INDEX NUMBER THEORY AND MEASUREMENT ECONOMICS

By W.E. Diewert February, 2015.

## Chapter 3: The Axiomatic Approach to Bilateral Index Number Theory

### 1. Bilateral Indexes and Some Early Tests

In this chapter, our goal will be to assume that the bilateral price index formula, $P(p^0,p^1,q^0,q^1)$, satisfies a sufficient number of "reasonable" tests or properties so that the functional form for P is determined.[1] The word "bilateral"[2] refers to the assumption that the function P depends only on the data pertaining to the two situations or periods being compared; i.e., P is regarded as a function of the two sets of price and quantity vectors, $p^0,p^1,q^0,q^1$, that are to be aggregated into a single number that summarizes the overall change in the N price ratios, $p_1^1/p_1^0,\ldots, p_N^1/p_N^0$.

We will take the perspective outlined in section 1 of Chapter 1; i.e., along with the price index $P(p^0,p^1,q^0,q^1)$, there is a companion quantity index $Q(p^0,p^1,q^0,q^1)$ such that the product of these two indices equals the value ratio between the two periods. Thus, throughout this section, we assume that P and Q satisfy the following product test:

$$(1)\quad p^1\bullet q^1/p^0\bullet q^0 = P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1).$$

Equation (1) means that as soon as the functional form for the price index P is determined, then (1) can be used to determine the functional form for the quantity index Q. However, a further advantage of assuming that the product test holds is that we can assume that the quantity index Q satisfies a "reasonable" property and then use (1) to translate this test on the quantity index into a corresponding test on the price index P.[3]

If N = 1, so that there is only one price and quantity to be aggregated, then a natural candidate for P is $p_1^1/p_1^0$ , the single price ratio, and a natural candidate for Q is $q_1^1/q_1^0$ , the single quantity ratio. When the number of commodities or items to be aggregated is greater than 1, then what index number theorists have done over the years is propose properties or tests that the price index P should satisfy. These properties are generally multi-dimensional analogues to the one good price index formula, $p_1^1/p_1^0$. Below, we list twenty tests that turn out to characterize the Fisher ideal price index.

We shall assume that every component of each price and quantity vector is positive; i.e., $p^t >> 0_N$ and $q^t >> 0_N$ [4] for t = 0,1. If we want to set $q^0 = q^1$ , we call the common quantity vector q; if we want to set $p^0 = p^1$ , we call the common price vector p.

[1] Much of the material in this Chapter is drawn from section sections 2 and 3 of Diewert (1992) and Diewert (1993a). For surveys of the axiomatic approach see Balk (1995) and von Auer (2001).

[2] Multilateral index number theory refers to the situation where there are more than two situations whose prices and quantities need to be aggregated.

[3] This observation was first made by Fisher (1911; 400-406). Vogt (1980) also pursued this idea.

[4] Notation: $q >> 0_N$ means that each component of the vector q is positive; $q \geq 0_N$ means each component of q is nonnegative and $q > 0_N$ means $q \geq 0_N$ and $q \neq 0_N$. Finally, $p\bullet q \equiv \sum_{n=1}^N p_nq_n$ denotes the inner product of the vectors p and q.

Our first two tests are not very controversial and so we will not discuss them.

T1: *Positivity* [5]: $P(p^0,p^1,q^0,q^1) > 0$.

T2: *Continuity* [6]: $P(p^0,p^1,q^0,q^1)$ is a continuous function of its arguments.

Our next two tests are somewhat more controversial.

T3: *Identity or Constant Prices Test* [7]: $P(p,p,q^0,q^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. The controversial part of this test is that the two quantity vectors are allowed to be different in the above test.[8]

T4: *Fixed Basket or Constant Quantities Test* [9]: $P(p^0,p^1,q,q) = \sum_{i=1}^N p_i^1 q_i / \sum_{i=1}^N p_i^0 q_i$.

That is, if quantities are constant during the two periods so that $q^0 = q^1 \equiv q$, then the price index should equal the expenditure on the constant basket in period 1, $\sum_{i=1}^N p_i^1 q_i$, divided by the expenditure on the basket in period 0, $\sum_{i=1}^N p_i^0 q_i$.

**Problem 1:** If the price index P satisfies Test T4 and P and Q jointly satisfy the product test, (1) above, then show[10] that Q must satisfy the identity test $Q(p^0,p^1,q,q) = 1$ for all strictly positive vectors $p^0,p^1,q$. This *constant quantities test* for Q is also somewhat controversial since $p^0$ and $p^1$ are allowed to be different.

## 2. Homogeneity Tests

The following four tests restrict the behavior of the price index P as the scale of any one of the four vectors $p^0,p^1,q^0,q^1$ changes.

---

5 Eichhorn and Voeller (1976, 23) suggested this test.

6 Fisher (1922; 207-215) informally suggested the essence of this test.

7 Laspeyres (1871; 308), Walsh (1901; 308) and Eichhorn and Voeller (1976; 24) have all suggested this test. Laspeyres came up with this test or property to discredit the ratio of unit values index of Drobisch (1871), which does not satisfy this test. This test is also a special case of Fisher's (1911; 409-410) price proportionality test.

8 Usually, economists assume that given a price vector p, the corresponding quantity vector q is uniquely determined. Here, we have the same price vector but the corresponding quantity vectors are allowed to be different.

9 The origins of this test go back at least two hundred years to the Massachusetts legislature which used a constant basket of goods to index the pay of Massachusetts soldiers fighting in the American Revolution; see Willard Fisher (1913). Other researchers who have suggested the test over the years include: Lowe (1823, Appendix, 95), Scrope (1833, 406), Jevons (1865), Sidgwick (1883, 67-68), Edgeworth (1925, 215) originally published in 1887, Marshall (1887, 363), Pierson (1895, 332), Walsh (1901, 540) (1921b; 544), and Bowley (1901, 227). Vogt and Barta (1997; 49) correctly observe that this test is a special case of Fisher's (1911; 411) proportionality test for quantity indexes which Fisher (1911; 405) translated into a test for the price index using the product test (1).

10 See Vogt (1980; 70).

T5: *Proportionality in Current Prices* [11]: $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector $p^1$. Most index number theorists regard this property as a very fundamental one that the index number formula should satisfy.

Walsh (1901) and Fisher (1911; 418) (1922; 420) proposed the related proportionality test $P(p,\lambda p,q^0,q^1) = \lambda$. This last test is a combination of T3 and T5; in fact Walsh (1901, 385) noted that this last test implies the identity test, T3.

In our next test, instead of multiplying all period 1 prices by the same number, we multiply all period 0 prices by the number λ.

T6: *Inverse Proportionality in Base Period Prices* [12]: $P(\lambda p^0,p^1,q^0,q^1) = \lambda^{-1}P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is 1/λ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector $p^0$.

The following two homogeneity tests can also be regarded as invariance tests.

T7: *Invariance to Proportional Changes in Current Quantities*: $P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if current period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 1 quantity vector $q^1$. Vogt (1980, 70) was the first to propose this test[13] and his derivation of the test is of some interest. Suppose the quantity index Q satisfies the quantity analogue to the price test T5; i.e., suppose Q satisfies $Q(p^0,p^1,q^0,\lambda q^1) = \lambda Q(p^0,p^1,q^0,q^1)$ for $\lambda > 0$. Then using the product test (1), we see that P must satisfy T7.

T8: *Invariance to Proportional Changes in Base Quantities* [14]: $P(p^0,p^1,\lambda q^0,q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if base period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 0 quantity vector $q^0$. If the quantity index Q satisfies the following counterpart to T8: $Q(p^0,p^1,\lambda q^0,q^1) = \lambda^{-1}Q(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$, then using (1), the

[11] This test was proposed by Walsh (1901, 385), Eichhorn and Voeller (1976, 24) and Vogt (1980, 68).

[12] Eichhorn and Voeller (1976; 28) suggested this test.

[13] Fisher (1911; 405) proposed the related test $P(p^0,p^1,q^0,\lambda q^0) = P(p^0,p^1,q^0,q^0) = \sum_{i=1}^N p_i^1 q_i^0/\sum_{i=1}^N p_i^0 q_i^0$.

[14] This test was proposed by Diewert (1992; 216).

corresponding price index P must satisfy T8. This argument provides some additional justification for assuming the validity of T8 for the price index function P.

T7 and T8 together impose the property that the price index P does not depend on the *absolute* magnitudes of the quantity vectors $q^0$ and $q^1$.

**3. Invariance and Symmetry Tests**

The next five tests are invariance or symmetry tests. Fisher (1922; 62-63, 458-460) and Walsh (1921b; 542) seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, 62-63) spoke of fairness but it is clear that he had symmetry properties in mind. It is perhaps unfortunate that he did not realize that there were more symmetry and invariance properties than the ones he proposed; if he had realized this, it is likely that he would have been able to provide an axiomatic characterization for his ideal price index, as will be done in section 6 below. Our first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed:

T9: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p^0,p^1,q^0,q^1)$$

where $p^{t*}$ denotes a permutation of the components of the vector $p^t$ and $q^{t*}$ denotes the same permutation of the components of $q^t$ for t = 0,1. This test is due to Irving Fisher (1922), and it is one of his three famous reversal tests. The other two are the time reversal test and the factor reversal test which will be considered below.

T10: *Invariance to Changes in the Units of Measurement* (commensurability test):

$$P(\alpha_1p_1^0,\ldots,\alpha_Np_N^0;\ \alpha_1p_1^1,\ldots,\alpha_Np_N^1;\ \alpha_1^{-1}q_1^0,\ldots,\alpha_N^{-1}q_N^0;\ \alpha_1^{-1}q_1^1,\ldots,\alpha_N^{-1}q_N^1) =$$
$$P(p_1^0,\ldots,p_N^0;\ p_1^1,\ldots,p_N^1;\ q_1^0,\ldots,q_N^0;\ q_1^1,\ldots,q_N^1) \text{ for all } \alpha_1 > 0, \ldots, \alpha_N > 0.$$

That is, the price index does not change if the units of measurement for each commodity are changed. The concept of this test was due to Jevons (1884; 23) and the Dutch economist Pierson (1896; 131), who criticized several index number formula for not satisfying this fundamental test. Fisher (1911; 411) first called this test *the change of units test* and later, Fisher (1922; 420) called it the *commensurability test*.

T11: *Time Reversal Test*: $P(p^0,p^1,q^0,q^1) = 1/P(p^1,p^0,q^1,q^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio; this test is satisfied (as are all of the other tests listed in this section). When the number of goods is greater than one, many commonly used price indices fail this test; e.g., the Laspeyres (1871) price index, $P_L$ defined earlier in Chapter 1, and the Paasche (1874) price index, $P_P$, both *fail* this fundamental test. The concept of the test was due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test, that he proposed that the entire concept of an index number should be abandoned.

More formal statements of the test were made by Walsh (1901; 368) (1921b; 541) and Fisher (1911; 534) (1922; 64).

Our next two tests are more controversial, since they are not necessarily consistent with the economic approach to index number theory. However, these tests are quite consistent with the weighted stochastic approach to index number theory discussed in Chapter 1.

T12: *Quantity Reversal Test* (quantity weights symmetry test): $P(p^0,p^1,q^0,q^1) = P(p^0,p^1,q^1,q^0)$.

That is, if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities are used to weight the prices in the index number formula, then the period 0 quantities $q^0$ and the period 1 quantities $q^1$ must enter the formula in a symmetric or even handed manner. Funke and Voeller (1978; 3) introduced this test; they called it the *weight property*.

The next test is the analogue to T12 applied to quantity indices:

T13: *Price Reversal Test* (price weights symmetry test)[15]:
$\{\sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) = \{\sum_{i=1}^N p_i^0 q_i^1 / \sum_{i=1}^N p_i^1 q_i^0\}/P(p^1,p^0,q^0,q^1)$.

Thus if we use (1) to define the quantity index Q in terms of the price index P, then it can be seen that T13 is equivalent to the following property for the associated quantity index Q:

$$Q(p^0,p^1,q^0,q^1) = Q(p^1,p^0,q^0,q^1).$$

That is, if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property T13 implies that these prices enter the quantity index in a symmetric manner.

## 4. Mean value tests

The next three tests are mean value tests.

T14: *Mean Value Test for Prices* [16]:
$$\min_i (p_i^1/p_i^0 : i=1,\ldots,N) \leq P(p^0,p^1,q^0,q^1) \leq \max_i (p_i^1/p_i^0 : i = 1,\ldots,N).$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is supposed to be some sort of an average of the N price ratios, $p_i^1/p_i^0$, it seems essential that the price index P satisfy this test.

The next test is the analogue to T14 applied to quantity indexes:

T15: *Mean Value Test for Quantities* [17]:

---

[15] This test was proposed by Diewert (1992; 218).
[16] This test seems to have been first proposed by Eichhorn and Voeller (1976; 10).

$$\min_i (q_i^1/q_i^0 : i=1,...,n) \leq \{V^1/V^0\}/ P(p^0,p^1,q^0,q^1) \leq \max_i (q_i^1/q_i^0 : i = 1,...,n)$$

where $V^t$ is the period t value aggregate $V^t \equiv \sum_{n=1}^N p_n^t q_n^t$ for t = 0,1. Using (1) to define the quantity index Q in terms of the price index P, we see that T15 is equivalent to the following property for the associated quantity index Q:

(2) $\min_i (q_i^1/q_i^0 : i=1,...,N) \leq Q(p^0,p^1,q^0,q^1) \leq \max_i (q_i^1/q_i^0 : i = 1,...,N)$.

That is, the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth $q_i^1/q_i^0$ of the individual quantities.

In Chapter 1,we argued that it was very reasonable to take an average of the Laspeyres and Paasche price indices as a single "best" measure of overall price change. This point of view can be turned into a test:

T16: *Paasche and Laspeyres Bounding Test* [18]: The price index P lies between the Laspeyres and Paasche indices, $P_L$ and $P_P$, defined by (3) and (4) below.

(3) $P_L \equiv p^1 \cdot q^0/p^0 \cdot q^0$ ;
(4) $P_P \equiv p^1 \cdot q^1/p^0 \cdot q^1$.

**Problem 2:** Consider a test where the implicit quantity index Q that corresponds to P via (1) is to lie between the Laspeyres and Paasche quantity indices, $Q_L$ and $Q_P$, defined by (5) and (6):

(5) $Q_L \equiv p^0 \cdot q^1/p^0 \cdot q^0$ ;
(6) $Q_P \equiv p^1 \cdot q^1/p^1 \cdot q^0$.

Show that the resulting test turns out to be *equivalent* to test T16 on P.

## 5. Monotonicity Tests

Our final four tests are monotonicity tests; i.e., how should the price index $P(p^0,p^1,q^0,q^1)$ change as any component of the two price vectors $p^0$ and $p^1$ increases or as any component of the two quantity vectors $q^0$ and $q^1$ increases.

T17: *Monotonicity in Current Prices*: $P(p^0,p^1,q^0,q^1) < P(p^0,p^2,q^0,q^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase, so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of $p^1$. This property was proposed by Eichhorn and Voeller (1976; 23) and it is a very reasonable property for a price index to satisfy.

T18: *Monotonicity in Base Prices*: $P(p^0,p^1,q^0,q^1) > P(p^2,p^1,q^0,q^1)$ if $p^0 < p^2$.

---

[17] This test was proposed by Diewert (1992; 219).
[18] Bowley (1901; 227) and Fisher (1922; 403) both endorsed this property for a price index.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of $p^0$ . This very reasonable property was also proposed by Eichhorn and Voeller (1976; 23).

T19: *Monotonicity in Current Quantities*: if $q^1 < q^2$, then
$\{\sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) < \{\sum_{i=1}^N p_i^1 q_i^2 / \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^2)$.

T20: *Monotonicity in Base Quantities*: if $q^0 < q^2$, then
$\{\sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) > \{\sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^2\}/P(p^0,p^1,q^2,q^1)$.

If we define the implicit quantity index Q that corresponds to P using (1), we find that T19 translates into the following inequality involving Q:

(7) $Q(p^0,p^1,q^0,q^1) < Q(p^0,p^1,q^0,q^2)$ if $q^1 < q^2$.

That is, if any period 1 quantity increases, then the implicit quantity index Q that corresponds to the price index P must increase. Similarly, we find that T20 translates into:

(8) $Q(p^0,p^1,q^0,q^1) > Q(p^0,p^1,q^2,q^1)$ if $q^0 < q^2$.

That is, if any period 0 quantity increases, then the implicit quantity index Q must decrease. Tests T19 and T20 are due to Vogt (1980, 70).

This concludes our listing of tests. In the next section, we ask whether any index number formula $P(p^0,p^1,q^0,q^1)$ exists that can satisfy all twenty tests.

## 6. The Fisher ideal index and the test approach

It can be shown that the only index number formula $P(p^0,p^1,q^0,q^1)$ which satisfies tests T1 - T20 is the Fisher ideal price index $P_F$ defined as the geometric mean of the Laspeyres and Paasche price indexes:[19]

(9) $P_F(p^0,p^1,q^0,q^1) \equiv [p^1\cdot q^0/p^0\cdot q^0]^{1/2} [p^1\cdot q^1/p^0\cdot q^1]^{1/2}$.

To prove this assertion, it is relatively straightforward to show that the Fisher index satisfies all 20 tests.

**Problem 3:** Show that $P_F$ satisfies tests T3, T4, T5, T6, T7, T8, T9, T10, T11, T12, T16, T17 and T18.

The more difficult part of the proof is to show that it is the *only* index number formula which satisfies these tests. This part of the proof follows from the fact that if P satisfies the positivity test T1 and the three reversal tests, T11-T13, then P must equal $P_F$. To see this, rearrange the terms in the statement of test T13 into the following equation:

---

[19] See Diewert (1992; 221).

$$(10)\ \{\sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^0\}/\{\sum_{i=1}^N p_i^0 q_i^1 / \sum_{i=1}^N p_i^1 q_i^0\}$$
$$= P(p^0,p^1,q^0,q^1) / P(p^1,p^0,q^0,q^1)$$
$$= P(p^0,p^1,q^0,q^1) / P(p^1,p^0,q^1,q^0) \quad \text{using T12, the quantity reversal test}$$
$$= P(p^0,p^1,q^0,q^1)\ P(p^0,p^1,q^0,q^1) \quad \text{using T11, the time reversal test.}$$

Now take positive square roots on both sides of (10) and we see that the left hand side of the equation is the Fisher index $P_F(p^0,p^1,q^0,q^1)$ defined by (9) and the right hand side is $P(p^0,p^1,q^0,q^1)$. Thus if P satisfies T1, T11, T12 and T13, it must equal the Fisher ideal index $P_F$.

**Problem 4:** Define the Fisher quantity index as the product of the square root of the Laspeyres and Paasche quantity indexes:

$$(11)\ \ Q_F(p^0,p^1,q^0,q^1) \equiv [p^0{\cdot}q^1/p^0{\cdot}q^0]^{1/2}\ [p^1{\cdot}q^1/p^1{\cdot}q^0]^{1/2}.$$

Show that the Q that corresponds to $P_F$ using the product test (1) is equal to the $Q_F$ defined by (11).

It turns out that $P_F$ satisfies yet another test, T21, which was Irving Fisher's (1921; 534) (1922; 72-81) third reversal test (the other two being T9 and T11):

T21: *Factor Reversal Test* (functional form symmetry test):
$$P(p^0,p^1,q^0,q^1)\ P(q^0,q^1,p^0,p^1) = \sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^0\ .$$

A justification for this test is the following one: if $P(p^0,p^1,q^0,q^1)$ is a good functional form for the price index, then if we reverse the roles of prices and quantities, $P(q^0,q^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument) and thus the product of the price index $P(p^0,p^1,q^0,q^1)$ and the quantity index $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ ought to equal the value ratio, $V^1 / V^0$. The second part of this argument does not seem to be valid and thus many researchers over the years have objected to the factor reversal test. However, if one is willing to embrace T21 as a basic test, Funke and Voeller (1978; 180) showed that the only index number function $P(p^0,p^1,q^0,q^1)$ which satisfies T1 (positivity), T11 (time reversal test), T12 (quantity reversal test) and T21 (factor reversal test) is the Fisher ideal index $P_F$ defined by (9). Thus the price reversal test T13 can be replaced by the factor reversal test in order to obtain a minimal set of four tests that lead to the Fisher price index.[20]

## 7. The Test Performance of Other Indexes

The Fisher price index $P_F$ satisfies all 20 of the tests listed in sections 1-5 above. Which tests do other commonly used price indexes satisfy? Recall the Laspeyres index $P_L$ defined by (3), the Paasche index $P_P$ defined by (4) and the Törnqvist index $P_T$ defined by:

$$(12)\ \ \ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N (1/2)(s_n^0 + s_n^1) \ln (p_n^1/p_n^0)$$

where $s_n^t$ is the expenditure share of commodity n in period t for t = 0,1 and n = 1,...,N.

---

[20] Other characterizations of the Fisher price index can be found in Funke and Voeller (1978) and Balk (1985) (1995).

Straightforward computations show that the Paasche and Laspeyres price indexes, $P_L$ and $P_P$, fail only the three reversal tests, T11, T12 and T13. Since the quantity and price reversal tests, T12 and T13, are somewhat controversial and hence can be discounted, the test performance of $P_L$ and $P_P$ seems at first sight to be quite good. However, the failure of the time reversal test, T11, is a severe limitation associated with the use of these indexes.

The Törnqvist price index $P_T$ fails nine tests: T4 (the fixed basket test), the quantity and price reversal tests T12 and T13, T15 (the mean value test for quantities), T16 (the Paasche and Laspeyres bounding test) and the 4 monotonicity tests T17 to T20. Thus the Törnqvist index is subject to a rather high failure rate.[21]

## 8. The Walsh Price Index: An Axiomatic Approach

Although the Laspeyres, Paasche, Fisher and Törnqvist price indexes are the most commonly used index number formulae,[22] there is one other price index that deserves mention at this point. Instead of looking for a "best" average of the two fixed basket indexes that correspond to the baskets chosen in either of the two periods being compared, we could instead look for a "best" average basket of the two baskets represented by the vectors $q^0$ and $q^1$ and then use this average basket to compare the price levels of periods 0 and 1.[23] Thus we ask that the nth quantity weight, $q_n$, to be an average or *mean* of the base period quantity $q_n^0$ and the period 1 quantity for commodity n $q_n^1$, say $m(q_n^0,q_n^1)$, for n = 1,2,…,N.[24] Price statisticians refer to this type of index as a *pure price index*[25] and it corresponds to Knibbs' (1924; 43) *unequivocal price index*. Under these assumptions, the pure price index can be defined as a member of the following class of index numbers:

$$(13)\quad P_K(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N p_n^1 m(q_n^0,q_n^1) / \sum_{j=1}^N p_j^0 m(q_j^0,q_j^1).$$

In order to determine the functional form for the mean function m, it is necessary to impose some *tests* or *axioms* on the pure price index defined by (13). Again we ask that $P_K$ satisfy the *time reversal test*, T11 above. Under this hypothesis, it is almost immediately obvious that the mean function m must be a *symmetric mean*; i.e., m must satisfy the following property: $m(a,b) = m(b,a)$ for all $a > 0$ and $b > 0$.

To see this, assume that $P_K$ satisfies the time reversal test. Using (13) and T11, this means that m must satisfy the following equation:

$$(14)\quad \sum_{n=1}^N p_n^1 m(q_n^0,q_n^1) / \sum_{j=1}^N p_j^0 m(q_j^0,q_j^1) = \sum_{n=1}^N p_n^1 m(q_n^1,q_n^0) / \sum_{j=1}^N p_j^0 m(q_j^1,q_j^0).$$

---

[21] However, we shall show later that the Törnqvist index approximates the Fisher index quite closely using "normal" time series data that is subject to relatively smooth trends. Hence under these circumstances, the Törnqvist index can be regarded as passing the 20 tests to a reasonably high degree of approximation.

[22] These are the four index number formulae that are standard options in the index command in the economics computer program SHAZAM; the "Divisia" index in SHAZAM is actually the (chained) Törnqvist price index $P_T$.

[23] Irving Fisher (1922) considered both averaging strategies in his classic study on index numbers. Walsh (1901) (1921a) concentrated on the second averaging strategy.

[24] Note that we have chosen the mean function $m(q_n^0,q_n^1)$ to be the same for each commodity n. We assume that m(a,b) has at least the following two properties: m(a,b) is a positive and continuous function, defined for all positive numbers a and b and $m(a,a) = a$ for all $a > 0$. For additional material on symmetric means, see Diewert (1993b; 361).

[25] See section 7 in Diewert (2001).

Assume that $N \geq 2$ and let $p_1^{\,1} = 1$, $p_2^{\,0} = 1$ and let the other $p_n^{\,t}$ approach 0. With these substitutions, equation (14) can be rewritten as follows:

(15) $m(q_1^{\,0},q_1^{\,1})/m(q_2^{\,0},q_2^{\,1}) = m(q_1^{\,1},q_1^{\,0})/m(q_2^{\,1},q_2^{\,0})$

and (15) must hold for all positive $q_1^{\,0}$, $q_1^{\,1}$, $q_2^{\,0}$ and $q_2^{\,1}$. Using the positivity of m, we can rearrange equation (15) into the following equation:

(16) $m(q_1^{\,0},q_1^{\,1})/m(q_1^{\,1},q_1^{\,0}) = m(q_2^{\,0},q_2^{\,1})/m(q_2^{\,1},q_2^{\,0})$.

Since the left hand side of (16) does not depend on $q_2^{\,0}$and $q_2^{\,1}$, it must remain constant as $q_2^{\,0}$ and $q_2^{\,1}$ vary and hence the right hand side of (16) must be constant as $q_2^{\,0}$ and $q_2^{\,1}$ vary. Thus letting $a = q_2^{\,0}$ and $b = q_2^{\,1}$, we have:

(17) $m(a,b)/m(b,a) = c > 0$ for all $a > 0$ and $b > 0$

where the positivity of c follows from the positivity of m. Now let a = b in (17) and using the mean property of m, m(a,a) = a, we have:

(18) $c = m(a,a)/m(a,a) = a/a =1$.

Now substitute (18) back into (17), and we get the desired result; i.e., m must satisfy m(a,b) = m(b,a).

The assumption that m must be a symmetric mean still does not pin down the functional form for the pure price index defined by (13) above. For example, the function m(a,b) could be the *arithmetic mean*, (1/2)a + (1/2)b, in which case (13) reduces to the *Marshall* (1887) *Edgeworth* (1925) *price index* $P_{ME}$, which was the pure price index preferred by Knibbs (1924; 56):

(19) $P_{ME}(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} p_n^{\,1}(1/2)(q_n^{\,0} + q_n^{\,1}) / \sum_{j=1}^{N} p_j^{\,0}(1/2)(q_j^{\,0} + q_j^{\,1})$ .

On the other hand, the function m(a,b) could be the *geometric mean*, $(ab)^{1/2}$, in which case (13) reduces to the *Walsh* (1901; 398) (1921a; 97) *price index*, $P_W$[26]:

(20) $P_W(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} p_n^{\,1}(q_n^{\,0}q_n^{\,1})^{1/2} / \sum_{j=1}^{N} p_j^{\,0}(q_j^{\,0}q_j^{\,1})^{1/2}$ .

However, there are many other possibilities for the mean function m, including the mean of order r, $[(1/2)a^r + (1/2)b^r\,]^{1/r}$ for $r \neq 0$. Obviously, in order to completely determine the functional form for the pure price index $P_K$, we need to impose at least one additional test or axiom on $P_K(p^0,p^1,q^0,q^1)$.

[26] Walsh endorsed $P_W$ as being the best index number formula: "We have seen reason to believe formula 6 better than formula 7. Perhaps formula 9 is the best of the rest, but between it and Nos. 6 and 8 it would be difficult to decide with assurance." C.M. Walsh (1921a; 103). His formula 6 is $P_W$ defined by (15) and his 9 is the Fisher ideal defined by (9) above. His formula 8 is the formula $p^1 \cdot q^1/p^0 \cdot q^0 Q_W(p^0,p^1,q^0,q^1)$, which is known as the implicit Walsh price index where $Q_W(p^0,p^1,q^0,q^1)$ is the Walsh quantity index defined later by (40). Thus although Walsh thought that his Walsh price index was the best functional form, his implicit Walsh price index and the "Fisher" formula were not far behind.

In order to obtain an additional axiom, we note that there is a problem with the use of the Marshall Edgeworth price index (19) in the context of using the formula to make international comparisons of prices. If the price levels of a very large country are compared to the price levels of a small country using formula (19), then the quantity vector of the large country may totally overwhelm the influence of the quantity vector corresponding to the small country.[27] In technical terms, the Marshall Edgeworth formula is not homogeneous of degree 0 in the components of both $q^0$ and $q^1$. To prevent this problem from occurring in the use of a pure price index $P_K(p^0,p^1,q^0,q^1)$ defined by (13), we ask that $P_K$ satisfy the *invariance to proportional changes in current quantities test*, Test T7 above. These two tests, the time reversal test T11 and the invariance test T7, enable us to determine the precise functional form for the pure price index $P_K$ defined by (13) above: the pure price index $P_K$ must be the Walsh index $P_W$ defined by (20).[28]

To prove this assertion, assume that the number of commodities N is greater than one. We have already noted that the time reversal test T11 implies that the mean function m must satisfy the symmetry property $m(a,b) = m(b,a)$. Substitution of (13) into the invariance test T7 yields the following equation, which must be valid for all $p^0 > 0_N$, $p^1 > 0_N$, $q^0 >> 0_N$, $q^1 >> 0_N$ and $\lambda > 0$:

$[\sum_{i=1}^N p_i^1 m(q_i^0,\lambda q_i^1)][\sum_{j=1}^N p_j^0 m(q_j^0,q_j^1)] = [\sum_{i=1}^N p_i^1 m(q_i^0,q_i^1)][\sum_{j=1}^N p_j^0 m(q_j^0,\lambda q_j^1)]$ or

(21) $\sum_{i=1}^N \sum_{j=1}^N p_i^1[m(q_i^0,\lambda q_i^1)m(q_j^0,q_j^1) - m(q_i^0, q_i^1)m(q_j^0,\lambda q_j^1)]p_j^0 = 0.$

Set all components of $p^1$ equal to 0 except the first component, $p_1^1$, which we set equal to 1. Set all components of $p^0$ equal to 0 except the second component, $p_2^0$, which we set equal to 1. Then (21) becomes:

(22) $m(q_1^0,\lambda q_1^1)m(q_2^0,q_2^1) - m(q_1^0, q_1^1)m(q_2^0,\lambda q_2^1) = 0.$

Let $a \equiv q_1^0$, $b \equiv q_1^1$, $c \equiv q_2^0$, $d \equiv q_2^1$. Then using these definitions and the positivity property of m, after some rearrangement, (22) becomes:

(23) $m(a,\lambda b)/m(a,b) = m(c,\lambda d)/m(c,d).$

The equation (23) holds for all positive a, b, c, d and $\lambda$. Now as a and b vary, the right hand side of (23) remains constant. Hence the left hand side of (23) must also be constant as a and b vary and so there exists a positive function of one variable, $f(\lambda)$ say, such that for all positive a, b and $\lambda$:

(24) $m(a,\lambda b)/m(a,b) = f(\lambda).$

Hence for all $a > 0$, $b > 0$ and $\lambda > 0$, we have:

(25) $m(a,\lambda b) = f(\lambda)m(a,b).$

[27] This is not likely to be a severe problem in the time series context where the change in quantity vectors going from one period to the next is likely to be small. A more serious problem with the Marshall Edgeworth formula is that the corresponding implicit quantity index, $(p^1\cdot q^1/p^0\cdot q^0)/P_{ME}(p^0,p^1,q^0,q^1)$, is not homogeneous of degree one in $q^1$.

[28] The following proof is taken from section 7 of Diewert (2001).

Substituting a =1 and b=1 into (24) yields:

(26) $f(\lambda) = m(1,\lambda 1)/m(1,1)$
$= m(1,\lambda)$ using the $m(a,a) = a$ property of m which implies $m(1,1) = 1$.

Substituting (26) back into (25) yields:

(27) $m(a,\lambda b) = m(1,\lambda)m(a,b)$.

Now set a = 1 in (27) and using (26), the resulting equation is:

(28) $f(\lambda b) = f(\lambda)f(b)$ for all $\lambda > 0$ and $b > 0$.

Since $f(b) = m(1,b)$ and using the continuity of m, f is a continuous function of one variable. But (28) is one of Cauchy's (1821) functional equations (see Eichhorn (1978; 3) for a more recent reference) and under our assumptions on the mean function m, has the solution:

(29) $f(\lambda) = \lambda^c$ for some constant $c \neq 0$.

In order to determine m, set b = 1 and evaluate (27):

(30) $m(a,\lambda) = m(1,\lambda)m(a,1)$
$= m(1,\lambda)m(1,a)$ using the symmetry property for m
$= f(\lambda)f(a)$ using (21) above.

Substitution of (29) into (30) yields the following functional form for m:

(31) $m(a,b) = a^c b^c$ for all $a > 0$ and $b > 0$.

Finally, set a = b in (31) and obtain

(32) $m(a,a) = a^{2c} = a$ using (31) and $m(a,a) = a$.

The second equality in (32) implies $c = ½$ and substituting this value for c back into (31) gives us the functional form for m; i.e., $m(a,b) = a^{1/2}b^{1/2}$.

In order to be of practical use by statistical agencies, an index number formula must be able to be expressed as a function of the base period expenditure shares, $s_n^0$, the current period expenditure shares, $s_n^1$, and the N price ratios, $p_n^1/p_n^0$. The Walsh price index defined by (20) above can be rewritten in this format:

(33) $P_W(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N p_n^1(q_n^0 q_n^1)^{1/2} / \sum_{j=1}^N p_j^0(q_j^0 q_j^1)^{1/2}$
$= \sum_{n=1}^N [p_n^1/(p_n^0 p_n^1)^{1/2}] (s_n^0 s_n^1)^{1/2} / \sum_{j=1}^N [p_j^0/(p_j^0 p_j^1)^{1/2}] (s_j^0 s_j^1)^{1/2}$
$= \sum_{n=1}^N (s_n^0 s_n^1)^{1/2} [p_n^1/p_n^0]^{1/2} / \sum_{j=1}^N (s_j^0 s_j^1)^{1/2} [p_j^0/p_j^1]^{1/2}$ .

The Walsh price index, $P_W$, fails four tests: T13, the price reversal test; T16, the Paasche and Laspeyres bounding test; T19, the monotonicity in current quantities test; and T20, the monotonicity in base quantities test.

The conclusion we draw from the above results is that from the viewpoint of the test approach to index numbers, the Fisher ideal price index $P_F$ appears to be "best" since it satisfies all 20 tests.[29] The Paasche and Laspeyres indexes are next best if we treat each test as being equally important. However, both of these indexes fail the very important time reversal test. The remaining two indexes, the Walsh and Törnqvist price indexes, both satisfy the time reversal test but the Walsh index emerges as being "better" since it passes 16 of our 20 tests whereas the Törnqvist only satisfies 11 tests.

Recall that our first approach to index number theory was to take an even handed average of the two primary fixed basket indexes: the Laspeyres and Paasche price indices. These two primary indexes are based on pricing out the baskets that pertain to the two periods under consideration. In a sense, they are extreme baskets. Taking an average of them led to the *Fisher ideal price index* $P_F$ defined by (9) above. Our second approach to index number theory was the stochastic approach and that approach led to the Törnqvist index $P_T$ defined by (12). Our third approach to index number theory was the test approach and that led to the Fisher ideal price index $P_F$ again. Our next approach that we considered in this section was to average the basket quantity weights and then price out this average basket at the prices pertaining to the two situations under consideration. This approach led to the *Walsh price index* $P_W$ defined by (20) above. All three of these indexes $P_F$, $P_T$ and $P_W$, can be written as a function of the base period expenditure shares, $s_n^0$, the current period expenditure shares, $s_n^1$, and the N price ratios, $p_n^1/p_n^0$. Assuming that the statistical agency has information on these three sets of variables, which of these three "best" indexes should be used? Experience with normal time series data has shown that these three indexes will not differ substantially and thus it is a matter of indifference which of these indexes is used in practice.[30] All three of these indexes are examples of *superlative indexes*, which will be defined in a subsequent chapter when we consider the economic approach. However, note that the Fisher, Walsh and Törnqvist indexes all treat the data pertaining to the two situations in a *symmetric* manner. Hill[31] commented on superlative price indexes and the importance of a symmetric treatment of the data as follows:

"Thus economic theory suggests that, in general, a symmetric index that assigns equal weight to the two situations being compared is to be preferred to either the Laspeyres or Paasche indices on their own. The precise choice of superlative index—whether Fisher, Törnqvist or other superlative index—may be of only secondary importance as all the symmetric indices are likely to approximate each other, and the underlying theoretic index fairly closely, at least when the index number spread between the Laspeyres and Paasche is not very great." Peter Hill (1993; 384).

---

[29] This assertion needs to be qualified: there are many other tests which we have not discussed and price statisticians could differ on the importance of satisfying various sets of tests. In particular, Reinsdorf and Triplett (2008) strongly criticize the test approach for the arbitrariness of the tests that are chosen to justify a particular index number formula. Some references which discuss other tests are Eichhorn and Voeller (1976), Balk (1995), Reinsdorf (2006), Vogt and Barta (1997) and von Auer (2001).

[30] Diewert (1978; 887-889) showed that these two indexes will approximate each other to the second order around an equal price and quantity point. Thus for normal time series data where prices and quantities do not change much going from the base period to the current period, the indexes will approximate each other quite closely. See problems 5 and 6 below for the proof.

[31] See also Hill (1988).

**Problem 5.** Consider the Laspeyres, Paasche, Fisher, Törnqvist and Walsh price indexes, $P_L$, $P_P$, $P_F$, $P_T$ and $P_W$ as functions of the four sets of variables, $p^0,p^1,q^0,q^1$. Show that all of the 4N first order partial derivatives of each of these 5 indexes are equal when evaluated at a point where the two price vectors are equal (so that $p^0 = p^1 \equiv p$) and where the two quantity vectors are equal (so that $q^0 = q^1 \equiv q$); i.e., show that

$$(34)\quad \nabla P_L(p,p,q,q) = \nabla P_P(p,p,q,q) = \nabla P_F(p,p,q,q) = \nabla P_T(p,p,q,q) = \nabla P_W(p,p,q,q).$$

**Comment:** It is easy to show that

$$(35)\quad P_L(p,p,q,q) = P_P(p,p,q,q) = P_F(p,p,q,q) = P_T(p,p,q,q) = P_W(p,p,q,q) = 1.$$

Equations (34) and (35) show that the Laspeyres, Paasche, Fisher, Törnqvist and Walsh indexes all approximate each other to the first order around an equal price and quantity point.

**Problem 6:**
(a) Show that $\nabla^2 P_L(p,p,q,q) \neq \nabla^2 P_P(p,p,q,q)$ and hence the Laspeyres and Paasche indexes do *not* approximate each other to the second order around an equal price and quantity point; i.e., their 4N by 4N matrices of second order partial derivatives are not all equal when evaluated at an equal price and quantity point.
(b) Show that $\nabla^2 P_L(p,p,q,q) \neq \nabla^2 P_F(p,p,q,q)$ and hence the Laspeyres and Fisher indexes do *not* approximate each other to the second order around an equal price and quantity point.
(c) Show that $\nabla^2 P_P(p,p,q,q) \neq \nabla^2 P_F(p,p,q,q)$ and hence the Paasche and Fisher indexes do *not* approximate each other to the second order around an equal price and quantity point.
(d) Show that $\nabla^2 P_F(p,p,q,q) = \nabla^2 P_W(p,p,q,q)$ and hence the Fisher and Walsh indexes *do* approximate each other to the second order around an equal price and quantity point.
(e) Show that $\nabla^2 P_F(p,p,q,q) = \nabla^2 P_T(p,p,q,q)$ and hence the Fisher and Törnqvist indexes *do* approximate each other to the second order around an equal price and quantity point.
**Comment:** Problems 5 and 6 show that the Fisher, Törnqvist and Walsh price indexes all approximate each other to the second order around an equal price and quantity point and hence these indexes are likely to be numerically very close to each other provided prices and quantities do not change "too much" between the two periods under consideration. These problems also show that the Paasche and Laspeyres indexes do not approximate the other three indexes to the second order around an equal price and quantity point.

**Problem 7:** Determine which of the 21 tests the Marshall Edgeworth index defined by (19) satisfies.

## 9. The Additivity Test

There is an additional test that many national income accountants regard as very important: the *additivity test*. This is a test or property that is placed on the implicit quantity index $Q(p^0,p^1,q^0,q^1)$ that corresponds to the price index $P(p^0,p^1,q^0,q^1)$ using the product test, $P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1) = p^1\cdot q^1/p^0\cdot q^0$. The additivity test states that the implicit quantity index has the following form:

(36) $Q(p^0,p^1,q^0,q^1) = \sum_{i=1}^N p_i^* q_i^1 / \sum_{m=1}^N p_m^* q_m^0$

where the common across periods *price* for commodity i, $p_i^*$ for i = 1,…,N, can be a function of all 4N prices and quantities pertaining to the two periods or situations under consideration, $p^0,p^1,q^0,q^1$. In the literature on making multilateral comparisons (i.e., comparisons between more than two situations), it is quite common to assume that the quantity comparison between any two regions can be made using the two regional quantity vectors, $q^0$ and $q^1$, and a common reference price vector, $p^* \equiv (p_1^*,…,p_N^*)$.[32]

Obviously, different versions of the additivity test can be obtained if we place further restrictions on precisely which variables each reference price $p_i^*$ depends.[33] The simplest such restriction is to assume that each $p_i^*$ depends only on the commodity i prices pertaining to each of the two situations under consideration, $p_i^0$ and $p_i^1$. If we further assume that the functional form for the weighting function is the same for each commodity, so that $p_i^* = m(p_i^0,p_i^1)$ for i = 1,…,N, then we are led to the *unequivocal quantity index* postulated by Knibbs (1924; 44).

The theory of the *unequivocal quantity index* (or the *pure quantity index*[34]) parallels the theory of the pure price index outlined in section 8 above. We give a brief outline of this theory. Let the pure quantity index $Q_K$ have the following functional form:

(37) $Q_K(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^N q_i^1\, m(p_i^0,p_i^1) / \sum_{k=1}^N q_k^0\, m(p_k^0,p_k^1)$.

We assume that the price vectors $p^0$ and $p^1$ are strictly positive and the quantity vectors $q^0$ and $q^1$ are nonnegative but have at least one positive component.[35] Our problem is to determine the functional form for the averaging function m if possible. To do this, we need to impose some tests or properties on the pure quantity index $Q_K$. As was the case with the pure price index, it is very reasonable to ask that the quantity index satisfy the *time reversal test*:

(38) $Q_K(p^1,p^0,q^1,q^0) = 1/Q_K(p^0,p^1,q^0,q^1)$.

---

[32] Hill (1993; 395-397) termed such multilateral methods *the block* approach while Diewert (1996; 250-251) (1999) used the term *average price approaches*. Diewert (1999; 19) used the term *additive multilateral system*. For axiomatic approaches to multilateral index number theory, see Balk (1996) and Diewert (1999).

[33] National income accountants typically demand that the reference price $p_i^*$ be either the base period price $p_i^0$ or the price of the current period $p_i^1$, which of course leads to the Laspeyres and Paasche quantity indexes. The reason why national income accountants prefer this very strong form of additivity is that if we use base period prices $p^0$ as our reference price vector, then the nominal and real value of the ith commodity coincide for each i in the base period (these values are $p_i^0q_i^0$) while the nominal and real values of the ith commodity in period 1 are set equal to $p_i^1q_i^1$ and $p_i^0q_i^1$ respectively. Thus nominal and real values for each commodity coincide in the base period and the commodity real values are additively comparable over the two periods using the Laspeyres quantity index. On the other hand, if we use current period prices $p^1$ as our reference price vector, then the nominal and real value of the ith commodity coincide for each i in the current period (these values are $p_i^1q_i^1$) while the nominal and real values of the ith commodity in period 0 are set equal to $p_i^1q_i^0$ and $p_i^1q_i^0$ respectively. Thus nominal and real values for each commodity coincide in the current period and the commodity real values are additively comparable over the two periods using the Paasche quantity index. Of course, the problem is that both sets of real values are equally plausible but in general, they will be different!

[34] Diewert (2001) used this term.

[35] We assume that m(a,b) has the following two properties: m(a,b) is a positive and continuous function, defined for all positive numbers a and b and m(a,a) = a for all a > 0.

As was the case with the theory of the unequivocal price index, it can be seen that if the unequivocal quantity index $Q_K$ is to satisfy the time reversal test (38), the mean function in (37) must be *symmetric*. We also ask that $Q_K$ satisfy the following *invariance to proportional changes in current prices test*.

(39) $Q_K(p^0,\lambda p^1,q^0, q^1) = Q_K(p^0,p^1,q^0,q^1)$ for all $p^0,p^1,q^0,q^1$ and all $\lambda > 0$.

The idea behind this invariance test is this: the quantity index $Q_K(p^0,p^1,q^0,q^1)$ should only depend on the *relative* prices in each period and it should not depend on the amount of inflation between the two periods. Another way to interpret test (39) is to look at what the test implies for the corresponding implicit price index, $P_{IK}$, defined using the product test. It can be shown that if $Q_K$ satisfies (39), then the corresponding implicit price index $P_{IK}$ will satisfy test T5 above, the *proportionality in current prices test*. The two tests, (38) and (39), enable us to determine the precise functional form for the pure quantity index $Q_K$ defined by (37) above: the *pure quantity index* or Knibbs' *unequivocal quantity index* $Q_K$ must be the Walsh quantity index $Q_W$[36] defined by:

(40) $Q_W(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^N q_i^1(p_i^0 p_i^1)^{1/2} / \sum_{k=1}^N q_k^0(p_k^0 p_k^1)^{1/2}$ .

Thus with the addition of two tests, the pure price index $P_K$ must be the Walsh price index $P_W$ defined by (20) and with the addition of the same two tests (but applied to quantity indexes instead of price indexes), the pure quantity index $Q_K$ must be the Walsh quantity index $Q_W$ defined by (40). However, note that the product of the Walsh price and quantity indexes is *not* equal to the expenditure ratio, $V^1/V^0$. Thus believers in the pure or unequivocal price and quantity index concepts *have to choose one of these two concepts*; they both cannot apply simultaneously.[37]

If the quantity index $Q(p^0,p^1,q^0,q^1)$ satisfies the additivity test (36) for some price weights $p_i^*$, then we can rewrite the percentage change in the quantity aggregate, $Q(p^0,p^1,q^0,q^1) - 1$, as follows:

$$
\begin{aligned}
(41)\quad Q(p^0,p^1,q^0,q^1) - 1 &= \{\textstyle\sum_{i=1}^N p_i^*q_i^1 / \sum_{m=1}^N p_m^*q_m^0\} - 1 \\
&= \{\textstyle\sum_{i=1}^N p_i^*q_i^1 - \sum_{i=1}^N p_i^*q_i^0\} / \sum_{m=1}^N p_m^*q_m^0 \\
&= \textstyle\sum_{i=1}^N w_i\{q_i^1 - q_i^0\}/q_i^0 \\
&= \textstyle\sum_{i=1}^N w_i\{(q_i^1/q_i^0) - 1\}
\end{aligned}
$$

where the *weight* for commodity i, $w_i$, is defined as a *hybrid expenditure share* using the reference price vector $p^*$ and the base period quantity vector $q^0$:

(42) $w_i \equiv p_i^*q_i^0/ \sum_{m=1}^N p_m^*q_m^0$ ; $\quad i = 1,\ldots,N.$

Note that the change in commodity i going from situation 0 to situation 1 is $q_i^1 - q_i^0$ and the percentage change in this commodity is $(q_i^1/q_i^0) - 1$. Thus the ith term on the right hand side of (41) is the *contribution of the percentage change in commodity i to the overall percentage change in the aggregate going from period 0 to 1*. Business analysts often want statistical agencies to

[36] This is the quantity index that corresponds to the price index 8 defined by Walsh (1921a; 101).

[37] Knibbs (1924) did not notice this point!

provide decompositions like (41) above *so that they can decompose the overall change in an aggregate into sector specific components of change*.[38] Thus there is a demand on the part of users for additive quantity indexes.

For the Walsh quantity index defined by (40), the ith weight is

(43) $w_{Wi} \equiv [p_i^0 p_i^1]^{1/2} q_i^0 / \sum_{m=1}^N [p_m^0 p_m^1]^{1/2} q_m^0$ ; $i = 1,\ldots,N.$

Thus the Walsh quantity index $Q_W$ has a percentage decomposition into component changes of the form (41) where the weights are defined by (43).

It turns out that the Fisher quantity index $Q_F$ defined by (11) also has an additive percentage change decomposition of the form given by (41). The ith weight $w_{Fi}$ for this Fisher decomposition is rather complicated and depends on the Fisher quantity index $Q_F(p^0,p^1,q^0,q^1)$ as follows[39]:

(44) $w_{Fi} \equiv [u_i^0 + \{Q_F\}^2 u_i^1]q_i^0/[1 + Q_F]$ ; $i = 1,\ldots,N$

where $Q_F$ is the value of the Fisher quantity index, $Q_F(p^0,p^1,q^0,q^1)$ and the *period t normalized price for commodity i*, $u_i^t$, is defined as the period i price $p_i^t$ divided by the period t expenditure on the aggregate:

(45) $u_i^t \equiv p_i^t / \sum_{m=1}^N p_m^t q_m^t$ ; $t = 0,1$ ; $i = 1,\ldots,N.$

Using the weights $w_{Fi}$ defined by (44) and (45), we obtain the following exact decomposition for the Fisher ideal quantity index[40]:

(46) $Q_F(p^0,p^1,q^0,q^1) - 1 = \sum_{i=1}^N w_{Fi}\{(q_i^1/q_i^0) - 1\}.$

*Thus the apparent lack of additivity of the Fisher quantity index does not prevent it from having an additive percentage change decomposition*. In fact, Problem 9 below shows that there is an even simpler additive representation for the Fisher quantity index.

**Problem 8:** Substitute (44) into (46) and solve the resulting equation for $Q_F$.

**Problem 9:** Consider the following N + 2 equations in the N + 2 unknowns, $Q_F$ and $P_F$ and $p_i^*$:

(i) $Q_F = \sum_{i=1}^N p_i^* q_i^1 / \sum_{m=1}^N p_m^* q_m^0$;
(ii) $P_F Q_F = \sum_{i=1}^N p_i^1 q_i^1 / \sum_{m=1}^N p_m^0 q_m^0$;
(iii) $p_i^* = (1/2)p_i^0 + (1/2)(p_i^1/P_F)$ for $i = 1,\ldots,N.$

[38] Business and government analysts also often demand an analogous decomposition of the change in price aggregate into sector specific components that add up.
[39] This decomposition was obtained by Diewert (2002) and Reinsdorf, Diewert and Ehemann (2002). For an economic interpretation of this decomposition, see Diewert (2002).
[40] To verify the exactness of the decomposition, substitute (44) into (46) and solve the resulting equation for $Q_F$. We find that the solution is equal to $Q_F$ defined by (11) above; see Problem 8.

Show that the $Q_F$ solution to the above equations is the Fisher ideal quantity index defined by (11). Thus (i) and (iii) show that the Fisher quantity index has an additive decomposition of the type defined by (36), which is due to Van IJzeren (1987; 6). The ith reference price $p_i^*$ is defined as $p_i^* \equiv (1/2)p_i^0 + (1/2)p_i^1/P_F(p^0,p^1,q^0,q^1)$ for i = 1,…,N and where $P_F$ is the Fisher price index. This decomposition was also independently derived by Dikhanov (1997). The Van IJzeren decomposition for the Fisher quantity index is currently being used by Bureau of Economic Analysis; see Moulton and Seskin (1999; 16) and Ehemann, Katz and Moulton (2002).

**Problem 10:** Show that percentage change decomposition of the Fisher quantity index $Q_F$ defined by (46) and the corresponding percentage change decomposition that can be obtained using Van IJzeren's additive representation of $Q_F$ approximate each other to the second order around any point where the period 0 and 1 price and quantity vectors are equal. This result was first derived by Reinsdorf, Diewert and Ehemann (2002).

Due to the symmetric nature of the Fisher price and quantity indexes, it can be seen that the Fisher price index $P_F$ defined by (9) also has the following additive percentage change decomposition:

(47) $P_F(p^0,p^1,q^0,q^1) - 1 = \sum_{i=1}^N w_{Fi}^* \{(p_i^1/p_i^0) - 1\}$

where the *commodity i price weight* $w_{Fi}^*$ is defined as

(48) $w_{Fi}^* \equiv [v_i^0 + \{P_F\}^2 v_i^1]p_i^0/[1 + P_F]$ ; i = 1,…,N

where $P_F$ is the value of the Fisher price index, $P_F(p^0,p^1,q^0,q^1)$ and the period t *normalized quantity for commodity i*, $v_i^t$, is defined as the period i quantity $q_i^t$ divided by the period t expenditure on the aggregate:

(49) $v_i^t \equiv q_i^t / \sum_{m=1}^N p_m^t q_m^t$ ; t = 0,1 ; i = 1,…,N.

The above results show that the Fisher price and quantity indexes satisfy the additivity test and as well, the percentage change in each of these indexes have at least two exact additive decompositions into components that give the contribution to the overall change in the price (or quantity) index of the change in each price (or quantity).[41]

## 10. The Circularity Test

If the identity test T3 is true, then the time reversal test T11 can be rewritten as follows:

(50) $1 = P(p^0,p^0,q^0,q^0) = P(p^0,p^1,q^0,q^1)P(p^1,p^0,q^1,q^0)$.

[41] But how can we choose which of these two additive decompositions is "best"? We could attempt to develop a test approach to answer this question or we could appeal to the economic approach to see if either of the two decompositions has an economic interpretation. However, in order to pursue this second approach, we much wait until we have studied the economic approach. Alternatively, we could just note that since the two decompositions turn out to be very close numerically, it will not matter much which decomposition we use in practice.

Thus if we start out with the prices $p^0$ in period 0 and go to the prices $p^1$ in period 1 but then return to the prices of period 0 in period 2, and if the tests T3 and T11 are satisfied, then the product of the price movement from period 0 to 1, $P(p^0,p^1,q^0,q^1)$, and the price movement from period 1 to 2, $P(p^1,p^0,q^1,q^0)$, turns out to equal 1, indicating that the *chained price index* in period 2 has returned to its period 0 level of 1. An obvious generalization of (50) would be to replace the assumption that the period 2 price and quantity vectors in the above formula are the same as the period 0 price and quantity vectors, $p^0$ and $q^0$, and allow for arbitrary period 2 price and quantity vectors, $p^2$ and $q^2$. With this replacement, (50) becomes:

(51) $P(p^0,p^2,q^0,q^2) = P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)$.

If an index number formula P satisfies (51), then we say that P satisfies the *circularity test*.[42]

What is the meaning of (51)? The index number on the left hand side of (51), compares prices in period 2 directly with prices in period 0 and $P(p^0,p^2,q^0,q^2)$ is called the *fixed base price index* for period 2. The *chained price index* for period 2, $P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)$, on the right hand side of (51) compares prices in period 2 with those in period 0 by first comparing prices in period 1 with those in period 0 (this is the *chain link index* $P(p^0,p^1,q^0,q^1)$) and multiplies that index by the chain link index that compares prices in period 2 to those of period 1, $P(p^1,p^2,q^1,q^2)$. If the index number formula P satisfies the circularity test (51), then it does not matter whether we use the *chained index* (the right hand side of (51)) to compare prices in period 2 with those of the base period 0 or if we use the *fixed base index* (the left hand side of (51)): *we get the same answer either way*. Obviously, it would be very nice if we could find an index number formula that satisfied the circularity test and had satisfactory axiomatic properties with respect to the other tests that we have considered.

Unfortunately, it turns out that index number formulae that satisfy the circularity test have other properties that make it very unsatisfactory. Consider the following result:

**Proposition:** Assume that the index number formula P satisfies the following tests: T1 (positivity), T2 (continuity), T3 (identity), T5 (proportionality in current prices), T10 (commensurability) and T17 (monotonicity in current prices) in addition to the circularity test above. Then P must have the following functional form due originally to Konüs and Byushgens[43] (1926; 163-166):[44]

(52) $P_{KB}(p^0,p^1,q^0,q^1) \equiv \prod_{i=1}^{N} [p_i^1 / p_i^0]^{\alpha_i}$

[42] The test name is due to Fisher (1922; 413) and the concept was originally due to Westergaard (1890; 218-219).
[43] Konüs and Byushgens show that the index defined by (52) is exact for Cobb-Douglas (1928) preferences; see also Pollak (1989; 23). The concept of an exact index number formula will be explained when we study the economic approach to index number theory.
[44] See also Eichhorn (1978; 167-168) and Vogt and Barta (1997; 47). Proofs of related results can be found in Funke, Hacker and Voeller (1979) and Balk (1995). This result vindicates Irving Fisher's (1922; 274) intuition who asserted that "the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*…" Fisher (1922; 275) went on to assert: "But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915, we need, theoretically at least, another set of weights. … Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another."

where the N constants $\alpha_i$ satisfy the following restrictions:

(53) $\sum_{i=1}^{N} \alpha_i = 1$ and $\alpha_i > 0$ for i = 1,…,N.

**Proof:** Rewrite the circularity test (51) in the following form:

(54) $P(p^*,p,q^*,q) = P(p^*,p^0,q^*,q^0)P(p^0,p,q^0,q)$.

Using T1, we can rewrite (54) as follows:

(55) $P(p^0,p,q^0,q) = P(p^*,p,q^*,q)/P(p^*,p^0,q^*,q^0)$.

Now hold $p^*$ and $q^*$ constant at some fixed values and define the function f(p,q) as follows:

(56) $f(p,q) \equiv P(p^*,p,q^*,q) > 0$ for all $p>>0_N$ and $q >> 0_N$

where the positivity of f(p,q) follows from T1. Substituting definition (56) back into (55) gives us the following representation for $P(p^0,p,q^0,q)$:

(57) $P(p^0,p,q^0,q) = f(p,q)/f(p^0,q^0)$.

Now let $p^0 = p$ in (57) and apply the identity test T3 to the resulting equation. We obtain:

(58) $1 = P(p,p,q^0,q) = f(p,q)/f(p,q^0)$ ; $p >> 0_N$; $q >> 0_N$; $q^0 >> 0_N$.

Define the function g(p) as

(59) $g(p) \equiv f(p,1_N) > 0$ $p >> 0_N$.

Now set $q^0$ in (58) equal to a vector of ones, $1_N$, and (58) becomes:

(60) $f(p,q) = f(p,1_N)$
$= g(p)$ using definition (59).

Thus f(p,q) cannot depend on q. Now substitute (60) back into (57) and we find that P must have the following representation if P satisfies the circularity test and the tests T1 and T3:

(61) $P(p^0,p,q^0,q) = g(p)/g(p^0)$ ; $p >> 0_N$; $p^0 >> 0_N$; $q >> 0_N$; $q^0 >> 0_N$.

Now apply the commensurability test, T10, to the P that is defined by (61) where we set $\alpha_i = (p_i^0)^{-1}$ for i = 1,...,N. Using the representation for P given by (61), we find that g must satisfy the following functional equation:

(62) $g(p^1)/g(p^0) = g(p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0)/g(1_N)$ ; $p^0 >> 0_N$; $p^1 >> 0_N$.

Define h(p) as follows:

(63) $h(p) \equiv g(p)/g(1_N) > 0$ ; $\quad p >> 0_N$

where the positivity of h follows from the positivity of g. Using definition (63), we have:

(64) $h(p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0) = g(p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0)/g(1_N)$ $\quad p^0 >> 0_N; p^1 >> 0_N$
$= g(p^1)/g(p^0)$ $\quad$ using (62)
$= [g(p^1)/g(1_N)]/[g(p^0)/g(1_N)]$ $\quad$ using T1
$= h(p^1)/h(p^0)$ $\quad$ using (63) twice.

Thus h must satisfy the following functional equation:

(65) $h(p^0)h(p_1^1/p_1^0,p_2^1/p_2^0,...,p_1^1/p_1^0) = h(p^1)$ ; $\quad p^0 >> 0_N; p^1 >> 0_N$.

Define the vector x as the vector $p^0$ and the vector y as $p_1^1/p_1^0,p_2^1/p_2^0,...,p_N^1/p_N^0$. Hence the product of the ith components of x and y is equal to the ith component of the vector $p^1$ and it can be seen that the functional equation (65) is equivalent to the following functional equation:

(66) $h(x_1y_1,x_2y_2,...,x_Ny_N) = h(x_1,x_2,...,x_N)h(y_1,y_2,...,y_N)$ ; $\quad x >> 0_N; y >> 0_N$.

Equation (66) becomes the following equation if we allow $x_1$ and $y_1$ to vary freely but fix all $x_i$ and $y_i$ at 1 for i = 2,3,...,N:

(67) $h(x_1y_1,1,...,1) = h(x_1,1,...,1)h(y_1,1,...,1)$ ; $\quad x_1 > 0; y_1 > 0$.

But (67) is an example of *Cauchy's* (1821) *fourth functional equation*. Using the T1 (positivity) and T2 (continuity) properties of P, which carry over to h, we see that the solution to (67) is:

(68) $h(x_1,1,...,1) = x_1^{c(1)}$

where c(1) is an arbitrary constant. In a similar fashion, (66) becomes the following equation if we allow $x_2$ and $y_2$ to vary freely but fix all other $x_i$ and $y_i$ at 1:

(69) $h(1,x_2y_2,1,...,1) = h(1,x_1,1,...,1)h(1,y_1,1,...,1)$ ; $\quad x_2 > 0; y_2 > 0$.

The solution to (69) is:

(70) $h(1,x_2,1,...,1) = x_2^{c(2)}$

where c(2) is an arbitrary constant. In a similar fashion, we find that

(71) $h(1,1,x_3,1,...,1) = x_3^{c(3)}$ ; ... ; $h(1,1,...,1,x_N) = x_N^{c(N)}$

where the c(i) are arbitrary constants. Using (66) repeatedly, we can show:

$$
\begin{aligned}
(72)\quad h(x_1,x_2,...,x_N) &= h(x_1,1,...,1)h(1,x_2,...,x_N) \\
&= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,...,x_N) \\
&= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,1,...,1)h(1,1,1,x_4,...,x_N) \\
&\;\;... \\
&= h(x_1,1,...,1)h(1,x_2,1,...,1)h(1,1,x_3,1,...,1)...h(1,1,1,...,1,x_N) \\
&= \prod_{i=1}^{N} x_i^{c(i)} \qquad \text{using (68), (70) and (71).}
\end{aligned}
$$

Thus we have determined the functional form for the function h. Now use (63) to determine the function g(p) in terms of h(p):

$$
\begin{aligned}
(73)\quad g(p) &= g(1_N)h(p) \\
&= g(1_N) \prod_{i=1}^{N} p_i^{c(i)}.
\end{aligned}
$$

Using (61), we can express P in terms of g as follows:

$$
\begin{aligned}
(74)\quad P(p^0,p^1,q^0,q^1) &= g(p^1)/g(p^0) \\
&= g(1_N) \prod_{i=1}^{N} (p_i^1)^{c(i)}/g(1_N) \prod_{i=1}^{N} (p_i^0)^{c(i)} \qquad \text{using (73)} \\
&= \prod_{i=1}^{N} (p_i^1/p_i^0)^{c(i)}.
\end{aligned}
$$

Now apply test T5, proportionality in current prices, to the P defined by (74). It is easy to see that this test implies that the constants c(i) must sum to 1.

Finally, apply test T17, monotonicity in current prices, to conclude that the constants c(i) must be positive. Hence we can set the c(i) equal to the $\alpha_i$ and we have proved the Proposition. Q.E.D.

Thus under fairly weak regularity conditions, *the only price index satisfying the circularity test is a weighted geometric average of all the individual price ratios*, the weights being constant through time.

An interesting special case of the family of indices defined by (52) occurs when the weights $\alpha_i$ are all equal. In this case, $P_{KB}$ reduces to the Jevons (1865) index:

$$
(75)\quad P_J(p^0,p^1,q^0,q^1) \equiv \prod_{i=1}^{N} [p_i^1/p_i^0]^{1/N} .
$$

The problem with the indexes defined by Konüs and Byushgens and Jevons is that the individual price ratios, $p_i^1/p_i^0$, have weights (either $\alpha_i$ or 1/N ) that are *independent* of the economic importance of commodity i in the two periods under consideration. Put another way, these price weights are independent of the quantities of commodity i consumed or the expenditures on commodity i during the two periods. Hence, these indexes are not really suitable for use by statistical agencies at higher levels of aggregation when expenditure share information is available.[45]

---

[45] However, if the expenditure shares are not changing much from period to period (or better yet, are constant), then by choosing the $\alpha_i$ to be these constant expenditure shares, the Konüs and Byushgens price index will reduce to the Theil price index $P_T$, defined by (12) above, which has good statistical properties.

The above results indicate that it is not useful to ask that the price index P satisfy the circularity test *exactly*. However, it is of some interest to find index number formulae that satisfy the circularity test to some degree of *approximation* since the use of such an index number formula will lead to measures of aggregate price change that are more or less the same no matter whether we use the chain or fixed base systems. Irving Fisher (1922; 284) found that deviations from circularity using his data set and the Fisher ideal price index $P_F$ defined by (9) above were quite small. This relatively high degree of correspondence between fixed base and chain indexes has been found to hold for other symmetrically weighted formulae like the Walsh index $P_W$ defined by (20) above.[46] Thus in most time series applications of index number theory where the base year in fixed base indexes is changed every 5 years or so, it will not matter very much whether the statistical agency uses a fixed base price index or a chain index, provided that a symmetrically weighted formula is used.[47] This of course depends on the length of the time series considered and the degree of variation in the prices and quantities as we go from period to period. The more prices and quantities are subject to large fluctuations (rather than smooth trends), the less the correspondence.[48]

It is possible to give a theoretical explanation for the approximate satisfaction of the circularity test for symmetrically weighted index number formulae. Another symmetrically weighted formula is the Törnqvist (1936) Theil (1967) index $P_T$ defined earlier by (12).[49] Alterman, Diewert and Feenstra (1999; 61) showed that if the logarithmic price ratios $\ln (p_i^t/p_i^{t-1})$ trend linearly with time t and the expenditure shares $s_i^t$ also trend linearly with time, then the Törnqvist index $P_T$ will satisfy the circularity test exactly.[50] Since many economic time series on prices and quantities satisfy these assumptions approximately, then the Törnqvist index $P_T$ will satisfy the circularity test approximately. As we can deduce from Problem 6 above, the Törnqvist index generally closely approximates the symmetrically weighted Fisher and Walsh indexes, so that for many economic time series, all three of these symmetrically weighted indices will satisfy the circularity test to a high enough degree of approximation so that it will not matter whether we use the fixed base or chain principle.

## 11. An Alternative Axiomatic Approach to Index Number Theory

One of Walsh's approaches to index number theory was an attempt to determine the "best" weighted average of the price relatives, $r_i$.[51] This is equivalent to using an axiomatic approach to

---

[46] See for example Diewert (1978; 894).

[47] More specifically, most superlative indexes (which are symmetrically weighted) will satisfy the circularity test to a high degree of approximation in the time series context. Superlative indexes will be defined later when we study the economic approach to index number theory. It is worth stressing that fixed base Paasche and Laspeyres indexes are very likely to diverge considerably over a 5 year period if computers (or any other commodity which has price and quantity trends that are quite different from the trends in the other commodities) are included in the value aggregate under consideration.

[48] See Szulc (1983) and Hill (1988).

[49] This formula was explicitly defined in Törnqvist and Törnqvist (1937).

[50] This exactness result can be extended to cover the case when there are monthly proportional variations in prices and the expenditure shares have constant seasonal effects in addition to linear trends; see Alterman, Diewert and Feenstra (1999; 65).

[51] Fisher also took this point of view when describing his approach to index number theory: "An index number of the prices of a number of commodities is an average of their price relatives. This definition has, for concreteness, been expressed in terms of prices. But in like manner, an index number can be calculated for wages, for quantities of goods

try and determine the "best" index of the form $P(r,v^0,v^1)$, where $v^0$ and $v^1$ are the vectors of expenditures on the n commodities during periods 0 and 1.[52] However, initially, rather than starting with indexes of the form $P(r,v^0,v^1)$, indexes of the form $P(p^0,p^1,v^0,v^1)$ will be considered, since this framework will be more comparable to the first bilateral axiomatic framework taken in sections 1-6 above. As will be seen below, if the invariance to changes in the units of measurement test is imposed on an index of the form $P(p^0,p^1,v^0,v^1)$, then $P(p^0,p^1,v^0,v^1)$ can be written in the form $P(r,v^0,v^1)$.

Recall that the product test was used in order to define the quantity index, $Q(p^0,p^1,q^0,q^1) \equiv V^1/V^0P(p^0,p^1,q^0,q^1)$, that corresponded to the bilateral price index $P(p^0,p^1,q^0,q^1)$. A similar product test holds in the present framework; i.e., given that the functional form for the price index $P(p^0,p^1,v^0,v^1)$ has been determined, then the corresponding *implicit quantity index* can be defined in terms of P as follows:

$$(76)\ Q(p^0,p^1,v^0,v^1) \equiv \sum_{n=1}^N v_n^1 / [\sum_{n=1}^N v_n^0\, P(p^0,p^1,v^0,v^1)].$$

In sections 1-6 above, the price and quantity indices $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ were determined *jointly*; i.e., not only were axioms imposed on $P(p^0,p^1,q^0,q^1)$ but they were also imposed on $Q(p^0,p^1,q^0,q^1)$ and the product test (1) was used to translate these tests on Q into tests on P. In what follows, only tests on $P(p^0,p^1,v^0,v^1)$ will be used in order to determine the "best" price index of this form. Thus there is a parallel theory for quantity indices of the form $Q(q^0,q^1,v^0,v^1)$ where it is attempted to find the "best" value weighted average of the quantity relatives, $q_n^1/q_n^0$.[53]

For the most part, the tests which will be imposed on the price index $P(p^0,p^1,v^0,v^1)$ in this section are counterparts to the tests that were imposed on the price index $P(p^0,p^1,q^0,q^1)$ in sections 1-6 above. It will be assumed that every component of each price and value vector is positive; i.e., $p^t >> 0_N$ and $v^t >> 0_N$ for t = 0,1. If it is desired to set $v^0 = v^1$, the common expenditure vector is denoted by v; if it is desired to set $p^0 = p^1$, the common price vector is denoted by p.

---

imported or exported, and, in fact, for any subject matter involving divergent changes of a group of magnitudes. Again, this definition has been expressed in terms of time. But an index number can be applied with equal propriety to comparisons between two places or, in fact, to comparisons between the magnitudes of a group of elements under any one set of circumstances and their magnitudes under another set of circumstances." Irving Fisher (1922; 3). However, in setting up his axiomatic approach, Fisher imposed axioms on the price and quantity indices written as functions of the two price vectors, $p^0$ and $p^1$, and the two quantity vectors, $q^0$ and $q^1$; i.e., he did not write his price index in the form $P(r,v^0,v^1)$ and impose axioms on indices of this type. Of course, in the end, his ideal price index turned out to be the geometric mean of the Laspeyres and Paasche price indices and as was seen in Chapter 1, each of these indices can be written as expenditure share weighted averages of the N price relatives, $r_n \equiv p_n^1/p_n^0$.

[52] Chapter 3 in Vartia (1976) considered a variant of this axiomatic approach.

[53] It turns out that the price index that corresponds to this "best" quantity index, defined as $P^*(q^0,q^1,v^0,v^1) \equiv \sum_{n=1}^N v_n^1/[\sum_{n=1}^N v_n^0\, Q(q^0,q^1,v^0,v^1)]$, will not equal the "best" price index, $P(p^0,p^1,v^0,v^1)$. Thus the axiomatic approach to be developed in this section generates separate "best" price and quantity indexes whose product does not equal the value ratio in general. This is a disadvantage of this second axiomatic approach to bilateral indices compared to the first approach studied in sections 1-6 above.

The first two tests are straightforward counterparts to the corresponding tests in section 1.

T1: *Positivity*: $P(p^0,p^1,v^0,v^1) > 0$.

T2: *Continuity*: $P(p^0,p^1,v^0,v^1)$ is a continuous function of its arguments.

T3: *Identity or Constant Prices Test*: $P(p,p,v^0,v^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the value vectors are. Note that the two value vectors are allowed to be different in the above test.

The following four tests restrict the behavior of the price index P as the scale of any one of the four vectors $p^0,p^1,v^0,v^1$ changes.

T4: *Proportionality in Current Prices*: $P(p^0,\lambda p^1,v^0,v^1) = \lambda P(p^0,p^1,v^0,v^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector $p^1$. This test is the counterpart to test T5 in section 2 above.

In the next test, instead of multiplying all period 1 prices by the same number, all period 0 prices are multiplied by the number λ.

T5: *Inverse Proportionality in Base Period Prices*:
$$P(\lambda p^0,p^1,v^0,v^1) = \lambda^{-1}P(p^0,p^1,v^0,v^1) \text{ for } \lambda > 0.$$

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is 1/λ times the old price index. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector $p^0$. This test is the counterpart to test T6 in section 2.

The following two homogeneity tests can also be regarded as invariance tests.

T6: *Invariance to Proportional Changes in Current Period Values*:
$$P(p^0,p^1,v^0,\lambda v^1) = P(p^0,p^1,v^0,v^1) \text{ for all } \lambda > 0.$$

That is, if current period values are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree zero in the components of the period 1 value vector $v^1$.

T7: *Invariance to Proportional Changes in Base Period Values*:
$$P(p^0,p^1,\lambda v^0,v^1) = P(p^0,p^1,v^0,v^1) \text{ for all } \lambda > 0.$$

That is, if base period values are all multiplied by the number $\lambda$, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,v^0,v^1)$ is (positively) homogeneous of degree zero in the components of the period 0 value vector $v^0$.

T6 and T7 together impose the property that the price index P does not depend on the *absolute* magnitudes of the value vectors $v^0$ and $v^1$. Using test T6 with $\lambda = 1/\sum_{i=1}^N v_i^1$ and using test T7 with $\lambda = 1/\sum_{i=1}^N v_i^0$, it can be seen that P has the following property:

$$(77)\ P(p^0,p^1,v^0,v^1) = P(p^0,p^1,s^0,s^1)$$

where $s^0$ and $s^1$ are the vectors of expenditure shares for periods 0 and 1; i.e., the ith component of $s^t$ is $s_i^t \equiv v_i^t/\sum_{k=1}^N v_k^t$ for t = 0,1. Thus the tests T6 and T7 imply that the price index function P is a function of the two price vectors $p^0$ and $p^1$ and the two vectors of expenditure shares, $s^0$ and $s^1$.

Walsh suggested the spirit of tests T6 and T7 as the following quotation indicates:

"What we are seeking is to average the variations in the exchange value of one given total sum of money in relation to the several classes of goods, to which several variations [i.e., the price relatives] must be assigned weights proportional to the relative sizes of the classes. Hence the relative sizes of the classes at both the periods must be considered." Correa Moylan Walsh (1901; 104).

Walsh also realized that weighting the ith price relative $r_i$ by the arithmetic mean of the value weights in the two periods under consideration, $(1/2)[v_i^0 + v_i^1]$ would give too much weight to the expenditures of the period that had the highest level of prices:

"At first sight it might be thought sufficient to add up the weights of every class at the two periods and to divide by two. This would give the (arithmetic) mean size of every class over the two periods together. But such an operation is manifestly wrong. In the first place, the sizes of the classes at each period are reckoned in the money of the period, and if it happens that the exchange value of money has fallen, or prices in general have risen, greater influence upon the result would be given to the weighting of the second period; or if prices in general have fallen, greater influence would be given to the weighting of the first period. Or in a comparison between two countries, greater influence would be given to the weighting of the country with the higher level of prices. But it is plain that *the one period, or the one country, is as important, in our comparison between them, as the other, and the weighting in the averaging of their weights should really be even*." Correa Moylan Walsh (1901; 104-105).

As a solution to the above weighting problem, Walsh (1901; 202) (1921a; 97) proposed the following *geometric Walsh price index*:

$$(78)\ P_{GW}(p^0,p^1,v^0,v^1) \equiv \prod_{n=1}^N [p_n^1/p_n^0]^{w(n)}$$

where the nth weight in the above formula was defined as

$$(79)\ w(n) \equiv (v_n^0 v_n^1)^{1/2} / \sum_{i=1}^N (v_i^0 v_i^1)^{1/2} = (s_n^0 s_n^1)^{1/2} / \sum_{i=1}^N (s_i^0 s_i^1)^{1/2}\ ; \qquad n = 1,\ldots,N.$$

The second equation in (79) shows that Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ can also be written as a function of the expenditure share vectors, $s^0$ and $s^1$; i.e., $P_{GW}(p^0,p^1,v^0,v^1)$ is homogeneous of degree 0 in the components of the value vectors $v^0$ and $v^1$ and so $P_{GW}(p^0,p^1,v^0,v^1)$

$= P_{GW}(p^0,p^1,s^0,s^1)$. Thus Walsh came very close to deriving the Törnqvist Theil index defined earlier by (12).[54]

The next 5 tests are *invariance* or *symmetry tests* and 4 of them are direct counterparts to similar tests in section 3 above. The first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed.

T8: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):
$P(p^{0*},p^{1*},v^{0*},v^{1*}) = P(p^0,p^1,v^0,v^1)$

where $p^{t*}$ denotes a permutation of the components of the vector $p^t$ and $v^{t*}$ denotes the same permutation of the components of $v^t$ for $t = 0,1$.

The next test asks that the index be invariant to changes in the units of measurement.

T9: *Invariance to Changes in the Units of Measurement* (commensurability test):
$P(\alpha_1p_1^0,\ldots,\alpha_Np_N^0;\ \alpha_1p_1^1,\ldots,\alpha_Np_N^1;\ v_1^0,\ldots,v_N^0;\ v_1^1,\ldots,v_N^1) =$
$P(p_1^0,\ldots,p_N^0;\ p_1^1,\ldots,p_N^1;\ v_1^0,\ldots,v_N^0;\ v_1^1,\ldots,v_N^1)$ for all $\alpha_1 > 0, \ldots, \alpha_N > 0$.

That is, the price index does not change if the units of measurement for each commodity are changed. Note that the expenditure on commodity i during period t, $v_i^t$, does not change if the unit by which commodity i is measured changes.

The last test has a very important implication. Let $\alpha_1 = 1/p_1^0, \ldots, \alpha_N = 1/p_N^0$ and substitute these values for the $\alpha_i$ into the definition of the test. The following equation is obtained:

$$(80)\ P(p^0,p^1,v^0,v^1) = P(1_N,r,v^0,v^1) \equiv P^*(r,v^0,v^1)$$

where $1_N$ is a vector of ones of dimension N and r is a vector of the price relatives; i.e., the ith component of r is $r_i \equiv p_i^1/p_i^0$. Thus if the commensurability test T9 is satisfied, then the price index $P(p^0,p^1,v^0,v^1)$, which is a function of 4N variables, can be written as a function of 3N variables, $P^*(r, v^0,v^1)$, where r is the vector of price relatives and $P^*(r, v^0,v^1)$ is defined as $P(1_N,r,v^0,v^1)$.

The next test asks that the formula be invariant to the period chosen as the base period.

T10: *Time Reversal Test*: $P(p^0,p^1,v^0,v^1) = 1/P(p^1,p^0,v^1,v^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is

---

[54] One could derive Walsh's index using the same arguments as Theil except that the geometric average of the expenditure shares $(s_n^0s_n^1)^{1/2}$ could be taken as a preliminary probability weight for the nth logarithmic price relative, ln $r_n$. These preliminary weights are then normalized to add up to unity by dividing by their sum. It is evident that Walsh's geometric price index will closely approximate Theil's index using normal time series data. More formally, regarding both indexes as functions of $p^0,p^1,v^0,v^1$, it can be shown that $P_{GW}(p^0,p^1,v^0,v^1)$ approximates $P_T(p^0,p^1,v^0,v^1)$ to the second order around an equal price (i.e., $p^0 = p^1$) and quantity (i.e., $q^0 = q^1$) point.

simply the single price ratio, this test will be satisfied (as are all of the other tests listed in this section).

The next test is a variant of the *circularity test*, that was introduced in section 10 above; see equation (51).

T11: *Transitivity in Prices for Fixed Value Weights*:

$$P(p^0,p^1,v^r,v^s)P(p^1,p^2,v^r,v^s) = P(p^0,p^2,v^r,v^s).$$

In this test, the expenditure weighting vectors, $v^r$ and $v^s$, are held constant while making all price comparisons. However, given that these weights are held constant, then the test asks that the product of the index going from period 0 to 1, $P(p^0,p^1,v^r,v^s)$, times the index going from period 1 to 2, $P(p^1,p^2,v^r,v^s)$, should equal the direct index that compares the prices of period 2 with those of period 0, $P(p^0,p^2,v^r,v^s)$. Obviously, this test is a many commodity counterpart to a property that holds for a single price relative.

The final test in this section captures the idea that the value weights should enter the index number formula in a symmetric manner.

T12: *Quantity Weights Symmetry Test*: $P(p^0,p^1,v^0,v^1) = P(p^0,p^1,v^1,v^0)$.

That is, if the expenditure vectors for the two periods are interchanged, then the price index remains invariant. This property means that if values are used to weight the prices in the index number formula, then the period 0 values $v^0$ and the period 1 values $v^1$ must enter the formula in a symmetric or even handed manner.

The next test is a *mean value test*.

T13: *Mean Value Test for Prices*:

$$\min_i (p_i^1/p_i^0 : i =1,\ldots,N) \le P(p^0,p^1,v^0,v^1) \le \max_i (p_i^1/p_i^0 : i = 1,\ldots,N).$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is to be interpreted as an average of the n price ratios, $p_i^1/p_i^0$, it seems essential that the price index P satisfy this test.

The next two tests in this section are *monotonicity tests*; i.e., how should the price index $P(p^0,p^1,v^0,v^1)$ change as any component of the two price vectors $p^0$ and $p^1$ increases.

T14: *Monotonicity in Current Prices*: $P(p^0,p^1,v^0,v^1) < P(p^0,p^2,v^0,v^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase (holding the value vectors fixed), so that $P(p^0,p^1,v^0,v^1)$ is increasing in the components of $p^1$ for fixed $p^0$, $v^0$ and $v^1$.

T15: *Monotonicity in Base Prices*: $P(p^0,p^1,v^0,v^1) > P(p^2,p^1,v^0,v^1)$ if $p^0 < p^2$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of $p^0$ for fixed $p^1$, $v^0$ and $v^1$.

The above tests are not sufficient to determine the functional form of the price index; for example, it can be shown that both Walsh's geometric price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by (78) and the Törnqvist Theil index $P_T(p^0,p^1,v^0,v^1)$ defined by (12)[55] satisfy all of the above axioms. Thus at least one more test will be required in order to determine the functional form for the price index $P(p^0,p^1,v^0,v^1)$.

The tests proposed thus far do not specify exactly how the expenditure share vectors $s^0$ and $s^1$ are to be used in order to weight say the first price relative, $p_1^1/p_1^0$. The next test says that only the expenditure shares $s_1^0$ and $s_1^1$ pertaining to the first commodity are to be used in order to weight the prices that correspond to commodity 1, $p_1^1$ and $p_1^0$.

T16. *Own Share Price Weighting*:

(81) $P(p_1^0,1,\ldots,1;p_1^1,1,\ldots,1;v^0;v^1) = f(p_1^0,p_1^1,v_1^0/\sum_{n=1}^N v_n^0, v_1^1/\sum_{n=1}^N v_n^1)$.

Note that $v_1^t/\sum_{k=1}^N v_k^t$ equals $s_1^t$, the expenditure share for commodity 1 in period t. The above test says that if all of the prices are set equal to 1 except the prices for commodity 1 in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity 1 and the two expenditure shares for commodity 1. The axiom says that a function of 2 + 2N variables is actually only a function of 4 variables.[56]

If test T16 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property:

(82) $P(1,\ldots,1,p_i^0,1,\ldots,1;1,\ldots,1,p_i^1,1,\ldots,1;v^0;v^1) = f(p_i^0,p_i^1,v_i^0/\sum_{n=1}^N v_n^0, v_i^1/\sum_{n=1}^N v_n^1)$ ; $i = 1,\ldots,N$.

Equation (82) says that if all of the prices are set equal to 1 except the prices for commodity i in the two periods, but the expenditures in the two periods are arbitrarily given, then the index depends only on the two prices for commodity i and the two expenditure shares for commodity i.

The final test that also involves the weighting of prices is the following one:

T17: *Irrelevance of Price Change with Tiny Value Weights*:

(83) $P(p_1^0,1,\ldots,1; p_1^1,1,\ldots,1; 0,v_2^0,\ldots,v_N^0; 0,v_2^1,\ldots,v_N^1) = 1$.

The test T17 says that if all of the prices are set equal to 1 except the prices for commodity 1 in the two periods, and the expenditures on commodity 1 are zero in the two periods but the expenditures

[55] Define $\ln P_T(p^0,p^1,v^0,v^1) \equiv \Sigma_{n=1}^N (1/2)(s_n^0 + s_n^1)\ln(p_n^1/p_n^0)$ where $s_n^t \equiv v_n^t/\Sigma_{i=1}^N v_i^t$ for n = 1,...,N and t = 0,1. Thus our "new" Törnqvist-Theil index is numerically equal to the earlier one defined by (12).

[56] In the economics literature, axioms of this type are known as separability axioms.

on the other commodities are arbitrarily given, then the index is equal to 1.[57] Thus, roughly speaking, if the value weights for commodity 1 are tiny, then it does not matter what the price of commodity 1 is during the two periods.

Of course, if test T17 is combined with test T8, the commodity reversal test, then it can be seen that P has the following property: for i = 1,…,N:

(84) $P(1,\ldots,1,p_i^0,1,\ldots,1;\ 1,\ldots,1,p_i^1,1,\ldots,1;\ v_1^0,\ldots,v_{i-1}^0,0,v_{i+1}^0,\ldots,v_N^0;\ v_1^1,\ldots,v_{i-1}^1,0,v_{i+1}^1,\ldots,v_N^1) = 1.$

Equation (84) says that if all of the prices are set equal to 1 except the prices for commodity i in the two periods, and the expenditures on commodity i are 0 during the two periods but the other expenditures in the two periods are arbitrarily given, then the index is equal to 1.

This completes the listing of tests for the weighted average of price relatives approach to bilateral index number theory. It turns out that the above tests are sufficient to imply a specific functional form for the price index as will be seen in the next section.

## 12. The Törnqvist Theil Price Index and the Third Test Approach to Bilateral Indexes

In the Appendix to this Chapter, it is shown that if the number of commodities N exceeds two and the bilateral price index function $P(p^0,p^1,v^0,v^1)$ satisfies the 17 axioms listed above , then P must be the Törnqvist Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined in footnote 56.[58] Thus the 17 properties or tests listed in section 11 provide an axiomatic characterization of the Törnqvist Theil price index, just as the 20 tests listed in sections1-5 provided an axiomatic characterization for the Fisher ideal price index.

Obviously, there is a parallel axiomatic theory for quantity indices of the form $Q(q^0,q^1,v^0,v^1)$ that depend on the two quantity vectors for periods 0 and 1, $q^0$ and $q^1$, as well as on the corresponding two expenditure vectors, $v^0$ and $v^1$. Thus if $Q(q^0,q^1,v^0,v^1)$ satisfies the quantity counterparts to tests T1 to T17, then Q must be equal to the Törnqvist Theil quantity index $Q_T(q^0,q^1,v^0,v^1)$, whose logarithm is defined as follows:

(85) $\ln Q_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} (1/2)(s_n^0 + s_n^1) \ln (q_n^1/q_n^0)$

where as usual, the period t expenditure share on commodity i, $s_i^t$, is defined as $v_i^t/\sum_{k=1}^{N} v_k^t$ for i = 1,…,N and t = 0,1.

Unfortunately, the implicit Törnqvist Theil price index, $P_{IT}(q^0,q^1,v^0,v^1)$ that corresponds to the Törnqvist Theil quantity index $Q_T$ defined by (85) using the product test is *not* equal to the direct

[57] Strictly speaking, since all prices and values are required to be positive, the left hand side of (83) should be replaced by the limit as the commodity 1 values, $v_1^0$ and $v_1^1$, approach 0.

[58] The Törnqvist Theil price index satisfies all 17 tests but the proof in the Appendix did not use all of these tests to establish the result in the opposite direction: tests 5, 13, 15 and one of 10 or 12 were not required in order to show that an index satisfying the remaining tests must be the Törnqvist Theil price index. For alternative characterizations of the Törnqvist Theil price index, see Balk and Diewert (2001) and Hillinger (2002).

Törnqvist Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined earlier in footnote 56. The product test equation that defines $P_{IT}$ in the present context is given by the following equation:

(86) $P_{IT}(q^0,q^1,v^0,v^1) \equiv \sum_{n=1}^{N} v_n^{1} / [\sum_{n=1}^{N} v_n^{0}\, Q_T(q^0,q^1,v^0,v^1)]$.

The fact that the direct Törnqvist Theil price index $P_T$ is not in general equal to the implicit Törnqvist Theil price index $P_{IT}$ defined by (86) is a bit of a disadvantage compared to the axiomatic approach outlined in section 6 above, which led to the Fisher ideal price and quantity indices as being "best". Using the Fisher approach meant that it was not necessary to decide whether one wanted a "best" price index or a "best" quantity index: the theory outlined in sections 1-6 determined both indices simultaneously. However, in the Törnqvist Theil approach outlined in this section, it is necessary to *choose* whether one wants a "best" price index or a "best" quantity index.[59]

Other tests are of course possible. A counterpart to Test T16 in section 4 , the Paasche and Laspeyres bounding test, is the following *geometric Paasche and Laspeyres bounding test*:

(87) $P_{GL}(p^0,p^1,v^0,v^1) \le P(p^0,p^1,v^0,v^1) \le P_{GP}(p^0,p^1,v^0,v^1)$ or
$P_{GP}(p^0,p^1,v^0,v^1) \le P(p^0,p^1,v^0,v^1) \le P_{GL}(p^0,p^1,v^0,v^1)$

where the logarithms of the geometric Laspeyres and geometric Paasche price indices, $P_{GL}$ and $P_{GP}$, are defined as follows:

(88) $\ln P_{GL}(p^0,p^1,v^0,v^1) \equiv \sum_{n=1}^{N} s_n^{0} \ln (p_n^{1}/p_n^{0})$ ;

(89) $\ln P_{GP}(p^0,p^1,v^0,v^1) \equiv \sum_{n=1}^{N} s_n^{1} \ln (p_n^{1}/p_n^{0})$

As usual, the period t expenditure share on commodity i, $s_i^t$, is defined as $v_i^t/\sum_{k=1}^{N} v_k^t$ for $i = 1,\ldots,N$ and $t = 0,1$.

**Problem 11:** Show that Törnqvist Theil price index $P_T(p^0,p^1,v^0,v^1)$ defined in footnote 56 satisfies the geometric Laspeyres and Paasche bounding test but the geometric Walsh price index $P_{GW}(p^0,p^1,v^0,v^1)$ defined by (78) does not satisfy it.

The geometric Paasche and Laspeyres bounding test was not included as a primary test in section 11 because a priori, it was not known what form of averaging of the price relatives (e. g., geometric or arithmetic or harmonic) would turn out to be appropriate in this test framework. The test (87) is an appropriate one if it has been decided that geometric averaging of the price relatives is the appropriate framework, since the geometric Paasche and Laspeyres indices correspond to "extreme" forms of value weighting in the context of geometric averaging and it is natural to require that the "best" price index lie between these extreme indices.

[59] Hillinger (2002) suggested taking the geometric mean of the direct and implicit Törnqvist Theil price indexes in order to resolve this conflict. Unfortunately, the resulting index is not "best" for either set of axioms that were suggested in this section. For more on Hillinger's approach to index number theory, see Hillinger (2003).

Walsh (1901; 408) pointed out a problem with his geometric price index $P_{GW}$ defined by (78), which also applies to the Törnqvist Theil price index $P_T(p^0,p^1,v^0,v^1)$: these geometric type indices do not give the "right" answer when the quantity vectors are constant (or proportional) over the two periods. In this case, Walsh thought that the "right" answer must be the *Lowe* (1823) *index*, which is the ratio of the costs of purchasing the constant basket during the two periods. Put another way, the geometric indices $P_{GW}$ and $P_T$ do not satisfy the fixed basket test, T4 in section 1 above. What then was the argument that led Walsh to define his geometric average type index $P_{GW}$? It turns out that he was led to this type of index by considering another test, which will now be explained.

Walsh (1901; 228-231) derived his test by considering the following very simple framework. Let there be only two commodities in the index and suppose that the expenditure share on each commodity is equal in each of the two periods under consideration. The price index under these conditions is equal to $P(p_1^0,p_2^0;p_1^1,p_2^1;v_1^0,v_2^0;v_1^1,v_2^1) = P^*(r_1,r_2;1/2,1/2;1/2,1/2) \equiv m(r_1,r_2)$ where $m(r_1,r_2)$ is a symmetric mean of the two price relatives, $r_1 \equiv p_1^1/p_1^0$ and $r_2 \equiv p_2^1/p_2^0$.[60] In this framework, Walsh then proposed the following *price relative reciprocal test*:

$$(90)\ m(r_1,r_1^{-1}) = 1.$$

Thus if the value weighting for the two commodities is equal over the two periods and the second price relative is the reciprocal of the first price relative $r_1$, then Walsh (1901; 230) argued that the overall price index under these circumstances ought to equal one, since the relative fall in one price is exactly counterbalanced by a rise in the other and both commodities have the same expenditures in each period. He found that the geometric mean satisfied this test perfectly but the arithmetic mean led to index values greater than one (provided that $r_1$ was not equal to one) and the harmonic mean led to index values that were less than one, a situation which was not at all satisfactory.[61] Thus he was led to some form of geometric averaging of the price relatives in one of his approaches to index number theory.

A generalization of Walsh's result is easy to obtain. Suppose that the mean function, $m(r_1,r_2)$, satisfies Walsh's reciprocal test, (90), and in addition, m is a homogeneous mean, so that it satisfies the following property for all $r_1 > 0$, $r_2 > 0$ and $\lambda > 0$:

$$(91)\ m(\lambda r_1,\lambda r_2) = \lambda m(r_1,r_2).$$

Let $r_1 > 0$, $r_2 > 0$. Then

$$\begin{aligned}(92)\ m(r_1,r_2) &= [r_1/r_1]\, m(r_1,r_2) \\ &= r_1\, m(r_1/r_1,r_2/r_1) && \text{using (91) with } \lambda \equiv 1/r_1 \\ &= r_1\, m(1,r_2/r_1) \\ &= r_1\, f(r_2/r_1)\end{aligned}$$

where the function of one (positive) variable f(z) is defined as

---

[60] Walsh considered only the cases where m was the arithmetic, geometric and harmonic means of $r_1$ and $r_2$.

[61] "This tendency of the arithmetic and harmonic solutions to run into the ground or to fly into the air by their excessive demands is clear indication of their falsity." Correa Moylan Walsh (1901; 231).

(93) $f(z) \equiv m(1,z)$.

Using (90):

(94) $1 = m(r_1,r_1^{-1})$
$= [r_1/r_1]\, m(r_1,r_1^{-1})$
$= r_1\, m(1,r_1^{-2})$ using (91) with $\lambda \equiv 1/r_1$.

Using definition (93), (94) can be rearranged into the following equation:

(95) $f(r_1^{-2}) = r_1^{-1}$ .

Letting $z \equiv r_1^{-2}$ so that $z^{1/2} = r_1^{-1}$, (95) becomes:

(96) $f(z) = z^{1/2}$ .

Now substitute (96) into (92) and the functional form for the mean function $m(r_1,r_2)$ is determined:

(97) $m(r_1,r_2) = r_1 f(r_2/r_1) = r_1(r_2/r_1)^{1/2} = r_1^{1/2}r_2^{1/2}$ .

Thus the geometric mean of the two price relatives is the only homogeneous mean that will satisfy Walsh's price relative reciprocal test.

There is one additional test that should be mentioned. Fisher (1911; 401) introduced this test in his first book that dealt with the test approach to index number theory. He called it the *test of determinateness as to prices* and described it as follows:

"A price index should not be rendered zero, infinity, or indeterminate by an individual price becoming zero. Thus, if any commodity should in 1910 be a glut on the market, becoming a 'free good', that fact ought not to render the index number for 1910 zero." Irving Fisher (1911; 401).

In the present context, this test could be interpreted as the following one: if any single price $p_i^0$ or $p_i^1$ tends to zero, then the price index $P(p^0,p,v^0,v^1)$ should not tend to zero or plus infinity. However, with this interpretation of the test, which regards the values $v_i^t$ as remaining constant as the $p_i^0$ or $p_i^1$ tends to zero, none of the commonly used index number formulae would satisfy this test. Hence this test should be interpreted as a test that applies to price indices $P(p^0,p^1,q^0,q^1)$ of the type that were studied in sections 1-5 above, which is how Fisher intended the test to apply. Thus Fisher's price determinateness test should be interpreted as follows: if any single price $p_i^0$ or $p_i^1$ tends to zero, then the price index $P(p^0,p,q^0,q^1)$ should not tend to zero or plus infinity. With this interpretation of the test, it can be verified that Laspeyres, Paasche and Fisher indexes satisfy this test but the Törnqvist Theil price index will not satisfy this test. Thus when using the Törnqvist Theil price index, *care must be taken to bound the prices away from zero in order to avoid a meaningless index number value*.

Walsh was aware that geometric average type indexes like the Törnqvist Theil price index $P_T$ or Walsh's geometric price index $P_{GW}$ defined by (78) become somewhat unstable[62] as individual price relatives become very large or small:

"Hence in practice the geometric average is not likely to depart much from the truth. Still, we have seen that when the classes [i. e., expenditures] are very unequal and the price variations are very great, this average may deflect considerably." Correa Moylan Walsh (1901; 373).

"In the cases of moderate inequality in the sizes of the classes and of excessive variation in one of the prices, there seems to be a tendency on the part of the geometric method to deviate by itself, becoming untrustworthy, while the other two methods keep fairly close together." Correa Moylan Walsh (1901; 404).

Weighing all of the arguments and tests presented in this chapter, it seems that there may be a slight preference for the use of the Fisher ideal price index as a suitable target index for a statistical agency that wishes to use the axiomatic approach, but of course, opinions can differ on which set of axioms is the most appropriate to use in practice.

**Appendix: The Törnqvist Theil Price Index and the Third Bilateral Test Approach**

Define $r_i \equiv p_i^1/p_i^0$ for $i = 1,\ldots,N$. Using T1, T9 and (80), $P(p^0,p^1,v^0,v^1) = P^*(r, v^0,v^1)$. Using T6, T7 and (77):

(A1) $P(p^0,p^1,v^0,v^1) = P^*(r,s^0,s^1)$

where $s^t$ is the period t expenditure share vector for $t = 0,1$.

Let $x \equiv (x_1,\ldots,x_N)$ and $y \equiv (y_1,\ldots,y_N)$ be strictly positive vectors. The transitivity test T11 and (A1) imply that the function $P^*$ has the following property:

(A2) $P^*(x,s^0,s^1)\, P^*(y,s^0,s^1) = P^*(x_1y_1,\ldots,x_Ny_N,s^0,s^1)$.

Using T1, $P^*(r,s^0,s^1) > 0$ and using T14, $P^*(r, s^0,s^1)$ is strictly increasing in the components of r. The identity test T3 implies that

(A3) $P^*(1_N,s^0,s^1) = 1$

where $1_N$ is a vector of ones of dimension N. Using a result due to Eichhorn (1978; 66), it can be seen that these properties of $P^*$ are sufficient to imply that there exist positive functions $\alpha_i(s^0,s^1)$ for $i = 1,\ldots,N$ such that $P^*$ has the following representation:

(A4) $\ln P^*(r,s^0,s^1) = \sum_{i=1}^N \alpha_i(s^0,s^1) \ln r_i$ .

The continuity test T2 implies that the positive functions $\alpha_i(s^0,s^1)$ are continuous. For $\lambda > 0$, the linear homogeneity test T4 implies that

[62] That is, the index may approach zero or plus infinity.

$$\begin{aligned}(A5)\ \ln P^*(\lambda r,s^0,s^1) &= \ln\lambda + \ln P^*(r,s^0,s^1)\\ &= \textstyle\sum_{i=1}^N \alpha_i(s^0,s^1)\ln\lambda r_i &&\text{using (A4)}\\ &= \textstyle\sum_{i=1}^N \alpha_i(s^0,s^1)\ln\lambda + \sum_{i=1}^N \alpha_i(s^0,s^1)\ln r_i\\ &= \textstyle\sum_{i=1}^N \alpha_i(s^0,s^1)\ln\lambda + \ln P^*(r,s^0,s^1) &&\text{using (A4).}\end{aligned}$$

Equating the right hand sides of the first and last lines in (A5) shows that the functions $\alpha_i(s^0,s^1)$ must satisfy the following restriction:

(A6) $\sum_{i=1}^N \alpha_i(s^0,s^1) = 1$

for all strictly positive vectors $s^0$ and $s^1$.

Using the weighting test T16 and the commodity reversal test T8, equations (84) hold. Equations (84) combined with the commensurability test T9 implies that P* satisfies the following equations:

(A7) $P^*(1,\ldots,1,r_i,1,\ldots,1;s^0;s^1) = f(1,s_i^0,s_i^1)$ ; $\quad i = 1,\ldots,N$

for all $r_i > 0$ where f is the function defined in test T16.

Substitute equations (A7) into equations (A4) in order to obtain the following system of equations:

(A8) $\ln P^*(1,\ldots,1,r_i,1,\ldots,1;s^0;s^1) = \ln f(1,s_i^0,s_i^1) = \alpha_i(s^0,s^1)\ln r_i$ ; $\quad i = 1,\ldots,N$

But equation i in (A8) implies that the positive continuous function of 2N variables $\alpha_i(s^0,s^1)$ is constant with respect to all of its arguments except $s_i^0$ and $s_i^1$ and this property holds for each i. Thus each $\alpha_i(s^0,s^1)$ can be replaced by the positive continuous function of two variables $\beta_i(s_i^0,s_i^1)$ for $i = 1,\ldots,N$.[63] Now replace the $\alpha_i(s^0,s^1)$ in equation (A4) by the $\beta_i(s_i^0,s_i^1)$ for $i = 1,\ldots,N$ and the following representation for $P^*$ is obtained:

(A9) $\ln P^*(r,s^0,s^1) = \sum_{i=1}^N \beta_i(s_i^0,s_i^1)\ln r_i$

Equations (A6) imply that the functions $\beta_i(s_i^0,s_i^1)$ also satisfy the following restrictions:

(A10) $\sum_{n=1}^N s_n^0 = 1$ ; $\sum_{n=1}^N s_n^1 = 1$ implies $\sum_{i=1}^N \beta_i(s_i^0,s_i^1) = 1$.

Assume that the weighting test T17 holds and substitute equations (84) into (A9) in order to obtain the following equations:

(A11) $\beta_i(0,0)\ln[p_i^1/p_i^0] = 0$ ; $\quad i = 1,\ldots,N$.

Since the $p_i^1$ and $p_i^0$ can be arbitrary positive numbers, it can be seen that (A11) implies

---

[63] More explicitly, $\beta_1(s_1^0,s_1^1) \equiv \alpha_1(s_1^0,1,\ldots,1;s_1^1,1,\ldots,1)$ and so on. That is, in defining $\beta_1(s_1^0,s_1^1)$, the function $\alpha_1(s_1^0,1,\ldots,1;s_1^1,1,\ldots,1)$ is used where all components of the vectors $s^0$ and $s^1$ except the first are set equal to an arbitrary positive number like 1.

(A12) $\beta_i(0,0) = 0$ ; $\qquad i = 1,\ldots,N.$

Assume that the number of commodities N is equal to or greater than 3. Using (A10) and (A12), Theorem 2 in Aczél (1987; 8) can be applied and the following functional form for each of the $\beta_i(s_i^0,s_i^1)$ is obtained:

(A13) $\beta_i(s_i^0,s_i^1) = \gamma s_i^0 + (1-\gamma)s_i^1$ ; $\qquad i = 1,\ldots,N$

where $\gamma$ is a positive number satisfying $0 < \gamma < 1$.

Finally, the time reversal test T10 *or* the quantity weights symmetry test T12 can be used to show that $\gamma$ must equal ½. Substituting this value for $\gamma$ back into (A13) and then substituting those equations back into (A9), the functional form for P* and hence P is determined as

(A14) $\ln P(p^0,p^1,v^0,v^1) = \ln P^*(r,s^0,s^1) = \sum_{n=1}^N (1/2)[s_n^0 + s_n^1] \ln (p_n^1/p_n^0)$ .

**References**


Aczél, J. (1987), *A Short Course on Functional Equations*, Dordrecht: Reidel Publishing Co.

Alterman, W.F., W.E. Diewert and R.C. Feenstra, (1999), *International Trade Price Indexes and Seasonal Commodities*, Bureau of Labor Statistics, Washington D.C.

Auer, L. von (2001), "An Axiomatic Checkup for Price Indices", working Paper No. 1/2001, Faculty of Economics and Management, Otto von Guericke University Magdeburg, Postfach 4120, 39016 Magdeburg, Germany.

Balk, B.M. (1985), "A Simple Characterization of Fisher's Price Index," *Statistische Hefte* 26, 59-63.

Balk, B.M. (1995), "Axiomatic Price Index Theory: A Survey", *International Statistical Review* 63, 69-93.

Balk, B.M. (1996), "A Comparison of Ten Methods for Multilateral International Price and Volume Comparisons", *Journal of Official Statistics* 12. 199-222.

Balk, B. M. and W. E. Diewert (2001), "A Characterization of the Törnqvist Price Index", *Economics Letters* 73, 279-281.

Bowley, A.L. (1901), *Elements of Statistics*, Westminster: Orchard House.

Cauchy, A.L. (1821), *Cours d'analyse de l'École Polytechnique*, Volume 1, *Analyse algébrique*, Paris.

Cobb, C. and P.H. Douglas (1928), "A Theory of Production", *American Economic Review* 18, 139-165.

Diewert, W.E. (1978), "Superlative Index Numbers and Consistency in Aggregation", *Econometrica* 46, 883-900; reprinted as pp. 253-273 in *Essays in Index Number Theory, Volume 1*, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland, 1993.

Diewert, W.E. (1992), "Fisher Ideal Output, Input and Productivity Indexes Revisited", *Journal of Productivity Analysis* 3, 211-248.

Diewert, W.E. (1993a), "The Early History of Price Index Research", pp. 33-65 in *Essays in Index Number Theory*, Volume 1, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland.

Diewert, W.E. (1993b), "Symmetric Means and Choice under Uncertainty"', pp. 355-433 in *Essays in Index Number Theory*, Volume 1, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland.

Diewert, W.E. (1996), "Price and Volume Measures in the National Accounts", pp. 237-285 in *The New System of National Economic Accounts*, J. Kendrick (ed.), Norwell, MA: Kluwer Academic Publishers.

Diewert, W.E, (1999), "Axiomatic and Economic Approaches to Multilateral Comparisons", pp. 13-87 in *International and Interarea Comparisons of Income, Output and Prices*, A. Heston and R.E. Lipsey (eds.), Chicago: The University of Chicago Press.

Diewert, W. E. (2001), "The Consumer Price Index and Index Number Purpose", *Journal of Economic and Social Measurement* 27, 167-248.

Diewert, W. E. (2002), "The Quadratic Approximation Lemma and Decompositions of Superlative Indexes", *Journal of Economic and Social Measurement* 28, 63-88.

Dikhanov, Y. (1997), "The Sensitivity of PPP-Based Income Estimates to Choice of Aggregation Procedures", mimeo, International Economics Department, The World Bank, Washington D.C., January.

Drobisch, M. W. (1871), "Ueber die Berechnung der Veränderungen der Waarenpreise und des Geldwerths", *Jahrbücher für Nationalökonomie und Statistik* 16, 143-156.

Edgeworth, F.Y. (1925), *Papers Relating to Political Economy*, Volume 1, New York: Burt Franklin.

Ehemann, C., A. J. Katz and B. R. Moulton (2002), "The Chain-Additivity Issue and the U.S. National Accounts", *Journal of Economic and Social Measurement* 28, 37-49.

Eichhorn, W. (1978), *Functional Equations in Economics*, Reading, MA: Addison-Wesley Publishing Company.

Eichhorn, W. and J. Voeller (1976), *Theory of the Price Index*, Lecture Notes in Economics and Mathematical Systems, Vol. 140, Berlin: Springer-Verlag.

Fisher, I. (1911), *The Purchasing Power of Money*, London: Macmillan.

Fisher, I. (1922), *The Making of Index Numbers*, Houghton-Mifflin, Boston.

Fisher, W.C. (1913), "The Tabular Standard in Massachusetts History*," Quarterly Journal of Economics* 27, 417-451.

Funke, H. and J. Voeller (1978), "A Note on the Characterization of Fisher's Ideal Index," pp. 177-181 in *Theory and Applications of Economic Indices*, W. Eichhorn, R. Henn, O. Opitz and R.W. Shephard (eds.), Würzburg: Physica-Verlag.

Funke, H., G. Hacker and J. Voeller (1979), "Fisher's Circular Test Reconsidered", *Schweizerische Zeitschrift für Volkswirtshaft und Statistik* 115, 677-687.

Hill, T.P. (1988), "Recent Developments in Index Number Theory and Practice", *OECD Economic Studies* 10, 123-148.

Hill, T.P. (1993), "Price and Volume Measures", pp. 379-406 in *System of National Accounts 1993*, Eurostat, IMF, OECD, UN and World Bank, Luxembourg, Washington, D.C., Paris, New York, and Washington, D.C.

Hillinger, C. (2002), "A General Theory of Price and Quantity Aggregation and Welfare Measurement", SEMECON, University of Munich, Ludwigstrasse 33/IV, D-80539 Munich, Germany.

Hillinger, C. (2003), "The Money Metric Price and Quantity Aggregation and Welfare Measurement", *Contributions to Macroeconomics* 3:1, Article7.

Jensen, J.L.W.V. (1906), "Sur les fonctions convexes et les inégalités entre les valeurs moyennes", *Acta Math.* 8,94-96.

Jevons, W.S., (1865), "The Variation of Prices and the Value of the Currency since 1782", *Journal of the Statistical Society of London* 28, 294-320; reprinted in *Investigations in Currency and Finance* (1884), London: Macmillan and Co., 119-150.

Jevons, W.S., (1884), "A Serious Fall in the Value of Gold Ascertained and its Social Effects Set Forth (1863)", pp. 13-118 in *Investigations in Currency and Finance*, London: Macmillan and Co.

Knibbs, Sir G.H. (1924), "The Nature of an Unequivocal Price Index and Quantity Index", *Journal of the American Statistical Association* 19, 42-60 and 196-205.

Konüs, A.A. and S.S. Byushgens (1926), "K probleme pokupatelnoi cili deneg", *Voprosi Konyunkturi* 2, 151-172.

Laspeyres, E. (1871), "Die Berechnung einer mittleren Waarenpreissteigerung", *Jahrbücher für Nationalökonomie und Statistik* 16, 296-314.

Lowe, J. (1823), *The Present State of England in Regard to Agriculture, Trade and Finance*, second edition, London: Longman, Hurst, Rees, Orme and Brown.

Marshall, A. (1887), "Remedies for Fluctuations of General Prices', *Contemporary Review* 51, 355-375.

Moulton, B.R., and E.P. Seskin (1999), "A Preview of the 1999 Comprehensive Revision of the National Income and Product Accounts", *Survey of Current Business* 79 (October), 6-17.

Paasche, H. (1874), "Über die Preisentwicklung der letzten Jahre nach den Hamburger Borsennotirungen", *Jahrbücher fur Nationalökonomie und Statistik* 12, 168-178.

Pierson, N.G. (1895), "Index Numbers and Appreciation of Gold", *Economic Journal* 5, 329-335.

Pierson, N.G. (1896), "Further Considerations on Index-Numbers," *Economic Journal* 6, 127-131.

Reinsdorf, M.B. (2006), "Axiomatic Price Index Theory", unpublished paper, Bureau of Economic Analysis, Washington D.C., May.

Reinsdorf, M.B. and J.E. Triplett (2008), "A Review of Reviews: Ninety Years of Professional Thinking About the Consumer Price Index", forthcoming in *Price Index Concepts and Measurement*, W.E. Diewert, J. Greenlees and C. Hulten (eds.), Chicago: University of Chicago Press.

Reinsdorf, M.B., W.E. Diewert and C. Ehemann (2002), "Additive Decompositions for the Fisher, Törnqvist and Geometric Mean Indexes", *Journal of Economic and Social Measurement* 28, 51-61.

Scrope, G.P. (1833), *Principles of Political Economy*, London: : Longman, Rees, Orme, Brown, Green and Longman.

Szulc, B.J. (1983), "Linking Price Index Numbers," pp. 537-566 *in Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

Sidgwick, H. (1883), *The Principles of Political Economy*, London: Macmillan.

Theil, H. (1967), *Economics and Information Theory*, Amsterdam: North-Holland.

Törnqvist, Leo (1936), "The Bank of Finland's Consumption Price Index," *Bank of Finland Monthly Bulletin* 10: 1-8.

Törnqvist, L. and E. Törnqvist (1937), "Vilket är förhållandet mellan finska markens och svenska kronans köpkraft?", *Ekonomiska Samfundets Tidskrift* 39, 1-39 reprinted as pp. 121-160 in *Collected Scientific Papers of Leo Törnqvist*, Helsinki: The Research Institute of the Finnish Economy, 1981.

Van IJzeren (1987), *Bias in International Index Numbers: A Mathematical Elucidation*, Dissertation for the Hungarian Academy of Sciences, Den Haag: Koninklijke Bibliotheek.

Vartia, Y. O. (1976), *Relative Changes and Index Numbers*, Helsinki: The Research Institute of the Finnish Economy.

Vogt, A. (1980), "Der Zeit und der Faktorumkehrtest als 'Finders of Tests'", *Statistische Hefte* 21, 66-71.

Vogt, A. and J. Barta (1997), *The Making of Tests for Index Numbers*, Heidelberg: Physica-Verlag.

von Auer, L. (2001), "An Axiomatic Checkup for Price Indices", working Paper No. 1/2001, Faculty of Economics and Management, Otto von Guericke University Magdeburg, Postfach 4120, 39016 Magdeburg, Germany.

Walsh, C.M. (1901), *The Measurement of General Exchange Value*, New York: Macmillan and Co.

Walsh, C.M. (1921a), *The Problem of Estimation*, London: P.S. King & Son.

Walsh, C. M. (1921b), "Discussion", *Journal of the American Statistical Association* 17, 537-544.

Westergaard, H. (1890), *Die Grundzüge der Theorie der Statistik*, Jena: Fischer.