 which are repeated here as equations (i) and (ii):

(i) $\pi_n = \sum_{k=1}^K [q_n^k/\sum_{j=1}^K q_n^j][p_n^k/P^k]$; $n = 1,...,N$;
(ii) $P^k = p^k \cdot q^k/\pi \cdot q^k$; $k = 1,...,K$.

(a). Show that if we substitute equations (ii) into equations (i), we obtain the following system of N equations involving just the components of the π vector:

(iii) $[I_N - C]\pi = 0_N$

where I_N is the N by N identity matrix and C and q are defined as follows:

(iv) $q \equiv \sum_{j=1}^K q^j$;
(v) $C \equiv \hat{q}^{-1} \sum_{k=1}^K \hat{p}^k q^k q^{kT}/p^k \cdot q^k$.

Note that $\hat{x}$ means: diagonalize the vector x into a diagonal matrix with the elements of x running down the main diagonal. Thus $\hat{q}^{-1}$ is the N by N diagonal matrix with the reciprocals of the components of the world total quantity vector q running down the main diagonal and $\hat{p}^k$ is the N by N diagonal matrix with the elements of the country k price vector p^k running down the main diagonal.

(b) Show that:

(vi) $q^T[I_N - C] = 0_N^T$.

Equation (vi) shows us two things: (1) The rows of $[I_N - C]$ are linearly dependent and hence the columns of $[I_N - C]$ are also linearly dependent and so nonzero π solutions to (iii) exist. (2) q^T is a nonzero left eigenvector of the matrix C with a corresponding eigenvalue equal to 1; i.e., we have $q^TC = q^T$. Assume that the N^2 elements of the N by N matrix C are all positive. Now we can apply the Perron (1907; 46) Theorem of matrix algebra (later generalized by Frobenius (1909; 514)) which says that the maximum eigenvalue of C is positive and has a strictly positive eigenvector. Moreover, this positive eigenvector is unique (up to a factor of proportionality) in that the matrix C cannot have any other nonnegative and nonzero eigenvectors; all other eigenvectors of C must have a negative component or a non-real component. This Theorem and the fact that $q >> 0_N$ and $q^TC = q^T$ is sufficient to imply that there exists a unique (up to a factor of proportionality) strictly positive solution π such that $C\pi = \pi$.[56]

5. (GK Computational Method 2). The GK system of N+K equations is the system defined by (7) and (8) in section 3 which are repeated here as equations (i) and (ii):

[56] Thus if one has access to a mathematical package that can compute eigenvalues and (right) eigenvectors for arbitrary square matrices, the maximal eigenvalue of C (which will turn out to equal 1) and the accompanying right eigenvector π can be obtained. Scale the components of π so that the first component equals 1 (so $\pi_1 = 1$) or so that $\pi \cdot q^1 = p^1 \cdot q^1$ (so that P^1 will equal 1). Then use equations (ii) above to calculate the country PPPs, the P^k for $k = 1,...,K$. Finally use equations (10) in section 3, $Q^k = p^k \cdot q^k/P^k$ for $k = 1,...,K$, in order to obtain the country volumes.

(i) $\pi_n = \sum_{k=1}^K [q_n^k/\sum_{j=1}^K q_n^j][p_n^k/P^k]$; $n = 1,...,N$;
(ii) $P^k = p^k \cdot q^k/\pi \cdot q^k$; $k = 1,...,K$.

(a). Suppose that all of the country price vectors p^k are proportional to each other so that there exist scalars $e^k > 0$ for $k = 1,...,K$ and a commodity price vector $p >> 0_N$ such that:

(iii) $p^k = e^k p$; $k = 1,...,K$.

Show that $P^k \equiv e^k$, $k = 1,...,K$ and $\pi \equiv p$ is a solution to (i) and (ii). (This problem illustrates *the law of one price*: if all goods and services are traded on international markets and there are zero transport costs plus competition, then country prices converted into a numeraire currency using market exchange rates should be equalized across the world and country k's PPP, P^k, should equal its exchange rate e^k.)

(b) Suppose that the country price vectors are given by $p^{k*} >> 0_N$ with corresponding quantity vectors $q^k >> 0_N$. Suppose further that $P^{k*} > 0$ for $k = 1,...,K$ and $\pi >> 0_N$ satisfy the following GK equations:

(iv) $\pi_n = \sum_{k=1}^K [q_n^k/\sum_{j=1}^K q_n^j][p_n^{k*}/P^{k*}]$; $n = 1,...,N$;
(v) $P^{k*} = p^{k*} \cdot q^k/\pi \cdot q^k$; $k = 1,...,K$.

Let $e^k > 0$ be a country k exchange rate that converts the country k vector of prices in domestic currency p^{k*} into a numeraire currency so that $p^k \equiv e^k p^{k*}$ is the country k vector of prices that expresses prices in the common currency for $k = 1,...,K$. *Show that* $P^k \equiv e^k P^{k*}$ for $k = 1,...,K$ and π are solutions to the GK equations (i) and (ii) above. (This problem shows that converting the domestic price vectors p^k into a common numeraire currency just shifts the GK PPPs by the exchange rate and leaves the vector of international prices π unchanged. Parts (a) and (b) of this problem show that if we convert the country price vectors into a common currency using the annual average exchange rates between the countries for the period under consideration and if the law of one price held, then the country PPPs, the P^k, should all equal 1 where we input the exchange rate adjusted prices into the GK equations (i) and (ii).)

Assume that the country price vectors p^k are expressed in a common currency. We now show how a solution to the GK equations (i) and (ii) above can be obtained by using an algorithm which will converge to the solution.

Iteration 1: Start out by setting the $P^k = 1$ in equations (i) and use these equations to define $\pi^{(1)} \equiv [\pi_1^{(1)},...,\pi_N^{(1)}]$ as follows:

(vi) $\pi_n^{(1)} \equiv \sum_{k=1}^K [q_n^k/\sum_{j=1}^K q_n^j][p_n^k]$; $n = 1,...,N$.

Once the components of $\pi^{(1)}$ have been defined by equations (vi), define the iteration 1 PPPs by using the following modifications of equations (ii):

(vii) $P^{k(1)} = p^k \cdot q^k / \pi^{(1)} \cdot q^k$; $\quad$ k = 1,...,K.

Iteration $i \geq 2$ of the algorithm assumes that the estimates $\pi^{(i-1)}$ and $P^{1(i-1)}, \ldots, P^{K(i-1)}$ have been constructed at iteration $i-1$.

Iteration i: Define $\pi^{(i)} \equiv [\pi_1^{(1)}, \ldots, \pi_N^{(1)}]$ and $P^{1(i)}, \ldots, P^{K(i)}$ as follows:

(vi) $\pi_n^{(i)} \equiv \sum_{k=1}^K [q_n^k / \sum_{j=1}^K q_n^j][p_n^k / P^{k(i-1)}]$; $\quad$ n = 1,...,N.
(vii) $P^{k(i)} = p^k \cdot q^k / \pi^{(i)} \cdot q^k$; $\quad$ k = 1,...,K.

(c) Implement the above algorithm for *5 iterations* using the data listed in Tables 3 and 4 in section 9 above. Show your estimates for π and the P^k for iterations 1 to 5. (Note that the country prices are expressed in a common currency in Table 4).

(d) Use the iteration 5 estimates for the P^k and then define your approximate GK consumption aggregates as $Q^k \equiv p^k \cdot q^k / P^{k(5)}$ for k = 1,...,8. Then form the country 1 to 7 ratios of consumption relative to country 8 consumption and compare your estimates to the GK estimates listed in Table 9 above.

6. Suppose K = 2, $p^k \equiv [p_1^k, .., p_N^k] >> 0_N$ and $q^k \equiv [q_1^k, .., q_N^k] >> 0_N$ for k = 1,2. Let P^1 and P^2 be GK PPPs for countries 1 and 2.

(a) Show that:

(i) $P^2/P^1 = \Sigma_{n=1}^N h(q_n^1, q_n^2) p_n^2 / \Sigma_{n=1}^N h(q_n^1, q_n^2) p_n^1$

where the harmonic mean h(a,b) of the positive numbers a and b is defined as

(ii) $h(a,b) \equiv [(1/2)a^{-1} + (1/2)b^{-1}] = 2ab/(a+b)$.

Hint: For K = 2, the N + 2 equations that determine the GK PPPs, P^1 and P^2, and the N international reference prices π_n for n = 1,...,N can be written as follows:

(iii) $\pi_n = [q_n^1/(q_n^1+q_n^2)][p_n^1/P^1] + [q_n^2/(q_n^1+q_n^2)][p_n^2/P^2]$; $\quad$ n = 1,...,N;
(iv) $\pi \cdot q^1 = p^1 \cdot q^1 / P^1$;
(v) $\pi \cdot q^2 = p^2 \cdot q^2 / P^2$.

Use the right hand sides of equations (iii) to form an expression for $\pi \cdot q^1$ and equate this expression to the right hand side of (iv). Call this equation (vi). Use the right hand sides of equations (iii) to form an expression for $\pi \cdot q^2$ and equate this expression to the right hand side of (v). Call this equation (vii). Multiply both sides of equation (v) by P^2 and you will obtain an equation with only P^2/P^1 as the unknown. Multiply both sides of equation (vi) by P^1 and you will obtain an equation with only P^1/P^2 as the unknown. It will turn out that both equations have the same P^2/P^1 solution that is equal to (i) above. Formula (i) was originally derived by Geary (1958; 98).

(b) Compare the GK bilateral formula defined by (i) with the Walsh and Marshall-Edgeworth bilateral price indexes. What do these three indexes have in common?

7. The equations that define the IDB PPPs, $P^1,...,P^K$, and the corresponding international reference prices $\pi_1,...,\pi_N$, are equations (i) and (ii) below where $s_n^k \equiv p_n^k q_n^k / p^k \cdot q^k$ is country k's expenditure share on commodity n and $p^k \equiv [p_1^k,...,p_N^k]$ and $q^k \equiv [q_1^k,...,q_N^k]$ are the country k price and quantity vectors:

(i) $\pi_n = [\sum_{k=1}^K s_n^k [p_n^k/P^k]^{-1} / \sum_{j=1}^K s_n^j]^{-1}$; $\qquad$ n = 1,...,N;
(ii) $P^k = p^k \cdot q^k / \pi \cdot q^k$; $\qquad$ k = 1,...,K.

(a) Show that equations (i) can be rewritten as follows:

(iii) $\pi_n = [\sum_{j=1}^K s_n^j] / [\sum_{k=1}^K (q_n^k P^k / p^k \cdot q^k)]$; $\qquad$ n = 1,...,N.

Note: if not all country prices and quantities are positive, then equations (iii) are preferred to equations (i) since the term $[p_n^k/P^k]^{-1}$ is not well defined if $p_n^k = 0$.

(b) Adapt the iterative algorithm described in Problem 5 above to the present situation. (Use equations (i) and (ii) above).

(c) Implement the algorithm you developed in part (b) for *5 iterations* using the data listed in Tables 2 and 4 in section 9 above. Show your estimates for π and the P^k for iterations 1 to 5.

(d) Use the iteration 5 estimates for the P^k and then define your approximate IDB consumption aggregates as $Q^k \equiv p^k \cdot q^k / P^{k(5)}$ for k = 1,...,8. Then form the country 1 to 7 ratios of consumption relative to country 8 consumption and compare your estimates to the IDB estimates listed in Table 9 above.

(e) Substitute equations (ii) into equations (iii) and show that you obtain the following system of equations involving only the π_n:

(iv) $\pi_n = [\sum_{j=1}^K s_n^j] / [\sum_{k=1}^K (q_n^k / \pi \cdot q^k)]$; $\qquad$ n = 1,...,N.

(f) Design an iterative algorithm to find the π_n using equations (iv) above. In order to find initial estimates for the π_n, use the initial π_n that you obtain at the first iteration of the method that you used in part (b) of this problem. Now implement this new algorithm for *5 iterations* using the data listed in Tables 2, 3 and 4 in section 9 above. Show your estimates for π for iterations 1 to 5. Once you have calculated $\pi^{(5)}$, calculate the corresponding estimates for the P^k using $P^{k(5)} = p^k \cdot q^k / \pi^{(5)} \cdot q^k$ for k = 1,...,8.

(g) Use the iteration 5 estimates for the P^k and then define your approximate IDB consumption aggregates as $Q^k \equiv p^k \cdot q^k / P^{k(5)}$ for k = 1,...,8. Then form the country 1 to 7 ratios of consumption relative to country 8 consumption and compare your estimates to the IDB estimates listed in Table 9 above.

(h) Which algorithm seems to converge more quickly; the part (b) algorithm or the part (f) algorithm?

References

Allen, R.C. and W.E. Diewert (1981), "Direct versus Implicit Superlative Index Number Formulae", *Review of Economics and Statistics* 63, 430-435.

Aten, B. and A. Heston (2009), "Chaining Methods for International Real Product and Purchasing Power Comparisons: Issues and Alternatives", pp. 245-273 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Balk, B.M. (1996), "A Comparison of Ten Methods for Multilateral International Price and Volume Comparisons", *Journal of Official Statistics* 12, 199-222.

Balk, B.M. (2008), *Price and Quantity Index Numbers: Models for Measuring Aggregate Change and Difference*, New York: Cambridge University Press.

Balk, B.M. (2009), "Aggregation Methods in International Comparisons: An Evaluation", pp. 59-85 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Barnett, W.A., W.E. Diewert and A. Zellner (2009), "Introduction to Measurement with Theory", *Macroeconomic Dynamics* 13, Supplement 2, 151-168.

Deaton, A. (2010), "Price Indexes, Inequality and the Measurement of World Poverty", Presidential Address, American Economic Association, Atlanta, January 2010.

Deaton, A. and A. Heston (2010), "Understanding PPPs and PPP Based National Accounts", *American Economics Journal: Macroeconomics* 2(4), 1-35.

Diewert, W.E. (1974), "Applications of Duality Theory," pp. 106-171 in M.D. Intriligator and D.A. Kendrick (ed.), *Frontiers of Quantitative Economics*, Vol. II, Amsterdam: North-Holland.

Diewert, W.E. (1976), "Exact and Superlative Index Numbers", *Journal of Econometrics* 4, 114-145.

Diewert, W.E. (1988), "Test Approaches to International Comparisons", pp. 67-86 in *Measurement in Economics: Theory and Applications of Economic Indices*, W. Eichhorn (ed.), Heidelberg: Physica-Verlag.

Diewert, W.E. (1992), "Fisher Ideal Output, Input and Productivity Indexes Revisited", *Journal of Productivity Analysis* 3, 211-248.

Diewert, W.E. (1999), "Axiomatic and Economic Approaches to International Comparisons", pp. 13-87 in *International and Interarea Comparisons of Income, Output and Prices*, A. Heston and R.E. Lipsey (eds.), Studies in Income and Wealth, Volume 61, Chicago: The University of Chicago Press.

Diewert, W.E. (2002), "Harmonized Indexes of Consumer Prices: Their Conceptual Foundations", *Swiss Journal of Economics and Statistics* 138, 547-637.

Diewert, W.E. (2004), "On the Stochastic Approach to Linking the Regions in the ICP", Department of Economics, Discussion Paper 04-16, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z1.

Diewert, W.E. (2009), "Similarity Indexes and Criteria for Spatial Linking", pp. 183-216 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Diewert, W.E. (2010a), "Understanding PPPs and PPP-Based National Accounts: Comment", *American Economic Journal: Macroeconomics* 2, 1-12.

Diewert, W.E. (2010b), "New Methodological Developments for the International Comparison Program", *Review of Income and Wealth*, Series 56, Special Issue 1, S11-S31.

Diewert, W.E. (2013a), "Methods of Aggregation above the Basic Heading Level within Regions", pp. 121-167 in *Measuring the Real Size of the World Economy: The Framework, Methodology and Results of the International Comparison Program—ICP*, Washington D.C.: The World Bank.

Diewert, W.E. (2013b), "Methods of Aggregation above the Basic Heading Level: Linking the Regions", pp. 169-196 in *Measuring the Real Size of the World Economy: The Framework, Methodology and Results of the International Comparison Program—ICP*, Washington D.C.: The World Bank.

Diewert, W.E. (2014), "Alternative Measures of OECD Output Growth and Inflation", Discussion Paper 14-01, Vancouver School of Economics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z1.

Dikhanov, Y. (1994), "Sensitivity of PPP-Based Income Estimates to Choice of Aggregation Procedures", The World Bank, Washington D.C., June 10, paper presented at 23rd General Conference of the International Association for Research in Income and Wealth, St. Andrews, Canada, August 21-27, 1994.

Dikhanov, Y. (1997), "Sensitivity of PPP-Based Income Estimates to Choice of Aggregation Procedures", Development Data Group, International Economics Department, The World Bank, Washington D.C., January.

Eltetö, O. and Köves, P. (1964), "On a Problem of Index Number Computation relating to international comparison", *Statisztikai Szemle* 42, 507-18. http://siteresources.worldbank.org/ICPINT/Resources/icppapertotal.pdf

Feenstra, R.C., R. Inklaar and M.P. Timmer (2013), "The Next Generation of the Penn World Table", NBER Working Paper No. 19255, Cambridge MA: NBER.

Feenstra, R.C., H. Ma and D.S. Prasada Rao (2009), "Consistent Comparisons of Real Incomes Across Time and Space", *Macroeconomic Dynamics* 13, Supplement 2, 169-193.

Fisher, I. (1922), *The Making of Index Numbers*, Boston: Houghton Mifflin.

Frobenius, G. (1909), "Über Matrizen aus positiven Elementen", *Sitzungberichte der königlich preussischen Akademie der Wissenschaften*, Part 1, 514-518.

Geary, R.G. (1958), "A Note on Comparisons of Exchange Rates and Purchasing Power between Countries", *Journal of the Royal Statistical Society Series A* 121, 97-99.

Gini, C. (1924), "Quelques considérations au sujet de la construction des nombres indices des prix et des questions analogues", *Metron* 4:1, 3-162.

Gini, C. (1931), "On the Circular Test of Index Numbers", *Metron* 9:9, 3-24.

Kravis, I.B., A. Heston and R. Summers (1982), *World Product and Income: International Comparisons of Real Gross Product*, Statistical Office of the United Nations and the World Bank, Baltimore: The Johns Hopkins University Press.

Hill, R.J. (1997), "A Taxonomy of Multilateral Methods for Making International Comparisons of Prices and Quantities", *Review of Income and Wealth* 43(1), 49-69.

Hill, R.J. (1999a), "Comparing Price Levels across Countries Using Minimum Spanning Trees", *The Review of Economics and Statistics* 81, 135-142.

Hill, R.J. (1999b), "International Comparisons using Spanning Trees", pp. 109-120 in *International and Interarea Comparisons of Income, Output and Prices*, A. Heston and R.E. Lipsey (eds.), Studies in Income and Wealth Volume 61, NBER, Chicago: The University of Chicago Press.

Hill, R.J. (2001), "Measuring Inflation and Growth Using Spanning Trees", *International Economic Review* 42, 167-185.

Hill, R.J. (2004), "Constructing Price Indexes Across Space and Time: The Case of the European Union", *American Economic Review* 94, 1379-1410.

Hill, R.J. (2009), "Comparing Per Capita Income Levels Across Countries Using Spanning Trees: Robustness, Prior Restrictions, Hybrids and Hierarchies", pp. 217-244 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Iklé, D.M. (1972), "A New Approach to the Index Number Problem", *The Quarterly Journal of Economics* 86:2, 188-211.

ILO/IMF/OECD/UNECE/Eurostat/The World Bank (2004), *Consumer Price Index Manual: Theory and Practice*, Peter Hill (ed.), Geneva: International Labour Office.

Khamis, S.H. (1972), "A New System of Index Numbers for National and International Purposes", *Journal of the Royal Statistical Society Series A* 135, 96-121.

Konüs, A.A. and S.S. Byushgens (1926), "K probleme pokupatelnoi cili deneg", *Voprosi Konyunkturi* 2, 151-172.

Kravis, I.B., A. Heston and R. Summers (1982), *World Product and Income: International Comparisons of Real Gross Product*, Statistical Office of the United Nations and the World Bank, Baltimore: The Johns Hopkins University Press.

Marris, R. (1984), "Comparing the Incomes of Nations: A Critique of the International Comparison Project", *Journal of Economic Literature* 22:1, 40-57.

McCarthy, P. (2013), "Extrapolating PPPs and Comparing ICP Benchmark Results", pp. 473-505 in *Measuring the Real Size of the World Economy: The Framework, Methodology and Results of the International Comparison Program—ICP*, Washington D.C.: The World Bank.

Neary, J.P. (2004), "Rationalizing the Penn World Tables: True Multilateral Indices for International Comparisons of Real Income", *The American Economic Review* 94, 1411-1428.

Perron, O. (1907), "Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus", *Mathematische Annalen* 64, 1-76.

Rao, D.S. Prasada (1990), "A System of Log-Change Index Numbers for Multilateral Comparisons", pp. 127-139 in *Comparisons of Prices and Real Products in Latin America*, J. Salazar-Carillo and D.S. Prasada Rao (eds.), New York: Elsevier Science Publishers.

Rao, D.S. Prasada (2009), "Generalised Eltetö-Köves-Szulc (EKS) and Country-Product-Dummy (CPD) Methods for International Comparisons", pp. 86-120 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Rao, D.S. Prasada (2013), "Computation of Basic Heading PPPs for Comparisons Between and Within Regions", pp. 93-120 in *Measuring the Real Size of the World Economy: The Framework, Methodology and Results of the International Comparison Program—ICP*, Washington D.C.: The World Bank.

Rao, D.S. Prasada, S. Shankar and G. Hajarghasht (2010), "A Minimum Distance and the Generalized EKS Approaches to Multilateral Comparisons of Prices and Real Incomes", unpublished paper, University of Queensland, Australia, May 7.

Sergeev, S. (2001), "Measures of the Similarity of the Country's Price Structures and their Practical Application", Conference on the European Comparison Program, U. N. Statistical Commission. Economic Commission for Europe, Geneva, November 12-14, 2001.

Sergeev, S. (2009), "Aggregation Methods Based on Structural International Prices", pp. 274-297 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Szulc, B. (1964), "Indices for Multiregional Comparisons", *Przeglad Statystyczny* 3, 239-254.

van Yzeren (Ijzeren), J. (1956), *Three Methods of Comparing the Purchasing Powers of Currencies*, Statistical Studies No. 7, The Hague: Central Bureau of Statistics.

van Ijzeren, J. (1983), *Index Numbers for Binary and Multilateral Comparison*, Statistical Studies 34, Central Bureau of Statistics, The Hague.

van Ijzeren, J. (1987), *Bias in International Index Numbers: A Mathematical Elucidation*, Eindhoven: Dissertatiedrukkerij Wibro.

Vogel, F. (2013), "The Ring Comparison: Linking the Regions", pp. 225-243 in *Measuring the Real Size of the World Economy: The Framework, Methodology and Results of the International Comparison Program—ICP*, Washington D.C.: The World Bank.

World Bank (2008), *2005 International Comparison Program: Tables of Final Results*, Washington D.C., February. http://siteresources.worldbank.org/ICPINT/Resources/ICP_final-results.pdf

World Bank (2013), *Measuring the Real Size of the World Economy: The Framework, Methodology and Results of the International Comparison Program—ICP*, Washington D.C.: The World Bank.

World Bank (2014), *Purchasing Power Parities and Real Expenditures of World Economies: Summary of Results and Findings of the 2011 International Comparison Program*, Washington D.C.: The World Bank.