A Robust Redesign of High School Match

Sam Il Myoung Hwang*

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Abstract

Many school districts allow parents to report their preference rankings over schools and assign as many students as possible to their first choices. However, this well-intended assignment policy, known as the Boston mechanism, creates incentives for parents to misreport their true preferences. I consider the problem of estimating parents’ preference parameters with reported rankings under this policy.

The challenge is that we do not know who misreports and how. Previous literature has made strong assumptions about the who and the how. I assume that everyone adheres to a simple rule: do not rank a popular school on your report unless you prefer it to less popular ones. I show that the simple rule, combined with an assumption regarding beliefs about popularities, partially identifies the preference parameters. I propose an estimator of the parameters and apply it to data from Seoul, Korea. I found that the estimated bounds on the parameters are tight. Counterfactual simulations show that the Boston mechanism is more efficient—but only by a small amount—than an alternative without incentives to misreport.

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1 Introduction

Consider a school district that assigns its students to its public schools through school choice programs. That is, it allows students to report a preference ranking over schools and make assignments based on the reported ranking. For a benevolent school district official, it is natural to assign as many students as possible to their first choices, and as many students who do not get their first choices as possible to their second choices, and so on.

However, such well-intended assignment policy, also known as the Boston mechanism, incentivizes students to misreport their true preferences. For example, consider a school district with measure 1 of students and three schools 1, 2, and 3 with \( \frac{1}{3} \) capacity each. Further assume that all students prefer 1 to 2 to 3 with varying degrees. Now consider a student who perceives almost no difference between school 1 and 2 but dislikes 3 by a wide margin. In addition, he believes everyone reports his or her preferences. If he reports truthfully, (he believes) he will be assigned to each school with equal probabilities. Then he would want to report 2 as his first choice, because the Boston mechanism most likely would assign him to 2 to increase the number of students assigned to their reported first choices.

The benevolent school district, informed that incentives to misreport exist in the Boston mechanism, might consider switching to an alternative student assignment mechanism without such incentives. But they would first have to answer the following question: “if we institute an alternative assignment mechanism without incentives to misreport, will we be able to assign students to schools they like to the degree that is possible under the Boston mechanism?” Unfortunately, they may not find the answer in theory literature, which is inconclusive about this issue.

Currently the best that the school district can do is to simulate outcomes under alternative assignment mechanisms with virtual cohorts of students whose preference resembles real ones. But the reported preference rankings may not be the same as true
preferences. Can the school district estimate (the distribution) of students’ true preferences? This is the question I address in this paper.

This question is challenging. Although we know a students might misreport his or her true preferences under the Boston mechanism, we do not know who misreports and exactly how they do it. Given that we do not know a student’s true preference we cannot determine just by looking at his reported ranking whether he is misreporting. Even if we know that the student has misreported, we can only guess at where in his preference the misreporting has occurred.

Those who have examined this issue previously have made strong assumptions about who misreports preferences and how. Hastings et al. (2009) assumes that everyone reports truthfully. He (2012) and Agarwal and Somaini (2014) assumes that everyone plays an equilibrium strategy. Calsamiglia et al. (2014) assumes that everyone either reports truthfully or maximizes their expected utility with a perfect knowledge of their assignment probabilities.

I take a different approach. Rather than committing to some stylized (mis-)reporting strategies, I assume that all students abide by a simple rule: do not rank a popular school on your report unless you prefer it to less popular ones. To illustrate what this rule is I return to the example (discussed above) of three schools. Suppose a student believes that school 1 is going to be more popular ex-post than 2 or 3. If this student is following the simple rule that I assume, he reports school 1 as his first choice only if he prefers 1 to the less popular 2 and 3.

Besides its intuitive appeal, this simple rule has as special cases all stylized (misreporting) strategies assumed in the previous literature. In addition, it allows other possibilities. For example, its special case includes a rule-of-thumb strategy that school district officials and non-economist observers of the Boston mechanism have repeatedly recommended for parents.\textsuperscript{2}

\textsuperscript{1}He (2012) also estimates the model where students play undominated strategies rather than an equilibrium strategy. However, he implicitly assumes point-identification with the weak assumption
\textsuperscript{2}Abdulkadiroğlu et al. (2006) documents West Zone Parent Group, an organization for parents in Boston participating in the school choice, discussing such strategy. Chen and Sönmez (2006) also cites
To use the simple rule as an identifying assumption, we need to assume that we know what students believe about the popularity of schools. The researcher may identify a set of beliefs that, he or she concludes, it is reasonable for students to have. For example, if a researcher has a strong prior that students’ beliefs about a current year’s popularity are influenced by the popularity realized in the past, then he may assume that students believe that schools that were popular in the past would be popular this year, too. Naturally, the smaller the set of potential beliefs, the stronger the identifying assumption becomes, at the risk of misspecification.

Let me illustrate the identification strategy in the context of the example just cited, wherein there are three schools, 1, 2, 3 and a measure 1 of students. Suppose that we observe two-thirds of the students reporting 1 as their first choice and the other one-third reporting 2 as theirs. Under the Boston mechanism, the former group of students is assigned to school 1 with probability $\frac{1}{2}$, while the latter group is assigned to 2 with probability 1. Now impose some assumption on the set of beliefs. For example, let us assume that all students (correctly) believed that school 1 would be more popular ex-post than school 2. But we allow students to (incorrectly) believe that the assignment probability for school 1 would be 0.48 or 0.53 rather than 0.5. This assumption, combined with the simple rule, implies that students who first-ranked school 1 on their reports must have preferred it to 2. Consequently, any distribution of preferences at which the probability of students preferring 1 over 2 is greater than two-thirds can potentially be the true distribution. There can potentially be many such distributions, so the true distribution might be only partially identified.

I propose an estimator of parameters of the preference distributions using recent results in the literature on the estimation of identified parameter sets (Romano et al. (2014)). I apply this method to evaluate a constrained Boston mechanism currently in use in Korea’s Seoul High School Match. I found that the limited structure provided by the simple rule and a reasonable assumption regarding beliefs is enough to pro-newspaper articles recommending similar strategy.
duce informative bounds on the parameters: when I assume that students either have an equilibrium belief or base their beliefs on assignment probabilities realized in the past, almost all parameters in my specification are unambiguously signed.

I also report the results of the counterfactual welfare simulations comparing the efficiency of the Seoul Match mechanism and its alternatives. Two of my results speak to policy makers and the market design literature, respectively. For policy makers, my result shows that an alternative mechanism called Top Trading Cycles is more efficient than the current mechanism under different assumptions about parents’ reporting strategies and in two different welfare measures. However, the magnitude of Top Trading Cycles’ advantage is not large: when measured in the minutes spent in commuting, the upper bound of the advantage of Top Trading Cycles is 2.7 minutes.

My welfare simulation results contribute to the debate in the market design literature regarding efficiency ranking of the Boston mechanism and another alternative called Deferred Acceptance. I find that the Boston mechanism provides higher ex-ante welfare than Deferred Acceptance in Seoul, which has been theoretically proved in special cases (Abdulkadiroğlu et al. (2011), Troyan (2012), Akyol (2013)). However, I find that the advantage of the Boston mechanism is small: the upper bound of the ex-ante welfare difference between the Boston mechanism and Deferred Acceptance translates to less than a minute spent in commuting.

Since the Abdulkadiroğlu and Sönmez (2003) study, thowe who have examined this issue has focused their efforts on comparing the Boston mechanism and Deferred Acceptance (Ergin and Sönmez (2006), Pathak and Sönmez (2008), Kojima (2008), Miralles (2008), Abdulkadiroğlu et al. (2011), Troyan (2012), Akyol (2013), Haeringer and Klijn (2009), Chen and Sönmez (2006), Pais and Pintér (2008), Calsamiglia et al. (2010), Chen et al. (2013), Featherstone and Niederle (2009), Kojima et al. (2011), He (2012), Agarwal and Somaini (2014), Calsamiglia et al. (2014)). In this paper I demonstrate that the efficiency difference between the Boston mechanism and Deferred Acceptance can be small.
This paper recalls those of Haile and Tamer (2003) and Hortac¸su and McAdams (2010), who used limited structures to estimate the upper and lower bounds on the parameters of interest in single-unit English auctions and multi-unit auctions, respectively. I extend their approach to the empirical analysis of the Boston mechanism. My argument is related to that made in the literature on the estimation of first price auction models (Guerre et al. (2000)), wherein the distribution of valuations of objects to be auctioned needs to be estimated taking into account the fact that bidders consider their probabilities of winning when making bids. This recalls students under the Boston mechanism, who consider their assignment probabilities when reporting their preference.

I use a recent result in the partial identification literature (Romano et al. (2014)) in proving the uniform consistency of the estimator I propose. I also add to the growing applied literature that uses the partial identification to estimate bounds on parameters of interest (Ciliberto and Tamer (2009), Holmes (2011), Ellickson et al. (2013), Ho and Pakes (forthcoming)).

This paper proceeds as follows: Section 2 introduces the model; Section 3 presents the identification and the estimation; section 4 applies the method to the Seoul High School Match; and section 5 concludes. All tables are collected in the appendix A. All proofs are collected in the appendix B.

2 Model

In this section I introduce notations and precisely define three reporting strategies: Truthful Reporting Strategy, Expected-utility-maximizing Reporting Strategy, and the Simple Rule. Thereafter I demonstrate that the Simple Rule is necessary for the other two stylized reporting strategies.

There is a measure one of students. Let \( I = [0, 1] \) denote the set of students and \( i \) a generic student. There are a finite number of schools and let \( S = \{s_1, \ldots, s_m\} \) denote
the set of schools and $s$ a generic school. Each school $s$ is endowed with a capacity, denoted by $q_s \in (0,1)$. No school has a capacity large enough to admit all students and the sum of the capacities is large enough to assign every student to some school, i.e. $\sum_{s \in S} q_s \geq 1$. For the rest of the paper, fix the set of schools $S$ and the capacity vector $(q_{s_1}, q_{s_2}, \ldots, q_{s_m})$.

Each student reports a preference ranking (PR) of length $m$, which is the number of schools.$^3$ Let $R$ denote the set of all PRs and $r$ denote a generic PR. $r_k$ is the $k^{th}$ ranked school on $r$. Number each PR in some order so that $r^i$ denotes the $i^{th}$ PR.

Let $\tilde{R}$ denote the set of distributions on the set of PRs and $\tilde{r}$ denote a generic distribution of PRs. The measure of students reporting $i^{th}$ PR, $r^i$, is denoted by $\tilde{r}^i$. Let $B \subseteq \tilde{R}$ denote a set of beliefs that the researcher assumes students may have about the ex-post distribution of PRs. Each student independently draws their belief from a common distribution $G^*$ on $B$.

Student $i$’s Von Neumann-Morgenstern (VNM) utility for being assigned to school $s$ is denoted by $u_{is} \in \mathbb{R}$. Let $\mu^*$ denote a probability measure on $\mathbb{R}^{|S|}$ and $F^*$ corresponding distribution function. Each student $i$ independently draws a vector of VNM utilities, $u_i \equiv (u_{i{s_1}}, u_{i{s_2}}, \ldots, u_{i{s_m}})$, from $F^*$. $F^*$ is the structural feature of interest.

A lottery is used to break ties between students at over-demanded schools. There are two types of tie-breaking scheme: under the single tie-breaking scheme, each student draws a number from the uniform distribution on $[0,1]$ and this number is used to break ties at all schools; under the multiple tie-breaking scheme, each student draws a number from the same distribution for each school and the lottery number for each student varies across schools. Results in this paper hold under both tie-breaking schemes. For expositional simplicity I assume the single tie-breaking scheme throughout the paper.

The algorithm for the Boston mechanism is as follows:

**The algorithm for the Boston Mechanism**

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$^3$ All the lemmas and propositions are true when the PRs are constrained, i.e. the number of schools one can rank on a PR is smaller than the number of schools
**Round -1** Students submit PRs.

**Round 0** Students draw lottery numbers independently from the uniform distribution on \([0, 1]\).

**Round 1** Assign students to their first choice schools. If the measure of students who reported \(s\) as their first choice is weakly greater than the capacity of \(s\), then assign those who have smaller lottery numbers first\(^4\). Set the capacities of such schools to \(0\) for the next round. Otherwise admit everyone and reduce the capacity of such schools accordingly.

**Round \(k\), \(2 \leq k \leq m\)** Consider only the remaining students and schools with remaining capacities. Assign students to their \(k^{th}\) choice schools. If the measure of remaining students who reported \(s\) as their \(k^{th}\) choice is weakly greater than the remaining capacity at \(s\), then assign those who have smaller lottery numbers first. Set the capacities of such schools to \(0\) for the next round. Otherwise admit everyone and reduce the capacity of such schools accordingly.

**Termination of the algorithm.** Terminate the algorithm when all students are assigned to some school.

For any given distribution of PRs, the largest lottery number that students drew and still got assigned to each school in each round of the algorithm can be calculated. I am going to call such number the cut-off for school \(s\) in round \(k\) given a distribution of PRs \(\tilde{r}\) and denote it by \(C(\tilde{r})_{(k,s)}\). Cut-offs contain sufficient information with which to calculate the assignment probabilities for each PR. For simplicity of notation, let \(\hat{C}\) denote a generic matrix of cut-offs and \(c^k_s\) denote the cut-off for school \(s\) in round \(k\), i.e. \(c^k_s \equiv \hat{C}_{(k,s)}\).

Given a cut-off matrix and PR, one can calculate the assignment probability to each school. Let \(\pi^k(\mathbf{r}, \hat{C})\) denote the probability of being assigned to \(k^{th}\) ranked school on \(\mathbf{r}\) given the cut-off matrix is \(\hat{C}\).

\(^4\)Results in this paper can easily be extended to the case where there are different priority levels across students as long as there are finite levels of priority. For notational and expositional simplicity, I assume that everyone has the same priority to each school.
Each student employs a **reporting strategy** to determine what PR to submit. A reporting strategy represents a thought process for a student with a given VNM utility vector and a belief about the ex-post distribution of PRs. More precisely, a reporting strategy \( \psi \) takes a VNM utility vector \( u_i \) and a belief about the ex-post distribution of PRs \( \hat{r} \) as inputs and recommends a set of PRs that one should submit with positive probabilities. Let \( \psi^i(u_i, \hat{r}) \) denote the probability recommended by \( \psi \) with which student \( i \) should submit the PR \( r^i \) given his VNM utility vector \( u_i \) and his belief \( \hat{r} \). We say \( r^i \) is **recommended by** \( \psi \) for \( u_i \) and \( \hat{r} \) if \( \psi^i(u_i, \hat{r}) > 0 \). Let \( \Psi \) denote the set of all possible reporting strategies. I assume that each student independently draws a reporting strategy from a common distribution \( H^* \) on \( \Psi \). Finally, we need a language to describe a relation between two reporting strategies. A reporting strategy is **necessary for** the other if whenever a PR is recommended by the latter, the former recommends it as well. More precisely, given two reporting strategies, \( \psi_1, \psi_2 \), we say \( \psi_1 \) is necessary for \( \psi_2 \) if a PR \( r \) is recommended by \( \psi_1 \) whenever \( r \) is recommended by \( \psi_2 \). I introduce the following three strategies:

**Definition 1** (Truthful Reporting Strategy). *If a student uses the Truthful Reporting Strategy, then he reports his \( k \)th favorite school as his \( k \)th choice. That is, given a VNM utility vector \( u_i \) and a belief \( \hat{r} \), a PR \( r \in R \) is recommended by the Truthful Reporting strategy \( \psi^{TR}(u_i, \hat{r}) \) if and only if*

\[
\begin{align*}
    r_1 &= \arg \max_{s \in S} u_{is} \quad \forall \gamma \in \{2, \ldots, \ell\}, \\
    r_\gamma &= \arg \max_{s \in S \setminus \{r_1, \ldots, r_{\gamma-1}\}} u_{is}
\end{align*}
\]

**Definition 2** (Expected-utility-maximizing (EU-Maximizing) Reporting Strategy). *If a student uses the EU-maximizing strategy, then he submits a PR that maximizes his expected utility given his belief about the ex-post distribution of PRs. That is, given a VNM utility vector \( u_i \) and a belief \( \hat{r} \) (and the associated cut-off matrix \( \hat{C} \)), a PR \( r \) is recommended by the EU-maximizing Reporting Strategy \( \psi^{EU}(u_i, \hat{r}) \) if and only if the expected utility of \( r \) is weakly
greater than those of all the other PRs:

$$\forall r' \in R, \sum_{k=1}^{m} \pi^k(r, \hat{C})u_{ir_k} \geq \sum_{k=1}^{m} \pi^k(r', \hat{C})u_{ir'}$$

**Definition 3** (Simple Rule). If a student uses the Simple Rule, then he does not rank a popular school over a less popular school if:

1. he believes that he would be assigned to the popular school with positive probability; AND

2. he prefers a less popular school.

Precisely, given a VNM utility vector $u_i$ and a belief $\hat{r}$ (and the associated cut-off matrix $\hat{C}$), a PR $r$ is recommended by the Simple Rule $\psi_{SR}(u_i, \hat{r})$ if and only if

$$\forall k = 1, \ldots, m \text{ such that } \pi^k(r, \hat{C}) > 0, \text{ if for some } s \in S \text{ is such that } \hat{c}^k_s > \hat{c}^k_{r_k}, \text{ then } u_{ir_k} \geq u_{is}$$

The following result establishes the relationship between these three rules.

**Proposition 1.** *The Simple Rule is necessary for Truthful Reporting Strategy*

**Proposition 2.** *The Simple Rule is necessary for EU-maximizing Strategy*

The proof is collected in the appendix B. The intuition for proposition 1 is simple. Note that to violate the Simple Rule, you must have $k^{th}$-ranked a school, say $s$, on your report such that a) you believe you would be assigned to with positive probability and b) you like less than some school, say $s'$, that is easier to get into in round $k$. Suppose you have such two schools. Then, due to the Truthful Reporting, you must have ranked $s'$ higher than above $s$. Suppose that you $k^{th}$-ranked $s'$ for some $k' < k$. The fact that $s'$ is available in round $k$ means the assignment to $s'$ is guaranteed in round $k'$, meaning that you are assigned to $s'$ with certainty and not get to apply to $s$. But this contradicts the fact that there is a positive probability of being assigned to $s$.

The proof of proposition 2 proceeds in two steps. First, I show that among the schools you ranked on your report, if there are two schools that you believe you will
be assigned with positive probabilities, you must have preferred the higher ranked one of the two. This is because if you in fact preferred the lower ranked one, then you should have raised its ranking and be better off. Continuing with this argument, I show that you prefer your $k^{th}$-ranked school to those that are ranked below and to which you believe you will be assigned with positive probabilities.

For the second step, suppose that you $k^{th}$-rank a school $s$ on your report that you believe you will be assigned to with a positive probability and that there is another school $s'$ that is less popular in round $k$ than $s$. To abide by the Simple Rule you must prefer $s$ to $s'$. In order to show that, I first prove that you should not prefer $s'$ to the best among all the schools you rank $k^{th}$ and below that you believe you would be assigned to with positive probabilities. If you did, then replacing $s$ with $s'$ would result in a first-order stochastically dominating lottery over schools for you, which contradicts the revealed preference. But as the first step showed, the best among all the schools you rank $k^{th}$ and below that you believe you would be assigned to with positive probabilities is the $k^{th}$ ranked school. Hence, it must be that you prefer $s$ to $s'$.

3 Identification and Estimation

3.1 Identification

In this section I introduce the data generating process and show that this data generating process is well-defined in the sense that any data could be generated through it. Then I show how to construct the moment inequalities that (partially) identify the distribution of VNM utility vectors.

To introduce the data generating process, a few more notations are necessary. Let $\hat{X}$ denote a finite set of observables on students. Let $P$ denote the data that we observe. More precisely, $P$ is a joint distribution on the set of PRs $R$ and the set of observables $\hat{X}$. Let $P$ denote the set of all possible joint distributions.

With the notations, the data generating process is as follows:
Data Generating Process 3.1

**Step 1.** Each student \(i\) independently draws his observables \(X_i\) from the marginal distribution of \(F^*\) on \(\hat{X}\)

**Step 2.** Given the observables \(X_i = X\), \(i\) independently draws her VNM utility vector \(u_i\) from the conditional distribution of \(u_i\) given \(X\), i.e. \(F^*|_{X_i=X}\)

**Step 3.** Each student \(i\) independently draws a belief \(\hat{r}_i\) from the distribution \(G^*\) on the set of beliefs \(B\)

**Step 4.** Each student \(i\) independently draws a reporting strategy \(\psi_i\) from the distribution \(H^*\) on the set of reporting strategies \(\Psi\)

**Step 5.** Given student \(i\)'s draw of the VNM utility vector, belief, and reporting strategy, randomly report one of the PRs recommended by the reporting strategy. Student \(i\)'s probability of choosing \(j^{th}\) PR \(r^j\) is equal to \(\psi_j^i(u_i, \hat{r}_i)\), which is the probability the reporting strategy assigns to \(r^j\)

The data generating process 3.1 makes each student an independent observation, which is necessary for the uniform consistency of confidence region to be constructed in the estimation section. It is also well-defined, as the following result shows, in the following sense: whatever the joint distribution \(P\) we observe, there exist primitives of the data generating process, namely \(F^*, G^*,\) and \(H^*\) such that generates the data.

**Lemma 1** (The Data Generating Process is Well-defined). *Given any data, there exists primitives of the model that generated it. More precisely, given any \(P \in \mathcal{P}\), there exists \(F^*, G^*,\) and \(H^*\) such that for each \(j^{th}\) PR \(r^j\), \(P(r_i = r) = \int \int \int \psi_j^i(u, \hat{r})dH^*dG^*dF^*\)

In the proof in the appendix I show that for any given data \(P\), if all students are playing a Truthful Reporting Strategy, then there exists a distribution of VNM utilities \(F^*\) that generates the joint distribution. Therefore, to ensure that the data generating process 3.1 is well-defined in the sense of Lemma 1, all we need is to have the Truthful Reporting Strategy in the set of reporting strategies. The following assumption guarantees that:
Assumption 1. A reporting strategy $\psi$ is in the set of reporting strategies $\Psi$ if and only if the Simple Rule is necessary for $\psi$.

This assumption ensures that both the Truthful Reporting Strategy and the EU-maximizing Strategy are included in the set of reporting strategies by the virtue of proposition 1 and proposition 2. However, note that this does not necessarily mean that there would be a positive measure of students who play the Truthful Reporting Strategy or the EU-maximizing Strategy because we remain agnostic about the distribution $H^*$ on the set of strategies. It is possible, for example, that the probability of drawing a EU-maximizing Reporting Strategy is zero under the true distribution of the reporting strategies $H^*$.

Now I show how to construct moment inequalities that partially identify the true distribution $F^*$. First, make an assumption about the set of beliefs depending on the researchers’ prior about students’ beliefs and/or the question one is trying to answer. The stronger assumption on the set of beliefs, the more moment inequalities one can construct, but at the risk of a misspecified model.

Second, choose two schools $s$ and $s'$ such that correspond to “popular” and “less popular” school in the definition of the Simple Rule (Definition 3):

**Condition 3-1. (Everyone believed that school $s$ would be popular in round $k$)** All students believed that assignments to school $s$ would not be impossible but not guaranteed in round $k$ of the algorithm for the Boston mechanism. For popular schools whose seats would be filled up in round 1, for example, $k$ would be equal to 1. For schools not as popular, $k$ could be larger than 1. Precisely, $s$ is such that there exists a $k \in \{1, \ldots, m\}$ for which for all $\hat{r} \in B$, $C(\hat{r})(k, s) \in (0, 1)$

**Condition 3-2. (Everyone believed that $s'$ would be less popular than $s$)** All students believed that $s'$ would be less popular than $s$ in the round $k$. More precisely, for

\[\text{Condition 3-1. (Everyone believed that school } s \text{ would be popular in round } k\text{)}\quad \text{All students believed that assignments to school } s \text{ would not be impossible but not guaranteed in round } k \text{ of the algorithm for the Boston mechanism. For popular schools whose seats would be filled up in round 1, for example, } k \text{ would be equal to 1. For schools not as popular, } k \text{ could be larger than 1. Precisely, } s \text{ is such that there exists a } k \in \{1, \ldots, m\} \text{ for which for all } \hat{r} \in B, C(\hat{r})(k, s) \in (0, 1)\]

\[\text{Condition 3-2. (Everyone believed that } s' \text{ would be less popular than } s \text{)}\quad \text{All students believed that } s' \text{ would be less popular than } s \text{ in the round } k. \text{ More precisely, for}\]

\[\text{Note that it does not have to be two schools: one may find multiple less popular schools than a school } s \text{ that satisfy conditions 3.1 through 3.3. Then in the moment inequality } s \text{ should be preferred to all the less popular schools jointly}\]
all \( \hat{r} \in B, C(\hat{r})_{(k,s')} > C(\hat{r})_{(k,s)}. \)

**Condition 3-3. (Some reported \( s \) as their \( k \)th choice)** A positive measure of students reported \( s \) as their \( k \)th choice and believed they would be assigned to \( s \) with positive probability. More precisely, \( P \left( r_i \in R_{(k,s)} \right) > 0 \) where

\[
R_{(k,s)} = \left\{ r \in R : (r_k = s) \cap \left( \forall \hat{r} \in B \pi^k (r, C(\hat{r})) > 0 \right) \right\}
\]

Here we are requiring that \( s \) and \( s' \) have to be “popular” and “less popular” for ALL beliefs in the set of beliefs. Such requirement guarantees that everyone believed \( s \) to be more popular than \( s' \). Naturally, the more assumption we impose on the set of beliefs, the less restrictive this requirement becomes. For example, we could assume that the set of beliefs is a singleton. A weaker assumption would combine the set of beliefs that includes the ex-post distribution AND those that are similar to it (although some assignment probabilities associated with the latter are not the same as those characteristic of the former).

Suppose we made an assumption on the set of beliefs that there are two schools \( s \) and \( s' \) that satisfy **Condition 3-1** through **Condition 3-3**. Then the assumption that everyone adheres to the Simple Rule implies that if a student \( i \) whose observables are equal to \( X \) reported school \( s \) as her \( k \)th choice AND she (and everyone else) believed she would be assigned to \( s \) with positive probability, then she must have preferred the school \( s \) to the less popular \( s' \). That is,

\[
1 \{ X_i = X \} \cdot 1 \left( (r_{ik} = s) \cap \left( \forall \hat{r} \in B \pi^k (r, C(\hat{r})) > 0 \right) \right) - 1 \{ X_i = X \} \cdot 1 \{ u_{is} \geq u_{is'} \} \leq 0
\]

This inequality is true for each realization of the random variables, i.e., the observables \( X_i \), the PRs \( r_i \), and VNM utilities \( u_{is} \) and \( u_{is'} \). Therefore, the inequality should hold in expectation. Therefore, we can construct the following moment inequality, which is merely the above inequality with expectation operator with respect to the joint distri-
bution of the observables and PRs, \( \mathbb{E}_P[\cdot] \). I also take the expectation on the indicator function \( 1 \{ u_{is} \geq u_{is}' \} \) with respect to the conditional distribution of the VNM utilities, \( F^*_{|X_i = x} \) because VNM utilities are unobservable. Such operation would turn the indicator function \( 1 \{ u_{is} \geq u_{is}' \} \) into the conditional probability \( Pr_{F^*_{|X_i = x}} (u_{is} \geq u_{is}') \). Now the moment inequality is as follows:

\[
\mathbb{E}_P \left[ 1 \{ X_i = X \} \cdot 1 \left\{ \left( r_{ik} = s \right) \cap \left( \forall r \in B \pi^k (r, C (r)) > 0 \right) \right\} \right. \\
- \left. 1 \{ X_i = X \} \Pr_{F^*_{|X_i = x}} (u_{is} \geq u_{is}') \right] \leq 0
\]

The interpretation of this moment inequality is simple. The probability of a student ranking a popular school over a less popular school should be smaller than the probability the popular school is actually preferred because there might be some students who prefer the popular school but did not rank them on their reports.

If moment inequalities such as these under some distribution \( F \), then \( F \) can potentially be the true distribution \( F^* \). To state the set of distribution \( F \) identified by the moment inequalities succinctly, let me introduce a few more notations. \( Y_i \) denotes student \( i \)'s observables and reported PR, that is, \( Y_i = (X_i, r_i) \). Suppose there are \( J \) moment inequalities and let \( j^{th} \) moment inequality evaluated at the distribution \( F \) be denoted by \( g_j(Y_i, F) \). Then, the set of distribution \( F \) that is identified by the \( J \) moment inequalities when the data (or the joint distribution on \( \hat{X} \) and \( R \)) is \( P \), denoted by \( \mathcal{F}_0(P) \), is

\[
\mathcal{F}_0(P) = \{ F : \text{For all } j = 1, \ldots, J \mathbb{E}_P \left[ g_j(Y_i, F) \right] \leq 0 \}
\]

For the convenience of notations in the subsequent section, I am going to denote the \( J \) dimensional vector of moment inequalities with \( g(Y_i, F) \). That is,

\[
g(Y_i, F) \equiv (g_1(Y_i, F), \ldots, g_J(Y_i, F))^t
\]
3.2 Estimation

In this section, I first parameterize the distribution on the VNM utility vectors $F$ and re-state the identified set. Then I propose a method to construct confidence regions for identifiable parameters that are uniformly consistent in level. Proposition 3 proves the uniform consistency of the constructed confidence region.

Suppose $F$ can be parameterized with a vector of parameters, $\theta$, that resides in some parameter space $\Theta$. Let $F_\theta$ denote the distribution of VNM utility vectors parameterized by $\theta$. Then the set of parameters identified the moment inequalities, denoted by $\Theta_0(P)$, can be stated as follows:

$$\Theta_0(P) = \{ \theta \in \Theta : \mathbb{E}_P [g(Y_i, \theta)] \leq 0 \}$$

where $g(Y_i, \theta)$ is equal to $g(Y_i, F_\theta)$ and the inequalities are interpreted component-wise.

We wish to construct confidence region for the parameters in this identified set.

Suppose a researcher has the following data:

- **Data 1.** Observables and reported PRs for $n$ students randomly selected from $I \in [0, 1]$, that is, $Y_i \equiv (X_i, r_i), \ i = 1, \ldots, n$

- **Data 2.** The cut-off matrix calculated from the ex-post distribution of PRs

For some prespecified significance level $\alpha \in (0, 1)$, we wish to construct random sets $C_n = C_n(Y_1, \ldots, Y_n)$ that is uniformly consistent in level, that is, $C_n$ satisfies the following condition:

$$\lim_{n \to \infty} \inf_{P \in \mathcal{P}} \inf_{\theta \in \Theta_0(P)} P(\theta \in C_n) = (1 - \alpha)$$

Before stating the algorithm to construct confidence regions, I introduce a set of notations to avoid cluttering in the exposition of the estimation algorithm. Let $w_i$ and

\[\text{We need data 2 to be able to calculate the assignment probabilities for each PRs. But this should not be very difficult because in all papers in this literature authors had access to preference reports for ALL students in the school district they are interested, and from such data they can calculate the ex-post cut-off matrix}\]
\( w_{i,j} \) denote the vectors of functions that constitute the moment inequalities and each component of it, i.e. \( w_i \equiv g(Y_i, \theta) \) and \( w_{i,j} \equiv g_j(Y_i, \theta) \). \( \mu(P) \) denotes the mean of \( w_i \), i.e. \( \mu(P) \equiv E_P[g(Y_i, \theta)] \) and \( \mu_j(P) \) denotes the mean of \( w_{i,j} \), i.e. \( \mu_j(P) = E_P[g_j(Y_i, \theta)] \) for \( j = 1, \ldots, J \). \( \hat{P}_n \) denotes the empirical distribution of the PRs and observables computed from the sample \( Y_i \ i = 1, \ldots, n \). Let \( \bar{w}_n = \mu(\hat{P}_n) \) and \( \bar{w}_{i,n} = \mu_j(\hat{P}_n) \). The notation \( \Sigma(P) \) denotes the covariance matrix of \( w_i \) when \( Y_i \) follows the distribution \( P \) and \( \sigma_j^2(P) \) denotes the variance of \( w_{i,j} \), i.e. \( \sigma_j^2(\hat{P}_n) \). The notation \( \Omega(P) \) denotes the correlation matrix of \( w_i \). Let \( \hat{\Omega}_n \) be a shorthand for the correlation matrix computed from the empirical distribution, i.e. \( \hat{\Omega} = \Omega(\hat{P}_n) \) and \( S_{i,j,n}^2 \) for the variance of \( w_{i,j} \) computed from \( \hat{P}_n \), i.e. \( S_{i,j,n}^2 = \sigma_j^2(\hat{P}_n) \). Finally, let \( S_n^2 \) denote the diagonal matrix of the empirical variances, i.e. \( S_n^2 = diag(S_{1,n}^2, \ldots, S_{d,n}^2) \).

Next, a test statistic \( T_n \) is required such that large values of \( T_n \) provide evidence against the null that the given \( \theta \in \Theta \) is in the confidence region. As in Romano et al. (2014), I consider a test statistic of the form

\[
T_n = T \left( S^{-1}_n \sqrt{n} \bar{w}_n, \hat{\Omega}_n \right)
\]

Now we are ready to state the algorithm to construct confidence regions. For each parameter vector \( \theta \), the following algorithm produces the test \( \phi_n(\theta) \) of the null \( H_\theta : E[g(Y_i, \theta)] \leq 0 \).

**Algorithm 3.2** (Romano et al. (2014))

**Step 1.** Set \( w_i = g(Y_i, \theta) \). For each \( i = 1, \ldots, n \), compute \( w_i \) as follows:

**Step 1-1.** \( \forall j = 1, \ldots, J \), compute the indicator functions associated with the \( j^{th} \) moment inequality. These can be directly calculated from the data.

**Step 1-2.** \( \forall j = 1, \ldots, J \), simulate the conditional probability associated with the \( j^{th} \) moment inequality. For example, \( \Pr_{F_\theta|X_i = X}(u_{is} \geq u_{is}') \) can be simulated by 1) drawing \( u_{is} \) and \( u_{is}' \) from \( F_{\theta|X_i = X} \) \( N \) times; 2) divide the number of times that \( u_{is} \geq u_{is}' \) by \( N \).
Step 2. Compute the bootstrap quantile $K^{-1}(1 - \beta, \hat{P}_n)$ for a fixed $\beta \in [0, \alpha]$, where $K_n(z, P)$ is defined as follows:

$$K_n(z, P) = P \left\{ \max_{1 \leq j \leq d} \frac{\sqrt{n}(\mu_j(P) - \bar{w}_{j,n})}{S_{j,n}} \leq z \right\}$$

Step 3. Using $K^{-1}(1 - \beta, \hat{P}_n)$ from step 2, compute $M_n(1 - \beta)$, which is defined as follows:

$$M_n(1 - \beta) = \left\{ \mu \in \mathbb{R}^k : \max_{1 \leq j \leq d} \frac{\sqrt{n}(\mu_j - \bar{w}_{j,n})}{S_{j,n}} \leq K_n^{-1}(1 - \beta, \hat{P}_n) \right\}$$

Step 4. Using $K^{-1}(1 - \beta, \hat{P}_n)$ from step 2, compute $\lambda^*$ defined as follows:

$$\lambda^*_j = \min \left\{ \frac{W_{j,n}S_{j,n}^{-1}(1 - \beta, \hat{P}_n)}{\sqrt{n}}, 0 \right\}, \ j = 1, \ldots, d$$

Step 5. Compute the bootstrap quantile $\hat{c}_n(1 - \alpha + \beta) = J_n^{-1}(1 - \alpha + \beta, \lambda^*, \hat{P}_n)$, where $J_n(z, \lambda, P)$ is defined as follows:

$$J_n(z, \lambda, P) = P \left\{ T(S_n^{-1}(\sqrt{n}(\bar{w}_n - \mu(P))) + S_n^{-1}\sqrt{n}\lambda, \hat{\Omega}_n) \leq z \right\}$$

Step 6. Compute $\phi_n(\theta) = \phi_n$, where $\phi_n$ is defined as follows:

$$\phi_n = 1 - 1 \left\{ \{ M_n(1 - \beta) \subseteq \mathbb{R}^d \} \cup \{ T_n \leq \hat{c}_n(1 - \alpha + \beta) \} \right\}$$

Consider $C_n = \{ \theta \in \Theta : \phi_n(\theta) = 0 \}$. Then $C_n$ satisfies the desired property:

**Proposition 3.** Assume the following conditions hold for the set of observables $\hat{X}$ and each joint distributions $P \in \mathcal{P}$ and each parameter $\theta \in \Theta$:

*Condition 3.4 (Variations in the Observables)* $|\hat{X}| > 1$ and for each $X \in \hat{X}$, $P(\{ i \in I : X_i = X \}) > 0$

*Condition 3.5 (Full Support Condition)* For all $\theta \in \Theta$ and any two schools $s, s'$, there
is a positive measure of students who prefer s to s'. That is,

$$\int_{\mathbb{R}^{|S|}} 1 \{ u_{is} \geq u_{is'} \} dF_{\theta} > 0$$

Then $C_n$ constructed through the algorithm 3.2 is a confidence region that is uniformly consistent at level $(1 - \alpha)$, i.e.

$$\lim \inf_{n \to \infty} \inf_{P \in \mathcal{P}} \inf_{\theta \in \Theta_0} P(\theta \in C_n) = (1 - \alpha)$$

For test statistics I recommend the “modified method of moments” test statistics among the ones suggested by Romano et al. (2014).

$$T_{n}^{mmm} = \sum_{j=1}^{d} \left( \frac{\sqrt{n} v_i w_{i,n}}{S_{j,n}} \right) \cdot 1 \{ w_{i,n} > 0 \}$$

4 Application: the (Constrained) Boston Mechanism in Seoul High School Match

In this section I introduce the institutional background of the Seoul High School Match (the Match henceforth) in 2012, the year for which I have data. Then I apply the proposed method to estimate the preference parameters and evaluate the efficiency of the current mechanism and its alternatives.

4.1 The Seoul High School Match

In middle schools 80181 seniors participate in the Match. Depending on where she lives, each student belongs to one of twelve choice-zones, each of which bundles together two to three administrative divisions of Seoul. Students can commute within or across choice-zones in public transportations such as subways or buses at a relatively small cost (the fare is less than a dollar). It takes approximately two hours traveling
from the city’s East end to the West end or from the North end to the South end.

In 2012 there are 216 schools in Seoul. Each school belongs to one of the choice-zones depending on its location. Schools are partitioned into sets on the basis of three criteria:

1. Is the school charter, public or private?
2. Does the school offer science-focused curriculum or not?
3. Is the school co-ed, boys-only or girls-only?

Table 1 shows the number of schools in each partition. The sum of the number of seats across all schools exceed the number of students. Science-focused schools are smaller than non-science-focused schools: the average number of seats at science-focused schools is 87 whereas that for non-science-focused schools is 413. Tuitions are equalized across all schools.

Students in each high school take a nationally administered standardized test in their junior year of high school and the results aggregated at school level are published on a website. The three statistics are published: the percentage of students who on the test scored above average, average, and below average. In this paper I use the sum of percentages of students who scored above average and average as the index for the quality of education at each school.

Each family reports their preference ranking for up to six schools. The difference between the preference reports in Seoul and the one considered in our model is that for each position on the preference report there may be a set of schools that one cannot rank on that position. In the Seoul system, for the first position students can rank a charter school only; for the second position a science school only; for the third and fourth position, non science-focused private or public schools only; and for the fifth and sixth position, non science-focused private or public schools in one’s choice zone only.

This constraint on the preference report may lead some students not to rank a charter and/or a science-focused school. For example, students who view a charter school
as an eligible option might not have ranked it on their report because they do not want to be assigned to a charter school before they can apply to their favorite non-science-focused public/private schools. Therefore, the fact that a small percentage of students ranked a charter and/or science school would not necessarily signify that students find charter/science-focused schools unacceptable. In the data we see that only 16% (see Table 2) of the students ranking charter and/or science schools.

Taking the reported preference rankings as input, the algorithm for the Boston mechanism is run to determine the assignment for each student. The difference between the algorithm for the Match and that considered in this paper is that in the former, for some rounds of the algorithm, a part of the seats at each school is not available for assignments. In other words, for the third and fourth round, only twenty percent of the seats at each school is available for assignments. while during the fifth and the sixth round, sixty percent of the seats. (The latter includes the twenty percent that were available for the previous two rounds and may have been filled up before the beginning of the fifth round).

This feature, combined with the constraint on the preference reports, might create incentives for students to shy away from schools that are far from home but of high quality (and hence are popular). Recall that students can rank non-science-focused private or public schools outside of their choice-zones only as their third and fourth choices. However, only 20% of the seats are available for assignments in round 3 and 4, so the probability of being assigned to popular schools in these rounds is very small. So as not to waste their third and fourth choices on popular schools that are outside of your choice zones, students might report within-choice-zone schools of mediocre quality as their third or fourth choices. In the data we do see that students ranked nearby schools much more often than far-away schools (see Table 3), but it is not clear to what extent this is due to students’ response to incentives or to an actual preference for proximity. With the structural estimates I am able to parse out the taste for quality from the taste for proximity.
At the end of the sixth round, unassigned students are administratively assigned to a school according to the following pre-announced rule:

...unassigned students will be randomly assigned to one of the schools in their choice zones and neighboring ones. Considerations will be given to schools they ranked, convenience of commutes, and their religions...

This is an ambiguous rule, except that it specifies the set of schools unassigned students might be assigned to administratively. However, while it is interesting to investigate why the Seoul school district has put forth such an unclear rule, this is an anomalous design feature specific to Seoul and not a general feature of the Boston mechanism. Therefore, I am going to abstract from the complication this rule might create by assuming that all students believe that their reports do not affect their assignment probabilities in the administrative assignment.

**Assumption 2.** In the Match, students believe that their PRs cannot affect their assignment probabilities in the administrative assignment.

### 4.2 Estimation Results

In this section I discuss my data and the parametric assumptions I make on the distribution on the set of VNM utility vectors. Then I present the confidence region in Table 4.

The data is \((X_i, r_i)\), for \(i = 1, \ldots, 80181\), where \(X_i = (x_{is,1}, \ldots, x_{is,m})\) and \(x_{is} \in \mathbb{R}^7\) consists of the following: for \(k \in \{1, \ldots, 7\}\), denoting \(k^{th}\) coordinate of \(x_{is}\) with \(x_{is,k}\),

- \(x_{is,1}\): the sum of percentage of students who scored above average or average on the nationally administered standardized test, or the quality index, for \(s\)
- \(x_{is,2}\): \(1\{s\text{ is a science-focused school}\}\)
- \(x_{is,3}\): \(1\{s\text{ is a charter school}\}\)
- \(x_{is,4}\): \(1\{s\text{ is a private school}\}\)
- \(x_{is,5}\): \(1\{(s\text{ is a boys-only school}) \cap (i\text{ is a boy})\}\)
$x_{is,6} = \mathbb{1}\{s \text{ is a girls-only school}\} \cap (i \text{ is a girl})$

$x_{is,7}$: distance in minutes from $i$’s home to $s$

Then I make the following assumption about the true distribution of VNM utility vector $u_i$ conditional on $X_i$.

**Assumption 3.** $u_i | X_i = X_i' \beta_i + \epsilon_i$ where $\beta_i = (\beta_{i1}, \ldots, \beta_{i7})' \in \mathbb{R}^7$ and $\epsilon_i = (\epsilon_{is1}, \ldots, \epsilon_{is216})' \in \mathbb{R}^{216}$

1. **(normalization)** $\forall i \in I, \beta_{i7} = 1$

2. **(random coefficients)** $(\beta_{i1}, \ldots, \beta_{i6})'$ has the distribution $N(\bar{\beta}, \Sigma)$ for some $\bar{\beta} \in \mathbb{R}^6$ and $\Sigma \in \mathbb{R}^{6 \times 6} \setminus \{\Sigma' \in \mathbb{R}^{6 \times 6} : \exists k, k' \in \{1, \ldots, 6\} \left(\Sigma'_{k,k} < 0 \right) \cup \left(\Sigma'_{k,k'} \neq \Sigma_{k',k'}\right)\}$

3. **(homoskedasticity and uncorrelatedness of unobservables)** $\epsilon_i$ has the distribution $N(0, \sigma_{\epsilon} I_m)$, where $I_m$ denotes $m \times m$ identity matrix.

Note that because of the normalization, the unit of parameter estimates and welfare is the minute spent in commuting.

As explained in the identification section, we need an assumption on the set of potential beliefs $B$. Here is the assumption I make for this data:

**Assumption 4.**


2. All students believe that if a school has been more popular than the other for the year 2011 and 2010, then it will be so as well in 2012.

Recall that the estimation is conducted with the data from year 2012. To not rule out the possibility that all students are playing an EU-maximizing Reporting Strategy with equilibrium beliefs, I assume that the ex-post distribution of PRs in 2012 is in the set of potential beliefs. The impetus for the second assumption is empirical. The popularity of schools realized in the Match for years 2011 and 2010 was published in year 2012.

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7∀$i \in \{1, \ldots, 80181\}$ and $\forall s \in S$, the quality index for $s$ and the distance from $i$’ home to $s$ are made unit-free by subtracting the average value of each variable over all schools and dividing the standard deviation of each variable over all schools. The purpose is to make the estimation of confidence region computationally easier. In the welfare simulations, the standard deviation of the distance variable is multiplied to each simulated VNM utility to convert the unit back to the minutes spent in computing.
a major newspaper in Korea. Consequently, students are likely to be aware of them. It is plausible that some students form their beliefs based on the ex-post popularity in previous years.

The Boston mechanism as implemented in Seoul is different from the one considered in the model section. Thus I focus on 1st, 2nd, 5th and 6th ranked schools for a clean application of the method. As will be seen, the confidence region is already quite tight so we are not losing much information by not using students’ 3rd and 4th ranked schools in the estimation.

There are 24458 moment inequalities and in the estimation I dropped a majority of them to account for the fact that I am imposing a strong parametric form on the distribution of VNM utility vectors. For \( j = 1, \ldots, 24458 \), the \( j \)th moment inequality \( \mathbb{E} [g_j(Y, \theta)] \) is dropped if:

**Criterion 1** if the number of students whose observables are equal to \( X \) associated with \( j \)th moment inequality is less than \( \bar{N} \); OR

**Criterion 2** if too few or too many students submitted PRs associated with \( j \)th moment inequality relative to the number of students whose observable is equal to the one associated with the same moment inequality. That is, letting \( X^j \) and \( r^j \) denote the observables and PR associated with \( j \)th moment inequality,

\[
\text{Drop if } \left| \frac{\{ i = 1, \ldots, 80181 : X_i = X^j, r_i = r^j \}}{\{ i = 1, \ldots, 80181 : X_i = X^j \}} \right| > \bar{u}_U \text{ or } < \bar{u}_L
\]

With \( \bar{N} = 145 \) and \( \bar{u}_L = 0.695 \) and \( \bar{u}_U = 0.405 \), 200 moment inequalities remain for the estimation. Table 4 presents the confidence region. The unit of each coefficient is a minute spent in commuting. The interpretation of the mean of the random coefficient is straightforward. For example, the interpretation of \( \bar{\beta}_{\text{QualityInde}} \) would be that if a school increases its quality index by one standard deviation, then for average students it is equivalent to the commute to the school shortening by 12.5 minutes to 14.4

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8Chosun Ilbo, one of the biggest newspaper in terms of circulation, published the cut-offs for the year 2011 and 2010.
minutes. However, the interpretation of the components of \( \hat{\Sigma} \) is difficult. For computational tractability, Cholesky-decomposed \( \Sigma \) for estimation and the \( \hat{\Sigma} \) is a lower triangle matrix such that \( \hat{\Sigma} \Sigma' = \Sigma \). Therefore, estimates of each component of \( \hat{\Sigma} \) is not the covariance of random coefficients.

Out of 28 parameters, only one, \( \hat{\Sigma}_{(6,3)} \), cannot be signed unambiguously. As expected, the sign of the mean of the coefficient on the quality index is positive, while those for the dummy variable for science-focused and charter schools is negative, reflecting the fact that less than one fifth of students ranked one on their reports.

### 4.3 Simulation Results

In this section I compare the efficiency of the current mechanism for the Match (Seoul mechanism henceforth) with three alternatives. The first alternative is the unconstrained Boston mechanism that has been explained above. I introduce the two alternatives, Deferred Acceptance and Top Trading Cycles, in the first subsection. Then I compare the efficiency of Seoul mechanism with the three alternatives and that of each alternative with the others. The first comparison has an obvious policy implication for the Seoul school district. The second comparison is of independent interest because the theory is not clear on whether the Boston mechanism is more or less efficient than Deferred Acceptance.

#### 4.3.1 Deferred Acceptance and Top Trading Cycles

Some economists have proposed Deferred Acceptance and Top Trading Cycles as alternatives to the Boston mechanism (Abdulkadiroğlu and Sönmez (2003), Ergin and Sönmez (2006), Kojima (2008), Chen and Sönmez (2006), Abdulkadiroğlu et al. (2006)) because these mechanisms possess at least one of the following properties that the Boston mechanism lacks. The first such property is stability. A mechanism is stable if in any assignment it produces, there is no unmatched student-school pair \((i, s)\) such that \(i\) prefers \(s\) to her assignment AND \(i\) has higher priority to \(s\) than another
student \( i' \) matched to \( s \). The stability has been considered a desirable property for a student assignment mechanism for two reasons. The first reason is practical; it allows the school district to avoid legal actions taken by justifiably dissatisfied parents whose children are not matched to the school. These students should have a higher priority than some students who in fact are matched to it. This can happen under the Boston mechanism: if a student \( i \) with low priority to a school \( s \) ranked it higher than another student \( i' \) with higher priority to \( s \) than \( i \), the Boston mechanism assigns \( i \) ahead of \( i' \). The second reason is that the stable mechanisms tend to be more sustainable than unstable ones.

The second property that the Boston mechanism lacks is strategy-proofness. Under strategy-proof mechanisms, it is weakly dominant for students to report their true preference rankings. Strategy-proofness has been considered desirable because it protects families not strategic enough to understand how non-strategy-proof mechanisms such as the Boston mechanism work and who fail to manipulate their true preference on their reports in a sophisticated way (Abdulkadiroğlu et al. (2006), Pathak and Sönmez (2008)).

Deferred Acceptance is both stable and strategy-proof (Abdulkadiroğlu and Sönmez (2003)). Here is the algorithm for Deferred Acceptance:

**Algorithm for Deferred Acceptance**

**Round -1.** Students report preference rankings

**Round 0.** A random lottery is drawn

**Round 1.** Students apply to their favorite schools. Each school tentatively admits students with highest priorities, and if there are ties within the same priority level it admits first the students with better lottery number. The remainder are rejected

**Round \( k, k \geq 2 \).** Students currently not admitted to any school apply to their favorite schools among those that have not yet rejected them. Among the students who have been tentatively admitted at the beginning of this round and those who applied this round, schools tentatively admit students with the highest priorities and let the lottery
number break ties. The remainder are rejected

Top Trading Cycles, on the other hand, is strategy-proof but not stable. The algorithm for Top Trading Cycles runs as follows:

**Algorithm for Top Trading Cycles**

**Round -1.** Students report preference rankings

**Round 0.** A random lottery is drawn

**Round 1.** Students “point” to their favorite schools and schools “point” to the student who has the highest priority. Find a cycle of students and schools that the student with the best lottery number initiates. The cycle represents a mutually beneficial exchange. Students in the cycle are assigned to the schools they point to, and these students and their seats are taken out of the algorithm

**Round k, k ≥ 2.** Remaining students “point” to their favorite schools that still have vacancies and schools that still have vacancies “point” to the student who has the highest priority. Find a cycle of students and schools that the student with the best lottery number initiates. The cycle represents a mutually beneficial exchange. Assign students in the cycle to the schools they point to, and take these students and their seats out of the algorithm

Unlike these two alternatives, The Boston mechanism are neither stable nor strategy-proof (Abdulkadiroğlu and Sönmez (2003)). Therefore, the alternatives clearly (although weakly) dominate the Boston mechanism in these two dimensions. However, theory is not clear on the efficiency ranking among the three alternatives. Several authors showed that the Boston mechanism is more efficient than Deferred Acceptance (Miralles (2008), Abdulkadiroğlu et al. (2011), Troyan (2012), Akyol (2013)) or the other way around (Ergin and Sönmez (2006), Kojima (2008)), but theory is not clear on the efficiency ranking under general environment. Therefore, the welfare comparison between the Boston mechanism and Deferred Acceptance is of independent interest. Top Trading Cycles is Pareto-efficient, but we do not have theory on how its ex-ante welfare fares with other alternatives.
4.3.2 Efficiency Comparison

To compare efficiency between mechanisms, the definition of welfare measures is necessary. The two welfare measures that have been used in the school choice literature are interim expected utility of each student and ex-ante welfare of a mechanism (Abdulkadiroğlu et al. (2011), Troyan (2012), Akyol (2013)). A precise definition with the current notation is to follow.

**Definition 1.** Interim expected utility of student $i$ whose VNM utility vector is $u_i$ and reports $r_i$ is

$$\sum_{k=1}^{m} \pi_k^{k}(\psi_i(u_i, \hat{r}_i), C(\hat{r}_i)) u_{ir_k}$$

**Definition 2.** Ex-ante welfare of a mechanism is

$$\int\int\int\int \sum_{k=1}^{m} \pi_k^{k}(\psi_i(u_i, \hat{r}_i), C(\hat{r}_i)) u_{ir_k} dHdGdF_{\theta^*}$$

However, we cannot directly compute the interim utility for each student because we do not observe her belief or reporting strategy. Similarly, the ex-ante welfare cannot be calculated either without knowing the distribution $G$ and $H$ on the set of beliefs and the set of reporting strategies. Therefore, I compute these measures under two strong assumptions.

**Assumption 5 (EU-maximizing Strategy and Equilibrium Beliefs).** The true distribution on the set of beliefs, $G^*$ is degenerate on the ex-post distribution of PRs; and the true distribution on the set of reporting strategies, $H^*$ is degenerate on the EU-maximizing Reporting Strategy.

**Assumption 6 (All students report truthfully).** $H^*$ is degenerate on Truthful Reporting Strategy.

Now I explain how I computed these measures. For each $\theta$ in the confidence re-
gion, I took the following steps to calculate the interim utility for each student:

**Step 1-int.** Draw $u_i$ for $i = 1, \ldots, 80181$ from $F_{\theta}$

**Step 2-int.** For each student choose a PR to report. For Deferred Acceptance and Top Trading Cycles that are strategy-proof, or the Boston mechanism when all students use the Truthful Reporting Strategy, then choose a PR that agrees with the true preferences. If students use the EU-maximizing Reporting Strategy with the ex-post distribution as their beliefs, then compute the (approximate) equilibrium\(^9\).

**Step 3-int.** Run each algorithm with many tie-breaking lotteries. Record students’ utilities from each final assignment.

**Step 4-int.** Average each students’ utilities over lotteries.

The statistics of interest based on the interim utility is the the percentage of students who weakly prefer a mechanism over others. For a given estimate of the parameters it is computed by repeating step 1 through 4, measuring the percentage of students who prefer the mechanism at the end of each step 4, and average over draws. The bounds are obtained from taking the minimum and maximum of the percentages over the estimates in the confidence region.

The steps for computing ex-ante welfare is similar:

**Step 1-ex.** same as **step 1-int**

**Step 2-ex.** same as **step 2-int**

**Step 3-ex.** same as **step 3-int**

**Step 4-ex.** same as **step 4-int**

**Step 5-ex.** Average the interim utility over students and record it.

**Step 6-ex.** Repeat **step 1-ex** through **step 5-ex** and average over draws.

The ex-ante welfare of a mechanism at each parameter estimate is obtained at the end of **Step 6-ex**. The upper and lower bounds that are presented in tables below are the maximum and minimum of the difference in ex-ante welfare over all parameter estimates.

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\(^9\)I computed an approximate equilibrium at which at least 95% of the students submit PRs for which their expected utility is within 5% of the maximum
Before discussing the results, I introduce the priority structure for the alternatives used in the simulation. The priority structure is another design parameter that can affect the efficiency of mechanisms. I experimented with several different definitions of walk-zone priorities and I chose the one that favors the alternatives the most. As a result, students have walk-zone priorities to four schools that are closest to their homes.

We first turn to the comparison of interim expected utilities (Table 5, 6). Note that there is no interim Pareto-dominance relation between any two mechanisms. However, some 70% students weakly prefer the unconstrained Boston mechanism to the Seoul mechanism. Top Trading Cycles is also favored by a majority of students. However, the results for Deferred Acceptance is ambiguous; when students play an equilibrium strategy under the Seoul mechanism, a majority of students weakly prefer the Seoul mechanism.

Surprisingly, the difference in ex-ante welfare between any two mechanisms is quite small (Table 8, 9). The three alternatives all provide higher ex-ante welfare than the Seoul mechanism when students report truthfully under the Seoul and the Boston mechanism. However, the difference measured in minutes of commuting is approximately two minutes. That is, students are willing to commute for two more minutes if an alternative mechanism replaces the current one. When students play an approximate equilibrium strategy under the Seoul and the Boston mechanism, only Top Trading Cycles is more efficient than the current mechanism, but the difference is less than a minute.

As for the comparison between the three alternatives, when students report truthfully under the Boston mechanism, close to 70% of students prefer the Boston mechanism, whereas when everyone plays an approximate equilibrium, then a majority prefers Deferred Acceptance and Top Trading Cycles (Table 7).

However, the difference in ex-ante welfare is again small between the Boston mechanism and the other two strategy-proof mechanisms (Table 10). Top Trading Cycles provides higher ex-ante welfare than the Boston mechanism regardless of the assump-
tions about $G$ and $H$, but the difference is less than a minute. When students play an equilibrium, the Boston mechanism provides higher ex-ante welfare than Deferred Acceptance, but the edge is less than half a minute. Lastly, a majority of students prefer Top Trading Cycles over Deferred Acceptance but the difference in the ex-ante welfare is less than a minute (Table 11).

5 Conclusion

In this paper I explore the possibility of estimating parents’ preference parameters without relying stylized reporting strategies that provide tight mapping between the primitives and the observed reports. More specifically, I assume that all parents follow a intuitively appealing simple rule when they choose what preference rankings to report and investigate how much information I can extract from the data with this limited structure. The results show that one can still do inference and run counterfactual simulations that are informative.

In future research I will investigate whether different student assignment mechanisms promote competitions among schools to different degrees. Kojima et al. (2011) has shown that in the large market any stable mechanism incentivizes schools to improve its quality, whereas the Boston mechanism and Top Trading Cycles fail to do so. To my knowledge, an empirical analysis of the supply side response to different mechanisms has yet been undertaken. Such analysis would allow an analysis of general equilibrium effect of different mechanisms.

References


Abdulkadiroğlu, Atila; Pathak, Parag A.; Roth, Alvin E., and Sönmez, Tayfun. Chang-


Chen, Yan; Jiang, Ming; Kesten, Onur; Robin, Stéphane, and Zhu, Min. A large scale school choice experiment. Working Paper, 2013.


### A Tables

<table>
<thead>
<tr>
<th></th>
<th>Public</th>
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<th></th>
<th>Charter</th>
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<td>Non-science</td>
<td>Science</td>
<td>Non-science</td>
<td>Science</td>
<td>Non-science</td>
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<td>4</td>
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<td>3</td>
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<tr>
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<td>55</td>
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<td>15</td>
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</table>

Table 1: Number of Schools by Type
<table>
<thead>
<tr>
<th>% of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ranked a charter but no science</td>
</tr>
<tr>
<td>Ranked a Science but no charter</td>
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<tr>
<td>Ranked a charter and science</td>
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<tr>
<td>Ranked neither</td>
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</table>

Table 2: Charter and science schools
<table>
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<th>Quality Index ≥ 60</th>
<th>Distance ≤ 25 min.</th>
<th>Distance &gt; 25 min.</th>
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<tr>
<td>86.25</td>
<td>58.93</td>
<td></td>
</tr>
<tr>
<td>9.19</td>
<td>5.14</td>
<td></td>
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</table>

Table 3: Taste for Quality and Proximity: % of students who ranked schools corresponding to each cell
<table>
<thead>
<tr>
<th>Variable</th>
<th>95% Confidence Region</th>
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<tbody>
<tr>
<td>$\hat{\beta}_{QualityIndex}$</td>
<td>[12.5, 14.4]</td>
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<tr>
<td>$\hat{\beta}_{Science}$</td>
<td>[-25.3, -22.87]</td>
</tr>
<tr>
<td>$\hat{\beta}_{Charter}$</td>
<td>[-16.2, -13.0]</td>
</tr>
<tr>
<td>$\hat{\beta}_{Private}$</td>
<td>[-1.7, -0.4]</td>
</tr>
<tr>
<td>$\hat{\beta}_{Boys-only}$</td>
<td>[20.4, 21.7]</td>
</tr>
<tr>
<td>$\hat{\beta}_{Girls-only}$</td>
<td>[4.5, 6.5]</td>
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<td>$\hat{\Sigma}_{(3,2)}$</td>
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<td>[14.7, 19.8]</td>
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<tr>
<td>$\hat{\Sigma}_{(5,2)}$</td>
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<td>$\hat{\Sigma}_{(5,3)}$</td>
<td>[10.5, 16.8]</td>
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<tr>
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<tr>
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<td>$\hat{\Sigma}_{(6,3)}$</td>
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<td>[-7.4, -4.3]</td>
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<tr>
<td>$\hat{\Sigma}_{(6,6)}$</td>
<td>[2.9, 9.2]</td>
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<tr>
<td>$\sigma_\epsilon$</td>
<td>[1.6, 2.6]</td>
</tr>
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</table>

Table 4: 95% Confidence Region of the Parameters

38
Table 5: Bounds on the percentage of students whose interim utility is weakly higher under each alternative relative to under the Seoul mechanism, when students report truthfully under the Seoul & the Boston mechanism

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Bounds</th>
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</thead>
<tbody>
<tr>
<td>Unconstrained BM</td>
<td>[72.3,72.9]</td>
</tr>
<tr>
<td>Deferred Acceptance</td>
<td>[54.3,54.9]</td>
</tr>
<tr>
<td>Top Trading Cycles</td>
<td>[56.6,56.8]</td>
</tr>
<tr>
<td>Mechanism</td>
<td>Percentage Range</td>
</tr>
<tr>
<td>---------------------------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>Unconstrained Boston Mechanism</td>
<td>[70.4,70.5]</td>
</tr>
<tr>
<td>Deferred Acceptance</td>
<td>[48.9,48.8]</td>
</tr>
<tr>
<td>Top Trading Cycles</td>
<td>[51.9,52.6]</td>
</tr>
</tbody>
</table>

Table 6: Bounds on the percentage of students whose interim utility is weakly higher under each alternative relative to under the Seoul mechanism, when students play an approximate equilibrium strategy under the Seoul & the Boston mechanism
<table>
<thead>
<tr>
<th></th>
<th>BM, Truthful Reporting</th>
<th>BM, Equilibrium Reporting</th>
</tr>
</thead>
<tbody>
<tr>
<td>DA</td>
<td>[38.5, 38.9]</td>
<td>[55.9, 56.5]</td>
</tr>
<tr>
<td>TTC</td>
<td>[38.2, 38.3]</td>
<td>[52.9, 52.8]</td>
</tr>
</tbody>
</table>

Table 7: Bounds on the percentage of students whose interim utility is weakly higher under DA and TTC than the Boston mechanism
<table>
<thead>
<tr>
<th></th>
<th>minutes spent in commuting</th>
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<tbody>
<tr>
<td>Unconstrained BM</td>
<td>[2.01,2.03]</td>
</tr>
<tr>
<td>Deferred Acceptance</td>
<td>[2.15,2.16]</td>
</tr>
<tr>
<td>Top Trading Cycles</td>
<td>[2.7748,2.7749]</td>
</tr>
</tbody>
</table>

Table 8: Bounds on the difference in the ex-ante welfare between each alternative and the Seoul mechanism when all students report truthfully under the Seoul and the Boston mechanism
Table 9: Bounds of the difference in the ex-ante welfare between each alternative and the Seoul mechanism when all students play an approximate equilibrium strategy under the Seoul and the Boston mechanism.

<table>
<thead>
<tr>
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<th>minutes spent in commuting</th>
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</thead>
<tbody>
<tr>
<td>Unconstrained BM</td>
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<tr>
<td>Deferred Acceptance</td>
<td>[-0.1,-0.15]</td>
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<tr>
<td>Top Trading Cycles</td>
<td>[0.47,0.51]</td>
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<tr>
<td>BM, Truthful Reporting</td>
<td>BM, Equilibrium Reporting</td>
</tr>
<tr>
<td>------------------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>DA</td>
<td>[0.13,0.15]</td>
</tr>
<tr>
<td></td>
<td>[-0.34,-0.29]</td>
</tr>
<tr>
<td>TTC</td>
<td>[0.75,0.76]</td>
</tr>
<tr>
<td></td>
<td>[0.28,0.33]</td>
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</tbody>
</table>

Table 10: Bounds of the difference in the ex-ante welfare between DA and TTC and the Boston mechanism, measured in units of minutes spent in commuting.
<table>
<thead>
<tr>
<th>% of students who weakly prefer TTC</th>
<th>[57.2, 57.4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ex-ante welfare of TTC – DA (unit: minutes)</td>
<td>[0.61, 0.62]</td>
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</table>

Table 11: Comparison of Interim Utilities and Ex-ante welfare between TTC and DA
B Proofs

First I prove lemmas that are not presented in the body of the paper. Here I generalize the model and let the length of PR, denoted by \( \ell \), to be less than the number of schools \( m \). Given a distribution of PRs, the set of schools available at the end of the algorithm for the Boston mechanism may be determined. Given a VNM utility vector \( \mathbf{u}_i \), I denote the utility from administrative assignment with \( \mathbf{u}_{ir_{r+1}} \) for any \( r \) (i.e. I assume that the administrative assignment is not affected by the PR one submits).

**Lemma 2.** Suppose the distribution of PRs is equal to \( \bar{r} \) and the tie-breaking scheme is STB. Then \( \pi^k(r, C(\hat{r})) = \max\{\hat{c}_r^k - \bar{c}_r^k, 0\} \) where \( \bar{c}_r^k \) is defined as follows:

\[
\bar{c}_r^1 = 0 \\
\bar{c}_r^k = \max\{\hat{c}_r^k, \bar{c}_r^{k-1}\} \text{ for } k \in \{2, \ldots, \ell\}
\]

**Proof.** Let \( r \in R \) and \( \bar{r} \in \hat{R} \) be given. Then \( \pi^1(r, C) = Pr(\rho_1 \leq \bar{c}_r^1) = \bar{c}_r^1 = \max\{\hat{c}_r^1 - \bar{c}_r^1, 0\} \). Now suppose that the claim is true for \( k \geq 1 \) and \( k \leq \ell - 1 \). Then showing that the claim is true for \( k + 1 \) will complete the proof. If you reached round \( k + 1 \), then it means \( \rho_1 > \max_{\gamma \in \{1, \ldots, k\}} \hat{c}_r^\gamma \), which is by definition \( \bar{c}_r^{k+1} \). If \( \hat{c}_r^{k+1} > \bar{c}_r^{k+1} \), then \( \pi^{k+1}(r, \hat{C}) = Pr(\rho_1 \in (\hat{c}_r^{k+1}, \bar{c}_r^{k+1}]) = \bar{c}_r^{k+1} - \bar{c}_r^{k+1} \). Otherwise, \( \pi^{k+1}(r, \hat{C}) = 0 \), finishing the proof. \( \square \)

**Lemma 3.** Suppose the distribution of PRs is equal to \( \bar{r} \) and the tie-breaking scheme is MTB. Then \( \pi^1(r, C(\hat{r})) = \hat{c}_s^k \) and \( \pi^k(r, C(\hat{r})) = \left( 1 - \sum_{\gamma=1}^{k-1} \pi^\gamma(r, \hat{r}) \right) \hat{c}_r^k \) for \( k \in \{2, \ldots, \ell\} \)

**Proof.** Let \( r \in R \) and \( \bar{r} \in \hat{R} \) be given. Then \( r^1(r, \hat{C}) = Pr(\rho_{r_1} \leq \hat{c}_r^1) = \hat{c}_r^1 \). Now suppose the claim is true for \( k \geq 1 \) and \( k \leq \ell - 1 \). Then showing that the claim is true for \( k + 1 \) will complete the proof. Suppose \( r_{k+1} = r_\gamma \) for some \( \gamma \in \{1, \ldots, k\} \). Then if \( \hat{c}_{r_\gamma}^\gamma \in (0, 1) \), then \( r_\gamma \) is not available in round \( k + 1 \) hence \( \hat{c}_{r_\gamma}^{k+1} = 0 \), therefore \( \pi^{k+1}(r, \hat{C}) = 0 \), which is also true according to the formula. If \( \hat{c}_{r_\gamma}^\gamma = 1 \), then \( \pi^\gamma(r, \hat{C}) = 1 - \sum_{\gamma' = 1}^{\gamma-1} \pi^\gamma'(r, \hat{c}) \), meaning that you do not reach round \( k + 1 \), which is a contradiction.

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Now suppose that \( r_{k+1} \neq r_\gamma \) for all \( \gamma \in \{1, \ldots, k\} \). Then

\[
\pi^{k+1}(r, \hat{C}) = Pr(Rejected \ by \ r_1, \ldots, r_k)Pr(Admitted \ to \ r_{k+1}|rejected \ by r_1, \ldots, r_k)
\]

However, under the MTB lottery number draws are independent across schools. And by definition \( Pr(rejected \ by \ r_1, \ldots, r_k) = 1 - \sum_{\gamma=1}^{k} \pi^{\gamma}(r, \hat{C}) \). Therefore, \( \pi^{k+1}(r, \hat{C}) = (1 - \sum_{\gamma=1}^{k} \pi^{\gamma}(r, \hat{C})) \hat{c}_{r_{k+1}}^{k+1} \), which completes the proof.

\[\Box\]

**Lemma 4.** Suppose a school \( s \) is available in round \( k \). Then if you rank \( s \) as your \((k-1)\)th choice, then you will be assigned to \( s \) with residual probabilities. More precisely, if \( \ell \geq 2 \), then \forall \gamma \in \{2, \ldots, \ell\}, if \( r \in R \) and \( \hat{r} \in \hat{R} \) is such that \( \pi^{k}(r, \hat{C}) > 0 \), then \( \forall r' \in R \) such that \( \forall \gamma \in \{\gamma' \in N : 1 \leq \gamma' \leq k-2\} \) (if nonempty) \( r'_\gamma = r_\gamma \) and \( r'_{k-1} = r_{k-1} \). \( \pi^{k-1}(r', \hat{C}) = 1 - \sum_{\gamma=1}^{k-2} \pi^{\gamma}(r, \hat{C}) \) if \( \forall \gamma \in \{\gamma' \in N : 1 \leq \gamma' \leq k-2\} \) is nonempty, 1 if empty. For any \( \ell \geq 1 \), if for some \( s \in S \), \( \pi^{\ell+1}(r, \hat{C}) > 0 \), then for \( r' \in R \) such that \( \forall \gamma \in \{\gamma' \in N : 1 \leq \gamma' \leq \ell-1\} \) (if nonempty) \( r'_\gamma = r_\gamma \) and \( r'_{\ell} = s \), then \( \pi^{\ell}(r', \hat{C}) = 1 - \sum_{\gamma=1}^{\ell-1} \pi^{\gamma}(r, \hat{C}) \) if \( \forall \gamma \in \{\gamma' \in N : 1 \leq \gamma' \leq \ell-1\} \) is nonempty, and 1 otherwise.

**Proof.** Suppose that a \( k \in \{2, \ldots, \ell\} \) with \( \ell \geq 2 \), \( r \in R \) and \( \hat{r} \in \hat{R} \) are such that \( \pi^{k}_{r_{k}}(r, \hat{r}) > 0 \). Then under either tie-breaking scheme, \( \hat{c}_{r_{k}}^{k} > 0 \). This means at the end of round \( k-1 \), school \( r_{k} \) had a positive measure of remaining capacity (otherwise \( \hat{c}_{r_{k}}^{k} = 0 \)). That means even if a student whose lottery number for \( r_{k} \) is equal to 1 would be admitted. Therefore by the definition of \( \hat{c}_{r_{k}}^{k-1}, \hat{c}_{r_{k}}^{k-1} = 1 \).

Now consider the suggested PR \( r' \). If \( \forall \gamma \in \{\gamma' \in N : 1 \leq \gamma' \leq k-2\} \) is empty, then \( k = 2 \) and because \( \hat{c}_{r_{k-1}}^{k-1} = 1 \), it follows that \( \pi^{k-1}_{r_{k-1}}(r', \hat{r}) = 1 \) as claimed. If nonempty, then by the definition of \( r' \), \( \forall \gamma \in \{\gamma' \in N : 1 \leq \gamma' \leq k-2\} \) \( \pi^{\gamma}_{r_{\gamma}}(r', \hat{r}) = \pi^{\gamma}_{r_{\gamma}}(r, \hat{r}) \). For the MTB scheme, the definition implies the claim because \( \hat{c}_{r_{k-1}}^{k-1} = 1 \). For the STB scheme, I need to show that

\[
\hat{c}_{r'}^{k-2} = \sum_{\gamma=1}^{k-2} \pi^{\gamma}_{r_{\gamma}}(r, \hat{r})
\]
Note that by definition, the LHS is equal to \( \max \{ \bar{c}_1^1, \ldots, \bar{c}_{k-2} \} \). Then the RHS is equal to
\[
\sum_{\gamma=1}^{k-2} \pi^{\gamma}_{r_{\gamma}}(r, \hat{r}) = \sum_{\gamma=1}^{k-2} \pi^{\gamma}_{r_{\gamma}}(r, \hat{r}) = \sum_{\gamma=1}^{k-2} \bar{c}_{r_{\gamma}} - \bar{c}_r
\]

Let \( \bar{K} = \{ \gamma \in \{1, \ldots, k-2\} : \pi^{\gamma}_{r_{\gamma}}(r, \hat{r}) > 0 \} \). Let \( \gamma_1 \) be the smallest element of \( \bar{K} \), \( \gamma_2 \) the second smallest element, and so on. Then
\[
\bar{c}_{r_{\gamma_1}} = 0 \text{ and } \bar{c}_{r_{\gamma_j}} = \bar{c}_{r_{\gamma_{j-1}}} \text{ for } j = 2, \ldots, |\bar{K}|
\]
The first part is true by the definition of \( \gamma_1 \) (otherwise there is \( \gamma_0 < \gamma_1 \) such that \( \pi^{\gamma_0}_{r_{\gamma_0}}(r, \hat{r}) > 0 \)). For the second part, by definition \( \bar{c}_{r_{\gamma_j}} \geq \bar{c}_{r_{\gamma_{j-1}}} \). If the inequality is strict, then there must be \( \gamma' \in \{ \gamma : \gamma_{j-1} < \gamma < \gamma_j \} \) such that \( \bar{c}_{r_{\gamma'}} > \bar{c}_{r_{\gamma_{j-1}}} \). Let \( \gamma^* \) be the minimum of such number. Then \( \pi^{\gamma^*}_{r_{\gamma^*}}(r, \hat{r}) = \bar{c}_{r_{\gamma^*}} = \bar{c}_{r_{\gamma^*}} - \bar{c}_{r_{\gamma_{j-1}}} > 0 \), which contradicts the definition of \( \gamma_{j-1} \) and \( \gamma_j \). This also means that \( \bar{c}_{r_{|\bar{K}|}} = \max_{j=1,\ldots,|\bar{K}|} \bar{c}_{r_{\gamma_j}} \).

\[
\sum_{\gamma=1}^{k-2} \pi^{\gamma}_{r_{\gamma}}(r, \hat{r}) = \sum_{j=1}^{|\bar{K}|} \pi^{\gamma}_{r_{\gamma_j}}(r, \hat{r}) = \sum_{j=1}^{|\bar{K}|} \bar{c}_{r_{\gamma_j}} - \bar{c}_r = \bar{c}_{r_{|\bar{K}|}} = \max \{ \bar{c}_1^1, \ldots, \bar{c}_{k-2} \}
\]
where the last equality follows from the definition of \( \gamma_{|\bar{K}|} \).

**Proof of Proposition 1.** Given a \( u \in \mathbb{R}^{|S|} \) and \( \hat{r} \in \hat{R} \), choose a \( r \) recommended by \( \psi^{TR}(u, \hat{r}) \). First suppose that \( \pi^1(r, \hat{C}) > 0 \). Then by the definition of \( \psi^{TR} \),
\[
\forall s \in S \setminus \{ r_1 \}, u_{ir_1} \geq u_{is}
\]
Then \( \forall s \in S \) such that \( \bar{c}_s^1 > \bar{c}_{r_1}^1 \), it follows that \( u_{is} \geq u_{is} \). Now suppose that for some \( k \in \{2, \ldots, \ell\} \), \( \pi^k(r, \hat{C}) > 0 \). Define \( S' = \{ s \in S : \bar{c}_s^k > \bar{c}_{r_k}^k \} \) and choose an \( s \in S' \). Suppose \( s \) is a ranked school, i.e., \( \exists k' \in \{1, \ldots, k-1, k+1, \ell\} \) such that \( r_{k'} = s \). Now sup-
pose that \( k' < k \). Then because \( \hat{c}_s^k > 0 \), it follows that 
\[
\pi^{k'}(r, \hat{C}) = \left( 1 - \sum_{\gamma=1}^{k-1} \pi^{\gamma}(r, \hat{C}) \right),
\]
contradicting the supposition that \( \pi^k(r, \hat{C}) > 0 \). Therefore, it must be that \( k' > k \). Then it follows that \( u_{ir_k} \geq u_{ir_{k'}} \) by the definition of \( \psi^{TR} \). If \( s \) is unranked, then it follows that 
\[
u_{ir_k} \geq u_{is} \text{ by the definition of } \psi^{TR}.
\]
Therefore, \( \forall s \in S', u_{ir_k} \geq u_{is} \).

**Proof of Proposition 2.** To prove Proposition 2, we need to prove the following two lemmas.

**Lemma 5.** Suppose that \( a \) for some \( u_i \in \mathbb{R}^{[S]} \) and \( \hat{r} \in \hat{R} \), \( r \) is recommended by \( \psi^{ST} \). Then if there exists some \( k \in \{1, \ldots, \ell\} \) such that

\[
\pi^k(r, \hat{C}) > 0 \text{ and } \{\gamma \in \{k+1, \ldots, \ell+1\} : \pi^{\gamma}(r, \hat{C}) > 0\} \neq \emptyset
\]

Then letting \( K \equiv \{\gamma \in \{k+1, \ldots, \ell+1\} : \pi^{\gamma}(r, \hat{C}) > 0\}, \forall \gamma \in K, u_{ir_k} \geq u_{ir_{k'}}.

**Proof.** I will prove the lemma by induction. Let \( \bar{k} \) be defined as \( \max\{\gamma \in \{1, \ldots, \ell\} : \pi^{\gamma}(r, \hat{C}) > 0 \text{ and } \{\gamma' \in \{\gamma + 1, \ldots, \ell + 1\} : \pi^{\gamma'}(r, \hat{C}) > 0\} \neq \emptyset\} \) and let \( \bar{K} \) be defined as \( \bar{K} \equiv \{\gamma \in \{\bar{k} + 1, \ldots, \ell + 1\} : \pi^{\gamma}(r, \hat{C}) > 0\}. \) Note that by the definition of \( \bar{k}, \bar{K} \) is a singleton. Let \( \bar{k}^* \) denote the element of \( \bar{K} \). Then consider an alternative PR \( r' \) such that \( \forall \gamma \in \{1, \ldots, \bar{k} - 1\} r'_{\gamma} = r_{\gamma} \) and \( r'_{\bar{k}^*} = r_{\bar{k}^*}. \) Then because \( \pi^{\bar{k}^*}(r, \hat{C}) > 0 \) and \( \bar{k}^* > \bar{k} \) by definition, it follows that 
\[
\pi^{\bar{k}}(r', \hat{C}) = 1 - \sum_{\gamma=1}^{\bar{k}-1} \pi^{\gamma}(r', \hat{C}) > 0.
\]
Then by revealed preference,

\[
U(r, \bar{r}) = \sum_{\gamma=1}^{\ell + 1} \pi^{\gamma}(r, \hat{C})u_{ir_\gamma} \geq \sum_{\gamma=1}^{\ell + 1} \pi^{\gamma}(r', \hat{C})u_{ir'_{\gamma}} = U(r', \bar{r})
\]

But because \( \forall \gamma \in \{1, \ldots, \bar{k} - 1\} r_{\gamma} = r'_{\gamma} \), we can cancel the first \( \bar{k} - 1 \) terms of the sum. Also, by the definition of \( \bar{k} \) and \( \bar{k}^* \), we get

\[
\pi^{\bar{k}}(r, \hat{C})u_{ir_k} + \pi^{\bar{k}}(r, \hat{C})u_{ir_{k'}} \geq \pi^{\bar{k}}(r', \hat{C})u_{ir_{k'}}
\]
Note that $\pi^k(r, \hat{C}) + \pi^{k^*}(r, \hat{C}) = \pi^k(r', \hat{C})$ because if $\hat{c}^{k^*}_{r^*} < 1$, then there must be some $\gamma \in \{k^* + 1, \ldots, \ell + 1\}$ such that $\pi^\gamma(r, \hat{C}) > 0$, contradicting the fact that $\hat{k}$ is the largest of such. Hence $\hat{c}^{k^*}_{r^*} = 1$. Hence $\pi^k(r, \hat{C}) + \pi^{k^*}(r, \hat{C}) = (1 - \sum_{\gamma=1}^{k-1} \pi^\gamma(r, \hat{C})) = \pi^k(r', \hat{C})$. Then it is obvious that if $u_{ir'_k} > u_{ir_k}$, then

$$\pi^k(r, \hat{C})u_{ir'_k} + \pi^{k^*}(r, \hat{C})u_{ir'_k} < \pi^k(r', \hat{C})u_{ir'_k}$$

which contradicts the inequality implied by the revealed preference. Therefore $u_{ir'_k} \geq u_{ir_k}$.

Now suppose the lemma is true for some $k \in \{1, \ldots, \ell\}$. Then by supposition it is true that $\pi^k(r, \hat{C}) > 0$ and $\forall \gamma' \in \{\gamma \in \{k+1, \ldots, \ell + 1\} : \pi^\gamma(r, \hat{C}) > 0\}$, $u_{ir_k} \geq u_{ir_{\gamma'}}$. Then let $k$ be defined as $\max\{\gamma \in \{1, \ldots, k-1\} : \pi^\gamma(r, \hat{C}) > 0\}$. Then consider an alternative PR $r'$ such that $\forall \gamma \in \{1, \ldots, k-1\}$ $r'_\gamma = r_\gamma$ and $r'_k = r_k$. Then because $\pi^k(r, \hat{C}) > 0$ and $k < k$, it follows that $\pi^k(r', \hat{C}) = 1 - \sum_{\gamma=1}^{k-1} \pi^\gamma(r', \hat{C})$. Then by revealed preference,

$$U(r, \hat{r}) = \sum_{\gamma=1}^{\ell+1} \pi^\gamma(r, \hat{C})u_{ir_\gamma} \geq \sum_{\gamma=1}^{\ell+1} \pi^\gamma(r', \hat{C})u_{ir'_{\gamma'}} = U(r', \hat{r})$$

But because $\forall \gamma \in \{1, \ldots, k - 1\}$, $r_\gamma = r'_{\gamma'}$, we can cancel the first $k - 1$ terms of the sum. Also, eliminating from the LHS all the terms for which $\pi^\gamma(r, \hat{C}) = 0$ for $\gamma \in \{k+1, \ldots, \ell + 1\}$,

$$\sum_{\gamma \in \{k, \ldots, \ell + 1\}}^{\ell+1} \pi^\gamma(r, \hat{C})u_{ir_\gamma} \geq \pi^k(r', \hat{C})u_{ir'_k}$$
But by supposition, \( \forall \gamma \in \{k, k + 1, \ldots, \ell + 1\} \) \( u_{ir_k} \geq u_{ir_\gamma} \). Hence

\[
\pi^k(r, \hat{C}) u_{ir_k} + \left( \sum_{\gamma = k+1}^{\ell+1} \frac{\pi^\gamma(r, \hat{C})}{\pi^\gamma(r, \hat{C})} \right) u_{ir_k} \geq \sum_{\gamma = k+1}^{\ell+1} \frac{\pi^\gamma(r, \hat{C})}{\pi^\gamma(r, \hat{C})} u_{ir_\gamma}
\]

Hence, putting the two inequalities together,

\[
\pi^k(r, \hat{C}) u_{ir_k} + \left( \sum_{\gamma = k+1}^{\ell+1} \frac{\pi^\gamma(r, \hat{C})}{\pi^\gamma(r, \hat{C})} \right) u_{ir_k} \geq \pi^k(r', \hat{C}) u_{ir'_k}
\]

But if \( u_{ir_k} > u_{ir'_k} \), then this inequality does not hold, which is a contradiction. Therefore, \( u_{ir_k} \geq u_{ir'_k} \). By proof by induction, the lemma holds. \( \square \)

**Lemma 6.** Suppose that for some \( u_i \in \mathbb{R}^{|S|} \) and \( \hat{r} \in \hat{R} \), \( r \) is recommended by \( \psi^{ST}(u_i, \hat{r}) \) and for some \( k \in \{1, \ldots, \ell\} \),

\[
\pi^k(r, \hat{C}) > 0
\]

Then \( \forall s \in S \) such that \( c^k_s > c^k_{r_k} \) \( u_{ir_k} \geq u_{is} \)

**Proof.** By Lemma 5,

\( \forall \gamma \in \{k + 1, \ldots, \ell + 1\} \) such that \( \pi^\gamma(r, \hat{C}) > 0 \), \( u_{ir_k} \geq u_{ir_\gamma} \)

Now suppose \( \exists s \in S \) such that \( c^k_s > c^k_{r_k} \). Then consider an alternative PR \( r' \) such that \( \forall \gamma \in \{1, \ldots, k - 1, k + 1, \ldots, \ell\} \) \( r'_\gamma = r_\gamma \) and \( r'_k = s \). Then by revealed preference,

\[
U(r, \hat{r}) = \sum_{\gamma = 1}^{\ell+1} \pi^\gamma(r, \hat{C}) u_{ir_\gamma} \geq \sum_{\gamma = 1}^{\ell+1} \pi^\gamma(r', \hat{C}) u_{ir'_\gamma} = U(r', \hat{r})
\]

We can cancel the first \( k - 1 \) terms by the definition of \( r' \). After eliminating terms
with $\pi^\gamma(r, \hat{C}) = 0$ and $\pi^\gamma(r', \hat{C}) = 0$ for $\gamma = \{k + 1, \ldots, \ell + 1\}$, we get the following inequality:

$$\sum_{\gamma \in \{k, \ldots, \ell + 1\}} \pi^\gamma(r, \hat{C}) u_{ir, \gamma} \geq \sum_{\gamma \in \{k, \ldots, \ell + 1\}} \pi^\gamma(r', \hat{C}) u_{ir', \gamma}$$

Let us first prove the lemma for single tie-breaking scheme. First recall that $\bar{\bar{c}}^\gamma_r$ is defined recursively as follows:

$$\bar{\bar{c}}^0_r = 0$$
$$\bar{\bar{c}}^\gamma_r = \max\{\bar{\bar{c}}^{\gamma - 1}_r, \bar{\bar{c}}^\gamma_{r'}\} \text{ for } \gamma = \{1, 2, \ldots, \ell\}$$

Because by supposition $\bar{\bar{c}}^k_r > \bar{\bar{c}}^k_{r'}$, it follows that $\forall \gamma \in \{k + 1, \ldots, \ell + 1\}, \bar{\bar{c}}^\gamma_r \geq \bar{\bar{c}}^\gamma_{r'}$. That means for each $\gamma \in \{k + 1, \ldots, \ell + 1\}$ such that $\pi^\gamma(r, \hat{C}) = \max\{\bar{\bar{c}}^\gamma_r - \bar{\bar{c}}^\gamma_{r'}, 0\} = 0$, $\pi^\gamma(r', \hat{C}) = 0$ as well. By the same token, $\forall \gamma \in \{k + 1, \ldots, \ell + 1\}$ such that $\pi^\gamma(r, \hat{C}) > 0$, $\pi^\gamma(r', \hat{C}) \leq \pi^\gamma(r, \hat{C})$. Note that by the definition of $r'$, the following equality holds:

$$\sum_{\gamma \in \{k, \ldots, \ell + 1\}} \pi^\gamma(r, \hat{C}) = 1 - \sum_{\gamma = 1}^{k - 1} \pi^\gamma(r, \hat{C}) = \sum_{\gamma \in \{k, \ldots, \ell + 1\}} \pi^\gamma(r', \hat{C})$$

which means that

$$\pi^k(r, \hat{C}) + \sum_{\gamma \in \{k + 1, \ldots, \ell + 1\}} \left(\pi^\gamma(r, \hat{C}) - \pi^\gamma(r', \hat{C})\right) = \pi^k(r', \hat{C})$$

and $\forall \gamma \in \{k + 1, \ldots, \ell + 1\}, \pi^\gamma(r, \hat{C}) - \pi^\gamma(r', \hat{C}) \geq 0$, with at least one inequality strict (because by supposition $\pi^k(r', \hat{C}) > \pi^k(r, \hat{C})$). Now going back to the revealed
preference inequality, we get the following inequality from rearranging the terms:

\[
\pi^k(r, \hat{C})u_{ir_k} + \sum_{\gamma \in \{k+1, \ldots, \ell+1\} \mid \pi^\gamma(r, \hat{C}) > 0} \left( \pi^\gamma(r, \hat{C}) - \pi^\gamma(r', \hat{C}) \right) u_{ir_{\gamma}} \geq \pi^k(r', \hat{C})u_{is}
\]

However, if \(u_{is} > u_{ir_k}\), it follows that \(\forall \gamma \in \{k+1, \ldots, \ell+1\}\) such that \(\pi^\gamma(r, \hat{C}) > 0\), \(u_{is} > u_{ir_{\gamma}}\) by Lemma refbiggerthanlower, hence the right hand side of this inequality is strictly larger than the left hand side, which contradicts the revealed preference. Therefore, \(u_{ir_k} \geq u_{is}\).

Now let us turn to multiple tie-breaking scheme. Because the supposition that \(\hat{c}_k^s > \hat{c}_k^r\), it follows from lemma 3, \(\pi^k(r', \hat{C}) > \pi^k(r, \hat{C})\). Again by lemma 3, \(\pi^\gamma(r, \hat{C}) \geq \pi^\gamma(r', \hat{C})\) for \(\gamma \in \{k+1, \ldots, \ell+1\}\). Then by a similar logic as the case of single tie-breaking, lemma follows.

\[
\text{Proof of Lemma 1. Let } \tilde{r}_0 \in \hat{R} \text{ denote the observed distribution of PRs and } \tilde{C}_0 = C(\tilde{r}_0) \text{ denote the cut-off matrix associated with } \tilde{r}_0. \text{ Define the set of subsets of } \mathbb{R}^{|S|}, \mathcal{A}, \text{ as follows: let } w : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\} \text{ denote a bijection, where } m \text{ denotes the number of schools. Let } W \text{ denote the set of all such bijections. Then,}
\]

\[a. \forall w \in W, \{u_j \in \mathbb{R}^{|S|} : u_{isw(1)} > u_{isw(2)} > u_{isw(3)} > \ldots > u_{isw(m)}\} \in \mathcal{A}
\]

\[b. \forall w \in W, \{u_j \in \mathbb{R}^{|S|} : u_{isw(j)} = u_{isw(k)} \text{ for some } j, k \in \{1, \ldots, m\}\} \in \mathcal{A}
\]

Note that \(|\mathcal{A}| < \infty\) and for each \(A, A' \in \mathcal{A}, A \cap A' = \emptyset\). Let \(X_i\) denote the smallest \(\sigma\)-algebra that contains \(\mathcal{A}\). Then define a probability measure as follows: first, let \(j : W \rightarrow \{1, \ldots, |R|\}\) denote the function that maps the bijection \(w\) above to a number between 1 and \(|R|\), inclusive. \(j(w)\) is the number attached to the PR \(r\) such that \(r_k =
\( s_w(k) \) for all \( k \in \{1, \ldots, m\} \). Then,

\[
\mu(\{u_i \in \mathbb{R}^{[S]} : u_{is_w(1)} > u_{is_w(2)} > u_{is_w(3)} > \ldots > u_{is_w(m)}\}) = \tilde{r}_{0,j}
\]

\[
\mu(\{u_i \in \mathbb{R}^{[S]} : u_{is_j} = u_{is_k} \text{ for some } j, k \in \{1, \ldots, m\}\}) = 0
\]

Let \( H \) be a degenerate distribution on the Truthful Reporting Strategy and let \( G \) be any distribution on the set of beliefs \( B \). Then,

\[
\int_{u \in \mathbb{R}^{[S]}} \int_{\hat{r} \in \hat{R}} \int_{\psi \in \Psi} \psi^j(u, \hat{r})dHdGd\mu = \int_{u \in \mathbb{R}^{[S]}} \int_{\hat{r} \in \hat{R}} \psi^{TR,j}(u, \hat{r})dGd\mu = \tilde{r}_0^j
\]

where the first equality follows from the fact that \( H \) is degenerate on \( \psi^{TR} \), and the second equality follows from the fact that the only VNM utility vectors that result in \( r^j \) under the Truthful Reporting Strategy is

\[
\{u \in \mathbb{R}^{[S]} : u_{w(1)} \leq \ldots \leq u_{w(m)} \text{ for } w \text{ such that } j(w) = j\}
\]

regardless of beliefs. However, by definition of \( \mu \) the VNM utilities with strict inequality gets positive measure. Third line follows from the definition of \( \mu \). \( \square \)

**Proof of Proposition 3.** I need to show that the condition of Theorem 3.1 in Romano et al. (2014) is satisfied. The condition is as follows: denoting the mean and variance, respectively, of \( g_j(Y, \theta) \) under \( P \) with \( \mu_j(\theta, P) \) and \( \sigma_j(\theta, P) \),

\[
\lim_{\lambda \to \infty} \sup_{P \in \mathcal{P}} \sup_{\theta \in \Theta_0(P)} \mathbb{E}_P \left[ \left( \frac{g_j(Y, \theta) - \mu_j(\theta, P)}{\sigma_j(\theta, P)} \right)^2 \times 1 \left\{ \frac{g_j(Y, \theta) - \mu_j(\theta, P)}{\sigma_j(\theta, P)} > \lambda \right\} \right] = 0
\]

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Because $\forall j = 1, \ldots, d$, $g_j(Y_i, \theta) \in [0, 1]$ and hence $\mu_j(\theta, P) \in [0, 1]$, if I can show that $\sigma_j(\theta, P)$ is bounded away from 0, the proof is complete.

Note that for any $P \in P$ and $\theta \in \Theta_0(P)$, $g_j(Y_i, \theta)$ is a discrete random variable. Let $(s, X) = v^{-1}(j)$. First for simplicity of notation, let

$$
\begin{align*}
\delta &= \Pr_{\theta} \left( u_{is} \geq \max_{s' \in \bigcap_{\mathcal{C} \in \mathcal{B}} \mathcal{S}(\mathcal{C}, s)} u_{is'} | X_i = X \right) \\
a &= P \left( \left\{ i \in I : (X_i = X) \bigcap \left( \bigcap_{\mathcal{C} \in \mathcal{B}} \overline{R}(\mathcal{C}, s) \right) \right\} \right) \\
b &= P \left( \left\{ i \in I : (X_i = X) \bigcap \left( \bigcap_{\mathcal{C} \in \mathcal{B}} R(\mathcal{C}, s) \right) \right\} \right)
\end{align*}
$$

Then $g_j(Y_i, \theta)$ has the following distribution:

- $1 - \delta$ with probability $a$
- $-\delta$ with probability $b$
- $0$ with probability $1 - a - b$

We need some variation for $\sigma_j(\theta, P) > 0$ to be true. Because of Condition 3.5, $\delta \in (0, 1)$. Hence neither of $1 - \delta$ or $\delta$ is equal to 0. And because of Condition 3.5, $a + b \in (0, 1)$, so at least one of $a$ and $b$ is positive and less than 1. Therefore, $\sigma_j(\theta, P)$ is never zero, completing the proof. \qed