Costly information acquisition in a speculative attack: Theory and experiments*

Michal Szkup† Isabel Trevino‡

Abstract

We solve and test experimentally a discrete global games model of speculative attack where agents choose, at a cost, the precision of the private signal they observe. We prove existence of a unique equilibrium in the coordination game and explore strategic incentives in information acquisition. In the experiment, we find that subjects follow the strategies suggested by the theory, but, contrary to our predictions, the number of attacks increases as subjects acquire more precise information. This implies that as the precision of the signal increases, the actions of subjects move towards efficiency and not risk dominance, which contradicts previous well known results of global games. Therefore, our results suggest that acquiring more precise information is beneficial for individual welfare, but it could be potentially harmful for society. (JEL C72, C90, D82)

1 Introduction

The last decade of the twentieth century and the beginning of this century were characterized by frequent episodes of currency crises. The existing models of currency crises at the time (the so called first generation models) were unable to offer an explanation for the underlying motives behind these largely unexpected events. This led to the development of a second generation of models of currency crises, of which Obstfeld (1996) remains a classic example. Obstfeld assumed that an economy can be characterized by a tripartite ordering of fundamentals: In bad states of the economy agents attack the currency and provoke a devaluation for sure. In good states agents restrain from attacking and devaluation never takes place, and in intermediate states there are multiple equilibria characterized by self-fulfilling beliefs. While this model seemed to explain the

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†New York University, m.szkup@nyu.edu.

‡New York University, Isabel.trevino@nyu.edu.

1Examples of these episodes include the EMR in 1992-93, Mexico in 1994, Asia in 1997-98, and Argentina in 2001.
occurrence of currency crises, it could only offer a limited amount of policy guidance due to the self-fulfilling nature of the outcome, which led to multiplicity of equilibria.

The paper by Morris and Shin (1998) was a breakthrough for the modeling of speculative attacks through its use of global games. As defined by Carlsson and van Damme (1993), a global game is a coordination game where the information structure is perturbed in such a way that the players no longer have certainty over payoffs and over the other players’ beliefs over payoffs. This implies that there is no longer common knowledge of the structure of the game. By relaxing one of the main assumptions of game theoretic models, the global games literature has proven to be very useful in characterizing macroeconomic situations, such as speculative attacks, where common knowledge of the underlying structure is an implausible assumption.

Morris and Shin (1998) use this global games setup to analyze a reduced form version of the model by Obstfeld (1996). In their model agents get a noisy private signal of the underlying state of the economy and have to decide simultaneously whether to attack the currency or not. If an agent chooses not to attack, he gets a nonnegative payoff with certainty. If he chooses to attack, his payoff depends on the underlying state of the economy and on the number of attacking agents. Morris and Shin (1998) show that the global games refinement pins down a unique rationalizable equilibrium characterized by the use of threshold strategies, where an agent attacks if he gets a signal above this threshold, and doesn’t attack otherwise.

The richness and applicability of the global games models have given rise to a vast literature, trying to understand under what conditions the uniqueness result holds, and how far one can extend the global games setup to better portray economic realities. In most of these applications the information structure has played a central role, and even small changes in the assumptions of the model have important consequences on equilibrium selection.\(^2\)

The development of these global games models has allowed economists to have a better understanding of the forces behind episodes of high speculation. Importantly, it is information that determines agents’ decisions to attack a currency or to make an investment. In most global games models, however, the information possessed by agents is given to them exogenously, rather than chosen. We develop a model that rectifies this shortcoming and that allows us to understand how agents choose their information sets in a global games context, and we test its predictions experimentally. In particular, we develop a global games model of speculative attacks where each agent has the possibility to choose the accuracy of his private information about the state of the economy. In our model, two agents covertly choose the precision of the private signal they will observe, at a cost, and then move on to play the coordination game of speculative attack, as in Morris and Shin (1998), using the information acquired in the first stage. In Szkup and Trevino (2012) we develop a similar model with a continuum of players, which allows for explicit analytical solutions and a

We believe that the study of costly private information acquisition in a speculative attack will bring the global games model of Morris and Shin (1998) closer to the macroeconomic phenomenon of interest. Investors involved in speculative attacks do not hold symmetric and exogenously given information about the economy, instead, they continuously make efforts to improve the information they possess, and they are willing to pay for it. Investment groups and individuals pay experts to extract more accurate information about the financial system in order to minimize losses, creating a market for information expertise and financial advising.\(^3\) Therefore, it seems both a natural and a necessary extension to add an initial stage to the speculative attack game where agents get to improve the precision of the information they receive.

Different authors have studied costly information acquisition in coordination games with imperfect information. In a similar spirit to our model, Hellwig and Veldkamp (2009) investigate what agents choose to observe when there is a cost to acquiring information and how information choices affect equilibrium outcomes in a beauty contest model. In a more general setup, Colombo, Feminis and Pavan (2012) study the welfare effects of endogenous information acquisition in a flexible framework of quadratic loss functions that include beauty contest models as a special case.\(^4\) In a global games context, some studies related to ours are Bannier and Heinemann (2005) in which the monetary authority chooses the precision of the private signals in order to minimize the probability of currency crises and Yang (2012) where information acquisition in coordination games is modeled using a rational inattention framework. In addition, Nikitin and Smith (2008) and Zwart (2008) study costly information acquisition in a Diamond and Dybvig type of model and in a liquidity run setup, respectively. However, in these last two studies information acquisition is modeled as a binary decision to acquire a private signal with a given precision or not to acquire a signal at all, which is in contrast to our setup where all agents observe private signals and have to choose their individual precision. This difference has important implications since a unique equilibrium is not guaranteed in the setup of Nikitin and Smith (2008) and Zwart (2008), and in our context we can easily find conditions to guarantee uniqueness.

Global games have proven themselves to be very relevant for modelling coordination games with incomplete information not only because they predict a unique equilibrium, but also because their predictions are consistent with observed behavior in the laboratory. Heinemann et al (2004, 2009) test the model by Morris and Shin (1998) experimentally and find that the global games solution is a good predictor of behavior, even for treatments with perfect information. Just like Heinemann et al (2004), we find that subjects follow the strategies predicted by the global games refinement, for any precision chosen. We generalize their results by showing how the incidence and success of speculative attacks varies with different precision levels when we endogenize information choices.

The experimental analysis of our model allows us to observe how information sets are generated

\(^3\)Examples of financial advising corporations include Merrill Lynch, Morgan Stanley, Goldman Sachs, etc.

\(^4\)Szkup and Trevino (2012) compares the results of costly information acquisition in a global game with existing results for beauty contest type of games.
and implemented in a speculative attack, and to derive more accurate policy implications from observed behavior. Our ultimate goal is to answer questions such as: In a world in which agents have the possibility to pay to improve the quality of the information they possess about the economy, does the incidence of speculative attacks increase or decrease? Do agents take advantage of this possibility and pay to improve their information? Can currency crises be prevented by having better informed market participants? What role do financial advisors play in determining speculative attack outcomes?

Our experimental results suggest that as agents get more precise information they coordinate more often on attacking the currency, and as a result attacks are more successful when agents hold more precise information. This result implies that as agents acquire more precise signals their threshold shifts towards efficiency and away from the risk dominant equilibrium, which contradicts well known results about behavior in global games in the limit, as the noise converges to zero. This departure from the theory leads to a welfare improvement with respect to the expected payoff in equilibrium and with respect to a constrained efficiency benchmark, which means that acquiring more precise information is beneficial to individual agents, but potentially harmful for the economy since it increases both the incidence and success of speculative attacks.

The paper is structured as follows. In section 2 we present and solve the theoretical model for two players (subsections 2.1 and 2.2), and review the theoretical predictions for the set of parameters used in the experiment (subsection 2.3). The experimental design and results are presented in section 3. We present a discussion of our results in section 4 and we conclude in section 5. All the proofs from the theory and some tables from the experimental analysis are relegated to the appendix.

2 Theory: Speculative attack with information acquisition

This is a discrete version of the two-stage model developed in Szkup and Trevino (2012) where agents privately choose the precision of their information in stage 1, and in stage 2 they observe this information and then play the speculative attack game. In this section we develop and solve the model for two players. Some of the qualitative results of Szkup and Trevino (2012) will hold in this discrete case, but we will prove important results for the two-player setup, such as uniqueness of equilibrium for the second stage and existence of equilibrium for the two stage game, since they require different proving techniques for the discrete case.

The framework for the speculative attack game follows closely the $2\times2$ model of Carlsson and van Damme (1993) (CvD93 henceforth), and a discrete version of Morris and Shin (1998) (MS98 henceforth). We introduce the model for a general class of parametric assumptions and in subsection 2.3 we state the equilibrium predictions of the model for the parameters used in the experiment.

Initially, the exchange rate is held fixed at some arbitrary level. If a high enough fraction of the agents attacks the currency, the currency is allowed to float at the underlying ("shadow") exchange
rate. The economy is characterized by $\theta$, the net gain from attacking the currency in case of a devaluation.\(^5\) Each agent is endowed with one unit of domestic currency and has to decide whether to change it for foreign currency (attack) or to keep it (not attack). Let $A(\theta)$ be the number of agents that attack when the level of fundamentals is $\theta$ and $A^*(\theta)$ be the minimum number of attacking agents needed to enforce a devaluation.\(^6\) Since a high $\theta$ implies weaker fundamentals, we assume that $A^*(\theta) < 0$. The payoff of holding domestic currency is normalized to zero. If an agent chooses to attack, he has to pay a transaction cost $T$, and gets payoff $\theta$ if at least $A^*(\theta)$ of players attack (i.e. if a devaluation occurs). If the attack is unsuccessful and the currency is not devalued, then the profit of an agent who chooses to attack is $-T$.

To be consistent with our experimental design we assume that there are two agents in the economy. With the use of the function $A^*(\theta)$ we can thus define two cutoff values for the state of the economy, $\{\theta, \bar{\theta}\}$, that will determine the number of agents that are needed to attack in order to provoke devaluation. Specifically, when $\theta \geq \bar{\theta} = A^*-1(1)$ one agent alone can provoke devaluation, and when $\theta < \theta = A^*-1(2)$, the economy is strong enough so that devaluation does not take place even if both agents attack. Therefore, for $\theta \in [\theta, \bar{\theta}]$ agents have to coordinate on the attack in order for devaluation to occur.

We assume that the state $\theta$ is not common knowledge. Instead, agents believe that the underlying state $\theta$ follows a normal distribution with mean $\mu_{\theta}$ and variance $\sigma_{\theta}^2$.\(^7\) Each agent $i = 1, 2$ observes a noisy private signal of the state $\theta$,

$$x_i = \theta + \sigma_i \varepsilon_i$$

for $i = 1, 2$ where $\sigma_i > 0$ and $\varepsilon_i$ is a random variable, normally distributed with mean zero and variance 1. We assume that $\varepsilon_i$ are iid across agents and we denote by $f(\cdot)$ the normal density of $\varepsilon_i$, and $F(\cdot)$ its cumulative distribution function.

Note that the precision of the signal that each agent receives is defined by its standard deviation, $\sigma_i$. As will be explained later on, information acquisition will take place through the choice of $\sigma_i$.

By Bayes’ Theorem, agent $i$’s posterior belief about $\theta$ follows $\theta | x_i \sim N\left(\tilde{\theta}_i, \tilde{\sigma}_i^2\right)$, where $\tilde{\theta}_i = \frac{\mu_{\theta} \sigma_i^2 + x_i \sigma_{\theta}^2}{\sigma_i^2 + \sigma_{\theta}^2}$ and $\tilde{\sigma}_i^2 = \frac{\sigma_{\theta}^2 \sigma_i^2}{\sigma_i^2 + \sigma_{\theta}^2}$.

Thus, the payoffs of the speculative attack for each one of the agents are defined as follows:

\(^5\) Let $\omega$ measure the strength of the country’s fundamentals and $f(\omega)$ be the associated shadow exchange rate. The $\theta$ is the difference between the shadow exchange rate and the fixed peg. We assume that the exchange rate is against a stronger currency, so that $f'(\omega) < 0$, implying that $\theta'(\omega) < 0$. This means that when the economy is strong (i.e. a devaluation is less likely to occur), the gains from speculating are smaller than when the economy is weak, so agents prefer to attack the currency for small values of $\omega$, or high values of $\theta$. Therefore, we can think of $\theta$ as an inverse measure of the strength of fundamentals.

\(^6\) Specifically, assume that the government derives a benefit $b > 0$ from defending the exchange rate at the fixed level. However, there is a cost to it, which depends on $\theta$ and on the size of the attack $A$, $c(\theta, A)$. We assume that $c_0(\theta, A) > 0$ and $c_A(\theta, A) > 0$. Following the literature on global games, we do not model explicitly the decision of the central bank here, but it underlies our analysis since we assume that the central bank observes the size of the attack $A$ and mechanically decides to devalue if $A \geq A^*(\theta)$, where $A^*(\theta)$ solves $b = c(\theta, A^*(\theta))$.

\(^7\) We make the additional technical assumption that $\mu_{\theta} \in [\underline{\theta}, \bar{\theta}]$. 

In the game with complete information (common knowledge of $\theta$), no information acquisition takes place and multiple equilibria arise:

- If $\theta \geq \bar{\theta}$, it is a dominant strategy to attack.
- If $\theta < \bar{\theta}$, it is a dominant strategy not to attack.
- If $\theta \in [\underline{\theta}, \bar{\theta}]$, there are two equilibria in pure strategies and one in mixed strategies.

CvD93 show that for $2 \times 2$ games without information acquisition, when agents don’t observe $\theta$ but only a noisy signal of it, there is a unique equilibrium characterized by symmetric threshold strategies $x^*$ where agents attack if and only if $x_i \geq x^*$, for $i = 1, 2$.

We analyze what happens in the game with information acquisition and noisy private signals by adding an initial stage to the game where agents get to choose the precision of the noisy signal they receive.\(^8\) This means that, starting from an initial common standard deviation $\sigma_0$, agents can either keep this level of noise or improve their precision by “buying” a lower $\sigma_i$ at a cost $C(\sigma_i)$. $C(\cdot)$ is assumed to be continuous, $C(\sigma_0) = 0$, $C'(\sigma_0) = 0$, $C''(\sigma) < 0$, and $C''''(\sigma) > 0$, for all $\sigma \in (0, \sigma_0)$. Because the precision is chosen in the first stage of the game, the cost of information acquisition becomes a sunk cost when players choose their action in the second stage. We assume that the level of precision that each agent chooses is private information, and hence not observed by the other player.\(^9\) Nevertheless, in equilibrium both agents hold correct beliefs about each other’s information choices.

We proceed to solve this game using backward induction.

### 2.1 Solving stage two: Uniqueness of equilibrium in the speculative attack game

At the beginning of the second stage each agent observes his own signal $x_i = \theta + \sigma_i \varepsilon_i$, where $\sigma_i$ corresponds to agent $i$’s choice of precision in the first stage. Note that the second stage alone can be thought of as a standard $2 \times 2$ global game with the notable difference that agents are potentially heterogenous with respect to the precision of their signals. This is a noticeable departure from the original model of CvD93 and from standard models of global games.\(^10\)

\(^8\)Recall that the precision is defined as the inverse of the variance of a random variable. Therefore, in what follows we will use either of the following terms to refer to the same situation: higher precision, lower variance, or lower standard deviation.

\(^9\)Not only is this a natural assumption for the phenomenon we are modelling, but it is also the usual assumption in the literature (see Persico, 2000, and Hellwig and Veldkamp, 2009).

\(^10\)Corsetti et al. (2004) allow agents to have asymmetric distributions of noise. However, in their setup the agents differ also with respect to other characteristics, such as size, leading to very different dynamics than the usually present in standard global games models.
After observing their own signals, agents update their beliefs about the fundamental $\theta$ and about the other agent’s type.\textsuperscript{11}

As in the classic global games setup, in the second stage agents follow symmetric threshold strategies,\textsuperscript{12} i.e. for $i = 1, 2$:

$$a(x_i; \sigma) = \begin{cases} 1 \text{ (attack)} & \text{iff } x_i \geq x^*_i(\sigma) \\ 0 \text{ (not attack)} & \text{iff } x_i < x^*_i(\sigma) \end{cases}$$

Notice that agents do not necessarily have the same thresholds because they are a function of their chosen precisions.

Given agent $i$’s posterior about $\theta$, he believes that player $j$ will attack if he observes a signal $x_j$ higher than his own threshold $x^*_i$. Therefore, from player $i$’s point of view, he believes player $j$ will attack with probability $1 - F\left(\frac{x^*_i - \theta}{\sigma_j}\right)$. This means that player $i = 1, 2$ will attack if and only if the expected payoff of attacking is higher than zero:

$$E[\theta \Pr(x_j \geq x^*_i \mid \theta \in [\bar{\theta}, \Theta]) \mid x_i] + E[\theta \mid x_i, \theta \in [\bar{\theta}, \infty)] \Pr(\theta \in [\bar{\theta}, \infty] \mid x_i) - T \geq 0$$

If $\theta < \bar{\theta}$ a devaluation cannot occur even if both players coordinate on the attack. If $\theta \geq \bar{\theta}$, one agent alone is capable of inducing a devaluation, and if $\theta \in [\bar{\theta}, \Theta)$ a devaluation occurs if and only if both agents attack. Assume that agent $j$ follows a threshold strategy with switching point $x^*_j$. Then, the expected payoff of attacking for agent $i = 1, 2$, conditional on observing signal $x_i$ and given that the other agent follows a threshold strategy with switching point $x^*_j$ is:

$$v_i(x_i, x^*_j; \sigma) = \frac{1}{\sigma_i} \int_\Theta^\bar{\theta} \theta f\left(\frac{\theta - \bar{\theta}_i}{\sigma_i}\right) \left(1 - F\left(\frac{x^*_j - \theta}{\sigma_j}\right)\right) d\theta + \frac{1}{\sigma_i} \int_\bar{\theta}^{\infty} \theta f\left(\frac{\theta - \bar{\theta}_i}{\sigma_i}\right) d\theta - T \quad (1)$$

**Lemma 1** The payoff for agent $i$ of attacking, $v_i(x_i, x^*_j; \sigma)$, is increasing in his own signal $x_i$, and decreasing in the other agent’s threshold $x^*_j$, for $i, j = 1, 2$, $i \neq j$.

To derive the equilibrium conditions of the model we need to analyze the optimal thresholds. A threshold $x^*_i$ is defined as the value of agent $i$’s signal $x_i$ for which he is indifferent between attacking and not attacking, taking as given the strategy of the other player. For $i = 1, 2$, $i \neq j$, taking $x^*_j$ as given, the optimal threshold of agent $i$, $x^*_i$, solves the following equation:

\textsuperscript{11}In this context the type of an agent is described by the signal he observes and his precision choice. Recall that we assume that agents hold correct beliefs about each other’s information choices, so after observing his own signal $x_i$, agent $i = 1, 2$ only forms beliefs about agent $j$’s signal, $x_j$.

\textsuperscript{12}We first focus on threshold strategies, and then verify that the resulting equilibrium is indeed the unique equilibrium of the second stage. In particular, we show that any equilibrium of the game will be in monotone strategies (i.e. thresholds), and that there is a unique dominance solvable equilibrium threshold. This is proven in Theorem 1 in the appendix.
\[ v_i(x^*_i, x^*_j; \sigma) = \frac{1}{\sigma_i} \int_{\theta}^{\infty} \theta f \left( \frac{\theta - \tilde{\theta}^*_i}{\sigma_i} \right) \left( 1 - F \left( \frac{x^*_j - \theta}{\sigma_j} \right) \right) d\theta + \frac{1}{\sigma_i} \int_{0}^{-\infty} \theta f \left( \frac{\theta - \tilde{\theta}^*_i}{\sigma_i} \right) d\theta - T \]

where \( \tilde{\theta}^*_i = \frac{\mu_i \sigma_i^2 + x^*_i \sigma^2}{\sigma_i^2 + \sigma_0^2} \).

**Definition 1** Given \( (\sigma_i, \sigma_j) \) an equilibrium in monotone strategies for the second stage of the game is a pure strategy profile \( a_i(x_i; \sigma) \) such that, for \( i = 1, 2 \)

\[
a_i(x_i; \sigma) = \begin{cases} 
1 & \text{if } x_i \geq x_i^*(\sigma) \\
0 & \text{if } x_i < x_i^*(\sigma)
\end{cases}
\]

where \( x_i^*(\sigma) \) solves

\[ v(x_i^*(\sigma), x_j^*(\sigma); \sigma) = 0 \]

Lemma 1 implies that the best response functions are well defined. To prove that there is a unique equilibrium in the second stage of the game, we need to show that there is a unique pair of thresholds that satisfies the above equilibrium conditions. This means that we need to show that there exists a unique combination of \( (x_1^*, x_2^*) \) that solves simultaneously equilibrium condition (2) for \( i = 1, 2 \). This is demonstrated in Theorem 1.\(^\text{13}\)

**Theorem 1** There exists a unique, dominance solvable equilibrium of the second stage of the game in which both players use threshold strategies characterized by \( (x_1^*, x_2^*) \) if either:\(^\text{14}\)

1. \( \frac{\sigma_i}{\sigma_0} < K_i(\vec{\theta}, \vec{\sigma}, \mu_0) \), \( i = 1, 2 \) holds, for any pair of \( (\sigma_1, \sigma_2) \),\(^\text{15}\) or

\(^{13}\)To prove this result we make use of the literature on monotone supermodular games. In Appendix 2, we show that the game specified above belongs to the class of Bayesian monotone supermodular games, as defined by Vives and van Zandt (2007). This feature of our model simplifies the proof of uniqueness in the second stage of the game. In particular, we extend the result of Vives and van Zandt (2007) to prove existence of a least and a greatest Bayesian Nash Equilibrium in monotone strategies in games with unbounded but integrable utility functions. This extension allows us to prove uniqueness of equilibrium in the second stage of our game by showing that the least and greatest BNE of our game coincide, and thus there is a unique equilibrium in threshold strategies.

\(^{14}\)This result implies that the game specified in the second stage is dominance solvable (Milgrom and Roberts, 1990). While the dominance solvability of symmetric binary global games is well understood (see Morris and Shin, 2003), our contribution is to show that it also applies to an asymmetric game where both thresholds are functions of each other. Corsetti et al (2004) also show dominance solvability in a binary action global game with asymmetric players. Nevertheless, in that setup the optimal threshold of a “large” player is independent of the threshold of the “small” players. Frankel, Morris, and Pauzner (2003) also show dominance solvability for players with bounded utility functions and asymmetric noise distributions with finite support. In our setup, utility functions are unbounded and noise distributions have infinite support.

\(^{15}\)Where \( K_i(\vec{\theta}, \vec{\sigma}, \mu_0) := \frac{1 - F \left( \frac{\theta = 0}{\sigma_i} \right) + \frac{\theta}{\sigma_i} / \left( \frac{\sigma_i^2}{\sigma_0^2} \right)^{\theta_1}}{\sigma_i^2 \left( 1 + \frac{\theta_1}{\sigma_i^2} \right)^{\theta_1} + \frac{\theta}{\sigma_i} / \left( \frac{\sigma_i^2}{\sigma_0^2} \right)^{\theta_1}} \).
2. $\sigma_\theta > \bar{\sigma}_\theta$, where $\bar{\sigma}_\theta$ is determined by the parameters of the model.

This result is very much in line with previous work on global games, like Hellwig (2002), who introduces a public signal to the model of MS98 and shows that uniqueness of equilibrium is preserved when the public signal is noisy enough with respect to the private signals. Moreover, in our model as $\sigma_\theta \to \infty$, the coordination game with asymmetric players has always a unique equilibrium. In our context we can think of the prior as being a public signal, since it carries information about the fundamental that is available to all agents at no cost. It is important to note that the conditions on precisions stated in Theorem 1 are only sufficient but not necessary for uniqueness of equilibrium in the second stage.\(^{16}\)

2.1.1 Limiting case

Now that we have proven uniqueness of equilibrium in the second stage of the game, we investigate what happens with equilibrium selection in the limit as noise vanishes. This result will be useful for the experiment, since it has been shown (Heinemann et al, 2004) that when agents play the speculative game without information acquisition in the laboratory, their behavior is consistent with the theoretical prediction as the noise of the private signals vanishes.

For $i = 1, 2$, $i \neq j$, we consider what happens when both $\sigma_i \to 0$ and $\sigma_j \to 0$ simultaneously and $\frac{x_i^*}{\sigma_j^*} \to c$, for some $c \in \mathbb{R}_+$, i.e. when the signal noise goes to zero for both agents while the ratio of their noises tends to some constant $c$.

**Lemma 2** Suppose that $\sigma_i \to 0$, $\sigma_j \to 0$ and $\frac{x_i^*}{\sigma_j^*} \to c$ where $c \in \mathbb{R}_+$. If the above game has a unique equilibrium, then this equilibrium converges to the risk-dominant equilibrium of the complete information game, i.e. $x_1^* \to 2T$ and $x_2^* \to 2T$.

Note that if $2T$ corresponds to a threshold such that if $\theta \geq 2T$, then attacking is the risk-dominant action, and if $\theta \leq 2T$, then not attacking is the risk dominant action. This result is of particular importance because it implies that the global games equilibrium converges to the risk-dominant equilibrium strategy of the complete information game in the case of players with

\[
K_2(\overline{\theta}, \mu) := \frac{1 - F\left(\frac{x_{\text{min}} - \theta_{\text{min}}}{\sigma_1}\right) - F\left(\frac{x_{\text{max}} - \theta_{\text{max}}}{\sigma_1}\right)}{\sqrt{2\pi} \sigma_1} F\left(\frac{\theta_{\text{max}} - \theta_{\text{min}}}{\sigma_2}\right) + \frac{1}{\sqrt{2\pi} \sigma_2} \left[ \frac{\theta_{\text{max}}}{\sigma_1} + \frac{\theta_{\text{min}}}{\sigma_2} \right]
\]

and $\mu_i := \mathbb{E} \left[ \frac{\sigma_2 \left( \theta_{\text{min}} - \frac{x_{\text{min}} + x_{\text{max}}}{\sigma_1^2 + \sigma_2^2} \right)}{\sigma_2^2} \right] + \lambda \left( 1 - F\left( \frac{\sigma_2 \left( \theta_{\text{max}} - \frac{x_{\text{max}} + x_{\text{min}}}{\sigma_1^2 + \sigma_2^2} \right)}{\sigma_2^2} \right) \right)$

where $\theta_{\text{min}}^*, \theta_{\text{max}}^*, \overline{\theta}_{\text{min}}$, $\overline{\theta}_{\text{max}}$, and $\theta_2$ are defined endogenously in detail in the appendix.

\(^{16}\)The conditions in Theorem 1 are sufficient but not necessary because, just as in Hellwig (2002) and Morris and Shin (2004), very precise public information aligns agents’ posterior beliefs. This allows players to better coordinate, reintroducing multiple equilibria into the model. Our conditions are in the same spirit as those of the above papers, however, due to the strategic effects of individual actions present in our model, they take more complex functional forms.
asymmetric signals. This feature was first pointed out by Carlsson and van Damme (1993) for $2 \times 2$ global games with symmetric noise structures in which iterated dominance forces the players to behave according to the risk dominant equilibrium (Harsanyi and Selten, 1988), given that the noise in the players’ observations is sufficiently small. We thus extend their limiting result for the case of unbounded utility functions and asymmetric noise distributions with infinite support.

2.2 Solving stage one: Optimal information acquisition

Given the equilibrium strategies in the second stage of the game, we move on to the first stage where agents get to choose the precision of the private signal they will observe, at a cost. Agents initially have a common prior about the underlying state of economy, $\theta \sim N(\mu_\theta, \sigma_\theta^2)$. During this first stage they can improve the precision of the signal they will receive in the second stage. If an agent does not improve his precision he will receive a signal $x_i \sim N(\theta, \sigma_\theta^2)$.

We assume that the ratio $\frac{\sigma_\mu}{\sigma_\theta}$ satisfies the uniqueness condition for the equilibrium in the second stage of the game derived in section 2.1.\footnote{In general, if the condition for uniqueness is not satisfied, the game played in the second stage will have three equilibria in monotone strategies (which can be Pareto ranked). If we assume that agents play a Pareto dominant (or Pareto dominated) equilibrium, as the precision of their private signals decreases and we enter the dominance region, the equilibrium threshold would experience a jump, creating a discontinuity in the first period utility at that point.}

The expected utility of agent $i$, who has standard deviation $\sigma_i$, and who believes that his opponent has standard deviation $\sigma_j$ and that his opponent holds correct beliefs about his own choice of $\sigma_i$, is given by:

$$U_i(\sigma_i; \sigma_j) = \int_{x_i \geq x^*_i} v_i(x_i, x^*_j; \sigma) \frac{1}{\sqrt{\sigma_i^2 + \sigma_\theta^2}} f \left( \frac{x_i - \mu_\theta}{\sqrt{\sigma_i^2 + \sigma_\theta^2}} \right) dx_i - C(\sigma_i) \quad (3)$$

where $v_i(\cdot)$ is defined in equation (1). $C(\sigma_i)$ is the cost associated with agent $i$’s choice of precision, and $\frac{1}{\sqrt{\sigma_i^2 + \sigma_\theta^2}} f \left( \frac{x_i - \mu_\theta}{\sqrt{\sigma_i^2 + \sigma_\theta^2}} \right)$ is the unconditional density of agent $i$’s signal, given his choice of $\sigma_i$.

We now define a pure strategy Bayesian Nash Equilibrium for the full game and proceed to analyze the problem faced by each agent.

Definition 2 (Equilibrium) A pure strategy Bayesian Nash Equilibrium is a pair of information choices $(\sigma^*_i, \sigma^*_j)$, optimal decision rules for the second stage $a_i(x_i; \sigma)$, and belief functions $\mu_i : (0, \sigma_0) \rightarrow [0, 1]$ such that for each $i = 1, 2$ we have:

- $U_i(\sigma^*_i, \sigma^*_j) \geq U_i(\sigma_i, \sigma^*_j) \quad \forall \sigma_i \in (0, \sigma_0)$

- The belief function $\mu_i$ satisfies $\mu_i(\sigma^*_j) = 1$ and $\mu_i(\sigma_j) = 0$ for $\sigma_j \neq \sigma^*_j$;
• For $i = 1, 2$, given the belief function $\mu_i$, agent $i$’s decision rule is given by

$$a_i(x_i; \sigma) = \begin{cases} 
1 & \text{if } x_i \geq x_i^*(\sigma_i, \sigma_j^*) \\
0 & \text{if } x_i < x_i^*(\sigma_i, \sigma_j^*)
\end{cases}$$

where $x_i^*(\sigma)$ solves

$$v(x_i^*(\sigma_i, \sigma_j^*), x_j^*(\sigma_i^*, \sigma_j^*); \sigma) = 0$$

The first condition is an optimality condition, it requires agent $i$ to have no incentives to deviate from his equilibrium precision choice. Note, however, that since each player’s precision choice is private, when considering deviations by agent $i$ we keep the strategy and beliefs of agent $j$ constant.

The second condition is a restriction on the belief function, namely, that an agent assigns positive probability only to the actual equilibrium choice of his opponent. Finally, the last condition requires each agent to act optimally in the second stage, even after unilateral deviations.

It is convenient at this stage to define the benefit of choosing $\sigma_i$, i.e. the gross payoff of choosing $\sigma_i$:

$$B_i(\sigma_i; \sigma_j) := \int_{x_i > x_i^*} v_i(x_i, x_j^*; \sigma) f(x_i; \sigma_i) dx_i$$

We make an extra assumption on the cost function: $\lim_{\sigma_i \to 0} B'_i(\sigma_i, \sigma_j) < \lim_{\sigma_i \to 0} C'(\sigma_i)$. This assumption ensures that agents will never find it optimal to choose infinitely precise signals.

We now establish existence of equilibrium for the full game. The proof involves proving two simple results that state that the best response functions are well defined and that agents improve their precision in any equilibrium of the game.

**Theorem 2 (Existence)** There exists a symmetric pure-strategy Bayesian Nash Equilibrium of the game with costly information acquisition.

In the present model with two agents, because of the lack of a closed form solution, it is difficult to derive parametric conditions to characterize comparative statics explicitly and to perform a thorough equilibrium characterization.\(^\text{18}\) We briefly discuss some results that arise from numerical simulations that will be helpful to understand the predictions of the model for the set of parameters used in the experiment.

The first observation is that for most parameter values there are increasing differences in precision choices, i.e. more information is always beneficial ($B'_i(\sigma_i, \sigma_j) > 0$) and the marginal benefit of increasing individual precisions is increasing in the precision of the other agent ($\frac{\partial B'_i(\sigma_i, \sigma_j)}{\partial \sigma_j} > 0$).

Therefore, there are strategic complementarities in information acquisition. Conditions for a unique symmetric equilibrium will depend on parametric assumptions for the prior and for the cost function.

\(^\text{18}\)To see a complete characterization of equilibrium and comparative statics for the game with a continuum of players, see Szkup and Trevino (2012).
The next observation is a comparative statics result. As presented in Metz (2002), Bannier and Heinemann (2005), and Szkup and Trevino (2012), in global games with public and private information it is not possible to derive comparative statics results that will hold for any general specification of parameters. In the game without costly information acquisition Metz (2002) and Bannier and Heinemann (2005) show how the probability of a currency crisis increases or decreases with the precision of public and private information, depending on the prior beliefs over fundamentals. Szkup and Trevino (2012) find that this is also true in an environment with costly information acquisition.

A similar logic applies to our model and extends to the effect of precision changes on equilibrium outcomes, i.e. holding agent $j$’s beliefs constant, the choice of a higher precision (lower $\sigma_i$) will have an effect on the unconditional distribution of agent $i$’s signal ($x_i$), on agent $i$’s equilibrium threshold ($x_i^*$), on agent $i$’s posterior beliefs about $\Theta$ ($\tilde{\Theta}_i$), and on the cost of acquiring information ($C(\sigma_i)$). Therefore, the overall effect on expected utility of a change in agent $i$’s precision will depend on the interaction between these effects. Our numerical simulations suggest that the mean and variance of the prior and the cost of attacking play an important role in determining how individual precisions affect optimal thresholds. In particular, the way in which optimal thresholds vary with the precisions of both agents seems to depend on the informativeness of the prior, $\sigma_\Theta$, for certain combinations of $\{T, \mu_\Theta\}$. However, for most parameter combinations thresholds are increasing functions of precisions. Specifically, when agents are optimistic about the state of the world and costs are low, i.e. when $\mu_\Theta$ is very high with respect to $T$, thresholds are increasing functions of both precisions, irrespective of $\sigma_\Theta$. When agents are pessimistic about the state of the world and costs are high, or when costs and prior beliefs are of about the same magnitude, i.e. when $\mu_\Theta$ is very low with respect to $T$, or when $\mu_\Theta$ is very similar to $T$, the direction in which precisions affect thresholds depends on the precision of the prior. For these cases, when the prior is very noisy (high $\sigma_\Theta$), the threshold of each agent is increasing with the precision of both agents. Nevertheless, when the precision of the prior is not too diffuse (but noisy enough so that a unique equilibrium in the second stage is ensured), the threshold of each player is a decreasing function of the precision of both agents.

In the next subsection we state the equilibrium predictions for the parameters used in our experiment.

2.3 Theoretical predictions for the experiment

The theoretical model is governed by a set of parameters $\Theta = \{\mu_\Theta, \sigma_\Theta, (\theta, \overline{\theta}), T, \{\sigma_i\}, \{C(\sigma_i)\}\}$. For the experiment presented in section 3, the parameters chosen are

$$\Theta = \{50, 50, (0, 100), 18, \{1, 3, 6, 10, 16, 20\}, \{6, 5, 4, 2, 1.5, 1\}\}$$

In particular:
• The fundamental $\theta$ is randomly drawn from a normal distribution with mean 50 and standard deviation of 50.

• The coordination region is for values of $\theta \in [0, 100)$.

• The cost of attacking is $T = 18$.

• A discrete choice set of precisions\(^{19}\) and the cost associated with each precision was presented in the form of a menu of 6 precision levels, standard deviations, and costs:

<table>
<thead>
<tr>
<th>Precision level</th>
<th>Standard deviation</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>1.5</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Precision choices

We decided not to have a default precision chosen for subjects in order to avoid status quo biases, and the reason to introduce a discrete choice set for precisions was to simplify the choice for subjects. We believe six is a reasonable number of options to observe dynamics in the level of informativeness that subjects choose, without losing statistical power.

Given these parametric assumptions we can characterize the predictions of the model in the form of three main hypotheses to be tested with our experiment:

**Hypothesis 1** *For any precision choices, subjects use threshold strategies.*

**Hypothesis 2** *Thresholds are increasing functions of precisions.*

**Hypothesis 3 (Equilibrium)** *Subjects coordinate on a precision level of 4 and set a symmetric threshold of 28.31.*

Hypothesis 1 establishes that subjects will use the type of strategy predicted by global games, for any precision choices, as shown in subsection 2.1. We have chosen a situation where the mean of the prior is high with respect to the cost of attacking to ensure that the effect of precisions on thresholds does not depend on the precision of the prior (see Szkup and Trevino, 2012), which allows us to formulate Hypothesis 2, which states that the optimal threshold in stage two for each

\(^{19}\)In the remaining of the paper we will refer to information choices as precision choices to be consistent with the language used in the implementation of the experiment. We will use the term precision as a qualitative measure of informativeness of the signals, i.e. we will compare levels of precision, and not magnitudes of standard deviations.
agent is increasing in the precision of both agents. Finally, Hypothesis 3 aims to test the unique symmetric equilibrium prediction for the parameters used in the experiment, which corresponds to subjects coordinating on their choice of precision level 4 and setting a symmetric threshold at 28.31. Implicit in this prediction is that precision choices are strategic complements, which leads agents to coordinate on both precisions and actions. As an additional feature of the model coming from section 2.1.1, note that the equilibrium prediction in the limit, as signal noise vanishes, is equal to $2T = 36$, which corresponds to the risk dominant equilibrium for our game. Previous experimental evidence shows that subjects coordinate on this limiting threshold when playing a global game without information acquisition (see Heinemann et al, 2004).\footnote{Note that in our case with 2 players the risk dominant equilibrium coincides with the prediction of global games in the limit, as the noise vanishes. Moreover, there is experimental evidence that shows that the risk dominant equilibrium is often selected in $2 \times 2$ coordination games (see Cabrales et al, 2000).}

We now proceed to present the experimental design for our model and summarize the main empirical findings.

3 The experiment

We present results of a series of laboratory experiments designed to test the implications of the model of speculative attack with costly information acquisition described in section 2. The experiment was conducted at the Center for Experimental Social Science at New York University during 2011 using the usual computerized recruiting procedures. All subjects were undergraduate students from New York University.\footnote{Instructions for all treatments can be found in the online appendix https://files.nyu.edu/it384/public/instructions_szkup_trevino.pdf}

The main experimental studies that relate to our paper are Cabrales et al (2007) and Heinemann et al (2004). In particular, our experimental design is closely related to the work of Heinemann et al (2004) (HNO04 henceforth), who test the predictions of the model by MS98 in the laboratory and find that, on average, 92% of the strategies observed are consistent with the use of undominated thresholds that coincide with the theoretical prediction of equilibrium in the limit, as noise vanishes. There are no studies, that we are aware of, that introduce costly information acquisition into a global games model and test the predictions experimentally.

3.1 Experimental design

We implemented a between subjects design that allowed us to directly compare the behavior of subjects across treatments. Each session consisted of 50 independent rounds.

There were three main treatments. In the first one, our Control treatment, subjects played the game without information acquisition (as in HNO04) with an exogenously fixed and commonly known precision for the signal distribution. This precision coincided with the equilibrium precision of the game with information acquisition (level 4). Since our experimental design differs from...
the one presented in HNO04 in the number of subjects in each group and in the distributional assumptions, we cannot directly compare our results with theirs, thus this first treatment serves as a benchmark for our analysis.

In the second treatment subjects played our speculative attack game with costly information acquisition, as presented by our model in section 2. In this treatment subjects chose from a set of discrete precisions with no default option, as described in Table 1, then received a signal and finally they chose an action. We refer to this as the treatment with Endogenous Precisions and Action choices (EPA).

Treatment 3 was introduced in order to observe the evolution of thresholds through time. In this treatment subjects played the speculative attack game with information acquisition using the strategy method for the second stage. In particular, the precision choice stage was the same as in the EPA treatment, but in the second stage subjects had to choose a cutoff value such that they would attack if their signal was higher than this cutoff and not attack if their signal was lower than the cutoff they report. Hence, this treatment allows us to observe thresholds directly, rather than infer them as in the EPA treatment. We refer to this as the treatment with Endogenous Precision and Strategy method choices (EPS).

On top of these three main treatments we also ran additional sessions of the Control treatment with exogenous high and low precisions to assess an alternative hypothesis to Hypothesis 2 that could explain departures from the theory, as will be explained in detail in the next subsection. We refer to these two extra treatments as Control High and Control Low. For each of these treatments there were sessions in which subjects chose actions directly, and sessions with the strategy method for thresholds. We also ran an additional session for the original Control treatment, with the equilibrium precision, using the strategy method.

Overall, we ran 14 sessions, leading to a total of 270 participants. Table 2 summarizes our experimental design.

<table>
<thead>
<tr>
<th>Treatment</th>
<th># Sessions</th>
<th># Subjects</th>
<th>Signal precision</th>
<th>Choice of action</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: Control</td>
<td>2</td>
<td>40</td>
<td>Level 4</td>
<td>Direct choice</td>
</tr>
<tr>
<td>2: EPA</td>
<td>2</td>
<td>40</td>
<td>Endogenous</td>
<td>Direct choice</td>
</tr>
<tr>
<td>3: EPS</td>
<td>3</td>
<td>44</td>
<td>Endogenous</td>
<td>Strategy method</td>
</tr>
<tr>
<td>Additional treatments:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Control - Strategy method</td>
<td>1</td>
<td>24</td>
<td>Level 4</td>
<td>Strategy method</td>
</tr>
<tr>
<td>Control High</td>
<td>2</td>
<td>38</td>
<td>Level 1</td>
<td>Direct choice</td>
</tr>
<tr>
<td>Control High - Strategy method</td>
<td>1</td>
<td>20</td>
<td>Level 1</td>
<td>Strategy method</td>
</tr>
<tr>
<td>Control Low</td>
<td>2</td>
<td>44</td>
<td>Level 6</td>
<td>Direct choice</td>
</tr>
<tr>
<td>Control Low - Strategy method</td>
<td>1</td>
<td>20</td>
<td>Level 6</td>
<td>Strategy method</td>
</tr>
</tbody>
</table>

Table 2: Experimental design

22 This feature of our treatment is related to the study of Duffy and Ochs (2011) who use the strategy method to elicit thresholds in coordination games. See Brandts and Charness (2011) for a survey on the strategy method.
Subjects were randomly matched in pairs at the beginning of the session and stayed with the same partner for all rounds. Each session lasted from 60 to 90 minutes and subjects earned on average $25, including a $5 show up fee.

To avoid framing effects, the game was explained using neutral terms. Subjects were told to choose between two actions $A$ or $B$, avoiding terminology such as “speculative attack”. To avoid bankruptcies, subjects entered each round with an endowment of 24 tokens from which they subtracted their precision cost. From Table 1 we can see that even if subjects choose the most precise information, the lowest payoff they can get in a round is zero (in case of an unsuccessful attack).

Before starting the first paying round, subjects had access to a practice screen where they could generate signals for the different precisions and they were given an explanation of the payoffs associated to each possible action, given their signal and the underlying state $\theta$. They could spend up to 5 minutes in the practice screen.

Each round of the game with information acquisition consisted of two decision stages:

1. Subjects chose from a menu of precisions (standard deviations $\sigma_i$) and associated costs (see Table 1).

2. Subjects observed a signal $x_i \sim N(\theta, \sigma_i^2)$ and simultaneously decided whether to attack ($A$) or not attack ($B$).

As stated above, treatment EPS differs from treatment EPA through the use of the strategy method in the second step.

For the Control treatment, where subjects do not choose precisions, they received their private signal drawn independently from the same normal distribution with mean $\theta$ and standard deviation of 10, which is the optimal level of precision dictated by the theory for that set of parameters. Once subjects observed their own signals, they decided simultaneously whether to attack or not.

After each round, each subject observed: his choice of precision, his own private signal, his choice of action, the realization of $\theta$, how many people in his pair chose $A$, whether $A$ was successful or not, and his individual payoff for the round. In addition, each subject could access the history of precision choices made over the previous rounds by pressing a button.

The computer randomly selected five of the rounds played and subjects were paid the average of the payoffs obtained in those rounds, using the exchange rate of 3 tokens per 1 US dollar.

The experiment was programmed and conducted with the software z-Tree (Fischbacher, 2007).

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23 We chose fixed pairs, as opposed to random pairs, to be able to study coordination of information choices over time. As is well understood in the experimental literature, there is a trade-off between these two matching protocols. Random pairs preserve better the spirit of a one shot game. However, due to the complexity of our setup, subjects might need to learn about the type of opponent they are facing and about the game itself. Our game is a double coordination game in actions and precision choices and we believe fixed pairs are better suited to study such environments. The complexity of our setup also makes it unlikely for agents to treat it as a repeated game. In particular, we find no evidence of agents using punishment schemes in their strategies across rounds.

24 The additional sessions of the Control Treatment were run with exogenously given precision levels 1 and 6 from Table 1.
3.2 Experimental results

The analysis of the results will first address the three hypotheses stated in section 2.3 that derive from the theory. This will be followed by a welfare analysis and some additional results that enrich the predictions of our model. The analysis is based on behavior in the last 25 rounds of the experiment, unless otherwise specified.

**Hypothesis 1**  
*For any precision choices, subjects use threshold strategies.*

We find strong support for Hypothesis 1 in the data for both the Control and the EPA treatments. First, we state the type of behavior that we define as arising from threshold strategies. Then, we analyze the results for the Control (since we can estimate thresholds directly), and then we move on to the EPA treatment where the use of threshold strategies depends on the stability of precision choices.

We say that a subject’s behavior is consistent with the use of threshold strategies if the subject uses either perfect or almost perfect thresholds. A subject uses a perfect threshold if he does not attack for low values of the signal and attacks for high values of the signal, with exactly one switching point. This effectively means that the set of signals for which a subject attacks and the set of signals for which he does not attack are disjoint. This type of behavior is illustrated in panel (a) of Figure 1, which has the signal a subject receives on the horizontal axis and a binary value (0 for not attack, 1 for attack) on the vertical axis. For almost perfect thresholds, we allow these two sets to overlap for at most three observations. This means that subjects do not attack for low signal values and attack for high signal values, but these two sets can intersect for at most three observations. Such behavior is portrayed in panel (b) of Figure 1 where we fit a logistic function to the observed data of a specific subject.

![Figure 1: Examples of perfect and almost perfect thresholds](image)

In our Control treatment, just like HNO04, we find that 92.5% of the behavior is consistent with the use of threshold strategies. In particular, we find that 67.5% of the subjects use perfect thresholds, and 25% use almost perfect thresholds.

Once we have identified the subjects who use threshold strategies, we use two different methods

---

25 HNO04 use a different metric to measure the use of threshold strategies and have 10 decision situations in each round of the experiment.
to estimate their mean thresholds. In the first method, we pool the data of all the threshold subjects in each treatment and fit a logistic function with random effects (RE) to determine the probability of attacking as a function of the observed signal. The cumulative logistic distribution function is defined as

$$Pr(A) = \frac{1}{1 + \exp(\alpha + \beta x_i)}$$

By fitting the pooled data to a logistic function, we can estimate the mean threshold of the group by finding the value of the signal for which subjects are indifferent between attacking and not attacking, i.e. the value of the signal for which they would attack with probability \( \frac{1}{2} \), which is given by \(-\frac{\alpha}{\beta}\). As pointed out by HNO04, the standard deviation of the estimated threshold, \( \frac{\pi}{\beta \sqrt{3}} \), is a measure of coordination and reflects variations within the group. We call this the Logit (RE) method.

For the second method we take the average, individual by individual, between the highest value of the signal for which a subject chooses not to attack and the lowest value of the signal for which he chooses to attack. This number approximates the value of the signal for which he switches from taking one action to taking the other action. Once we have estimated individual thresholds using this approach, we take the mean and standard deviation of the thresholds in the group. We refer to this estimate as the Mean Estimated Threshold (MET) of the group.

The estimated mean thresholds for the Control treatment are shown in the middle column of table A.1 in the appendix, corresponding to a medium precision. Standard deviations are reported in parenthesis.

The estimated mean thresholds for medium precision are not statistically different from each other and coincide with the risk dominant equilibrium predicted by the theory, i.e. the optimal threshold in the limit, as noise in the private signal vanishes. This result is also consistent with the findings of HNO04 who find that in the game without costly information acquisition the threshold selected in the limit, as the noise vanishes, best explains the data (in our case, because we have 2 players, this threshold corresponds to the risk dominant equilibrium of the underlying complete information game, see Lemma 2). Therefore, we can conclude that the results of HNO04 extend to our framework.

We now proceed to analyze the use of thresholds for the EPA treatment when subjects choose their precision. Since optimal thresholds predicted by the theory depend on the precision chosen, we cannot aggregate the actions taken by a subject when he chooses different levels of precision. Here we must ask whether, conditional on the choice of precision, subjects use a threshold strategy. Therefore, to illustrate the use of threshold strategies we first establish stability of individual precision choices and then analyze actions for the precision level that was mostly chosen by each subject.

---

26 For the EPA treatment we pool the data of subjects according to the level of precision chosen.

27 In particular, we cannot reject the hypothesis that the threshold estimated with the logit is different from the risk dominant threshold to the 1% level of significance.

28 If individual precision choices were unstable over the last 25 periods, it would be difficult to condition on a precision choice since it would be constantly changing.
subject.

We find that, on average, in the last 25 rounds of the experiment subjects choose the same level of precision for more than 22 out of the last 25 periods and that the most popular precision choice is the equilibrium level 4. To illustrate this result, in figure 2 we report the transition matrix of precision choices in the last 25 rounds.\textsuperscript{29} The entry $a_{ij}$ of the matrix shows the probability of choosing precision level $j$ in $t+1$, given that a subject chose precision level $i$ at $t$, for $i, j \in [1, 6]$ and $t > 25$.

\[
\begin{pmatrix}
Prec 1 & Prec 2 & Prec 3 & Prec 4 & Prec 5 & Prec 6 \\
Prec 1 & 0.95 & 0.03 & 0 & 0 & 0 & 0.02 \\
Prec 2 & 0.08 & 0.74 & 0.08 & 0.05 & 0 & 0.05 \\
Prec 3 & 0 & 0.02 & 0.87 & 0.09 & 0 & 0.02 \\
Prec 4 & 0 & 0 & 0.04 & 0.92 & 0.02 & 0.02 \\
Prec 5 & 0.01 & 0.01 & 0.10 & 0.15 & 0.58 & 0.15 \\
Prec 6 & 0.01 & 0 & 0.02 & 0.04 & 0.03 & 0.90 \\
\end{pmatrix}
\]

Figure 2: Transition matrix of precision choices in the last 25 rounds, EPA and EPS treatments

By looking at the diagonal entries of the transition matrix, we can see that most precision levels (except for level 5) are absorbent states.\textsuperscript{30} This effectively means that precision choices are stable over the last 25 rounds, i.e. that subjects on average choose their precision consistently with respect to their own previous choices.

Given this stability result, we can characterize subjects by their preferred precision choice. We show the percentage of subjects that choose each precision level for the last 25 rounds of the experiment in Table 3.\textsuperscript{31}

<table>
<thead>
<tr>
<th>Precision level</th>
<th>Standard deviation</th>
<th>Cost</th>
<th>Precision choices in last 25 rounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>6</td>
<td>14.7%</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>3.7%</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>4</td>
<td>18.4%</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>2</td>
<td>36.9%</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>1.5</td>
<td>3.9%</td>
</tr>
<tr>
<td>6</td>
<td>20</td>
<td>1</td>
<td>22.4%</td>
</tr>
</tbody>
</table>

Table 3: Precision choices in the last 25 rounds, for EPA and EPS treatments

\textsuperscript{29}This includes choices in treatments EPA and EPS. We aggregate the data because the distributions of precision choices are not statistically different between these two treatments. This was expected since the treatment effect is in the second stage of the game.

\textsuperscript{30}Very few subjects choose precision level 5 and their behavior in the second stage is mostly random.

\textsuperscript{31}Precision choices in the first rounds are not very dissimilar to the precision choices portrayed in table 3. In particular, if we compare the choices of the first 5 rounds with the choices of the last 5 rounds of the experiment we observe the following proportion of choices, by precision level (the first number corresponds to the first 5 rounds and the second to the last 5 rounds). Level 1: 16.2% vs 13.8%; level 2: 11.9% vs 4%; level 3: 25% vs 19%; level 4: 21.4% vs 36.8%; level 5: 4.8% vs 4%; level 6: 20.7% vs 22.4%. We observe the highest shift in precision choices to be in favor of the equilibrium precision level 4.
We can now analyze subject behavior for the EPA treatment. We identify types of subjects: those who use thresholds, those who use a degenerate strategy, and those who act randomly. The use of thresholds can be either by perfect or almost perfect thresholds, as defined above. We say that a subject uses a degenerate strategy if his choice of action is constant and does not vary with the signals (i.e. always attack or never attack). For a subject that exhibits random behavior, the choice of action is independent of his signal realization.

As shown in table 4, we find that 100% of subjects choosing precision levels 1, 2, or 3 use threshold strategies. For precision level 4, all but one subject use threshold strategies, which corresponds to 93.75% of the subjects. There is only one subject choosing precision level 5, whose behavior is random. For precision level 6, 75% of the subjects use threshold strategies and 25% use degenerate strategies. This suggests that when subjects invest in more precise information their behavior is consistent with the use of threshold strategies more often, and when subjects choose not to invest in precision, they sometimes follow a degenerate action.

<table>
<thead>
<tr>
<th>Precision</th>
<th>Thresholds</th>
<th>Degenerate strategies</th>
<th>Random behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>100%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>100%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>93.75%</td>
<td>-</td>
<td>6.25%</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>-</td>
<td>100%</td>
</tr>
<tr>
<td>6</td>
<td>75%</td>
<td>25%</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4: Strategies followed in the second stage, for EPA treatment

In total, 90% of the subjects use threshold strategies for their most preferred precision choice. This implies that the global games strategy prediction is robust also under endogenous information.

Given these results, the next step is to analyze how the individual decision to attack depends on the level of precision chosen by each subject.

**Hypothesis 2**

*Thresholds are increasing functions of precisions.*

In order to test this hypothesis we need to establish a number of results. First, we analyze how the individual decision to attack depends on individual precision choices. Once we understand behavior at the individual level, we move on to the pair level to establish how the joint decision to attack (and hence the success of attacks) depends on the precision level chosen by the members of the pair. Just as in Hypothesis 1, we establish a notion of stability of precision choices within a pair. We do this for treatments EPA and EPS.

To look at the individual decision to attack, in Figure 3 we plot the cumulative distribution function (pooled over all subjects in treatment EPA) that illustrates the probability of attacking for any signal realization, for each precision level chosen. The value of the signal for which subjects attack with probability 0.5 determines the value of the signal for which they are indifferent between attacking and not attacking, which is how we define a threshold. Looking at the intersection of the curves with the 0.5 line in the vertical axis from left to right, we can see that the values
of the estimated thresholds are larger for less precise levels of information.\footnote{This is true except for precision level 2, but less than 4\% of the subjects chose this level in the EPA treatment.} This suggests that the subjects who acquire more precise information attack more often. We also see that as we move towards lower precision levels the slopes are decreasing, indicating that the dispersion of the observations within each precision group increases.

![Figure 3: Probability of attack, by precision choices in EPA treatment](image)

In order to better understand this phenomenon, we also run two regressions (one for the EPA and one for the EPS treatments) to determine the statistical effect that each level of precision has on the decision to attack (for EPA) and on the reported thresholds (for EPS).

For EPA, we estimate a random effects logit where the dependent variable is the decision to attack (1) or not attack (0) and the independent variables are dummies for the six precision levels, interacted with the signal realizations. We interact precisions and signals because the decision to attack is determined by the value of the signal, and the information choice affects how precise the signal is. As shown in Table A.2 in the appendix, we find that all coefficients for the interacted variables are positive and significant to the 1\% level, and that the magnitudes of the coefficients decrease for less precise levels of information. The positive coefficients mean that subjects attack for higher signal realizations, for all precision levels, which is consistent with the monotonicity implied by threshold strategies. The fact that the magnitudes of the coefficients decrease for less precise information implies that this signal effect is stronger when subjects observe very precise signals, and it is less strong when subjects observe noisier signals. This effectively means that agents attack more often when observing more precise signals than when observing less precise signals. If we categorize precision levels into high precision (levels 1 and 2), medium precision (levels 3 and 4), and low precision (level 6),\footnote{We do not include precision level 5 since there is only one subject choosing it, who behaves randomly in the second stage.} we find that the coefficients for high precision are statistically smaller...
than the coefficients for medium precision (at the 10% level of significance), and that the coefficients of medium precision are statistically smaller than the coefficient for low precision (at the 1% level of significance). We can therefore conclude that subjects attack more often for higher precisions.

We find further evidence of this effect in the EPS treatment by analyzing how reported thresholds vary with precision choices. Table A.3 in the appendix reports the results of a random effects OLS regression where the dependent variable is the threshold reported by subjects, and the independent variables are dummies for each level of precision. Each of these dummies takes the value of 1 if the subject chooses this precision level, and 0 otherwise. We find that reported thresholds depend positively and significantly on the level of precision chosen. In particular, the magnitudes of the coefficients for each precision level increase as we move towards less precise information, suggesting that less precise information gives rise to higher thresholds. This means that a subject with lower precision attacks less often than a subject with higher precision, which corroborates the findings in the EPA treatment that having more precise signals leads to a higher incidence of attacks. Notice that these results are at odds with the theoretical predictions for the set of parameters used in the experiment and with Hypothesis 2, which states that higher precisions lead to higher thresholds.

In order to fully understand if Hypothesis 2 holds, we need to focus on the behavior at the pair level because thresholds depend on the precision choices of both members of the pair. Therefore, we need to establish first a notion of convergence in precision within a pair to understand how joint precision choices affect the incidence and success of attacks. Precision choices within a pair can be separated in four categories, which are illustrated in Figure 4. We define individual convergence when a subject chooses the same precision level for the last 25 rounds, with at most three deviations. We say that a pair exhibits non-stable behavior if at least one of its members does not converge individually in his precision choice (panel (a)). A pair that has stability but not convergence is a pair in which both members converge individually in their own precision choices, but the levels at which they converge are more than one level apart (panel (b)). We define weak convergence as pairs in which both members converge individually to a level of precision and these two precision levels are at most one level away from each other (panel (c)). We say that a pair exhibits full convergence if both members converge individually to the same level of precision for the last 25 rounds of play (panel (d)).

![Figure 4: Examples of types of convergence in precision](image-url)
Table A.4 in the appendix shows all the combinations of precision choices made by the different pairs in our experiment (for both the EPA and EPS treatments). The diagonal entries correspond to the pairs that exhibit full convergence.

In what follows, we use the notion of weak convergence and we restrict our attention to pairs that coordinate on high precision (levels 1 and 2), medium precision (levels 3 and 4), and low precision (levels 5 and 6). Note that weak convergence includes full convergence. Table 5 summarizes the combinations of precision choices across pairs according to this notion of weak convergence, and we can see that if we use this qualitative characterization we find that approximately two thirds of the pairs in the endogenous precision treatments exhibit weak convergence in precision choices. Moreover, the majority of pairs converge to medium precisions, which corresponds to the theoretical prediction.

<table>
<thead>
<tr>
<th></th>
<th>High</th>
<th>Medium</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td><strong>10.00%</strong></td>
<td>13.81%</td>
<td>3.05%</td>
</tr>
<tr>
<td>M</td>
<td><strong>40.00%</strong></td>
<td>16.67%</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td></td>
<td>16.48%</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Weak convergence of precision choices, EPA and EPS treatments

Given this notion of convergence in precision choices, we can now characterize how the incidence and success of attacks is determined by precision choices. In tables 6 and 7 we report, for the EPA and EPS treatments respectively, the number of attacks and the number of successful attacks as proportions of the total number of situations when subjects are in the coordination region \( \theta \in [0, 100) \), for each combination of precision choices within pairs.\(^{34}\) The number of attacks corresponds to the instances when at least one of the pair members decides to attack and \( \theta \in [0, 100) \) and the number of successful attacks corresponds to the number of coordinated attacks (i.e. both subjects attack) and \( \theta \in [0, 100) \).

In order to characterize how the incidence and success of attacks varies with precisions, we look at the diagonal entries that correspond to the cases where subjects converge in their precision choices. From tables 6 and 7 we can see that in both treatments the incidence and success of speculative attacks increases with the precision at which subjects converge, which is in line with our findings at the individual level and establishes that our data does not find support for Hypothesis 2. We provide further evidence for this result in table A.5 in the appendix where we report the results of a random effects probit regression where we test how the success of an attack is determined by the level of precision to which each pair weakly converges for the EPA treatment.

One possible explanation for this departure from the theory is that subjects do not take the cost of precision as a sunk cost, as the theory suggests. Instead, subjects that choose high precision levels might take into account the high cost that they have paid for information when they make their decision to attack, in the sense that if they have invested in more precise information they

\(^{34}\)We thank readers of a previous version for suggesting these tables.
Table 6: Incidence and success of attacks, by precision choices, EPA treatment

<table>
<thead>
<tr>
<th></th>
<th>Number of attacks</th>
<th>Total situations</th>
<th>Number of successful attacks</th>
<th>Total situations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High</td>
<td>Medium</td>
<td>Low</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Number of attacks</td>
<td>Total situations</td>
<td>Number of successful attacks</td>
<td>Total situations</td>
</tr>
<tr>
<td></td>
<td>23</td>
<td>29 (79.31%)</td>
<td>2</td>
<td>2 (100%)</td>
</tr>
<tr>
<td></td>
<td>41</td>
<td>44 (93.18%)</td>
<td>2</td>
<td>4 (100%)</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>29 (62.07%)</td>
<td>2</td>
<td>2 (100%)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2 (100%)</td>
<td>18</td>
<td>29 (62.07%)</td>
</tr>
<tr>
<td></td>
<td>35</td>
<td>44 (79.54%)</td>
<td>2</td>
<td>2 (100%)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>5 (26.47%)</td>
<td>35</td>
<td>44 (79.54%)</td>
</tr>
<tr>
<td></td>
<td>59</td>
<td>115 (51.30%)</td>
<td>5</td>
<td>5 (26.47%)</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>68 (26.47%)</td>
<td>59</td>
<td>115 (51.30%)</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2 (100%)</td>
<td>18</td>
<td>68 (26.47%)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>5 (26.47%)</td>
<td>59</td>
<td>115 (51.30%)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>5 (26.47%)</td>
<td>18</td>
<td>68 (26.47%)</td>
</tr>
</tbody>
</table>

Table 7: Incidence and success of attacks, by precision choices, EPS treatment

<table>
<thead>
<tr>
<th></th>
<th>Number of attacks</th>
<th>Total situations</th>
<th>Number of successful attacks</th>
<th>Total situations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High</td>
<td>Medium</td>
<td>Low</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Number of attacks</td>
<td>Total situations</td>
<td>Number of successful attacks</td>
<td>Total situations</td>
</tr>
<tr>
<td></td>
<td>35</td>
<td>37 (94.59%)</td>
<td>7</td>
<td>19 (36.84%)</td>
</tr>
<tr>
<td></td>
<td>43</td>
<td>52 (81.85%)</td>
<td>34</td>
<td>47 (91.89%)</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>52 (81.85%)</td>
<td>33</td>
<td>47 (91.89%)</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>19 (26.47%)</td>
<td>3</td>
<td>19 (26.47%)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>19 (26.47%)</td>
<td>3</td>
<td>19 (26.47%)</td>
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<td></td>
<td>3</td>
<td>19 (26.47%)</td>
<td>3</td>
<td>19 (26.47%)</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>19 (26.47%)</td>
<td>3</td>
<td>19 (26.47%)</td>
</tr>
</tbody>
</table>

attack more often to make up for the high cost paid.\textsuperscript{35} Under this view, one could think that subjects might use high precisions to signal to their opponents that they are willing to set lower thresholds and thus make the most out of the high cost paid.

In order to understand whether or not this is a plausible explanation for our results, we ran extra sessions with exogenous high (level 1) and low (level 6) levels of precision (as in the Control treatment) to isolate the active role of precision choice on threshold formation. We ran sessions where subjects choose actions (as in the Control) and sessions with the strategy method to elicit thresholds directly. We report the estimated thresholds for these sessions in Table A.1 in the appendix.

We find that the thresholds of subjects that are given either medium or low precision exogenously are not statistically different from each other, but when subjects face a high precision they choose a significantly smaller threshold than in any of the other two precisions, just as in the case with endogenous precision choices. This implies that the sunk cost hypothesis (i.e. that subjects fail to understand that precision cost is a sunk cost or that they signal their intentions by choosing high precisions) might not explain our findings. Instead, these extra sessions suggest that subjects seem to respond to more precise information in the opposite direction from what is predicted by the theory in a systematic way, i.e. when high precision is chosen by them or is given to them exogenously. We will get back to this result in our discussion in section 4.

\textsuperscript{35}There is numerous evidence from psychology that suggests that subjects are willing to take riskier choices to make up for sunk costs, see for example Arkes and Blumer (1985). In economics Thaler (1980) uses prospect theory to explain the sunk cost fallacy.
Hypothesis 3 (Equilibrium)  

Subjects coordinate on a precision level of 4 and set a symmetric threshold of 28.31.

In Table 8 we compare, for high, medium, and low precision levels, the mean estimated thresholds (MET) and the random effects logit estimations from the EPA treatment, and the mean reported thresholds (MRT) from the EPS treatment with the equilibrium predictions of the theory. Remember that we define weak convergence to high precision as pairs that converge to precision levels 1 or 2, medium precision as pairs that converge to precision levels 3 or 4, and low precision as pairs that converge to precision levels 5 or 6. Therefore, for each precision level (high, medium, or low) we include the two equilibrium predictions that correspond to each of the precision levels (1 and 2, 3 and 4, or 5 and 6), as well as the risk dominant equilibrium, i.e. the threshold prediction when signal noise converges to zero. Standard deviations are reported in parenthesis.

<table>
<thead>
<tr>
<th></th>
<th>High precision</th>
<th>Medium precision</th>
<th>Low precision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logit (RE) (EPA)</td>
<td>24.99</td>
<td>30.30</td>
<td>74.62</td>
</tr>
<tr>
<td></td>
<td>(7.44)</td>
<td>(12.32)</td>
<td>(21.48)</td>
</tr>
<tr>
<td>MET (EPA)</td>
<td>25.29</td>
<td>27.84</td>
<td>50.65</td>
</tr>
<tr>
<td></td>
<td>(9.27)</td>
<td>(17.65)</td>
<td>(28.65)</td>
</tr>
<tr>
<td>MRT (EPS)</td>
<td>19.77</td>
<td>38.23</td>
<td>32.13</td>
</tr>
<tr>
<td></td>
<td>(1.14)</td>
<td>(13.24)</td>
<td>(12.19)</td>
</tr>
<tr>
<td>Equilibrium prediction ( x^* )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Info 1</td>
<td>35.31</td>
<td>33.88</td>
<td>31.61</td>
</tr>
<tr>
<td>Info 2</td>
<td>33.31</td>
<td>38.31</td>
<td>28.31</td>
</tr>
<tr>
<td>Info 3</td>
<td></td>
<td></td>
<td>22.82</td>
</tr>
<tr>
<td>Info 4</td>
<td></td>
<td></td>
<td>18.73</td>
</tr>
<tr>
<td>Info 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Info 6</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8: Estimated thresholds and equilibrium predictions

We find that the logit and MET estimates for medium precision are not statistically different from the predictions from the theory for the equilibrium precision level 4. This means that the subjects that coordinate on medium precision levels behave on average in accordance to the unique equilibrium suggested by the theory, unlike those who converge to either a high or low precision.

Table 8 and Figure 5 illustrate exactly how subjects with high and low precisions set thresholds in the opposite direction of what is predicted by the theory. We can see in Figure 5, which plots the theoretical thresholds and the mean estimated thresholds of the EPA treatment for each precision level, that the theory predicts thresholds to increase with higher precisions, while the experimental data suggests the opposite. However, note that the theoretical predictions and the mean estimated thresholds coincide for the equilibrium precision (level 4). This means that subjects who coordinate to choose precision level 4 understand the trade off between precision and the cost associated to it by choosing a medium level of precision, and then apply this information optimally in the speculative attack game by choosing the threshold level that maximizes their expected profits, given that their opponent also chooses a medium level of precision.
3.2.1 Payoff efficiency

We now analyze the payoffs realized in our experiment to establish whether the actions taken by subjects increased their payoffs with respect to the equilibrium actions predicted by the theory and to two other benchmarks of surplus extraction. In particular, Table 9 compares the average realized payoffs of subjects for the treatments with endogenous precision choices (EPA and EPS, separated by precision choices) to three benchmarks. The first one is the average expected payoffs that would have arisen if subjects had followed the equilibrium strategy for each realization of the state $\theta$ observed in the different sessions of the experiment. The second benchmark illustrates a constrained efficiency situation where agents truthfully reveal their signals and jointly choose actions to extract the maximum surplus, for each realization of $\theta$ observed in the experiment. This means that subjects would still face fundamental uncertainty and would have to purchase precisions for their signals to deal with it, but they would not face strategic uncertainty (it is as if a planner could choose actions for both pair members to extract the maximum surplus). In this case, we find that the optimal precision choice would be 6. The payoffs we report for these two first benchmarks are built using expected values of signals and not the observed signal realizations, because for each benchmark a specific precision is assumed to be chosen (levels 4 and 6, respectively) and each precision gives rise to a different distribution of signals. Therefore, to calculate the payoffs corresponding to these benchmarks we only take into account the realized values of $\theta$ and disregard the signals observed because they correspond to different distributions.

Finally, the third benchmark corresponds to the average payoffs that would have arisen if subjects had chosen a “first-best” action under complete information, i.e. if they had attacked whenever they could get a positive payoff ($\theta > 18$). This first-best definition is similar to the setup where a social planner can observe the realizations of $\theta$ and prescribe the actions that would

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Notice that we cannot calculate hypothetical payoffs by taking the realized signals and imposing the equilibrium threshold for subjects who do not choose the equilibrium precision because each precision leads to a different realization of signals. Therefore, our equilibrium benchmark is the expected payoff that would arise for each observed $\theta$ if both players were to draw signals with the equilibrium precision and set the equilibrium threshold.
maximize the payoffs of both agents without informational constraints. Therefore, signals are not taken into account for this last benchmark since it is the realization of each \( \theta \) that determines the actions, so we assume that subjects do not pay a cost to improve the precision of their signals. Standard deviations are reported in parenthesis.

<table>
<thead>
<tr>
<th>Treatment</th>
<th>High precision</th>
<th>Medium precision</th>
<th>Low precision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Realized payoffs</td>
<td>32.94</td>
<td>38.76</td>
<td>26.16</td>
</tr>
<tr>
<td></td>
<td>(7.25)</td>
<td>(7.37)</td>
<td>(8.61)</td>
</tr>
<tr>
<td>Expected equilibrium payoffs</td>
<td>31.06***</td>
<td>39.40</td>
<td>30.83***</td>
</tr>
<tr>
<td></td>
<td>(6.65)</td>
<td>(0.84)</td>
<td>(6.66)</td>
</tr>
<tr>
<td>Expected constrained efficient payoffs</td>
<td>31.21***</td>
<td>39.76**</td>
<td>30.97***</td>
</tr>
<tr>
<td></td>
<td>(6.83)</td>
<td>(0.75)</td>
<td>(6.84)</td>
</tr>
<tr>
<td>First-best complete information payoffs</td>
<td>35.25***</td>
<td>43.62***</td>
<td>35.02***</td>
</tr>
<tr>
<td></td>
<td>(6.61)</td>
<td>(0.71)</td>
<td>(6.62)</td>
</tr>
</tbody>
</table>

Statistically different from realized payoffs at the ***1%; **5%; *10% level of significance

Table 9: Average payoffs

Clearly, the highest possible payoffs among all of these benchmarks would come from the first-best with complete information, followed by the constrained efficiency, and finally followed by the equilibrium play. The results in Table 9 show that subjects who choose a high level of precision increase their payoff with respect to the constrained efficiency and the equilibrium play in the EPA treatment. For the EPS treatment we can see that realized and equilibrium payoffs are not statistically different, so we can conclude that subjects who choose a high precision see a Pareto improvement with respect to equilibrium due to the departure from the theory that leads them to set lower thresholds and attack more often.

For subjects with medium and low precisions equilibrium payoffs are higher than realized payoffs. Nevertheless, for the EPA treatment, realized payoffs for subjects with medium precision are at most 13.4% less than the corresponding equilibrium payoffs, whereas realized payoffs for subjects with low precisions are up to 45.9% less than the corresponding equilibrium payoffs. This effectively means that choosing a low precision leads to the highest loss in individual payoffs with respect to equilibrium.

In our game, an increase in the incidence of speculative attacks for subjects who coordinate on a high precision improves individual investors’ payoffs because this high precision allows them to be more aggressive in their decision to attack, while staying confident that the attack will most likely be successful. This analysis sheds some light on the findings of the previous section that show a departure from the theory in the way that precisions determine thresholds. As Table 9 indicates,

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37 Even if we have shown that subjects who choose equilibrium precisions also set equilibrium thresholds, we can still observe differences in realized payoffs with respect to the expected equilibrium payoffs. This is due to the fact that we are comparing payoffs from the realization of \( \theta \) for a small sample to expected payoffs that would arise according to the distribution of a population.
it is beneficial for subjects who coordinate on a high precision to set a lower threshold than what is predicted by the theory, as long as both pair members do not deviate to lower precisions. This observation suggests the emergence of an implicit agreement towards cooperation, since both pair members pay the cost to get the most precise signal in order to coordinate on attacking for a wider range of fundamentals. In theory, the highest precision level is not the equilibrium precision because there is an individual incentive to deviate and save some of the cost, while still getting the benefit of having a very well informed pair member that will make more accurate decisions. Nevertheless, by choosing the highest precision both pair members get more accurate signals, thus reducing the probability of making the mistake of attacking for bad fundamentals. Clearly, such a cooperative agreement results in a higher payoff than the one predicted in equilibrium.

3.2.2 Welfare

Even though individual investors extract a higher surplus by choosing a higher precision, it is important to keep in mind that for the specific context of our model speculative attacks can be detrimental for society, as has been shown by the numerous episodes of currency crises in the past two decades. This means that an increase in individual investor’s welfare are not necessarily beneficial to other members of society (non-investors). If one were to model society’s welfare in this context, we could think that the interests of society are aligned to those of the central bank. In this sense, for very bad states of the economy ($\theta \geq 100$) it is not optimal for the central bank to defend the currency, so a devaluation is the best outcome. For very good states of the economy ($\theta < 0$) the central bank always defends the currency, so a devaluation never takes place. However, when devaluations are driven by self-fulfilling beliefs (coordinated attacks for $\theta \in [0, 100]$) they can be detrimental to society because, in principle, the central bank would have been able to defend the currency and avoid the instability caused by the devaluation if an attack had not occurred. If one were to analyze social welfare in this context, a measure could be constructed with our data by looking at the number of successful attacks as a fraction of total realizations of $\theta \in [0, 100]$. This would serve as an index of society’s welfare loss (the higher the number of coordinated attacks, the higher the loss in welfare for society). Tables 6 and 7 report these indices for treatments EPA and EPS, respectively, for each level of precision. Looking at the indices for each treatment, we can see that there is a higher welfare loss for society when investors coordinate on a higher precision and that the smallest loss of welfare for society arises when investors choose lower precisions, since a high precision leads to a higher rate of successful attacks. This can be seen by comparing the rate of successful attacks for the EPA treatment to be 62.07% for high precision, 51.3% for medium precision, and 14.7% for low precision. Likewise, for the EPS treatment we observe the rate of successful attacks to be 91.89% for high precision, as opposed to 66.44% and 62.34% for medium and low precisions, respectively. By looking at these numbers it is evident that the interests of individual investors and the rest of society are not aligned in this model, so that when speculators coordinate on a high precision they improve their individual welfare by extracting
a higher surplus, but this translates into a reduction of social welfare.\textsuperscript{38} This illustrates the trade-off between investor’s and society’s welfare in episodes of high speculation and endogenous precision choices.

3.2.3 Additional results

So far we have analyzed the data set with respect to the predictions of the theoretical model. We now look at some additional results that focus on the stability of thresholds, both individually and at the pair level. In particular, we find that variations in individual thresholds converge to zero for pairs that converge to high levels of precision, and that subjects in pairs that converge to high precision levels also converge in their thresholds.

We analyze the convergence of reported thresholds from the EPS treatment across rounds, for pairs that converge to high, medium, and low levels of precision. First, we analyze the evolution of individual thresholds. That is, we take the average period-to-period differences (in absolute value) of the individual reported thresholds in each of the last 25 periods of the experiment, for subjects in pairs that coordinate on high, medium, and low precision. We calculate, for each subject, the difference in absolute value between the threshold chosen in one period with respect to the previous period, and then we take the average of these differences across subjects from pairs that coordinate on each precision level. This is portrayed in Figure 6. In this sense, the vertical bar at period $t$ illustrates how much, on average, a subject changed the value of his own threshold in period $t$ with respect to the threshold he reported in period $t - 1$. We find that pairs that coordinate on high precision levels converge individually in their thresholds, since the mean difference is consistently zero, except for the last round. For medium precision levels the mean difference is less than 7 in all periods, and for low precision levels it is less than 7 in all but one period. Therefore, convergence in individual thresholds is less stark as we move towards less precise levels of information.

![Figure 6: Convergence of individual thresholds, EPS treatment](image)

Given this behavior, we also plot the difference in absolute value between the reported thresholds

\textsuperscript{38}It is interesting to note that our results can also be applied to different contexts, such as an investment decision or a social revolt. In these cases the interests of society would be aligned to the interests of the agents in the model. For example, a successful investment would be beneficial to both individual investors and society.
of the members in each pair, for each precision level. In Figure 7 we see, for each period and each precision level, the average difference in absolute value between the reported thresholds of both members of each pair, i.e. we observe by how much subjects coordinate on the same threshold with their pair member. Recall that subjects do not receive feedback about the threshold reported by their pair member, but only observe whether action $A$ (attack) was successful or not. We can see that subjects from pairs that converge to high precision levels seem to coordinate also on their thresholds, and that this is less clear as we move towards less precise levels of information. Figure 7 suggests thus that convergence to a unique threshold is more likely as subjects coordinate on more precise information.

![Graphs showing convergence of thresholds within a pair, EPS treatment](image)

**Figure 7**: Convergence of thresholds within a pair, EPS treatment

## 4 Discussion

We have analyzed the behavior of subjects in an experiment of speculative attacks with costly information acquisition, modeled in section 2. Possibly the most striking result is that, opposite to the predictions of the theory, subjects who choose and coordinate on the highest precision set lower thresholds than those subjects who coordinate on lower precisions. This result prevails even when subjects are given exogenously high and low precisions, which suggests that this observation is independent of the cost of acquiring information.

As presented in section 2.1.1 and consistent with the previous well known result of global games (see Carlsson and van Damme, 1993), the theory predicts that in the limit, as the noise in private signals vanishes, the unique equilibrium of the game will coincide with the risk dominant equilibrium of the underlying complete information game. For the parameters used in the experiment, this risk dominant equilibrium corresponds to a threshold of 36. Our model predicts that the six precision levels from which subjects have to choose in our experiment give rise to thresholds in a similar fashion, i.e. higher precisions correspond to higher thresholds that eventually converge to 36. However, we observe that as subjects in our experiment converge to choosing higher precisions, they set thresholds that are further away from risk dominance and closer to efficiency. As long as both pair members choose a high precision, setting a low threshold, clearly, is payoff improving,
which gives them incentives to continue with this strategy (see table 9). We find that this result is robust to exogenously endowing subjects with a high and low precision, which suggests that this is not an effect due to the cost of information.

This result is also closely related to the results of HNO04 when they study experimentally a global game without costly information acquisition in the treatment with complete (perfectly precise) information. In particular, when subjects are given the value of the true state and have to decide whether to attack or not, they use threshold strategies, which is not consistent with the theoretical prediction of multiplicity of equilibria. Moreover, the threshold used by subjects in the complete information treatment is lower than in their treatment with noisy private signals, allowing them to coordinate more often, i.e. under complete information they set thresholds that are closer to efficiency than to risk dominance. Therefore, it seems that as subjects approach common knowledge they process information in a different way than when they face a high noise in their signals.

In order to reconcile the observations derived from our experiment with the theoretical reasoning underlying the game, one possible avenue of investigation is to think about the way in which a higher precision of signals influences the reasoning that leads to the formation of thresholds. Our experimental results not only show that higher precisions lead to lower thresholds, but they also show that the evolution of individual thresholds and the coordination of thresholds within pairs over time is much more stable for pairs that converge to a high precision level. In our model we assume that agents are rational in their belief formation process and this leads them to choose actions as predicted by the theory. However, this might not be the way in which subjects in the experiment reason when making a decision.\footnote{One might think that our results reflect the fact that subjects could be relying less on the prior than what is assumed by the theory, therefore putting more weight on the private signal. However, doing this is equivalent to behaving as if the precision of the private signal was higher than the actual precision chosen, in which case the theory would predict an even higher threshold. Thus, such “overconfidence” cannot explain our results.}

Different models of cognition have given rise to several explanations to observed departures from the theoretical predictions in strategic settings, such as level-$k$ and cognitive hierarchy (see Nagel, 1995, Costa-Gomes and Crawford, 2006, to name a few). These models suggest that in experimental settings subjects might not be able to perform the fixed point reasoning required in equilibrium, i.e. they do not form higher order beliefs in the way that is predicted by the theory. Instead, they might exhibit a bounded depth of reasoning and they best respond to their beliefs about the other agent’s bounded depth of reasoning. In particular, according to these models, if an agent’s depth of reasoning has bound $k$ it means that he can iterate the best response correspondence at most $k$ times and he believes that his opponent’s level of reasoning is at most $k - 1$. In a global games context (without costly information acquisition), Kneeland (2012) develops a theoretical model of level-$k$ to unify experimental evidence about the role of private and public information on thresholds, following the results of HNO04. It is shown that even under complete information, if agents follow a level-$k$ type of reasoning, threshold behavior is rationalizable, which explains the
results of HNO04 for their complete information treatment.

In our model, if one were to think of a similar explanation it would require to model different “types” of levels of reasoning for each precision, i.e. instead of having a generic level-$k$ we would have to talk about a level-$k$ ($\sigma$) model where the specific characteristics of each level would depend on the level of precision acquired. For example, one could think that a given level-$k$ ($\sigma_H$) for a high precision $\sigma_H$ would correspond to deeper levels of reasoning than the level-$k$ ($\sigma_L$), for a low precision $\sigma_L$. In this sense, the speed of convergence to common knowledge (as $k \to \infty$) would depend on the precision level chosen.

However, the predictions of level-$k$ models highly depend on the assumptions about the behavior of level-0 agents (non-strategic types) and on the assumptions about the beliefs of the level of one’s opponent (e.g. being exactly level $k-1$ or being drawn from a distribution of types lower or equal to level $k-1$). For example, in the global games model of Kneeland (2012) thresholds might be increasing or decreasing on the bound $k$, depending on the assumptions made for level-0 types.

Alternatively, we could also think about other belief-based models that would seem more appropriate to global games to explain the departure of our data from the theoretical model, such as disentangling the way in which beliefs are formed about the reduction in fundamental and strategic uncertainty due to a higher precision. While a systematic investigation of these issues is beyond the scope of this paper, we deem this endeavour as an exciting avenue for future research.

5 Conclusion

We have characterized equilibrium selection in a global games model with a discrete number of agents and an endogenous information structure. We show theoretically that when agents choose privately the precision of their signal, uniqueness of equilibrium in the coordination game can be guaranteed as long as the precision of the prior is diffuse enough with respect to the precisions of the private signals, which is consistent with previous results in the case of symmetric noise distributions. For most ranges of parameters, strategic complementarities in information acquisition arise, leading to symmetric precision choices in equilibrium.

The experimental results of our model give us further insight about the behavior during speculative attacks in which subjects are allowed to choose the precision of the signals they observe, at a cost. We find that over 30% of the subjects behave as is predicted in equilibrium since they understand the trade off between precision and cost, and thus choose a medium level of precision and coordinate on this level within their group. These subjects seem to apply correctly the information they acquire on their decision in the speculative attack by using a threshold that coincides with the unique equilibrium predicted by the theory.

However, we also find subjects who acquire high and low precision levels. In our model, the theory suggests that the incidence of speculative attacks should decrease as agents coordinate on higher levels of precision, i.e. that better informed agents should be more careful and attack when
they have more certainty of a successful outcome. This theoretical prediction suggests that having more informed agents would decrease the incidence and success of speculative attacks. Nevertheless, our experimental results show the opposite. In our study, subjects that acquire more precise information attack significantly more often than those who acquire less precise information. As a result, the overall success of attacks increases for better informed market participants: they achieve higher individual payoffs by coordinating on successful attacks for a wider range of fundamentals. One possible interpretation for this departure from the theory could be that subjects fail to take the cost of precision as a sunk cost, and thus attack more often when they invest in better information. The results from extra sessions that address this hypothesis suggest that subjects attack more often when faced with a higher precision, even if this precision is given to them exogenously at no cost. This leads us to believe that subjects who coordinate on high levels of precision might actually be forming an implicit agreement by which they decide to bear a high cost in order to get a very accurate signal that would minimize the probability of attacking in bad states. It is shown that this cooperative agreement actually leads to higher average payoffs than the expected payoffs that would have arisen if both pair members had behaved according to the equilibrium.

Our experimental results show that having better quality of information leads to a significant increase in individual investor’s welfare since it allows them to extract higher payoffs than equilibrium play. This feature of our experimental analysis has important policy implications in the context of a speculative attack. In this particular context, even if this behavior improves the welfare of individuals, an increased incidence and success of speculative attacks is detrimental for society, as proven by the numerous episodes of currency crises in the last two decades. In this sense, the results of the experiment illustrate the trade off between individual and social welfare in episodes of speculative attack when agents can improve their private information at a cost. This finding sheds light on the importance for policy making of taking into account the role that private information acquisition plays on speculative attack outcomes. Nevertheless, our results can also be applied to different contexts (e.g. an investment decision or a social revolt), in which case different qualitative implications on welfare might apply.
References


[34] Strzalecki, T., “Depth of reasoning and higher order beliefs” mimeo.


### A Appendix 1 - Tables and figures

<table>
<thead>
<tr>
<th></th>
<th>High precision</th>
<th>Medium precision</th>
<th>Low precision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logit (RE) (EPA)</td>
<td>27.61</td>
<td>40.16</td>
<td>35.79</td>
</tr>
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<td></td>
<td>(5.86)</td>
<td>(9.13)</td>
<td>(9.00)</td>
</tr>
<tr>
<td>MET (EPA)</td>
<td>27.42</td>
<td>40.37</td>
<td>36.23</td>
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<tr>
<td></td>
<td>(19.16)</td>
<td>(18.77)</td>
<td>(23.36)</td>
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<tr>
<td>MRT (EPS)</td>
<td>33.58</td>
<td>37.34</td>
<td>37.83</td>
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<tr>
<td></td>
<td>(16.75)</td>
<td>(18.98)</td>
<td>(15.47)</td>
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<td>Equilibrium prediction $x^*$</td>
<td>35.31</td>
<td>28.31</td>
<td>18.73</td>
</tr>
<tr>
<td>Risk dominant equilibrium</td>
<td>36</td>
<td>36</td>
<td>36</td>
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Table A.1: Estimated thresholds and equilibrium predictions for treatments without information acquisition

<table>
<thead>
<tr>
<th>Variable</th>
<th>Attack {0,1}</th>
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<tbody>
<tr>
<td>Precision 1*signal</td>
<td>0.178***</td>
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<tr>
<td></td>
<td>(0.03)</td>
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<tr>
<td>Precision 2*signal</td>
<td>0.191***</td>
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<td></td>
<td>(0.05)</td>
</tr>
<tr>
<td>Precision 3*signal</td>
<td>0.096***</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
</tr>
<tr>
<td>Precision 4*signal</td>
<td>0.088***</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
</tr>
<tr>
<td>Precision 5*signal</td>
<td>0.062***</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
</tr>
<tr>
<td>Precision 6*signal</td>
<td>0.057***</td>
</tr>
<tr>
<td></td>
<td>(0.01)</td>
</tr>
<tr>
<td>Constant</td>
<td>-3.559***</td>
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<tr>
<td></td>
<td>(0.46)</td>
</tr>
<tr>
<td>N</td>
<td>1000</td>
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</tbody>
</table>

Clustered (by subject) standard errors in parentheses; * significant at 10%; ** significant at 5%; *** significant at 1%

Table A.2: Attack as a function of precision, EPA treatment
<table>
<thead>
<tr>
<th>Variable</th>
<th>Reported threshold</th>
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</thead>
<tbody>
<tr>
<td>Precision 2</td>
<td>6.25</td>
</tr>
<tr>
<td></td>
<td>(4.00)</td>
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<tr>
<td>Precision 3</td>
<td>10.65***</td>
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<td></td>
<td>(3.48)</td>
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<td>Precision 4</td>
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<td></td>
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<td>Precision 5</td>
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<td></td>
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<td>Precision 6</td>
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<td>Constant</td>
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<td>(5.14)</td>
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<tr>
<td>N</td>
<td>1100</td>
</tr>
</tbody>
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Clustered (by subject) standard errors in parentheses; * significant at 10%; ** significant at 5%; *** significant at 1%

Table A.3: Reported threshold as a function of precision, EPS treatment

<table>
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<th>Choice of pair member 2</th>
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<tr>
<td></td>
<td>Prec 1</td>
</tr>
<tr>
<td>Prec 1</td>
<td>7.24%</td>
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<tr>
<td>Prec 2</td>
<td>0.57%</td>
</tr>
<tr>
<td>Prec 3</td>
<td>5.43%</td>
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<td>Prec 4</td>
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<tr>
<td>Prec 5</td>
<td>0.10%</td>
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<tr>
<td>Prec 6</td>
<td></td>
</tr>
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Table A.4: Combination of precision choices, EPA and EPS treatments
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<thead>
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<th>Variable</th>
<th>Success of attack ({0,1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>High precision*signal</td>
<td>0.194***</td>
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<tr>
<td></td>
<td>(0.061)</td>
</tr>
<tr>
<td>Medium precision*signal</td>
<td>0.119***</td>
</tr>
<tr>
<td></td>
<td>(0.022)</td>
</tr>
<tr>
<td>Low precision*signal</td>
<td>0.044***</td>
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<tr>
<td></td>
<td>(0.01)</td>
</tr>
<tr>
<td>Constant</td>
<td>-4.534***</td>
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<td></td>
<td>(1.052)</td>
</tr>
<tr>
<td>N</td>
<td>500</td>
</tr>
</tbody>
</table>

Clustered (by pairs) standard errors in parentheses; * significant at 10%; ** significant at 5%; *** significant at 1%

Table A.5: Success of attack as a function of precision, EPA treatment
Appendix 2 - Relation to monotone supermodular games

We first prove that the game specified in the second stage of the game belongs to the class of monotone supermodular games as defined by Vives and van Zandt (2007). Following their notation, define $N = \{1, 2\}$ as the set of players indexed by $i$. Let the type space of player $i$ be a measurable space $(\Omega_i, \mathcal{F}_i)$. Denote by $(\Omega_0, \mathcal{F}_0)$ a state space that is capturing the residual uncertainty. We let $\mathcal{F}_{-i}$ be the product $\sigma$-algebra $\otimes_{k \neq i} \mathcal{F}_k$. Let player $i$’s interim beliefs be given by a function $p_i : \Omega_i \rightarrow M_{-i}$, where $M_{-i}$ is the set of probability measures on $(\Omega_{-i}, \mathcal{F}_{-i})$. Finally, let $A_i = \{0, 1\}$ be the action set of player $i$, $A$ be the set of action profiles and $u_i : A \times \Omega \rightarrow \mathbb{R}$ be the payoff function.

**Definition 3** A game belongs to the class of monotone supermodular games if

1. The utility function $u_i(a_i, a_{-i}, \omega)$ is supermodular in own actions, $a_i$, and has increasing differences in $(a_i, a_{-i})$ and in $(a_i, \omega)$.

2. The belief map $p_i : \Omega_i \rightarrow M_{-i}$ is increasing with respect to a partial order on $M_{-i}$ of first-order stochastic dominance.

In our case, the type space is defined as follows: $\Omega_0 = \mathbb{R}$, $\Omega_i = \mathbb{R}$ for $i = 1, 2$, where $\varpi_0 = \theta$, $\varpi_i = \vartheta_i$, $\varpi_j = \vartheta_j$ and $\mathcal{F}_i = B(\mathbb{R})$, a Borel $\sigma$-algebra on $\mathbb{R}$, $i = 0, 1, 2$. The set of probability measures $M_{-i}$ is simply the set of joint normal probability distributions over $(\Omega_{-i}, \mathcal{F}_{-i})$ conditional on the realization of $\varpi_i$. The belief mapping $p_i : \Omega_i \rightarrow M_{-i}$ maps $\vartheta_i$ into the posterior distribution of $(\theta, \vartheta_j)$ using Bayes’ rule. Finally, the underlying utility function for agent $i$ is given by

$$u(a_i, a_j, \theta) = 1_{\{a_i = 1\}} \left[ \theta \left[ 1_{\{\theta \in [\vartheta_i, \vartheta_j]\}} 1_{\{a_j = 1\}} + 1_{\{\theta > \vartheta_j\}} \right] - T \right]$$

and the expected utility of agent $i$ is:

$$v_i(a_i, a_j, \theta) = 1_{\{a_i = 1\}} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta \left[ 1_{\{\theta \in [\vartheta_i, \vartheta_j]\}} 1_{\{s_j(\vartheta_i) = 1\}} + 1_{\{\theta > \vartheta_j\}} \right] \frac{1}{\sigma_i} \frac{1}{\sigma_j} f \left( \frac{\theta - \vartheta_i}{\sigma_i} \right) f \left( \frac{\sigma_j (\vartheta_j - \vartheta_j)}{\sigma_j^2} \right) d\vartheta_j d\theta \right] - T$$

where $s_j : \Omega_j \rightarrow A_j$ is a measurable strategy of player $j$.

The fact that global games belong to the class of monotone supermodular games was noted first by Vives (2005) and Vives and Van Zandt (2007). The following lemma shows that our game in the second stage satisfies the above definition of monotone supermodular games.

---

40 In a global games setting, we usually interpret $(\Omega_i, \mathcal{F}_i)$ to be the space of possible signals that agent $i$ receives, while $(\Omega_0, \mathcal{F}_0)$ corresponds to the measurable space of the underlying parameter of the game.
Lemma 3 The game specified in the second stage of the speculative attack game with information acquisition belongs to the class of monotone supermodular games.

Proof. Consider the ex-ante payoff function of agent $i$:

$$
1_{\{a_i=1\}} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta \left[ 1_{\{\theta \in [\hat{\theta}_j, \tilde{\theta}_j]\}} \left\{ 1_{\{\theta_i > \hat{\theta}_j\}} + 1_{\{\theta_i = \hat{\theta}_j\}} \right\} \right] \frac{1}{\sigma_i \sigma_j} f \left( \frac{\theta - \hat{\theta}_i}{\sigma_i} \right) f \left( \frac{\sigma_j (\hat{\theta}_j - \tilde{\theta}_j)}{\sigma_j^2} \right) d\theta_j d\theta - T \right)
$$

We see that the underlying utility function is simply

$$
u(a_i, a_j, \theta) = 1_{\{a_i=1\}} \left[ \theta \left( 1_{\{\theta \in [\hat{\theta}_j, \tilde{\theta}_j]\}} \left\{ 1_{\{a_j = 1\}} + 1_{\{a_j = 0\}} \right\} - T \right) \right]
$$

By example 2.6.2 in Topkis (1998) we conclude that this function is supermodular. To show that $u$ has increasing differences in $(a_i, a_j)$ we need to show that

$$
u(1,0,\theta) - \nu(0,0,\theta) \leq \nu(1,1,\theta) - \nu(0,1,\theta)
$$

But this follows immediately from the definition of $u$, namely

$$
u(1,0,\theta) - \nu(0,0,\theta) = \theta 1_{\{\theta \in [\hat{\theta}_j, \tilde{\theta}_j]\}} - T \text{ and } \nu(1,1,\theta) - \nu(0,1,\theta) = \theta 1_{\{\theta \in [\hat{\theta}_j, \tilde{\theta}_j]\}} - T
$$

so that

$$
u(1,0,\theta) - \nu(0,0,\theta) \leq \nu(1,1,\theta) - \nu(0,1,\theta)
$$

The fact that $u$ has increasing differences in $(a_i, \theta)$ follows immediately from the above expressions. Increasing $\theta$ always weakly increases the difference between attacking and not attacking, regardless of action of the other player.

To show that the belief mapping is increasing with respect to first-order stochastic dominance it is enough to show that $(\theta, \hat{\theta}_1, \hat{\theta}_2)$ are affiliated. Let $\theta'' > \theta', \hat{\theta}_1'' > \hat{\theta}_1', \hat{\theta}_2'' > \hat{\theta}_2'$ and denote by $f$ the joint PDF of $(\theta, \hat{\theta}_1, \hat{\theta}_2)$. We need to show that $f$ is log-supermodular (see Milgrom and Weber, 1982). Without loss of generality it is enough to show that

$$f(\theta'', \hat{\theta}_1', \hat{\theta}_2') f(\theta', \hat{\theta}_1', \hat{\theta}_2') \geq f(\theta'', \hat{\theta}_1', \hat{\theta}_2') f(\theta', \hat{\theta}_1', \hat{\theta}_2') f(\theta'', \hat{\theta}_1, \hat{\theta}_2)$$

Note that

$$f(\theta'', \hat{\theta}_1, \hat{\theta}_2) f(\theta', \hat{\theta}_1, \hat{\theta}_2) = f(\hat{\theta}_1' | \theta'') f(\hat{\theta}_2' | \theta'') f(\theta'') f(\hat{\theta}_1' | \theta') f(\hat{\theta}_2' | \theta') f(\theta')$$

and similarly for $f(\theta', \hat{\theta}_1', \hat{\theta}_2')$.
Therefore, \( \hat{\theta}_1, \hat{\theta}_2 \) are conditionally independent. Hence, it is enough to show that

\[
f(\hat{\theta}_1''|\theta'') f(\hat{\theta}_2'|\theta') \geq f(\hat{\theta}_1''|\theta') f(\hat{\theta}_2'|\theta')
\]

or that \( f(\hat{\theta}_2|\theta) \) has an increasing marginal likelihood ratio. Since \( f(\hat{\theta}_2|\theta) \sim N \left( \frac{\sigma_2^2 \theta + \sigma_2^2 \mu_2}{\sigma_2^2 \sigma_2}, \frac{1}{\sigma_2^2} \right) \), it follows that \( f(\hat{\theta}_2|\theta) \) has a monotone likelihood ratio and hence \( f(\theta'', \hat{\theta}_1'', \hat{\theta}_2'') f(\theta, \hat{\theta}_1', \hat{\theta}_2') \geq f(\theta', \hat{\theta}_1', \hat{\theta}_2') f(\theta'', \hat{\theta}_1'', \hat{\theta}_2'') \).

Therefore, \( (\theta, \hat{\theta}_1, \hat{\theta}_2) \) are affiliated and the belief mapping is increasing with respect to first-order stochastic dominance. □

In order to prove our main result in this section, we prove the following lemma.

**Lemma 4** The following are true about the utility function \( u(.; ; .) \):

1. \( u \) is bounded from below;

2. \( u \) is integrable with respect to \( \mu_F \), a Baire measure implied by \( F \) - the distribution function of \( \theta \);

3. There exists a function \( h \), integrable w.r.t. \( \mu_F \), such that \( |u| < h \).

**Proof.** It is easy to see that \( u \) is bounded from below by \(-T\). To prove (3) note that

\[
\int |u| \, d\mu_F = \int \left| 1'_{\{a_i=1\}} \left[ \theta \left( 1'_{\{\theta \in [\underline{\theta}, \bar{\theta}]\}} 1'_{\{a_j=1\}} + 1'_{\{\theta > \bar{\theta}\}} \right) - T \right] \right| \, d\mu_F
\]

\[
\leq \int |\theta - T| \, d\mu_F \leq \int |\theta| \, d\mu_F + \int |-T| \, d\mu_F < \infty
\]

where \( \int |\theta| \, d\mu_F < \infty \) since \( \int \theta \, d\mu_F = \mu < \infty \). This shows that \( |u| \) is integrable and since \( u \) is measurable it follows that \( u \) is also integrable (hence (2) is true).

Finally let \( h(\theta) = |\theta| + |T| \). Then \( |u| < h \) for all \( a_i, a_j \) and \( \theta \) since

\[
\left| 1'_{\{a_i=1\}} \left[ \theta \left( 1'_{\{\theta \in [\underline{\theta}, \bar{\theta}]\}} 1'_{\{a_j=1\}} + 1'_{\{\theta > \bar{\theta}\}} \right) - T \right] \right|
\leq \left| 1'_{\{a_i=1\}} \left[ \theta \left( 1'_{\{\theta \in [\underline{\theta}, \bar{\theta}]\}} 1'_{\{a_j=1\}} + 1'_{\{\theta > \bar{\theta}\}} \right) \right] \right| + |T|
\leq |\theta| + |T|
\]

We argued above that \( \int |\theta| \, d\mu_F + \int |T| \, d\mu_F < \infty \) and hence (3) holds. □

We proceed now by extending the following result from van Zandt and Vives (2007) for unbounded utility functions.

**Proposition 1** Assume that a game \( \Gamma \) belongs to the class of monotone supermodular games. Furthermore, assume that the following hold:

1. Each \( \Omega_k \) is endowed with a partial order,
2. \( A_i \) is a complete lattice,

3. \( \forall a_i \in A_i, \ u_i(a_i, \cdot) : \Omega \to \mathbb{R} \) is measurable,

4. \( u_i \) is bounded.

5. \( u_i \) is continuous in \( a_i \)\(^{41}\)

Then, there exist a least and a greatest Bayesian Nash Equilibrium of the game \( \Gamma \) and each one of them is in monotone strategies.

**Proof.** See van Zandt and Vives (2007). ■

Note that in our setup the underlying utility function \( u(\cdot) \) is unbounded, namely as \( \theta \to \infty \), \( u(\theta) \to \infty \). Thus, we cannot apply the above proposition to our problem directly. In the following corollary, we show that, as long as conditions (1) – (3) of Proposition 1 hold, \( u \) is bounded from below, and the distribution of \( \theta \) satisfies assumptions A1 – A5, we can still reach the conclusion of the Proposition of Vives and van Zandt (2007) under some further assumptions. The strategy is to use the Dominated Convergence Theorem in the proof of existence of a greatest and a least Bayesian Nash Equilibria instead of the Bounded Convergence Theorem and to use the fact that \( u \) is bounded from below to show that the best reply mapping is well defined.

**Corollary 1** Assume that the game \( \Gamma \) belongs to the class of monotone supermodular games. Furthermore, assume that assumptions (1) – (3) of Proposition 1 are satisfied, \( u \) is bounded from below, and let \( v_i \) satisfy the following assumption:

\[(1C) \text{ There exists a measurable function } h \text{ s.t. } h \text{ is integrable with respect to } p(t_{-i}|t_i) \text{ for all } t_i, \text{ and all } t_{-i} \text{ and } |u| < h. \]

Then there exists a least and a greatest Bayesian Nash Equilibrium of the game \( \Gamma \) and each one of them is in monotone strategies.

**Proof.** We prove this corollary in two steps. First, assuming that the greatest best reply mapping \( \overline{\beta}_i \) is well-defined, increasing, and monotone, we show that the greatest Bayesian Nash Equilibrium (BNE) exists. Then, we show that under the above conditions \( \overline{\beta}_i \) is indeed well-defined, increasing, and monotone.

**Step 1:** Suppose that \( \overline{\beta}_i \) is well-defined, increasing and monotone and \( u \) satisfies assumption (1C). Then we can repeat the argument of van Zandt and Vives (2007) to show that there is a greatest and least BNE in monotone strategies. We can relax the boundedness assumption, since under assumption (1C) we can interchange the order of limit and integration due to the Lebesgue Dominated Convergence Theorem. Since this is the only step in that proof that requires boundedness of the utility function, we are done.

**Step 2:** Here we need to establish that \( \overline{\beta}_i \) is well-defined and increasing. Then, the monotonicity of \( \overline{\beta}_i \) will follow from Proposition 11 in van Zandt and Vives (2007). The tricky part of this step

\[^{41}\text{When } A_i \text{ is finite this condition is vacuous.}\]
is to show that $\beta_i$ is well-defined, and more precisely that it is a measurable function of $\varpi_i$. For this purpose we extend the proof of Lemma 9 in Ely and Peski (2006) to cover general measurable functions. The rest of argument follows from van Zandt (2010).

Fix $a_i \in A_i$ and define $U_i(\varpi_i, \varpi_j) := u_i(a_i, s_j(\varpi_j), \varpi_i, \varpi_{-i})$. We need to show that a function $\pi_i : A_i \times \Omega_i \to \mathbb{R}$ defined by

$$\pi_i(a_i, \varpi_i) = \int_{\Omega_{-i}} U_i(\varpi_i, \varpi_{-i})dp(\varpi_{-i} | \varpi_i)$$

is measurable in $t_i$. To prove this we use a result by Ely and Peski (2006):

**Lemma (Ely and Peski)** Let $A$ and $B$ be measurable sets and $g : A \times B \to [0, 1]$ be a jointly measurable map. If $m : A \to \Delta B$ (where $\Delta B$ denotes the set of probability measures defined on $B$) is measurable, then the map $L^g : A \to \mathbb{R}$ defined as $L^g(a) = \int g(a, \cdot)dm(a)$ is measurable.

Note however, that the proof of their lemma is essentially unchanged if we allow $g : A \times B \to \mathbb{R}$, as long as $g$ is measurable and bounded from below. In this case, there exists an increasing sequence of simple functions $g_n$ such that $g_n \uparrow g$, so by the extended Monotone Convergence Theorem (Ash, 2000) we have $\int g_n dv \to \int gdv$ for a measure $v$ defined on $A \times B$. Hence we conclude that $\pi_i : A_i \times \Omega_i \to \mathbb{R}$ is a measurable function of $\varpi_i$. The rest of the proof follows directly from van Zandt (2010) section 7.5. Monotonicity of $\beta_i$ follows from Proposition 11 in van Zandt and Vives (2007).
C Appendix 3 - Proofs

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In order to facilitate notation when solving the model, we rewrite the condition for threshold strategies in terms of the posteriors that agents hold about the fundamental \( \theta \), as in Hellwig (2002). Note that this is straightforward since the posterior about \( \theta \) held by agent \( i \), \( \widehat{\theta_i} \), is a linear, strictly increasing function of the signal he observes, \( x_i \). Therefore, agent \( i \) will attack whenever his posterior belief about \( \theta \), given his signal realization of \( x_i \), is higher than the posterior of \( \theta \) that corresponds to agent \( i \)'s optimal threshold:

\[
a(x_i; \sigma) = \begin{cases} 1 & \text{iff } \widehat{\theta_i} \geq \widehat{\theta_i}^*(\sigma) \\ 0 & \text{iff } \widehat{\theta_i} < \widehat{\theta_i}^*(\sigma) \end{cases}
\]

where \( \widehat{\theta_i}^* = \frac{\mu \sigma_i^2 + x_i^* \sigma_0^2}{\sigma_i^2 + \sigma_0^2} \). In order to write the condition of agent \( j \) in terms of his posterior belief, notice that

\[
\frac{x_j^*-\theta}{\sigma_j} = \frac{\sigma_j (\widehat{\theta_j}^*-\overline{\theta_j})}{\sigma_j^2}
\]

where \( \overline{\theta_j} = \frac{\sigma_j^2 \mu + \sigma_0^2 \mu_0}{\sigma_j^2 + \sigma_0^2} \).

The expected payoff of attacking for agent \( i = 1, 2 \), conditional on observing signal \( x_i \) and given that the other agent follows a threshold strategy with switching point \( \widehat{\theta_j}^* \) is:

\[
v_i(x_i, x_j^*; \sigma) = \frac{1}{\sigma_i} \int_{\overline{\theta_i}}^{\widehat{\theta_i}} \theta f\left( \frac{\theta-\widehat{\theta_i}}{\sigma_i} \right) \left( 1 - F\left( \frac{\sigma_j (\widehat{\theta_j}^*-\overline{\theta_j})}{\sigma_j^2} \right) \right) d\theta + \frac{1}{\sigma_i} \int_{\overline{\theta_i}}^{\infty} \theta f\left( \frac{\theta-\widehat{\theta_i}}{\sigma_i} \right) d\theta - T \tag{4}
\]

This notation is used for all proofs in this appendix.

**Lemma 1**  The payoff for agent \( i \) of attacking, \( v_i(x_i, x_j^*; \sigma) \), is increasing in his own signal \( x_i \), and decreasing in the other agent’s threshold \( x_j^* \), for \( i, j = 1, 2, i \neq j \).

**Proof.**  (1)

Note that \( \widehat{\theta_i} \) is an increasing function of \( x_i \), i.e. \( \frac{\partial \widehat{\theta_i}}{\partial x_i} > 0 \). Thus, it is enough to show that the payoff of attacking is increasing with respect to the posterior mean of \( \theta \), \( \widehat{\theta_i} \).

Taking a partial derivative of (4) wrt \( \widehat{\theta_i} \) yields:

\[
- \int_{\overline{\theta}}^{\widehat{\theta_i}} \frac{1}{\sigma_i^2} f\left( \frac{\theta-\widehat{\theta_i}}{\sigma_i} \right) \left( 1 - F\left( \frac{\sigma_j (\widehat{\theta_j}^*-\overline{\theta_j})}{\sigma_j^2} \right) \right) d\theta - \int_{\widehat{\theta_i}}^{\infty} \frac{1}{\sigma_i^2} f\left( \frac{\theta-\widehat{\theta_i}}{\sigma_i} \right) d\theta
\]
Consider the second term and apply integration by parts. Then,

\[- \int_{\tilde{\theta}}^{\infty} \frac{1}{\sigma_i} f' \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) d\theta = - \left[ \frac{1}{\sigma_i} f \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) \right]_{\tilde{\theta}}^{\infty} + \int_{\tilde{\theta}}^{\infty} \frac{1}{\sigma_i} f \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) d\theta\]

\[= - \left[ \frac{1}{\sigma_i} f \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) \right]_{\tilde{\theta}}^{\infty} + \left( 1 - F \left( \frac{\tilde{\theta}_i}{\sigma_i} \right) \right)\]

Since \( f(\cdot) \) is a normal density, this simplifies to:

\[\frac{\tilde{\theta}}{\sigma_i} f \left( \frac{\tilde{\theta}_i}{\sigma_i} \right) + \left( 1 - F \left( \frac{\tilde{\theta}_i}{\sigma_i} \right) \right) > 0 \tag{5}\]

Now consider the first term. Again apply integration by parts:

\[- \int_{\tilde{\theta}}^{\infty} \frac{1}{\sigma_i} f' \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) \left( 1 - F \left( \frac{\sigma_j (\tilde{\theta}_j^* - \tilde{\theta}_j)}{\sigma_j^2} \right) \right) d\theta =\]

\[= - \left[ \frac{1}{\sigma_i} f \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) \left( 1 - F \left( \frac{\sigma_j (\tilde{\theta}_j^* - \tilde{\theta}_j)}{\sigma_j^2} \right) \right) \right]_{\tilde{\theta}}^{\infty} + \int_{\tilde{\theta}}^{\infty} \frac{1}{\sigma_i} f \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) d\theta + \int_{\tilde{\theta}}^{\infty} \frac{1}{\sigma_i} f \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) \left( 1 - F \left( \frac{\sigma_j (\tilde{\theta}_j^* - \tilde{\theta}_j)}{\sigma_j^2} \right) \right) d\theta \tag{6}\]

Note that the first term of the above expression is of unknown sign while the second and third terms are unambiguously positive.

Putting (5) and (6) together we get:

\[\frac{\tilde{\theta}}{\sigma_i} f \left( \frac{\tilde{\theta}_i}{\sigma_i} \right) + \left( 1 - F \left( \frac{\tilde{\theta}_i}{\sigma_i} \right) \right) - \frac{\tilde{\theta}}{\sigma_i} f \left( \frac{\tilde{\theta}_i}{\sigma_i} \right) \left( 1 - F \left( \frac{\sigma_j (\tilde{\theta}_j^* - \tilde{\theta}_j)}{\sigma_j^2} \right) \right) + \int_{\tilde{\theta}}^{\infty} \frac{1}{\sigma_i} f \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) d\theta + \int_{\tilde{\theta}}^{\infty} \frac{1}{\sigma_i} f \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) \left( 1 - F \left( \frac{\sigma_j (\tilde{\theta}_j^* - \tilde{\theta}_j)}{\sigma_j^2} \right) \right) d\theta > 0\]
where determine \( b \) (these equilibria are the same, i.e. that there is a unique equilibrium in threshold strategies.

As proven in the first section of the appendix, the coordination game belongs to the class of monotone supermodular games and therefore we know that there are a least and a greatest

\[ \begin{align*}
\text{Proof.} \quad & \text{Take a partial derivative of (4) wrt } \tilde{\theta}_j, \text{ which is always negative:} \\
& -\int \frac{\theta - \tilde{\theta}_i}{\sigma_i \sigma_j} f \left( \frac{\theta - \tilde{\theta}_i}{\sigma_i} \right) f \left( \frac{\sigma_j (\tilde{\theta}_j - \tilde{\theta}_j)}{\sigma_j^2} \right) d\theta < 0
\end{align*} \]

Therefore, it holds true for \( x^*_j \) as well. ■

**Theorem 1** There exists a unique, dominance solvable equilibrium of the second stage of the game in which both players use threshold strategies characterized by \((x^*_1, x^*_2)\) if either:

1. \( \frac{\sigma_j}{\sigma_0} < K_i(\theta, \bar{\theta}, \mu_0), i = 1, 2 \) holds, for any pair of \((\sigma_1, \sigma_2)\), or
2. \( \sigma_0 > \bar{\sigma}_0 \), where \( \bar{\sigma}_0 \) is determined by the parameters of the model.

**Proof.** As proven in the first section of the appendix, the coordination game belongs to the class of monotone supermodular games and therefore we know that there are a least and a greatest Bayesian Nash Equilibria in monotone strategies. To prove the theorem, we only need to show that these equilibria are the same, i.e. that there is a unique equilibrium in threshold strategies.

For ease of exposition, we will perform the analysis in terms of thresholds over posterior beliefs, \((\tilde{\theta}_1, \tilde{\theta}_2)\). Uniqueness of these thresholds imply uniqueness of thresholds over signals \((x^*_1, x^*_2)\).

Let, \( s_i(\tilde{\theta}_i^*) \) be a threshold strategy of player \( i \) with switching point \( \tilde{\theta}_i^* \) such that \( s_i(\tilde{\theta}_i^*) = 1 \) (attack) if \( \tilde{\theta}_i \geq \tilde{\theta}_i^* \) and \( s_i(\tilde{\theta}_i^*) = 0 \) (not attack) if \( \tilde{\theta}_i < \tilde{\theta}_i^* \), where \( \tilde{\theta}_i \) is the posterior belief that agent \( i \) holds about \( \theta \) after observing signal \( x_i \), for \( i = 1, 2 \). Then the equilibrium conditions are given by the following equations:

\[ \begin{align*}
v_1(\tilde{\theta}_1^*, \tilde{\theta}_2^*; \sigma) & = \frac{1}{\sigma_1} \int_{\theta_1}^{\bar{\theta}} \theta f \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) \left( 1 - F \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_2^*)}{\sigma_2^2} \right) \right) d\theta + \frac{1}{\sigma_1} \int_{\bar{\theta}}^{\tilde{\theta}_1} \theta f \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) d\theta - T = 0 \\
v_2(\tilde{\theta}_2^*, \tilde{\theta}_1^*; \sigma) & = \frac{1}{\sigma_2} \int_{\theta_2}^{\bar{\theta}} \theta f \left( \frac{\theta - \tilde{\theta}_2^*}{\sigma_2} \right) \left( 1 - F \left( \frac{\sigma_1 (\tilde{\theta}_1 - \tilde{\theta}_1^*)}{\sigma_1^2} \right) \right) d\theta + \frac{1}{\sigma_2} \int_{\bar{\theta}}^{\tilde{\theta}_2} \theta f \left( \frac{\theta - \tilde{\theta}_2^*}{\sigma_2} \right) d\theta - T = 0
\end{align*} \]

(8)

(9)

where \( \tilde{\theta}_i^* = \frac{\sigma_j^2 \mu_{\theta_i} - \sigma_{i, \theta}^2}{\sigma_i^2 + \sigma_{i, \theta}^2}, \bar{\sigma}_i = \sqrt{\frac{\sigma_i^2 \sigma_{i, \theta}^2}{\sigma_i^2 + \sigma_{i, \theta}^2}} \) and \( \tilde{\theta}_i = \frac{\sigma_j^2 \mu_{\theta_i} + \sigma_{i, \theta}^2}{\sigma_i^2 + \sigma_{i, \theta}^2} \) for \( i = 1, 2 \). Note that both equations determine \( \tilde{\theta}_j^* \) in terms of \( \tilde{\theta}_j \). Without loss of generality we analyze the behavior of \( \tilde{\theta}_2^* \) as a function
of $\tilde{\theta}_1^*$ in the $(\tilde{\theta}_1, \tilde{\theta}_2)$ space and rewrite equations (8) and (9) as:

$$
v_1(\tilde{\theta}_1^*, w_1(\tilde{\theta}_1^*; \sigma); \sigma) = 0 \tag{10}
$$

$$
v_2(w_2(\tilde{\theta}_1^*; \sigma), \tilde{\theta}_1^*; \sigma) = 0 \tag{11}
$$

where $\tilde{\theta}_2^* = w_i(\tilde{\theta}_1^*; \sigma)$ for $\tilde{\theta}_2^*$ as defined by the equation that characterized agent $i$'s payoff function, for $i = 1, 2$. Then any $\tilde{\theta}_1$ that solves simultaneously both equations defines an equilibrium threshold for player 1 and the associated threshold for player 2 is simply given by $\tilde{\theta}_2 = w_1(\tilde{\theta}_1^*; \sigma)$.

Consider first equation (10). Define $\tilde{\theta}_1^*$ as the unique solution to the following equation:

$$
\int_{\theta}^{\infty} \frac{1}{\sigma_1} f \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) d\theta - T = 0
$$

Similarly, denote by $\tilde{\theta}_2^*$ the unique solution to the following equation:

$$
\int_{\theta}^{\infty} \frac{1}{\sigma_1} f \left( \frac{\theta - \tilde{\theta}_2^*}{\sigma_1} \right) d\theta - T = 0
$$

The first of the above conditions corresponds to the situation when player 2 never attacks while the second condition corresponds to the situation where player 2 always chooses to take the risky action. Note that $-\infty < \theta_1^* < \tilde{\theta}_1^* < \infty$ and therefore it follows that $\tilde{\theta}_1^*$ is finite (and $\tilde{\theta}_1^* \in [\overline{\theta}_1^*, \tilde{\theta}_1^*]$).

Recall that by lemma 1, the LHS of (10) is increasing in $\tilde{\theta}_1^*$ and decreasing in $\tilde{\theta}_2^*$. It follows then that as $\tilde{\theta}_1^* \to \overline{\theta}_1^*, \tilde{\theta}_2^* \to -\infty$ and as $\tilde{\theta}_1^* \to \tilde{\theta}_1^*, \tilde{\theta}_2^* \to \infty$. Therefore $w_1(\tilde{\theta}_1^*; \sigma)$ has asymptotes at $\theta_1^*$ and $\tilde{\theta}_1^*$. Similarly define $\theta_2^*$ and $\overline{\theta}_2^*$ for agent two. By lemma 1 we conclude that $w_2(\tilde{\theta}_1^*; \sigma)$ is bounded above by $\overline{\theta}_2^*$ and below by $\theta_2^*$. Finally, let $\theta_{\min}^* = \min \{\theta_1^*, \theta_2^*\}$ and $\theta_{\max}^* = \max \{\theta_1^*, \theta_2^*\}$ so that $\theta_{\min}^*$ is the smallest and $\theta_{\max}^*$ is the largest threshold that can be rationalized.

Using the implicit function theorem we can find the derivative of $w_1(\tilde{\theta}_1^*; \sigma)$ w.r.t. $\tilde{\theta}_1^*$:

$$
\frac{dw_1(\tilde{\theta}_1^*)}{d\tilde{\theta}_1^*} = \frac{\int_{\theta}^{\overline{\theta}} \frac{1}{\sigma_1} f \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) \frac{1}{\sigma_2} f \left( \frac{\sigma_2(\tilde{\theta}_2^* - \tilde{\theta}_2)}{\sigma_2^2} \right) d\theta + V_1}{\sigma_2^2 + \sigma_\theta^2} \int_{\theta}^{\overline{\theta}} \frac{1}{\sigma_1} f \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) \frac{1}{\sigma_2} f \left( \frac{\sigma_2(\tilde{\theta}_2^* - \tilde{\theta}_2)}{\sigma_2^2} \right) d\theta} > 0
$$
where

\[ \tilde{V}_1 = 1 - F\left(\frac{\tilde{\theta}_1^*}{\sigma_1}\right) + \frac{1}{\sigma_1} \int_{\tilde{\theta}_1^*}^{\theta} f\left(\frac{\theta}{\sigma_1}\right) \left(1 - F\left(\frac{\sigma_2\left(\tilde{\theta}_2^* - \frac{\sigma_2^2\tilde{\theta}_2^*}{\sigma_2^2 + \sigma_1^2}\right)}{\sigma_2^2}\right)\right) \]

\[ + \int_{\tilde{\theta}_1^*}^{\theta} \frac{1}{\sigma_1} f\left(\frac{\theta}{\sigma_1}\right) \left(1 - F\left(\frac{\sigma_2\left(\tilde{\theta}_2 - \frac{\sigma_2^2\tilde{\theta}_2}{\sigma_2^2 + \sigma_1^2}\right)}{\sigma_2^2}\right)\right) d\theta > 0 \]

is strictly positive (since both \(\tilde{\theta}_1^*\) and \(\tilde{\theta}_2^*\) are finite - see the discussion above).

Similarly, we calculate the derivative of \(w_2(\tilde{\theta}_1^*; \sigma)\) w.r.t. \(\tilde{\theta}_1^*:\)

\[ \frac{dw_2(\tilde{\theta}_1^*)}{d\tilde{\theta}_1^*} = \frac{\sigma_2^2 + \sigma_1^2}{\sigma_2^2} \int_{\tilde{\theta}_1^*}^{\theta} f\left(\frac{\theta}{\sigma_2}\right) \frac{1}{\sigma_1} f\left(\frac{\sigma_1(\tilde{\theta}_1^* - \theta)}{\sigma_2}\right) d\theta > 0 \]

where \(\tilde{V}_2\) a strictly positive constant and is defined in analogously to \(\tilde{V}_1\).

Note that a sufficient condition for uniqueness is

\[ \frac{dw_1(\tilde{\theta}_1^*)}{d\tilde{\theta}_1^*} > \frac{dw_2(\tilde{\theta}_1^*)}{d\tilde{\theta}_1^*} > 0 \]

This translates in the following inequality:

\[ \frac{\sigma_2^2 + \sigma_1^2}{\sigma_2^2} \int_{\tilde{\theta}_1^*}^{\theta} f\left(\frac{\theta}{\sigma_2}\right) \frac{1}{\sigma_1} f\left(\frac{\sigma_2(\tilde{\theta}_2^* - \theta)}{\sigma_2}\right) d\theta + \tilde{V}_1 > \frac{\sigma_2^2 + \sigma_1^2}{\sigma_2^2} \int_{\tilde{\theta}_1^*}^{\theta} f\left(\frac{\theta}{\sigma_2}\right) \frac{1}{\sigma_1} f\left(\frac{\sigma_1(\tilde{\theta}_1^* - \theta)}{\sigma_2}\right) d\theta + \tilde{V}_2 \]

Doing some algebraic manipulations, we get that the expression above is equivalent to

\[ \frac{\tilde{V}_1}{\frac{\sigma_2^2 + \sigma_1^2}{\sigma_2^2} \int_{\tilde{\theta}_1^*}^{\theta} f\left(\frac{\theta}{\sigma_2}\right) \frac{1}{\sigma_1} f\left(\frac{\sigma_2(\tilde{\theta}_2^* - \theta)}{\sigma_2}\right) d\theta + \tilde{V}_2} \]

\[ \frac{\tilde{V}_1}{\frac{\sigma_2^2 + \sigma_1^2}{\sigma_2^2} \int_{\tilde{\theta}_1^*}^{\theta} f\left(\frac{\theta}{\sigma_2}\right) \frac{1}{\sigma_1} f\left(\frac{\sigma_1(\tilde{\theta}_1^* - \theta)}{\sigma_2}\right) d\theta} \]

\[ \left(\frac{\sigma_2}{\sigma_2^2} + \frac{\sigma_1^2}{\sigma_2^2} + \frac{\sigma_1}{\sigma_2^2} \frac{\sigma_1^2}{\sigma_2^2} \right) \tilde{V}_1 \tilde{V}_2 \]

\[ \sigma_2^2 + \sigma_1^2 + \sigma_1^2 \sigma_2^2 \sigma_1^2 \sigma_2^2 \]

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A sufficient condition for this inequality to hold is to have:

\[
\frac{\sigma_1^2}{\sigma_\theta^2} < \frac{\tilde{v}_1}{\int_\theta \theta \frac{1}{\sigma_1} \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) \frac{1}{\sigma_2} \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_2)}{\sigma_2^2} \right) d\theta} \]

and

\[
\frac{\sigma_2^2}{\sigma_\theta^2} < \frac{\tilde{v}_2}{\int_\theta \theta \frac{1}{\sigma_2} \left( \frac{\theta - \tilde{\theta}_2}{\sigma_2} \right) \frac{1}{\sigma_1} \left( \frac{\sigma_1 (\tilde{\theta}_1 - \tilde{\theta}_1)}{\sigma_1^2} \right) d\theta} \]

Take the first expression, for agent 1 (the result is analogous for agent 2). We want to find a lower bound for the RHS, i.e. a lower bound for the numerator of the RHS and an upper bound for the denominator of the RHS.

We first look at the denominator \(\int_\theta \theta \frac{1}{\sigma_1} \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) \frac{1}{\sigma_2} \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_2)}{\sigma_2^2} \right) d\theta\). By doing some algebraic manipulations and with the help of the properties of a normal distribution, we can rewrite this expression as

\[
\frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \int_\theta \theta \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \left( \frac{\tilde{\theta}_1^* - \Omega}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \left( \frac{\tilde{\theta}_2^* - \Omega}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right) d\theta \leq \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \left[ \frac{\theta_{\max} (\sigma_1^2 + \sigma_2^2 + \tilde{\sigma}_1^2 \tilde{\sigma}_2^2)}{\sigma_1^2 + \sigma_2^2} \right] \frac{\tilde{\sigma}_1^2 \tilde{\sigma}_2^2}{\sigma_1^2 + \sigma_2^2} \mu_\theta \]

where \(\Omega \equiv \frac{(\sigma_1^2 + \sigma_2^2) \tilde{\theta}_2^* - \sigma_1^2 \mu_\theta}{\sigma_1^2 + \sigma_2^2}\).

We now look at the numerator \(\tilde{v}_1\). Note that

\[
\frac{1}{\sigma_1} \int_\theta \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) \left( 1 - F \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_2)}{\sigma_2^2} \right) \right) d\theta \geq \left[ \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_1^*)}{\sigma_2^2} \right][1 - F \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_2)}{\sigma_2^2} \right)]
\]

where \(\tilde{\theta}_2 \equiv \frac{\sigma_2^2 \tilde{\theta}_2 + \sigma_1 \sigma_2 \mu_\theta}{\sigma_1^2 + \sigma_2^2}\) and therefore

\[
\tilde{v}_1 > 1 - F \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) - \left[ \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_1^*)}{\sigma_2^2} \right][1 - F \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_2)}{\sigma_2^2} \right)] + \frac{1}{\sigma_1} \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) \left[ \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_2)}{\sigma_2^2} \right][1 - F \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_2)}{\sigma_2^2} \right)]
\]

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We can bound the last two terms of the RHS of the above expression by

\[
\frac{1}{\sigma_1^2} f\left( \frac{\theta - \theta^*_{\min}}{\sigma_1} \right) \left[ \bar{\theta} F\left( \frac{\sigma_2}{\sigma_1} \left( \frac{\theta^*_{\min} - \sigma_2^2 \theta + \sigma_2^2 \mu_\theta}{\sigma_1^2 + \sigma_2^2} \right) \right) + \theta \left( 1 - F\left( \frac{\sigma_2}{\sigma_1} \left( \frac{\theta^*_{\max} - \sigma_2^2 \theta + \sigma_2^2 \mu_\theta}{\sigma_1^2 + \sigma_2^2} \right) \right) \right) \right]
\]

Therefore, sufficient conditions for uniqueness are:

\[
\frac{\sigma_2^2}{\bar{\sigma}}^2 < \frac{1 - F\left( \frac{\bar{\theta} - \theta^*_{\min}}{\sigma_2} \right) - \left[ F\left( \frac{\bar{\theta} - \theta^*_{\min}}{\sigma_1} \right) - F\left( \frac{\theta^*_{\min} - \theta^*_{\max}}{\sigma_1} \right) \right] F\left( \frac{\sigma_2}{\sigma_1} \left( \frac{\theta^*_{\max} - \theta^*_{\min}}{\sigma_1} \right) + \frac{1}{\sigma_1} f\left( \frac{\theta - \theta^*_{\min}}{\sigma_1} \right) \right)}{1 - \frac{1}{\sqrt{2\pi}} \left( \frac{2}{\sqrt{2\pi}} + \frac{\bar{\sigma}_2 \left( \frac{\bar{\theta} - \theta^*_{\min}}{\sigma_2} \right) - \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right) \right)}
\]

\[
\frac{\sigma_1^2}{\bar{\sigma}^2} < \frac{1 - F\left( \frac{\bar{\theta} - \theta^*_{\max}}{\sigma_2} \right) + \frac{1}{\sigma_2} f\left( \frac{\bar{\theta} - \theta^*_{\max}}{\sigma_2} \right) \right)}{1 - \frac{1}{\sqrt{2\pi}} \left( \frac{2}{\sqrt{2\pi}} + \frac{\bar{\sigma}_1 \left( \frac{\bar{\theta} - \theta^*_{\max}}{\sigma_1} \right) - \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \right) \right)}
\]

Where \( \kappa_i := \bar{\theta} F\left( \frac{\sigma_2}{\sigma_1} \left( \frac{\theta^*_{\min} - \sigma_2^2 \theta + \sigma_2^2 \mu_\theta}{\sigma_1^2 + \sigma_2^2} \right) \right) + \theta \left( 1 - F\left( \frac{\sigma_2}{\sigma_1} \left( \frac{\theta^*_{\max} - \sigma_2^2 \theta + \sigma_2^2 \mu_\theta}{\sigma_1^2 + \sigma_2^2} \right) \right) \right) \)

If such conditions hold, \( 0 < \frac{d w_2(\bar{\theta}_{1,\sigma})}{d\bar{\theta}_{1}} < \frac{d w_1(\bar{\theta}_{1,\sigma})}{d\bar{\theta}_{1}} \) \( \forall \bar{\theta}_{1} \in [\theta^*, \bar{\theta}^*_{1}] \). This means that the least and greatest Bayesian Nash equilibria of the game, as described by our Corollary in the first section of the appendix, coincide. Therefore, there is a unique equilibrium in thresholds strategies. This proves the first part of the theorem.

The proof for the second part of the theorem follows directly from the proof of the above result. Namely, recall that to prove uniqueness we have to find conditions under which the functions \( w_1(\bar{\theta}_{1}^*) \) and \( w_2(\bar{\theta}_{1}) \) (there were defined above) are such that

\[
\frac{d w_1(\bar{\theta}_{1})}{d\bar{\theta}_{1}} \leq \frac{d w_2(\bar{\theta}_{1})}{d\bar{\theta}_{1}}
\]

Note that as \( \sigma_\theta \to \infty \) we have

\[
\lim_{\sigma_\theta \to \infty} \frac{\sigma_2^2 + \sigma_1^2}{\sigma_2^2} \to 1 \quad \text{and} \quad \lim_{\sigma_\theta \to \infty} \frac{\sigma_2^2 + \sigma_2^2}{\sigma_2^2} \to 1
\]
Therefore
\[
\frac{dw_1(\tilde{\theta}_1^*)}{d\tilde{\theta}_1^*} = \int_{\tilde{\theta}_1^*}^{\tilde{\theta}_1} \frac{1}{\sigma_1} \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) f \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_1)}{\sigma_2^2} \right) d\theta + \tilde{V}_1
\]
\[
\quad = \int_{\tilde{\theta}_1}^{\tilde{\theta}_1} \frac{1}{\sigma_1} \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) f \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_1)}{\sigma_2^2} \right) d\theta
\]
\[
\quad > 1
\]
and
\[
\frac{dw_2(\tilde{\theta}_1^*)}{d\tilde{\theta}_1^*} = \frac{\sigma_1^2}{\sigma_1} f \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) f \left( \frac{\sigma_1 (\tilde{\theta}_1 - \theta)}{\sigma_1^2} \right) d\theta
\]
\[
\quad = \int_{\tilde{\theta}_1}^{\tilde{\theta}_1} \frac{1}{\sigma_2} \left( \frac{\theta - \tilde{\theta}_1}{\sigma_2} \right) f \left( \frac{\sigma_1 (\tilde{\theta}_1 - \theta)}{\sigma_1^2} \right) d\theta
\]
\[
\quad < 1
\]
since, as we argued in the proof of the first part of the theorem, \( \tilde{V}_1 \) and \( \tilde{V}_2 \) are strictly positive. By continuity of the above expressions we conclude that for any \( \sigma_1 \) and \( \sigma_2 \) there exists \( \sigma_\theta (\sigma_1, \sigma_2) \) such that if \( \sigma_\theta > \sigma_\theta (\sigma_1, \sigma_2) \) we have a unique equilibrium in the coordination game.

The above bound depends on the information acquisition choices made by players. However, it is easy to show existence of a uniform bound. To do so, recall first that
\[
\tilde{V}_1 = 1 - F \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) + \tilde{\theta}_1 f \left( \frac{\theta - \tilde{\theta}_1^*}{\sigma_1} \right) F \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_1^*)}{\sigma_2^2} \right)
\]
\[
\quad + \sigma_2 f \left( \frac{\theta - \tilde{\theta}_1}{\sigma_2} \right) \left( 1 - F \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_1)}{\sigma_2^2} \right) \right)
\]
\[
\quad + \int_{\tilde{\theta}_1}^{\tilde{\theta}_1} \frac{1}{\sigma_1} \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) \left( 1 - F \left( \frac{\sigma_2 (\tilde{\theta}_2 - \tilde{\theta}_2)}{\sigma_2^2} \right) \right) d\theta > 0
\]
and in particular
\[
\tilde{V}_1 > 1 - F \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) > 0
\]
Recall that \( \tilde{\theta}_1 = \sqrt{\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}} \) and that \( \sigma_1^2 \leq \sigma_0^2 \) where \( \sigma_0^2 \) is the precision of a private signal if player 1 did not acquire any information. Let \( \sigma_\theta^2 \) be an arbitrary lower bound on \( \sigma_\theta^2 \) such that \( \sigma_\theta^2 > 0 \).

If at \( \sigma_\theta^2 = \sigma_\theta^2 \) we have \( \frac{dw_1(\tilde{\theta}_1)}{d\tilde{\theta}_1} > \frac{dw_2(\tilde{\theta}_1)}{d\tilde{\theta}_1} \) then we are done. Otherwise, we have to show that there exists a bound on \( \sigma_\theta^2 \) that is higher than \( \sigma_\theta^2 \) for which \( \frac{dw_1(\tilde{\theta}_1)}{d\tilde{\theta}_1} > \frac{dw_2(\tilde{\theta}_1)}{d\tilde{\theta}_1} \) independent of values of
\( \sigma_1 \) and \( \sigma_2 \).

To do so we start by finding uniform bounds on \( \tilde{\theta}_1^* \). Note that, for any \( \sigma_1 \) and \( \sigma_\theta \), the lowest threshold that agent 1 can possibly choose (which we denote by \( \theta_1^*(\sigma_\theta, \sigma_1) \)) is determined by equation

\[
\int_0^\infty \frac{1}{\sigma_1^2} \theta f \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) d\theta = T
\]

which corresponds to the situation in which the other agent always chooses to attack (and where we suppressed the dependence of \( \tilde{\theta}_1^* \) on \((\sigma_\theta, \sigma_1))\). This can be written as

\[
\theta_1^* \left( 1 - F \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) \right) + \tilde{\sigma}_1 f \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) = T
\]

By the implicit function theorem we have

\[
\frac{\partial \theta_1^*}{\partial \tilde{\sigma}_1} = \frac{\theta_1^* f \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) - f \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) + \frac{(\theta - \tilde{\theta}_1)^2}{\sigma_1^2} f \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right)}{1 - F \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) + \tilde{\sigma}_1 f \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) + \frac{(\theta - \tilde{\theta}_1)^2}{\sigma_1^2} f \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right)} < 0
\]

So as \( \tilde{\sigma}_1 \) increases \( \theta_1^* \) decreases which implies that an increase in \( \sigma_\theta^2 \) decreases \( \theta_1^* \) while an increase in \( \sigma_1 \) increases \( \theta_1^* \). In the same way we can show that the highest threshold that agent 1 can possibly choose (denoted by \( \bar{\theta}_1 \)) is also decreasing in \( \tilde{\sigma}_1 \). Therefore, \( \theta_1^* \) is minimized at \( \sigma_1 = \sigma_0 \) and \( \sigma_\theta \to \infty \). This implies that

\[
\bar{V}_1 > 1 - F \left( \frac{\bar{\theta} - \bar{\theta}_1}{\sigma_1} \right) > K_1
\]

where

\[
K_1 \equiv 1 - F \left( \frac{\bar{\theta} - \lim_{\sigma_\theta \to \infty} \theta_1^*(\sigma_\theta, \sigma_0)}{\tilde{\sigma}_1} \right)
\]

This establishes a bound on \( \bar{V}_1 \) that is independent of \( \sigma_\theta, \sigma_0 \). Note that by symmetry this is also a lower bound on \( \bar{V}_2 \).

We now show that we can also bound other term appearing in the expression for the derivative uniformly. But

\[
\int_\theta^\infty \theta \frac{1}{\tilde{\sigma}_2^2} f \left( \frac{\theta - \bar{\theta}_2^*}{\tilde{\sigma}_2} \right) \frac{1}{\sigma_1} f \left( \frac{\sigma_1(\bar{\theta}_1^* - \bar{\theta}_1)}{\tilde{\sigma}_1^2} \right) d\theta < \frac{1}{2\pi} \left( \tilde{\theta} - \theta \right) \frac{1}{\tilde{\sigma}_2 \sigma_1}
\]

Below, when we discuss the first stage of the game we show that the benefit from acquiring information tends to zero as \( \sigma_i \to 0 \) and therefore, given our assumptions on the cost function, i.e. \( C'(\sigma_i) > 0 \) and \( \lim_{\sigma_i \to \infty} C'(\sigma_i) \to \infty \), there is a bound on the precision choice, call it \( \sigma_i^{\min} \), such
that agent $i$ will never choose to acquire a lower standard deviation than $\sigma_i^{\text{min}}$. Therefore,

$$
\int_\vartheta \frac{1}{\sigma_2} \left( \frac{\theta - \hat{\theta}_2}{\hat{\sigma}_2} \right) \frac{1}{\sigma_1} \left( \frac{\sigma_1 (\hat{\theta}_1^* - \hat{\theta}_1)}{\hat{\sigma}_1^2} \right) < \frac{1}{2\pi} [\bar{\theta} - \theta] \frac{1}{\sigma_2 \sigma_1^2} \\
\leq \frac{1}{\sigma_2} \frac{1}{\sigma_1^\text{min}} \frac{1}{2\pi} [\bar{\theta} - \theta]
$$

Finally, note that $\hat{\sigma}_2$ is increasing in $\sigma_\theta$ and $\sigma_2$ and so $\hat{\sigma}_2$ is minimized at $\sigma_\theta = \sigma_\theta^B$ (our exogenous lower bound on $\sigma_\theta$) and $\sigma_2 = \sigma_2^{\text{min}}$ and denote by $\hat{\sigma}_2^{\text{min}}$ the posterior standard deviation of $\theta$ for player 2 when $\sigma_\theta = \sigma_\theta^B$ and $\sigma_2 = \sigma_2^{\text{min}}$. Then

$$
\int_\vartheta \frac{1}{\sigma_2} \left( \frac{\theta - \hat{\theta}_2^*}{\hat{\sigma}_2} \right) \frac{1}{\sigma_1} \left( \frac{\sigma_1 (\hat{\theta}_1^* - \hat{\theta}_1)}{\hat{\sigma}_1^2} \right) \leq K_2
$$

where

$$
K_2 = \frac{1}{\hat{\sigma}_2^{\text{min}}} \frac{1}{\sigma_1^{\text{min}}} \frac{1}{2\pi} [\bar{\theta} - \theta]
$$

Therefore, we have

$$
\frac{d\tilde{w}_1(\tilde{\theta}_1)}{d\tilde{\theta}_1} = \frac{\sigma_2^2 + \sigma_2^2}{\sigma_\theta^2 + \sigma_2^2} \frac{1}{\sigma_1^2} \int_\vartheta \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) \frac{1}{\sigma_2} \left( \frac{\sigma_1 (\tilde{\theta}_1^* - \tilde{\theta}_1)}{\sigma_1^2} \right) d\theta + \tilde{V}_1 \\
= \frac{\sigma_2^2 + \sigma_2^2}{\sigma_\theta^2 + \sigma_2^2} \frac{1}{\sigma_1^2} \int_\vartheta \left( \frac{\theta - \tilde{\theta}_1}{\sigma_1} \right) \frac{1}{\sigma_2} \left( \frac{\sigma_1 (\tilde{\theta}_1^* - \tilde{\theta}_1)}{\sigma_1^2} \right) d\theta \\
\geq \frac{\sigma_2^2 + \sigma_2^2}{\sigma_\theta^2 + \sigma_2^2} \frac{1}{\sigma_1^2} \frac{\sigma_2^2}{\sigma_1^2} \frac{K_1}{\sigma_2^2} \\
\geq \frac{\sigma_2^2 + \sigma_2^2}{\sigma_\theta^2 + \sigma_2^2} \frac{1}{\sigma_1^2} \frac{\sigma_2^2}{\sigma_1^2} \frac{K_2}{\sigma_2^2}
$$

where we used the fact that $\sigma_2^2 \leq \sigma_\theta^2$. Note that as $\sigma_\theta^2 \to \infty$ we have $\frac{\sigma_2^2}{\sigma_\theta^2 + \sigma_2^2} \to \frac{\sigma_2^2}{\sigma_2^2} \frac{K_1}{\sigma_2^2} \to 1$ and therefore there exists a bound on $\sigma_\theta^2$, call it $\sigma_\theta^{2,P1}$ such that if $\sigma_\theta^2 > \sigma_\theta^{2,P1}$ then $\frac{d\tilde{w}_1(\tilde{\theta}_1)}{d\tilde{\theta}_1} > 1$ irrespective of $\sigma_1$ and $\sigma_2$.\footnote{We use the superscript $P1$ to emphasize the fact that this restriction follows from equilibrium condition of player 1.}

Following the same steps as above we can show that there exists a bound on $\sigma_\theta^2$, which we denote by $\sigma_\theta^{2,P2}$, such that if $\sigma_\theta^2 > \sigma_\theta^{2,P2}$ then $\frac{d\tilde{w}_1(\tilde{\theta}_1)}{d\tilde{\theta}_1} < 1$ and $\sigma_\theta^{2,P2}$ is independent of $\sigma_1$ and $\sigma_2$.

Setting $\sigma_\theta = \max \left\{ \sigma_\theta^{2,P1}, \sigma_\theta^{2,P2} \right\}$ proves the second part of the theorem. \[\blacksquare\]

**Lemma 2** Suppose that $\sigma_1 \to 0$, $\sigma_2 \to 0$ and $\sigma_\theta^2 \to c$ where $c \in \mathbb{R}_+$. If the above game has a unique equilibrium then this equilibrium converges to the risk-dominant equilibrium of the complete
information game, i.e. \( x^*_i \to 2T \) and \( x^*_j \to 2T \).

**Proof.** Agent \( i \)’s expected payoff of attacking is given by:

\[
\int_{-\infty}^{\infty} 1_{\{\theta \in [\bar{a}, \bar{b}]\}} \theta \left( 1 - F \left( \frac{x_j^* - \theta}{\sigma_j} \right) \right) d\alpha_{\sigma_i} + \int_{\bar{a}}^{\infty} \theta d\alpha_{\sigma_i} - T
\]

where \( \alpha_{\sigma_i} \) is a measure implied by the CDF of a Normal distribution with mean \( \hat{\theta}_i \) and variance \( \hat{\sigma}_i \) (i.e. \( \alpha_{\sigma_i} \) is a Baire measure implied by \( F \)). Then the characteristic function of \( \alpha_{\sigma_i} \) is given by:

\[
\phi_{\alpha_{\sigma_i}}(t) = e^{it\hat{\theta}_i - \frac{t^2\sigma_i^2}{2}}
\]

where \( \hat{\theta}_i = \frac{\sigma_i^2 x_i + \sigma_i^2 \mu_i}{\sigma_i^2 + \sigma_i^2} \), \( \hat{\sigma}_i^2 = \frac{\sigma_i^2 \sigma_i^2}{\sigma_i^2 + \sigma_i^2} \) and \( x_i = \theta + \sigma_i \varepsilon_i \). Note that, as \( \sigma_i \to 0 \) then \( \hat{\sigma}_i \to 0 \), and so

\[
\lim_{\sigma_i \to 0} \phi_{\alpha_{\sigma_i}}(t) = e^{itx_i}
\]

But \( e^{itx_i} \) is the characteristic function of a probability distribution with mass 1 at \( \theta \), which we denote by \( \delta_\theta \). By the Levy-Cramer Continuity theorem (see e.g. Varadhan 2001) this implies that \( \alpha_{\sigma_i} \to \delta_{x_i} \) as \( \sigma_i \to 0 \), i.e. as the standard deviation converges to 0, the Normal distribution converges to the degenerate distribution with all the mass centered at \( x_i \).

Consider first the limit of \( \int_{\bar{a}}^{\infty} \theta d\alpha_{\sigma_i} \) as \( \sigma_1 \to 0 \):

\[
\lim_{\sigma_1 \to 0} \int_{\bar{a}}^{\infty} \theta d\alpha_{\sigma_i} = \left( 1 - F \left( \frac{\bar{\theta} - \hat{\theta}_i}{\hat{\sigma}_i} \right) \right) + \sigma_1 f \left( \frac{\bar{\theta} - \hat{\theta}_i}{\hat{\sigma}_i} \right) = \begin{cases} 
1 & \text{if } x_i > \bar{\theta} \\
\frac{1}{2}x_i & \text{if } x_i = \bar{\theta} \\
0 & \text{if } x_i < \bar{\theta}
\end{cases}
\]

Let \( h(\theta; \sigma_j) \equiv 1_{\{\theta \in [\bar{a}, \bar{b}]\}} \theta \left( 1 - F \left( \frac{x^*_j - \theta}{\sigma_j} \right) \right) \) and denote by \( x^*_{j, \infty} \) the limit of \( x^*_j \) as \( \sigma_i \to 0, \sigma_j \to 0 \) and \( \frac{\sigma_i}{\sigma_j} \to c \in \mathbb{R} \). Note that \( h \) is a bounded, Borel measurable function and is continuous except for a finite set of measure zero. Moreover,

\[
\lim_{\sigma_j \to 0} h(\theta; \sigma_j) \to \begin{cases} 
\theta & \text{if } \theta \in \left( x^*_{j, \infty}, \bar{\theta} \right] \\
\frac{1}{2} \theta & \text{if } \theta = x^*_{j, \infty} \\
0 & \text{otherwise}
\end{cases}
\]

where we assumed that \( x^*_{j, \infty} \in [\bar{\theta}, \bar{\theta}] \).

\[\text{43} \] It follows then by Levy-Cramer Continuity Theorem we argue below that only \( x^*_{j, \infty} \in [\bar{\theta}, \bar{\theta}] \) is consistent with equilibrium.
have

\[
\lim_{\sigma_i \to 0} \int_{-\infty}^{\infty} h(\theta) \, d\alpha_{i_i} \quad \rightarrow \quad \int_{-\infty}^{\infty} h(\theta) \, d\delta_i = \int_{-\infty}^{\infty} 1_{\theta \in [x_i^*, \theta]}(\theta) \left( 1 - F \left( \frac{x_i^* - \theta}{\sigma_j} \right) \right) \, d\delta_i
\]

\[
= \begin{cases} 
  x_i^* & \text{if } x_i^* \in \left[ x_j^{*\infty}, \bar{\theta} \right] \\
  \frac{1}{2}x_i^* & \text{if } x_i^* = x_j^{*\infty} \\
  0 & \text{if } x_i^* \notin \left[ x_j^{*\infty}, \bar{\theta} \right]
\end{cases}
\]

The optimal threshold is defined as the value of a signal that makes agent $i$ indifferent between attacking and not attacking, i.e. it has to solve

\[
v_i(x_i^*, x_j^*; \sigma_i, \sigma_j) = 0
\]

Since the same is true for agent $j$ it has to be the case (by the symmetry of the problem) that in the limit $x_i^{*\infty} = x_j^{*\infty}$ and so $x_i^{*\infty}$ solves

\[
\frac{1}{2}x_i^{*\infty} - T = 0
\]

\[
x_i^{*\infty} = 2T
\]

It is easy to verify that $2T$ corresponds to a threshold such that if $\theta \geq 2T$ then attacking is a risk-dominant action and if $\theta < 2T$ not attacking is a risk-dominant function.

Above we assumed that $x_j^{*\infty} \in [\bar{\theta}, \bar{\theta}]$. Suppose that this is not the case. If $x_j^{*\infty} > \bar{\theta}$ then following the same argument as before we would find that in the limit

\[
x_i^{*\infty} = \bar{\theta} < x_j^{*\infty}
\]

which contradicts optimality of $x_j^{*\infty}$ (if agent $i$ follows a threshold $x_i^{*\infty} = \bar{\theta}$ then the optimal threshold for agent $j$ is $x_j^{*\infty} = \bar{\theta}$). A symmetric argument establishes that it cannot be the case that $x_j^{*\infty} < \bar{\theta}$. ■

**Theorem (Existence)** There exists a symmetric pure-strategy Bayesian Nash Equilibrium of the game with information acquisition.

We first prove two simple claims and two corollaries that will make the proof of existence straightforward.

**Claim 1** $B_i(\sigma_i; \sigma_j)$ is a decreasing function of $\sigma_i$, that is $B_i(\sigma_i; \sigma_j) \leq 0$.

**Proof.** Since $u(\theta, a_i)$ has the single crossing property in $(\theta, a_i)$, and that the signal $x_i$ and the unknown parameter $\theta$ are affiliated, the claim then follows from Theorem 1 in Persico (2000). ■

**Claim 2** The marginal benefit of increasing the precision of agent $i$ converges to zero as the signal noise for agent $i$ vanishes, i.e. $\lim_{\sigma_i \to 0} \frac{\partial}{\partial \sigma_i} B_i(\sigma_i; \sigma_j) = 0$. 

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This proof is very lengthy and can be obtained from the authors by request. It requires to show that in the limit $\frac{\partial}{\partial \sigma_i} x_i^*(\sigma_i, \sigma_j)$ is bounded and to verify that all the integrals in the expression for the marginal benefit converge to zero.

From the above results we have the following immediate corollaries:

**Corollary 1**  
The best response functions for both agents are well defined.

**Proof.** Since the cost function is strictly decreasing in $\sigma_i$ and tends to infinite as $\sigma_i \to 0$, and since $B_i(\sigma_i, \sigma_j)$ is positive and stays bounded for each $\sigma_j$, we know that for each $\sigma_j$ there is a unique choice of $\sigma_i$, holding beliefs of both players constant. ■

**Corollary 2**  
In any equilibrium of the game both agents choose to acquire information (increase the precision of their signals).

**Proof.** This follows from the fact that the marginal cost of acquiring information is continuous and equal zero at $\sigma_i = \sigma_0$, together with the fact that the marginal benefit of lowering $\sigma_i$ is strictly positive for $\sigma_i > 0$.

**Existence.** Suppose that agent $j$ believes that whenever he chooses a precision $\sigma_j$, agent $i$ will make the same choice. Holding agent $j$’s beliefs fixed, we showed above that the best response function

$$\sigma_i^*(\sigma_j) = \max_{\sigma_i \in [0, \sigma_0]} U_i(\sigma_i, \sigma_j)$$

is well defined. Since $U_i(\sigma_i, \sigma_j)$ is continuous in $\sigma_i$ and $\sigma_j$, by the Theorem of the Maximum we conclude that $\sigma_i^*(\sigma_j)$ is a continuous function. $\sigma_i^*(\cdot)$ is also a self-map: $\sigma_i^* : [0, \sigma_0] \to [0, \sigma_0]$. Hence, by Brower’s Fixed Point Theorem, $\sigma_i^*(\cdot)$ has a fixed point. This implies that there exists a $\sigma_j$ such that if agent $j$ believes that agent $i$ chooses $\sigma_i = \sigma_j$ agent $i$ will find it optimal to choose such a $\sigma_i$, that is $\sigma_i^*(\sigma_j) = \sigma_j$. ■

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44Note that above we established that $B_{i1} \leq 0$ and $\lim_{\sigma_i \to 0} B_{i1} = 0$. It can be shown that $B_{i1}(s_i, \sigma_j) \neq B_{i1}(s_i', \sigma_j)$ $\forall s_i \neq s_i'$. It follows then that $B_{i1}(\sigma_i, \sigma_j) < 0$ whenever $\sigma_i > 0$. That is, decreasing $\sigma_i$ (increasing the precision) strictly increases the gross payoff for agent $i$. 

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