ECONOMICS 581: LECTURE NOTES

CHAPTER 9: The Measurement of Waste and Marginal Excess Burdens

W. Erwin Diewert May 2, 2008.

1. Introduction

What are the losses to the economy due to inefficient schemes of taxation? This question has a long history in economics.[1] Some of the early contributors to this literature are Hotelling (1938), Allais (1943) (1973) (1977), Boiteux (1951), Debreu (1951) (1954) and Harberger (1964). In the present chapter, we will attempt to provide answers to the above question. For the most part, we follow the treatment in Diewert (1983),[2] which of course relies on the earlier authors mentioned above.

In section 2, we start out with the problem of evaluating the costs of tax distortions that occur within the production sector of the economy. Examples of such distortions are commodity taxes that fall on intermediate inputs or capital taxes that are not neutral across industrial sectors or subsidies to specific industries. Since most international trade flows through the production sector of an economy, tariffs on imports or export subsidies or taxes also qualify as distortions that fall within the production sector of an economy.[3] The evaluation of losses due to taxes that fall within the production sector is conceptually simpler than the evaluation of the losses due to taxes that fall between the interface of the production sector and the household sector. The problem is that countries who want to provide a lot of government services are usually forced to resort to distortive commodity taxes that fall on the outputs produced and inputs used by the economy's production sector and so some deadweight cost due to taxation is inevitable in these high tax, high public services economies. However, *there is no reason to impose taxes that fall within the production sector*. Such taxes will inevitably reduce the aggregate output or increase the aggregate input used by the production sector; i.e., such internal to the production sector *will destroy the productive efficiency of the economy*.

A problem now arises: how exactly should we measure the loss of output due to the tax distortions? Allais (1943) (1973) (1977) favoured the following strategy: pick a numeraire commodity, determine the net output of this numeraire commodity in the tax distorted equilibrium and note what the net output of the production sector is for all of the remaining commodities in the economy. Now solve a nonlinear programming problem that maximizes the amount of the numeraire commodity that could be produced by the

---

[1] Related questions are: what are the losses to the economy of monopolistic behavior or of union wage premiums? The techniques to be studied in this chapter can address these issues as well.

[2] In this chapter, we assume the existence of sectoral fixed factors whereas Diewert (1983) considered a model with constant returns to scale sectoral production functions.

[3] This treatment of international trade follows that of Kohli (1978) (1990) (1991) (2003) and Woodland (1982) in assuming that imports flow through the domestic production sector and are "transformed" (perhaps only by adding transportation, wholesaling and retailing margins) by the domestic production sector. The recent textbook by Feenstra (2004; 76) also uses this approach.

production sector, subject to producing at least the same net amounts of the other goods and services that were produced in the tax distorted equilibrium. The increase in the resulting (hypothetical) production of the numeraire commodity serves to measure the deadweight cost of the tax distortions that fall within the production sector. This is a very useful measure of the costs of distortions but it does suffer from the fact that different measures of loss will arise if we choose different commodities as the numeraire commodity. Thus Debreu (1951) proposed a variant of the Allais loss measure: instead of singling out any particular commodity as the numeraire commodity, consider the vector of fixed endowments of commodities that exists in the economy. Now solve a programming problem that extracts the maximum number of multiples of this endowment vector such that with the remaining endowment vector, the economy could still produce the observed vector of net outputs produced by the production sector in the tax distorted equilibrium.[4]

As an alternative to the Allais-Debreu reference quantity based measures of loss due to distortions, one could follow the example of Boiteux (1951) and Farrell (1957; 255)[5] and use price based measures. In what follows, we will use a mixture of the price based and quantity based measures of loss. We will follow Allais, and choose our numeraire commodity to be *foreign exchange* but we define foreign exchange generated by the production sector to be the value of exports less the value of imports at international prices. Thus foreign exchange is really a price type aggregate where we choose foreign prices for exports and imports as our reference prices, which remain fixed throughout the analysis.[6] We feel that foreign exchange is a very natural choice for the numeraire good since it is easy to interpret and it is broadly substitutable with domestic commodities.

Before we proceed to our usual algebraic morass, it will be useful to list Debreu's (1951; 285) *three sources of waste* in the allocation of resources:

- Waste due to the *underemployment* of available physical resources (e.g., unemployed workers);
- Waste due to *technical inefficiency* in production by an establishment or firm (a firm is technically inefficient if it produces less output or uses more input than is required) and
- Waste due to the *imperfection of economic organization*; i.e., different economic units face different prices for the same input or output and this causes aggregate output to fall or aggregate input to rise compared to a situation where all economic units face the same price for identical outputs and inputs.

[4] Diewert (1983, 172) further generalized Debreu's measure of loss by proposing an arbitrary basket of commodities rather than the endowment basket. However, the resulting generalization is not particularly helpful in empirical applications because it gives no guidance on how this arbitrary basket should be chosen.

[5] See Farrell's (1957; 255) price efficiency measure. See also Figure 3.2 in Hicks (1940) and adapt the Paasche variation of Hicks (1941-42; 128) from the consumer context to the producer context.

[6] Thus we are making the small country assumption that is common in international trade theory; i.e., the home country is so small that its demands for imports and supplies of exports do not affect world prices for internationally traded goods.

As Debreu (1951; 286) noted, the third type of loss is the most interesting (and the hardest for the layman to understand) and hence is the source of loss for which a numerical evaluation would be the most useful. This is the type of loss that we will focus on in this chapter.

2. Productive Efficiency and the Costs of Distortions

Suppose the production sector of an economy consists of 2 industries or sectors; this is the simplest situation that can illustrate the problems that we are about to explore. There are two classes of commodities that each industry produces or uses:

- M internationally traded goods and services and
- N domestically traded goods and services.

The technology sets for industries 1 and 2 are the sets S^1 and S^2. If (y^i,x^i) belongs to S^i, then y^i is a feasible vector of net exports for industry i (if $y_m^i > 0$, then industry i can export internationally traded commodity m and if $y_m^i < 0$, then industry i uses the mth internationally traded commodity as an input) and x^i is a feasible vector of net domestic outputs (if $x_n^i > 0$, then industry i can produce domestic commodity n and if $x_n^i < 0$, then industry i uses the nth domestic commodity as an input). We will assume that the technology sets S^1 and S^2 are bounded, closed convex sets that contain $(0_M,0_N)$ as feasible net output vectors. Given a nonnegative international price vector $p \geq 0_M$ and a nonnegative domestic price vector $w \geq 0_M$, we can define the *industry i profit function* π^i as follows:

$$(1)\ \pi^i(p,w) \equiv \max_{y,x} \{p^Ty + w^Tx : (y,x)\in S^i\} ; \qquad i = 1,2.$$

In what follows, we will assume that the profit functions are twice continuously differentiable.

We now consider a tax distorted equilibrium where the foreign price vector for internationally traded goods and services is $p >> 0_M$. However, the domestic government imposes various tariffs, export subsidies and other trade taxes on each industry so that industry 1 faces the international price vector $p+\tau^1$ and industry 2 faces the international price vector $p+\tau^2$, where τ^1 and τ^2 are the industry specific international distortion vectors.[7] We also assume that the government imposes various taxes and subsidies on domestic outputs and inputs and these taxes and subsidies need not be uniform across industries. Thus we assume that industry 1 faces the domestic price vector w^*+t^1 and industry 2 faces the domestic price vector w^*+t^2, where t^1 and t^2 are the industry specific domestic tax and subsidy distortion vectors. Using Hotelling's (1932) Lemma, we can

[7] Suppose M = 2 and the first international good is an export and the second one is an import. If the government imposes a uniform specific export tax $\tau_1 > 0$ on the first commodity, then $\tau_1^1 = \tau_1^2 = -\tau_1$; i.e., domestic producers get *less* than the international price for the exported good. If the government imposes a uniform tariff on imports of the second internationally traded commodity equal to $\tau_2 > 0$, then $\tau_2^1 = \tau_2^2 = \tau_2$; i.e., domestic producers pay *more* than the international price for the imported good.

calculate the net export vectors of industries 1 and 2 in the tax distorted equilibrium, y^{1*} and y^{2*}, as follows:

(2) $y^{1*} = \nabla_p \pi^1(p+\tau^1, w^*+t^1)$; $y^{2*} = \nabla_p \pi^2(p+\tau^2, w^*+t^2)$.

Again using Hotelling's Lemma, we can calculate the *net domestic supply vectors of industries* 1 and 2 in the tax distorted equilibrium, x^{1*} and x^{2*}, as follows:

(3) $x^{1*} = \nabla_w \pi^1(p+\tau^1, w^*+t^1)$; $x^{2*} = \nabla_w \pi^2(p+\tau^2, w^*+t^2)$.

Now we can calculate the *net amount of foreign exchange* f^* that the production sector generated in the tax distorted equilibrium as the sum of the value of net exports at international prices of each industry:

(4) $f^* \equiv p^T y^{1*} + p^T y^{2*}$
$\quad = p^T \nabla_p \pi^1(p+\tau^1, w^*+t^1) + p^T \nabla_p \pi^2(p+\tau^2, w^*+t^2)$ using (2).

In a similar fashion, we can calculate the *net amounts of domestic commodities* x^* that the production sector produced in the tax distorted equilibrium:[8]

(5) $x^* \equiv x^{1*} + x^{2*}$
$\quad = \nabla_w \pi^1(p+\tau^1, w^*+t^1) + \nabla_w \pi^2(p+\tau^2, w^*+t^2)$ using (3).

As an aside, we note that the observed production vectors are feasible; i.e., we have

(6) $(y^{1*}, x^{1*}) \in S^1$ and $(y^{2*}, x^{2*}) \in S^2$.

Now we consider a hypothetical equilibrium where the government eliminates all of the tax and commodity subsidy distortions; i.e., τ^1, t^1, τ^2, and t^2 are all reduced to zero vectors. Under this new tax regime, we consider the problem of maximizing the amount of foreign exchange that the production sector can produce, given that it has to produce at least the distorted equilibrium vector of net outputs of domestic commodities x^* defined by (5). This foreign exchange constrained maximization problem is the following nonlinear programming problem:

(7) $f^\circ \equiv \max_{y^1, x^1, y^2, x^2} \{p^T y^1 + p^T y^2 : x^1 + x^2 \geq x^* ; (y^1, x^1) \in S^1 ; (y^2, x^2) \in S^2\}$
$\quad \geq p^T y^{1*} + p^T y^{2*}$ since $y^{1*}, x^{1*}, y^{2*}, x^{2*}$ is a feasible solution for the problem (7)
$\quad = f^*$ using definition (4).

Thus the optimal amount of foreign exchange that the production sector can generate if the tax distortions are eliminated, f°, is always equal to or greater than the amount of

---

[8] If $x_n^* > 0$, then domestic commodity n was produced by the production sector in the tax distorted equilibrium; if $x_n^* < 0$, then domestic commodity n was being utilized as an input by the aggregate production sector in the tax distorted equilibrium and finally, if $x_n^* = 0$, then domestic commodity n was an intermediate input in the tax distorted equilibrium.

foreign exchange generated in the observed tax distorted equilibrium, f^*. If we define the deadweight loss L due to the tax distortions as the optimal amount less the observed amount of foreign exchange, we have, using (7):[9]

(8) $L \equiv f^\circ - f^* \geq 0$.

Typically, the deadweight loss due to tax distortions within the production sector, L, will be strictly positive.

It is possible to use duality theory and the Karlin (1959) Uzawa (1958) saddle point theorem to express the nonlinear programming problem defined by the first line of (7) in an instructive dual form, which we will use in the following section; i.e., we have:

$$(9)\ f^\circ \equiv \max_{y^1,x^1,y^2,x^2} \{p^Ty^1 + p^Ty^2 : x^1 + x^2 \geq x^* ; (y^1,x^1)\in S^1 ; (y^2,x^2)\in S^2\}$$
$$= \min_w \max_{y^1,x^1,y^2,x^2} \{p^Ty^1 + p^Ty^2 + w^T[x^1 + x^2 - x^*] ; (y^1,x^1)\in S^1; (y^2,x^2)\in S^2; w \geq 0_N\}$$
using the Karlin Uzawa saddle point theorem
$$= \min_w \max_{y^1,x^1,y^2,x^2} \{p^Ty^1+w^Tx^1+p^Ty^2+w^Tx^2-w^Tx^* ; (y^1,x^1)\in S^1; (y^2,x^2)\in S^2; w \geq 0_N\}$$
$$= \min_w \{\pi^1(p,w) + \pi^2(p,w) - w^Tx^* ; w \geq 0_N\}$$
using definitions (1).

The first order necessary conditions for an interior solution for the minimization problem in the last line of (9) are:

$$(10)\ \nabla_w\pi^1(p,w^\circ) + \nabla_w\pi^2(p,w^\circ) - x^* = 0_N.$$

We will assume that a strictly positive solution $w^\circ >> 0_N$ to equations (10) exists. From chapter 8, we know that this w° also solves the minimization problem in the last line of (9).

The above material, while showing that there will generally be costs to the tax and subsidy distortions, does not indicate the probable magnitude of the cost. In the following section, we develop a second order approximation to the loss, which could be used to determine the magnitude of the loss and the factors which might make it larger or smaller.

3. A Second Order Approximation to the Costs of Distortions in the Production Sector: Preliminary Results

Let z be a scalar variable that takes on values between 0 and 1 and consider the following economic model:

$$(11)\ f(z) = p^T\nabla_p\pi^1(p+z\tau^1,w(z)+zt^1) + p^T\nabla_p\pi^2(p+z\tau^2,w(z)+zt^2) ;$$
$$(12)\ x^* = \nabla_w\pi^1(p+z\tau^1,w(z)+zt^1) + \nabla_w\pi^2(p+z\tau^2,w(z)+zt^2).$$

---

[9] For a generalization of this result to many constant returns to scale sectors, see Diewert (1983; 162).

Equations (11) and (12) are N+1 equations in the N+2 variables f, w and z. We regard f and w as being endogenous variables and z as the exogenous variable so that f = f(z) and w = w(z). In order to economize on notation, at times, we shall write $\pi^1(p+z\tau^1,w(z)+zt^1)$ as $\pi^1(z)$ and $\pi^2(p+z\tau^2,w(z)+zt^2)$ as $\pi^2(z)$.

When z = 1, equations (11) and (12) become the following equations:

(13) $f(1) = p^T\nabla_p\pi^1(p+\tau^1,w(1)+t^1) + p^T\nabla_p\pi^2(p+\tau^2,w(1)+t^2)$;
(14) $x^* = \nabla_w\pi^1(p+\tau^1,w(1)+t^1) + \nabla_w\pi^2(p+\tau^2,w(1)+t^2)$.

If we set $f(1) = f^*$ and $w(1) = w^*$, it can be seen that equations (13) and (14) become equations (4) and (5), the equations that defined the tax distorted equilibrium. On the other hand, if we set z = 0, equations (11) and (12) become the following equations:

(15) $f(0) = p^T\nabla_p\pi^1(p,w(0)) + p^T\nabla_p\pi^2(p,w(0))$;
(16) $x^* = \nabla_w\pi^1(p,w(0)) + \nabla_w\pi^2(p,w(0))$.

If we set $f(0) = f^\circ$ defined by (9) and $w(0) = w^\circ$ defined by (10), then upon defining $y^{1\circ} \equiv \nabla_p\pi^1(p,w^\circ)$ and $y^{2\circ} \equiv \nabla_p\pi^2(p,w^\circ)$ so that

(17) $f^\circ = p^Ty^{1\circ} + p^Ty^{2\circ}$;

then it can be seen that equations (15) and (16) become equations (17) and (10), which are the equations that characterize the undistorted equilibrium. Thus equations (11) and (12) map the undistorted equilibrium into the observed tax distorted equilibrium as z travels from 0 to 1. Our goal in this section is to find a second order Taylor series approximation to f(1) around the point f(0); i.e., we want to evaluate the following approximation:

(18) $f^* = f(1) \cong f(0) + f'(0)[1-0] + (1/2)f''(0)[1-0]^2$
$= f^\circ + f'(0) + (1/2)f''(0)$.

Thus we need to determine the first and second derivatives $f'(0)$ and $f''(0)$.

Looking at equations (11) and (12) which determine f(z), it can be seen that f(z) does not appear in equation (12). Hence (12) can be differentiated with respect to z and the resulting equation will determine the vector of domestic price derivatives, $w'(z)$. Differentiating (12) with respect to z leads to the following equation:

(19) $[\nabla_{ww}{}^2\pi^1(p+z\tau^1,w(z)+zt^1) + \nabla_{ww}{}^2\pi^2(p+z\tau^2,w(z)+zt^2)]w'(z)$
$= -\nabla_{wp}{}^2\pi^1(p+z\tau^1,w(z)+zt^1)\tau^1 - \nabla_{wp}{}^2\pi^2(p+z\tau^2,w(z)+zt^2)\tau^2$
$-\nabla_{ww}{}^2\pi^1(p+z\tau^1,w(z)+zt^1)t^1 - \nabla_{ww}{}^2\pi^2(p+z\tau^2,w(z)+zt^2)t^2$.

We assume that the N by N matrix on the left hand side of (19) has an inverse; i.e., assume

(20) $[\nabla_{ww}{}^2\pi^1(p+z\tau^1,w(z)+zt^1) + \nabla_{ww}{}^2\pi^2(p+z\tau^2,w(z)+zt^2)]^{-1}$ exists.

Writing $\nabla_{ww}{}^2\ \pi^1(p+z\tau^1,w(z)+zt^1)$ more succinctly as $\nabla_{ww}{}^2\pi^1(z)$, etc., and using assumption (20), (19) becomes the following equation:

$$(21)\ w'(z) = -\,[\nabla_{ww}{}^2\pi^1(z)+\nabla_{ww}{}^2\pi^2(z)]^{-1}[\nabla_{wp}{}^2\pi^1(z)\tau^1+\nabla_{wp}{}^2\pi^2(z)\tau^2+\nabla_{ww}{}^2\pi^1(z)t^1+\nabla_{ww}{}^2\pi^2(z)t^2].$$

Now look at equations (11) and (12), which are the basic equations that define our model. Premultiply both sides of (12) by $w(z)^T$ and add the resulting equation to equation (11) in order to obtain the following equation:

$$(22)\ f(z) + w(z)^Tx^* = \pi^1(p+z\tau^1,w(z)+zt^1) + \pi^2(p+z\tau^2,w(z)+zt^2) - z\{\tau^{1T}\nabla_p\pi^1(p+z\tau^1,w(z)+zt^1) + \tau^{2T}\nabla_p\pi^2(p+z\tau^2,w(z)+zt^2) + t^{1T}\nabla_w\pi^1(p+z\tau^1,w(z)+zt^1) + t^{2T}\nabla_w\pi^2(p+z\tau^2,w(z)+zt^2)\}$$

where we have used the linear homogeneity of the profit functions in prices and Euler's theorem on homogeneous function in order to derive (22). Now differentiate (22) with respect to z. Making use of (12), the resulting equation simplifies to the following one:

$$(23)\ f'(z) = -\,z\{\tau^{1T}\nabla_{pp}{}^2\pi^1(z)\tau^1 + \tau^{1T}\nabla_{pw}{}^2\pi^1(z)t^1 + t^{1T}\nabla_{wp}{}^2\pi^1(z)\tau^1 + t^{1T}\nabla_{ww}{}^2\pi^1(z)t^1\} - z\{\tau^{2T}\nabla_{pp}{}^2\pi^2(z)\tau^2 + \tau^{2T}\nabla_{pw}{}^2\pi^2(z)t^2 + t^{2T}\nabla_{wp}{}^2\pi^2(z)\tau^2 + t^{2T}\nabla_{ww}{}^2\pi^2(z)t^2\} - z[\tau^{1T}\nabla_{pw}{}^2\pi^1(z) + \tau^{2T}\nabla_{pw}{}^2\pi^2(z) + t^{1T}\nabla_{ww}{}^2\pi^1(z) + t^{2T}\nabla_{ww}{}^2\pi^2(z)]w'(z).$$

Now substitute (21) into (23) in order to obtain the following formula for the derivative $f'(z)$:

$$(24)\ f'(z) = -\,z\{\tau^{1T}\nabla_{pp}{}^2\pi^1(z)\tau^1 + \tau^{1T}\nabla_{pw}{}^2\pi^1(z)t^1 + t^{1T}\nabla_{wp}{}^2\pi^1(z)\tau^1 + t^{1T}\nabla_{ww}{}^2\pi^1(z)t^1\} - z\{\tau^{2T}\nabla_{pp}{}^2\pi^2(z)\tau^2 + \tau^{2T}\nabla_{pw}{}^2\pi^2(z)t^2 + t^{2T}\nabla_{wp}{}^2\pi^2(z)\tau^2 + t^{2T}\nabla_{ww}{}^2\pi^2(z)t^2\} + z[\tau^{1T}\nabla_{pw}{}^2\pi^1(z) + \tau^{2T}\nabla_{pw}{}^2\pi^2(z) + t^{1T}\nabla_{ww}{}^2\pi^1(z) + t^{2T}\nabla_{ww}{}^2\pi^2(z)]\times [\nabla_{ww}{}^2\pi^1(z)+\nabla_{ww}{}^2\pi^2(z)]^{-1}[\nabla_{wp}{}^2\pi^1(z)\tau^1+\nabla_{wp}{}^2\pi^2(z)\tau^2+\nabla_{ww}{}^2\pi^1(z)t^1+\nabla_{ww}{}^2\pi^2(z)t^2].$$

Needless to say, formula (24) looks rather intimidating. However, since each term on the right hand side is multiplied by z, it is easy to see that

$$(25)\ f'(0) = 0.$$

Recall that the loss of foreign exchange due to the tax distortions L was defined by (8) above. Making use of the second order approximation formula (18) and our result (25), it can be seen that to the accuracy of a second order approximation, the loss is approximately equal to:

$$(26)\ L \equiv f(0) - f(1) \cong -\,(1/2)f''(0).$$

Differentiating (24) with respect to z and evaluating the resulting derivatives at z = 0 leads to the following formula for $-f''(0)$:

$$
\begin{aligned}
(27)\ -f''(0) = {} & \tau^{1T}\nabla_{pp}{}^{2}\pi^{1}(0)\tau^{1} + \tau^{1T}\nabla_{pw}{}^{2}\pi^{1}(0)t^{1} + t^{1T}\nabla_{wp}{}^{2}\pi^{1}(0)\tau^{1} + t^{1T}\nabla_{ww}{}^{2}\pi^{1}(0)t^{1} \\
& + \tau^{2T}\nabla_{pp}{}^{2}\pi^{2}(0)\tau^{2} + \tau^{2T}\nabla_{pw}{}^{2}\pi^{2}(0)t^{2} + t^{2T}\nabla_{wp}{}^{2}\pi^{2}(0)\tau^{2} + t^{2T}\nabla_{ww}{}^{2}\pi^{2}(0)t^{2} \\
& - [\tau^{1T}\nabla_{pw}{}^{2}\pi^{1}(0) + \tau^{2T}\nabla_{pw}{}^{2}\pi^{2}(0) + t^{1T}\nabla_{ww}{}^{2}\pi^{1}(0) + t^{2T}\nabla_{ww}{}^{2}\pi^{2}(0)]\times \\
& [\nabla_{ww}{}^{2}\pi^{1}(0)+\nabla_{ww}{}^{2}\pi^{2}(0)]^{-1}[\nabla_{wp}{}^{2}\pi^{1}(0)\tau^{1}+\nabla_{wp}{}^{2}\pi^{2}(0)\tau^{2}+\nabla_{ww}{}^{2}\pi^{1}(0)t^{1}+\nabla_{ww}{}^{2}\pi^{2}(0)t^{2}].
\end{aligned}
$$

In order to interpret the right hand side of (27), we define the following symmetric matrices:

$$
(28)\ A \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } B \equiv \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}
$$

where

$$(29)\ A_{11} \equiv \nabla_{pp}{}^{2}\pi^{1}(0)\ ;\ A_{12} \equiv \nabla_{pw}{}^{2}\pi^{1}(0)\ ;\ A_{21} \equiv \nabla_{wp}{}^{2}\pi^{1}(0)\ ;\ A_{22} \equiv \nabla_{ww}{}^{2}\pi^{1}(0) \text{ and}$$
$$(30)\ B_{11} \equiv \nabla_{pp}{}^{2}\pi^{2}(0)\ ;\ B_{12} \equiv \nabla_{pw}{}^{2}\pi^{2}(0)\ ;\ B_{21} \equiv \nabla_{wp}{}^{2}\pi^{2}(0)\ ;\ B_{22} \equiv \nabla_{ww}{}^{2}\pi^{2}(0).$$

Using the above definitions, it can be seen that (27) can be rewritten as follows:

$$
\begin{aligned}
(31)\ -f''(0) = {} & [\tau^{1T},t^{1T}]A[\tau^{1T},t^{1T}]^{T} + [\tau^{2T},t^{2T}]B[\tau^{2T},t^{2T}]^{T} \\
& - [\tau^{1T},t^{1T},\tau^{2T},t^{2T}]C[\tau^{1T},t^{1T},\tau^{2T},t^{2T}]^{T}
\end{aligned}
$$

where the matrix C is defined in terms of the block elements of A and B as follows:

$$
(32)\ C \equiv \begin{bmatrix} A_{12} \\ A_{22} \\ B_{12} \\ B_{22} \end{bmatrix} \left[A_{22} + B_{22}\right]^{-1} \left[A_{21} \quad A_{22} \quad B_{21} \quad B_{22}\right].
$$

Since A and B are positive semidefinite symmetric matrices, we know the first two terms on the right hand side of (31) are nonnegative. We also know that A_{22} and B_{22} are positive semidefinite and since we have assumed that $A_{22} + B_{22}$ has an inverse, we know that $[A_{22} + B_{22}]^{-1}$ is positive definite. Note that C is equal to $D[A_{22} + B_{22}]^{-1}D^{T}$ where D is defined in (32) and it is easy to show that C is positive semidefinite. However, the minus sign in front of the third term on the right hand side of (32) means that the third term is nonpositive. Hence $-f''(0)$ is equal to the sum of three terms where the first two are positive or zero and the third term is negative or zero and hence at this stage, the sign of $-f''(0)$ appears to be indeterminate. To show that the first two nonnegative terms outweigh the third nonpositive term is not a trivial exercise. In the following section, we digress briefly into the intricacies of matrix algebra to show that $-f''(0)$ is indeed nonnegative.

4. Some Results from Matrix Algebra

It can be seen that $-f''(0)$ will be nonnegative if we can show that $A^* + B^* - C$ is positive semidefinite where C is defined by (32) and A^* and B^* are defined as follows:

$$(33)\ A^* \equiv \begin{bmatrix} A & 0_{M+N\times M+N} \\ 0_{M+N\times M+N} & 0_{M+N\times M+N} \end{bmatrix}\ ;\ \ B^* \equiv \begin{bmatrix} 0_{M+N\times M+N} & 0_{M+N\times M+N} \\ 0_{M+N\times M+N} & B \end{bmatrix}$$

where A and B are the positive semidefinite symmetric M+N by M+N matrices defined by (28)-(30). We will establish this result by proving a series of Propositions.

Proposition 1: If A is a positive semidefinite symmetric matrix, then the matrix E defined below by (34) is also a positive semidefinite symmetric matrix:

$$(34)\ E \equiv \begin{bmatrix} A & A \\ A & A \end{bmatrix}.$$

Proof: Since A is symmetric, it is easy to see that E is also symmetric. Define the matrix F as follows:

$$(35)\ F \equiv \begin{bmatrix} I_{M+N} & -I_{M+N} \\ 0_{M+N\times M+N} & I_{M+N} \end{bmatrix}.$$

Since F is block triangular, the determinant of F is equal to the product of the determinants of the diagonal blocks so the determinant of F is 1. Now calculate:[10]

$$(36)\ F^TEF = \begin{bmatrix} A & 0_{M+N\times M+N} \\ 0_{M+N\times M+N} & 0_{M+N\times M+N} \end{bmatrix}.$$

Since A is positive semidefinite, we see that F^TEF is also positive semidefinite. But since the determinant of F is nonzero, F^TEF is a similarity transformation of E and preserves the rank and definiteness properties of E. Thus E is also a positive semidefinite matrix (that has the rank of A). Q.E.D.

Proposition 2: If A is a positive semidefinite symmetric matrix, then the matrix G defined below by (37) is also a positive semidefinite symmetric matrix:

$$(37)\ G \equiv \begin{bmatrix} A & 0_{M+N\times M+N} & A \\ 0_{M+N\times M+N} & 0_{M+N\times M+N} & 0_{M+N\times M+N} \\ A & 0_{M+N\times M+N} & A \end{bmatrix}.$$

[10] Essentially, we are applying the Jacobi diagonalization procedure to E in block form.

Proof: Define the matrix F as follows:

$$(38)\ F \equiv \begin{bmatrix} I_{M+N} & 0_{M+N\times M+N} & -I_{M+N} \\ 0_{M+N\times M+N} & I_{M+N} & 0_{M+N\times M+N} \\ 0_{M+\times M+N} & 0_{M+N\times M+N} & I_{M+N} \end{bmatrix}.$$

It can be seen that the determinant of F is 1. Now compute:

$$(39)\ F^{T}GF = \begin{bmatrix} A & 0_{M+N\times M+N} & 0_{M+N\times M+N} \\ 0_{M+N\times M+N} & 0_{M+N\times M+N} & 0_{M+N\times M+N} \\ 0_{M+\times M+N} & 0_{M+N\times M+N} & 0_{M+N\times M+N} \end{bmatrix}$$

which is positive semidefinite. Hence G is also positive semidefinite. Q.E.D.

Proposition 3: If B is a positive semidefinite symmetric matrix, then the matrix H defined below by (40) is also a positive semidefinite symmetric matrix:

$$(40)\ H \equiv \begin{bmatrix} 0_{M+N\times M+N} & 0_{M+N\times M+N} & 0_{M+N\times M+N} \\ 0_{M+N\times M+N} & B & B \\ 0_{M+N\times M+N} & B & B \end{bmatrix}.$$

Proof: Define the matrix F as follows:

$$(41)\ F \equiv \begin{bmatrix} I_{M+N} & 0_{M+N\times M+N} & 0_{M+N\times M+N} \\ 0_{M+N\times M+N} & I_{M+N} & -I_{M+N} \\ 0_{M+\times M+N} & 0_{M+N\times M+N} & I_{M+N} \end{bmatrix}.$$

It can be seen that the determinant of F is 1. Now compute:

$$(42)\ F^{T}HF = \begin{bmatrix} 0_{M+N\times M+N} & 0_{M+N\times M+N} & 0_{M+N\times M+N} \\ 0_{M+N\times M+N} & B & 0_{M+N\times M+N} \\ 0_{M+\times M+N} & 0_{M+N\times M+N} & 0_{M+N\times M+N} \end{bmatrix}$$

which is positive semidefinite. Hence H is also positive semidefinite. Q.E.D.

Proposition 4: If A and B are positive semidefinite symmetric matrices, then the matrix D defined below by (43) is also a positive semidefinite symmetric matrix:

$$(43)\ D \equiv \begin{bmatrix} A & 0_{M+N\times M+N} & A \\ 0_{M+N\times M+N} & B & B \\ A & B & A+B \end{bmatrix}.$$

Proof: D is equal to the sum of the positive semidefinite matrices G and H defined above by (37) and (40) and hence must be positive semidefinite. Q.E.D.

Let A and B be the positive semidefinite partitioned matrices defined by (28) above. Inserting these partitioned matrices into the positive semidefinite matrix D defined above by (43) leads to the following expression for D:

$$(44)\ D = \begin{bmatrix} A_{11} & A_{12} & 0_{M\times M} & 0_{M\times N} & A_{11} & A_{12} \\ A_{21} & A_{22} & 0_{N\times M} & 0_{N\times N} & A_{21} & A_{22} \\ 0_{M\times M} & 0_{M\times N} & B_{11} & B_{12} & B_{11} & B_{12} \\ 0_{N\times M} & 0_{N\times N} & B_{21} & B_{22} & B_{21} & B_{22} \\ A_{11} & A_{12} & B_{11} & B_{12} & A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} & A_{21}+B_{21} & A_{22}+B_{22} \end{bmatrix}.$$

Delete the fifth block of columns and the fifth block of rows from D and call the resulting matrix E:

$$(45)\ E \equiv \begin{bmatrix} A_{11} & A_{12} & 0_{M\times M} & 0_{M\times N} & A_{12} \\ A_{21} & A_{22} & 0_{N\times M} & 0_{N\times N} & A_{22} \\ 0_{M\times M} & 0_{M\times N} & B_{11} & B_{12} & B_{12} \\ 0_{N\times M} & 0_{N\times N} & B_{21} & B_{22} & B_{22} \\ A_{21} & A_{22} & B_{21} & B_{22} & A_{22}+B_{22} \end{bmatrix}.$$

E will also be a positive semidefinite symmetric matrix. Assume that the N by N matrix $A_{22} + B_{22}$ has an inverse; i.e., assume that

(46) $[A_{22} + B_{22}]^{-1}$ exists.

Now take $-A_{12}\,[A_{22} + B_{22}]^{-1}$ times the last block of rows in (45) and add the resulting block of rows to the first block of rows in (45). Then take $-A_{22}\,[A_{22} + B_{22}]^{-1}$ times the last block of rows in (45) and add the resulting block of rows to the second block of rows in (45). Then take $-B_{12}\,[A_{22} + B_{22}]^{-1}$ times the last block of rows in (45) and add the resulting block of rows to the third block of rows in (45). Finally, take $-B_{22}\,[A_{22} + B_{22}]^{-1}$ times the last block of rows in (45) and add the resulting block of rows to the fourth block of rows in (45). The resulting transformed matrix will have four blocks of zero matrices in the rows above the southeast corner block, $A_{22} + B_{22}$. The transformed matrix is equal to F^TE, where F^T represents the various block row operations that we have applied to E. Thus F^TE is equal to the following matrix:

$$(47)\ F^TE = \begin{bmatrix} & & K & & \begin{matrix} 0_{M\times N} \\ 0_{N\times N} \\ 0_{M\times N} \\ 0_{N\times N} \end{matrix} \\ A_{21} & A_{22} & B_{21} & B_{22} & A_{22}+B_{22} \end{bmatrix} .$$

Now take $-A_{21}\,[A_{22} + B_{22}]^{-1}$ times the last block of columns on the right hand side of (47) and add the resulting block of columns to the first block of columns on the right hand side of (47). Then take $-A_{22}\,[A_{22} + B_{22}]^{-1}$ times the last block of columns in (47) and add the resulting block of columns to the second block of columns in (47). Then take $-B_{21}\,[A_{22} + B_{22}]^{-1}$ times the last block of columns in (47) and add the resulting block of columns to the third block of columns in (47). Finally, take $-B_{22}\,[A_{22} + B_{22}]^{-1}$ times the last block of columns in (47) and add the resulting block of columns to the fourth block of columns on the right hand side of (47). The resulting matrix is equal to the right hand side of (48) and it is also equal to F^TEF; i.e., we have

$$(48)\ F^TEF = \begin{bmatrix} & & K & & \begin{matrix} 0_{M\times N} \\ 0_{N\times N} \\ 0_{M\times N} \\ 0_{N\times N} \end{matrix} \\ 0_{N\times M} & 0_{N\times N} & 0_{N\times M} & 0_{N\times N} & A_{22}+B_{22} \end{bmatrix} .$$

Since E is positive semidefinite, so is the matrix K. Performing the above block matrix operations, we find that K is equal to the following matrix:

$$(49)\ K = \begin{bmatrix} A & 0_{M+N\times M+N} \\ 0_{M+N\times M+N} & 0_{M+N\times M+N} \end{bmatrix} + \begin{bmatrix} 0_{M+N\times M+N} & 0_{M+N\times M+N} \\ 0_{M+N\times M+N} & B \end{bmatrix} - C$$

where C is defined by (32). Since K is positive semidefinite, we have shown that $-\,f'(0)$ defined by (31) in the previous section is nonnegative; i.e., to the accuracy of a second order approximation, it is always wasteful[11] to tax international trade transactions or to have taxes on domestic inputs and outputs that cause domestic producers to face different domestic relative prices. Another useful observation that emerges from formula (31) and the analysis in this section is that the loss of efficiency grows as the *squares* of the

[11] More accurately, it can never be beneficial under our assumptions; a fact which we already knew from the inequality (8) in section 2. The advantage of the present methodology is that given elasticity information, we can obtain approximate estimates of the loss in productive efficiency due to the tax distortions.

distorting tax rates. Thus for small tax rates the (approximate) loss will be small but it will grow quadratically if we increase all of the tax rates in a proportional fashion.[12]

What is the intuition behind the third term on the right hand side of (31); i.e., how can we interpret the term $-[\tau^{1T},t^{1T},\tau^{2T},t^{2T}]C[\tau^{1T},t^{1T},\tau^{2T},t^{2T}]^T$ which *reduces* the losses of efficiency that are inherent in the first two terms, $[\tau^{1T},t^{1T}]A[\tau^{1T},t^{1T}]^T + [\tau^{2T},t^{2T}]B[\tau^{2T},t^{2T}]^T$? The intuition is as follows: the imposition of the taxes τ^1,t^1,τ^2,t^2 induces the two industries to allocate resources in an inefficient manner which leads to the approximate losses , $[\tau^{1T},t^{1T}]A[\tau^{1T},t^{1T}]^T$ and $[\tau^{2T},t^{2T}]B[\tau^{2T},t^{2T}]^T$. However, these losses ignore the constraint (5); i.e., the constraint that the two domestic production vectors x^1 and x^2 are forced to add up to x^*. This constraint reduces the initial losses represented by the A and B terms by the term involving the C matrix.

In the following section, we introduce a consumer sector into our model and adapt the results in the previous section to this more complete framework.

Problems

1. Suppose that the tax distortion vectors are proportional to the international prices p and the undistorted domestic price vector w(0) or w°; i.e., we have for some scalars λ_1 and λ_2:

(a) $\tau^1 = \lambda_1 p$; $t^1 = \lambda_1 w°$;
(b) $\tau^2 = \lambda_2 p$; $t^2 = \lambda_2 w°$.

We assume that λ_1 and λ_2 are small enough so that $p+\tau^1, w°+t^1, p+\tau^2$ and $w°+t^2$ are all positive price vectors. Show that under these conditions that $f''(0)$ defined by (31) above is equal to 0.

2. Suppose that there are no tax distortions on internationally traded commodities so that

(a) $\tau^1 = \tau^2 = 0_M$.

Suppose further that each industry is treated in an equitable manner so that the domestic tax rates are the same for each industry; i.e., we have

(b) $t^1 = t^2 = t$.

We assume that any negative components of t are small enough so that $w^* + t >> 0_N$.
(i) Calculate $f''(0)$ defined by (31) under assumptions (a) and (b).
(ii) Provide an economic interpretation for the result you derived in part (i).

5. The Measurement of Waste when there are Households and Producers

[12] This result is similar to the deadweight loss result that we obtained in section 13 of chapter 3. For results that are similar to our present results, see Diewert (1983; 165-171).

In this section, we extend the production sector model that was explained in section 2 to an economy with households. In order to keep the notation manageable, we will consolidate the production sector into a single aggregate production sector so we are implicitly assuming that each producer faces the same vector of tax distorted prices.[13] Let the producer's production possibilities set be S and define the aggregate profit function π in the usual way:

(50) $\pi(p,w) \equiv \max_{y,x} \{p^Ty + w^Tx : (y,x) \in S\}$;

As usual, we will assume that the profit function is twice continuously differentiable.

Turning now to the consumer side of the model, we assume that there are two households in the economy (or classes of identical households). The extension of our model to many households is reasonably straightforward. We assume that the consumer has preferences over domestic goods and services. Let household h's utility function $f^h(c)$ be a continuous quasiconcave function defined over a closed convex consumption possibilities set C say and define household h's expenditure function $e^h(u^h,w)$ in the usual way, where u^h is in the range of f^h and $w >> 0_N$ is a strictly positive vector of domestic prices:

(51) $e^h(u^h,w) \equiv \min_c \{w^Tc : f^h(c) \geq u^h ; c \in C\}$; h = 1,2.

We assume that each $e^h(u^{h*},w)$ is twice continuously differentiable with respect to w when utility is set equal to the level of utility u^{h*} that household h achieves in an observed tax distorted equilibrium.

We now consider a tax distorted equilibrium where the foreign price vector for internationally traded goods and services is $p >> 0_M$. However, as in section 2, the domestic government imposes various tariffs, export subsidies and other trade taxes on international trade so that the production sector faces the international price vector p plus the tax distortion vector τ. With respect to domestic transactions, we assume that the production sector faces the vector of domestic prices $w^* >> 0_N$ and that each household h faces the domestic price vector w^*+t for h = 1,2 so that all households are treated in the same manner for tax purposes. Using Hotelling's (1932) Lemma, we can calculate the net export vector of the production sector, y^*, in the usual way:

(52) $y^* = \nabla_p\pi(p+\tau,w^*)$.

Again using Hotelling's Lemma, we can calculate the *net domestic supply vector* in the tax distorted equilibrium, x^*, as follows:

(53) $x^* = \nabla_w\pi(p+\tau,w^*)$.

---

[13] If this were not the case, then the production sector should be disaggregated into sectors where each sector faces the same vector of tax distortions.

As was the case in our earlier model, we can calculate the *net amount of foreign exchange* f^* that the production sector generated in the tax distorted equilibrium as the sum of the value of net exports at international prices:

$$(54)\ f^* \equiv p^T y^* = p^T \nabla_p \pi(p+\tau, w^*) \qquad \text{using (52).}$$

Let c^{1*} and c^{2*} be the observed net consumption vectors of the two households[14] in the tax distorted equilibrium and let u^{1*} and u^{2*} be the corresponding levels of utility that each household attained. In addition to demanding foreign exchange, we assume that the government also requires the vector $g \equiv [g_1,...,g_N]^T$ of domestic commodities in order to produce government goods and services. Thus the supply equals demand equations for domestic commodities in the tax distorted equilibrium are as follows:

$$(55)\ x^* = c^{1*} + c^{2*} + g.$$

Using Hotelling's Lemma and Shephard's Lemma, equation (55) can be rewritten as follows:

$$(56)\ \nabla_w \pi(p+\tau, w^*) = \nabla_w e^1(u^{1*}, w^*+t) + \nabla_w e^2(u^{2*}, w^*+t) + g.$$

We use the observed household utility levels u^{1*} and u^{2*} in order to define the *household consumption possibilities sets*, C^{1*} and C^{2*}, as follows:

$$(57)\ C^{h*} \equiv \{c : f^h(c) \geq u^{h*} ; c \in C\} ; \qquad h = 1,2.$$

Thus C^{h*} is the set of net consumption vectors that will give household h at least the utility level u^{h*} that household h achieved in the observed tax distorted equilibrium. Note that the quasiconcavity of the utility functions f^h implies that the sets C^{h*} are convex. Note also that the observed net consumption vector for household h, c^{h*}, belongs to C^{h*} for each h. The observed production vectors are also feasible; i.e., we have

$$(58)\ (y^*, x^*) \in S ; c^{1*} \in C^{1*} \text{ and } c^{2*} \in C^{2*}.$$

Now we consider a hypothetical equilibrium where the government eliminates all of the tax and commodity subsidy distortions; i.e., τ and t are reduced to zero vectors. Under this new tax regime, we consider the problem of maximizing the amount of foreign exchange that the production sector can produce, given that the supply equals demand equations (55) are respected and households are allocated net consumption vectors that allow them to attain their tax distorted equilibrium levels of utility. This foreign exchange constrained maximization problem is the following nonlinear programming problem:

---

[14] If $c_n^{h*} > 0$, then domestic commodity n was demanded by household h in the tax distorted equilibrium; if $c_n^{h*} < 0$, then domestic commodity n was being supplied by household h as a factor of production in the tax distorted equilibrium.

$$(59)\ f^\circ \equiv \max_{y,x,c^1,c^2} \{p^Ty : x \geq c^1 + c^2 + g\ ;\ (y,x)\in S\ ;\ c^1\in C^{1*};\ c^2\in C^{2*}\}$$
$$\geq p^Ty^* \quad \text{since } y^*,x^*,c^{1*},c^{2*} \text{ is a feasible solution for the problem using (58)}$$
$$= f^* \quad \text{using definition (54).}$$

Thus the optimal amount of foreign exchange that the economy sector can generate if the tax distortions are eliminated, f°, is always equal to or greater than the amount of foreign exchange generated in the observed tax distorted equilibrium, f^*. If we define the deadweight loss L due to the tax distortions as the optimal amount less the observed amount of foreign exchange, we have, using (59):[15]

$$(60)\ L \equiv f^\circ - f^* \geq 0.$$

As was the case in section 2, it is possible to use duality theory and the Karlin (1959) Uzawa (1958) saddle point theorem to express the nonlinear programming problem defined by the first line of (59) in an instructive dual form, which we will use later; i.e., we have:

$$(61)\ f^\circ \equiv \max_{y,x,c^1,c^2} \{p^Ty : x \geq c^1 + c^2 + g\ ;\ (y,x)\in S\ ;\ c^1\in C^{1*};\ c^2\in C^{2*}\}$$
$$= \min_w \max_{y,x,c^1,c^2} \{p^Ty + w^T[x - c^1 - c^2 - g]\ ;\ (y,x)\in S\ ;\ c^1\in C^{1*};\ c^2\in C^{2*};\ w \geq 0_N\}$$
$$\text{using the Karlin Uzawa saddle point theorem}$$
$$= \min_w \max_{y^1,x^1,y^2,x^2} \{p^Ty+w^Tx-w^Tc^1-w^Tc^2-w^Tg\ ;\ (y,x)\in S\ ;\ c^1\in C^{1*};\ c^2\in C^{2*};\ w \geq 0_N\}$$
$$= \min_w \{\pi(p,w) - e^1(u^{1*},w) - e^2(u^{2*},w) - w^Tg\ ;\ w \geq 0_N\} \quad \text{using definitions (50) and (51).}$$

The first order necessary conditions for an interior solution for the minimization problem in the last line of (61) are:

$$(62)\ \nabla_w\pi(p,w^\circ) - \nabla_we^1(u^{1*},w^\circ) - \nabla_we^2(u^{2*},w^\circ) - g = 0_N.$$

We will assume that a strictly positive solution $w^\circ >> 0_N$ to equations (62) exists. From chapter 8, we know that this w° also solves the minimization problem in the last line of (9).

In the remainder of this section, we will adapt the analysis presented in section 3 above and develop a second order approximation to the loss of foreign exchange L defined by (60).

Let z be a scalar variable that takes on values between 0 and 1 and consider the following economic model:

$$(63)\ f(z) = p^T\nabla_p\pi(p+z\tau,w(z))\ ;$$
$$(64)\quad g = \nabla_w\pi(p+z\tau,w(z)) - \nabla_we^1(u^{1*},w(z)+zt) - \nabla_we^2(u^{2*},w(z)+zt).$$

[15] For a generalization of this result to many constant returns to scale sectors, see Diewert (1983; 162).

Equations (63) and (64) are N+1 equations in the N+2 variables f, w and z. We regard f and w as being endogenous variables and z as the exogenous variable so that f = f(z) and w = w(z). In order to economize on notation, at times, we shall write $\pi(p+z\tau,w(z))$ as $\pi(z)$ and $e^h(u^{h*},w(z)+zt)$ as $e^h(z)$ for h = 1,2.

When z = 1, equations (63) and (64) become the following equations:

$$(65)\ f(1) = p^T\nabla_p\pi(p+\tau,w(1))\ ;$$
$$(66)\quad g\ = \nabla_w\pi(p+\tau,w(1)) - \nabla_w e^1(u^{1*},w(1)+t) - \nabla_w e^2(u^{2*},w(1)+t).$$

If we set $f(1) = f^*$ and $w(1) = w^*$, it can be seen that equations (65) and (66) become equations (54) and (55), the equations that defined the tax distorted equilibrium. On the other hand, if we set z = 0, equations (63) and (64) become the following equations:

$$(67)\ f(0) = p^T\nabla_p\pi(p,w(0))\ ;$$
$$(68)\quad g\ = \nabla_w\pi(p,w(0)) - \nabla_w e^1(u^{1*},w(0)) - \nabla_w e^2(u^{2*},w(0)).$$

If we set $f(0) = f^\circ$ defined by (61) and $w(0) = w^\circ$ defined implicitly by (62), then upon defining $y^\circ \equiv \nabla_p\pi(p,w^\circ)$ so that

$$(69)\ f^\circ = p^Ty^\circ\ ;$$

then it can be seen that equations (67) and (68) become equations (69) and (62), which are the equations that characterize the undistorted equilibrium. Thus equations (63) and (64) map the undistorted equilibrium into the observed tax distorted equilibrium as z travels from 0 to 1. Our goal now is to find a second order Taylor series approximation to f(1) around the point f(0); i.e., we want to evaluate the following approximation:

$$(70)\ f^* = f(1) \cong f(0) + f'(0)[1-0] + (1/2)f''(0)[1-0]^2$$
$$= f^\circ + f'(0) + (1/2)f''(0).$$

Thus we need to determine the first and second derivatives $f'(0)$ and $f''(0)$.

Looking at equations (63) and (64) which determine f(z), it can be seen that f(z) does not appear in equation (64). Hence (64) can be differentiated with respect to z and the resulting equation will determine the vector of domestic price derivatives, $w'(z)$. Differentiating (64) with respect to z leads to the following equation:

$$(71)\ [\nabla_{ww}^2\pi(p+z\tau,w(z)) - \nabla_{ww}^2 e^1(u^{1*},w(z)+zt) - \nabla_{ww}^2 e^2(u^{2*},w(z)+zt)]w'(z)$$
$$= -\nabla_{wp}^2\pi(p+z\tau,w(z))\tau + \nabla_{ww}^2 e^1(u^{1*},w(z)+zt)t + \nabla_{ww}^2 e^2(u^{2*},w(z)+zt)t$$

We assume that the N by N matrix on the left hand side of (71) has an inverse; i.e., assume

(72) $[\nabla_{ww}{}^2\pi(p+z\tau,w(z)) - \nabla_{ww}{}^2 e^1(u^{1*},w(z)+zt) - \nabla_{ww}{}^2 e^2(u^{2*},w(z)+zt)]^{-1}$ exists.

Writing $\nabla_{ww}{}^2\pi(p+z\tau,w(z))$ more succinctly as $\nabla_{ww}{}^2\pi(z)$, etc., and using assumption (72), (71) becomes the following equation:

(73) $w'(z)$
$= -[\nabla_{ww}{}^2\pi(z)-\nabla_{ww}{}^2 e^1(z))-\nabla_{ww}{}^2 e^1(z)]^{-1}[\nabla_{wp}{}^2\pi(z)\tau - \nabla_{ww}{}^2 e^1(z)t - \nabla_{ww}{}^2 e^2(z)t]$.

Now look at equations (63) and (64), which are the basic equations that define our model. Premultiply both sides of (64) by $w(z)^T$ and add the resulting equation to equation (63) in order to obtain the following equation:

(74) $f(z) + w(z)^Tg = \pi(p+z\tau,w(z)) - e^1(u^{1*},w(z)+zt) - e^2(u^{2*},w(z)+zt)$
$- z\{\tau^T\nabla_p\pi(p+z\tau,w(z)) - t^T\nabla_w e^1(u^{1*},w(z)+zt) - t^T\nabla_w e^1(u^{1*},w(z)+zt)\}$

where we have used the linear homogeneity of the profit and expenditure functions in prices and Euler's theorem on homogeneous function in order to derive (74). Now differentiate (74) with respect to z. Making use of (64), the resulting equation simplifies to the following one:

(75) $f'(z) = -z\{\tau^T\nabla_{pp}{}^2\pi(z)\tau - t^T\nabla_{ww}{}^2 e^1(z)t - t^T\nabla_{ww}{}^2 e^2(z)t\}$
$- z[\tau^T\nabla_{pw}{}^2\pi(z) - t^T\nabla_{ww}{}^2 e^1(z) - t^T\nabla_{ww}{}^2 e^2(z)]w'(z)$.

Now substitute (73) into (75) in order to obtain the following formula for the derivative $f'(z)$:

(76) $f'(z) = -z\{\tau^T\nabla_{pp}{}^2\pi(z)\tau - t^T\nabla_{ww}{}^2 e^1(z)t - t^T\nabla_{ww}{}^2 e^2(z)t\}$
$+ z[\tau^T\nabla_{pw}{}^2\pi(z) - t^T\nabla_{ww}{}^2 e^1(z) - t^T\nabla_{ww}{}^2 e^2(z)]\times$
$[\nabla_{ww}{}^2\pi(z)-\nabla_{ww}{}^2 e^1(z))-\nabla_{ww}{}^2 e^1(z)]^{-1}[\nabla_{wp}{}^2\pi(z)\tau - \nabla_{ww}{}^2 e^1(z)t - \nabla_{ww}{}^2 e^2(z)t]$.

Since each term on the right hand side of (76) is multiplied by z, it is easy to see that

(77) $f'(0) = 0$.

Recall that the loss of foreign exchange due to the tax distortions L was defined by (60) above. Making use of the second order approximation formula (70) and the result (77), it can be seen that to the accuracy of a second order approximation, the loss is approximately equal to:

(78) $L \equiv f(0) - f(1) \cong -(1/2)f''(0)$.

Differentiating (76) with respect to z and evaluating the resulting derivatives at z = 0 leads to the following formula for $-f''(0)$:

(79) $-f''(0) = \tau^T\nabla_{pp}{}^2\pi(0)\tau - t^T\nabla_{ww}{}^2 e^1(0)t - t^T\nabla_{ww}{}^2 e^2(0)t$

$$- [\tau^T\nabla_{pw}{}^2\,\pi(0) - t^T\nabla_{ww}{}^2\,e^1(0) - t^T\nabla_{ww}{}^2\,e^2(0)]\times$$
$$[\nabla_{ww}{}^2\,\pi(0) - \nabla_{ww}{}^2\,e^1(0)\,) - \nabla_{ww}{}^2\,e^2(0)]^{-1}[\nabla_{wp}{}^2\,\pi(0)\tau - \nabla_{ww}{}^2\,e^1(0)t - \nabla_{ww}{}^2\,e^2(0)t].$$

In order to interpret the right hand side of (79), we define the following symmetric matrices:[16]

$$(80)\ A \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B_{22} \equiv B_{22}{}^1 + B_{22}{}^2$$

where

$$(81)\ A_{11} \equiv \nabla_{pp}{}^2\,\pi(0)\ ;\ A_{12} \equiv \nabla_{pw}{}^2\,\pi(0)\ ;\ \ A_{21} \equiv \nabla_{wp}{}^2\,\pi(0)\ ;\ A_{22} \equiv \nabla_{ww}{}^2\,\pi(0)\ \text{ and}$$
$$(82)\ B_{22}{}^h \equiv -\nabla_{ww}{}^2\,e^h(0)\ ;\ h = 1,2.$$

It can be seen that A is the positive semidefinite producer substitution matrix at the undistorted equilibrium and B_{22} is the negative of the sum of the two consumer substitution matrices and hence is also a positive semidefinite matrix. Using definitions (80)-(82), it can be seen that (79) can be rewritten as follows:

$$(83)\ -f''(0) = \tau^TA_{11}\tau + t^TB_{22}t - [\tau^TA_{12} + t^TB_{22}][A_{22}+B_{22}]^{-1}[A_{21}\tau + B_{22}t] \geq 0.$$

Since A_{11} and B_{22} are positive semidefinite matrices, it is clear that the first two terms on the right hand side of (83), $\tau^TA_{11}\tau$ and $t^TB_{22}t$, are nonnegative. However, since $A_{22}+B_{22}$ is positive definite, it can be seen that the third term on the right hand side of (83) is nonpositive. However, we can show that the first two terms outweigh the third term using Proposition 4 in section 4; i.e., recalling (31) and (32) above, set $\tau^1 = \tau$, $t^1 = 0_N$, $\tau^2 = 0_M$, $t^1 = t$, A is defined by (81) and define B_{22} as $B_{22}{}^1+B_{22}{}^2$ where the $B_{22}{}^h$ are defined by (82) and finally define $B_{11} \equiv 0_{MxM}$, $B_{12} \equiv 0_{MxN}$ and $B_{21} \equiv 0_{NxM}$ and apply Proposition 4.

The above model is a blend of results due to Allais (1943) (1973) (1977), Boiteux (1951) and Diewert (1983).

Problems

3. Derive a counterpart to (83) when there are H households instead of only 2 households.

4. (Difficult) Consider the model presented in this section but now assume that household h faces the price vector $w^* + t^h$ for h = 1,2 in the tax distorted equilibrium where $t^1 \neq t^2$. Derive a counterpart to the inequality in (83) under these conditions (if possible).

6. Marginal Excess Burdens: The Case of a Change in Transfers

[16] The matrix 0_{MxM} is an M by M matrix of zeros, etc.

We now consider a simplification of the model presented in the previous section in that we consider a one household economy. We also know that it is inefficient to tax international trade transactions (if the country cannot affect international prices) and so we will also assume that the vector of trade taxes or subsidies τ is equal to 0_M. We now assume that the government can tax fixed factors and pure profits. This turns out to be equivalent to assuming that the government owns the production sector and gives the household a transfer T as its nonlabour income. We also assume that the government spends a certain amount of its tax revenues on internationally traded goods (or equivalently, reduces or increases its foreign debt) and so we replace f in the previous sections by g_0, the government's foreign exchange requirements.

In this section and in the next section, we will look at the effects of changes in taxes and transfers on the utility of the consumer. The basic idea is that the government increases a tax rate by a marginal amount, gets increased revenues and we look at how many extra units of foreign exchange the government could buy with the increased tax revenues. This is a *benefit* to the economy in that the government could reduce its foreign debt or use the extra foreign exchange to purchase imports that could be used to expand public services. But there is a *cost* to the economy as well: presumably, the increased taxation will lead to a loss of consumer utility (defined only over private goods). Thus we need to look at the marginal loss in (money metric) utility that is caused by the marginal tax increase and divide this loss by the marginal amount of extra foreign exchange that the tax increase makes possible. This ratio is called the *marginal excess burden of the tax change*.[17]

In the present section, we need to decrease the transfer T (which is equivalent to increasing profit taxes) and this will free up some foreign exchange and decrease utility at the same time. We need to calculate these derivatives. The basic equations in our model are the following ones:

(84) $$g_0 = p^T\nabla_p\pi(p,w) ;$$
(85) $$e(u,w+t) = T ;$$
(86) $$\nabla_w e(u,w+t) + g = \nabla_w\pi(p,w).$$

Equation (84) says that the government spends the foreign exchange that is raised by the private production sector. Equation (85) equates household spending e(u,w+t) (less labour earnings if labour is in the household utility function) on domestic goods and services to the transfer income T that it receives from the government. Equations (86) set household net demand for domestic goods and services, $\nabla_w e(u,w+t)$, plus government net demands for domestic goods, g, equal to market sector net supplies of domestic commodities, $\nabla_p\pi(p,w)$. T is regarded as the exogenous variable and the household utility level u, government net spending on foreign goods and services g_0 and the vector

[17] There are many ways that have been used in the literature to define marginal excess burden. Frequently, one looks at the extra revenue that is raised by the tax as the benefit and the loss of utility as the cost. However, we prefer to convert the extra revenue into foreign exchange in order to have a concept that is not dependent on the degree of domestic inflation.

of domestic producer prices w are regarded as endogenous variables so we have N + 2 equations in N + 2 unknowns.

In order to obtain the government budget constraint, premultiply equation (86) by the transpose of w+t and add equation (84) to the resulting equation. We obtain the following equation, using the linear homogeneity properties of $\pi(p,w)$ and $e(u,w)$ in prices:

$$(87)\ g_0 + e(u,w+t) + (w+t)^T g = \pi(p,w) + t^T\nabla_w\pi(p,w).$$

Using equation (85) to replace household expenditure $e(u,w+t)$ by the transfer T leads to the following government budget constraint:

$$(88)\ g_0 + T + (w+t)^T g = \pi(p,w) + t^T\nabla_w\pi(p,w).$$

On the left hand side of (88), we have the government's expenditures on foreign exchange, g_0, plus government transfers to households, T, plus government expenditures on domestic goods and services (at consumer prices), $(w+t)^T g$. On the right hand side of (88), we have the government's sources of revenue: profits $\pi(p,w)$ plus commodity tax revenue, $t^T\nabla_w\pi(p,w)$.

Looking at equations (84)-(86), it can be seen that g_0 appears only in equation (84) but not in (85) and (86). Hence once the vector of derivatives of domestic prices with respect to T have been determined, $w'(T) = [\partial w_1(T)/\partial T,..., \partial w_N(T)/\partial T]^T$, we can determine the derivative of government spending on foreign exchange with respect to T, $g_0'(T) = \partial g_0(T)/\partial T$, by differentiating (84) with respect to T to obtain the following equation:

$$(89)\ g_0'(T) = p^T\nabla_{pw}^2\pi(p,w)w'(T) = p^T S_{pw}\, w'(T)$$

where $S_{pw} \equiv \nabla_{pw}^2\pi(p,w)$ is the producer substitution matrix between internationally traded commodities and domestic commodities. In order to obtain expressions for the derivatives $u'(T)$ and $w'(T)$, differentiate equations (85) and (86) with respect to T. We obtain the following system of equations:

$$(90)\ \begin{bmatrix} 1 & c^T \\ b & S \end{bmatrix} \begin{bmatrix} u'(T) \\ w'(T) \end{bmatrix} = \begin{bmatrix} 1 \\ 0_N \end{bmatrix}$$

where c, b and S are defined as follows:

$$(91)\ c \equiv \nabla_w e(u,w+t)\ ;\ b \equiv \nabla_{wu}^2\, e(u,w+t)\ ;\ S \equiv \textstyle\sum_{ww} - S_{ww}.$$

Thus c is the initial net consumption vector for domestic commodities of the household sector, b is the vector of derivatives of household domestic demand with respect to utility u and S is the economy's aggregate domestic goods substitution matrix, which in turn is equal to the negative semidefinite consumer substitution matrix $\sum_{ww} \equiv \nabla_{ww}^2\, e(u,w+t)$ less the positive semidefinite producer substitution matrix for domestic commodities $S_{ww} \equiv \nabla_{ww}{}^2\pi(p,w)$. In deriving (90), we have also assumed money metric utility scaling for the single household which implies the following restrictions:

(92) $\partial e(u,w+t)/\partial u = 1$;
(93) $(w+t)^T\nabla_{wu}{}^2 e(u,w+t) = (w+t)^T b = \partial e(u,w+t)/\partial u = 1$.

The linear homogeneity of e(u,w+t) in its price variables implies the following restrictions on the consumer substitution matrix:

(94) $\sum_{ww} (w+t) = 0_N$.

The linear homogeneity of $\pi(p,w)$ in the components of p,w implies the following restrictions on the producer substitution matrices :

(95) $p^T S_{pw} + w^T S_{ww} = 0_N$.

We need a formula for the inverse of the N+1 by N+1 matrix which appears in the left hand side of (90). We will find it convenient to develop two expressions for this inverse matrix.

In order to develop our first expression for this inverse matrix, we make the following assumption:

(96) $[S - bc^T]^{-1}$ exists.

It can be verified that assumption (96) is necessary and sufficient for the inverse matrix to exist and under this assumption, we have the following formula for the inverse:

$$(97)\quad \begin{bmatrix} 1 & c^T \\ b & S \end{bmatrix}^{-1} = \begin{bmatrix} 1 + c^T[S - bc^T]^{-1}b, & -c^T[S - bc^T]^{-1} \\ -[S - bc^T]^{-1}b, & [S - bc^T]^{-1} \end{bmatrix}.$$

For our second expression for the inverse matrix, we make the following assumption:

(98) $S^{-1} = [\sum_{ww} - S_{ww}]^{-1}$ exists.

Assumption (98) implies that the negative semidefinite aggregate substitution matrix for domestic commodities S is actually negative definite. This will not be a restrictive assumption in practice but it is a stronger assumption than assumption (96). We will also require the following assumption in order to exhibit our second formula for the inverse matrix:

(99) $c^T S^{-1} b \neq 1$.

Using assumptions (98) and (99), we have the following formula for the inverse matrix:

$$(100)\quad \begin{bmatrix} 1 & c^T \\ b & S \end{bmatrix}^{-1} = \begin{bmatrix} (1-c^TS^{-1}b)^{-1}, & -(1-c^TS^{-1}b)^{-1}c^TS^{-1} \\ -(1-c^TS^{-1}b)^{-1}S^{-1}b, & S^{-1}+S^{-1}b(1-c^TS^{-1}b)^{-1}c^TS^{-1} \end{bmatrix}.$$

Problems

5. Verify that formula (97) is true under assumption (96).
6. Verify that formula (100) is true under assumptions (98) and (99).

If the consumer's preferences are homothetic so that

(101) $e(u,w) = ue(1,w)$,

then it can be seen that the vectors b and c defined in (91) are related as follows:

(102) $c = \nabla_w e(u,w+t) = u\nabla_w e(1,w+t) = u\nabla_{wu}^2 e(u,w+t) = ub$.

We shall assume that the consumer's initial utility level u is positive in what follows.

Proposition 5: Suppose that the consumer's preferences are homothetic so that the consumer's dual expenditure function satisfies (101) above. Assume in addition that assumptions (98) and (99) hold and there is money metric utility scaling at the initial prices w+t. Then we have the following restrictions on the elements of the inverse matrix defined by (100):

(103) $0 < (1-c^TS^{-1}b)^{-1} < 1$;
(104) $0 < -c^TS^{-1}b(1-c^TS^{-1}b)^{-1} < 1$.

Proof: Under the assumption of homothetic preferences, $c = ub$ where $u > 0$. Since S is negative definite, so is S^{-1} and thus we have:

(105) $c^TS^{-1}b = ub^TS^{-1}b < 0$

where we have used (93) which implies that $b \neq 0_N$. Thus $-\,c^TS^{-1}b$ is a positive number and $1 - c^TS^{-1}b$ is a positive number greater than one, which implies (103) and (104). Note that the expressions in (103) and (104) are positive fractions which sum to one. Q.E.D.

Now return to equations (90). Under assumptions (98) and (99), we have the following expressions for the derivatives of utility and domestic prices with respect to a change in the transfer T:

(106) $u'(T) = (1-c^TS^{-1}b)^{-1}$;
(107) $w'(T) = -S^{-1}b(1-c^TS^{-1}b)^{-1}$.

Normally, we would expect $u'(T)$ to be positive; i.e., if the government increases the transfer to the consumer, we would normally expect household utility to increase (unless the induced changes in domestic prices w are very perverse). Under the assumptions of homothetic preferences, Proposition 5 above shows that in this case, $u'(T)$ is a positive fraction.

Note that the first equation in (90) implies the following equation:

$$(108)\ u'(T) = 1 - c^T w'(T).$$

Obviously, once we have expressions for $u'(T)$ and $w'(T)$, we could differentiate equation (84) in order to obtain an expression for $g_0'(T)$. However, a more instructive formula for $g_0'(T)$ emerges if we differentiate the government budget constraint (88):

$$(109)\ g_0'(T) + 1 + g^T w'(T) = \nabla_w \pi(p,w)^T w'(T) + t^T \nabla_{ww}{}^2 \pi(p,w) w'(T).$$

Using demand equals supply equations (86) along with definition (91) for c leads to the following simplification of equation (109):

$$(110)\ g_0'(T) = -1 + c^T w'(T) + t^T \nabla_{ww}{}^2 \pi(p,w) w'(T)$$
$$(111)\qquad = -\,u'(T) + t^T \nabla_{ww}{}^2 \pi(p,w) w'(T) \qquad \text{using (108).}$$

Equations (108) and (111) are key results. If the domestic commodity tax rates t are small, then (111) shows that $u'(T)$ and $g_0'(T)$ will be approximately equal in magnitude but of opposite signs. Under the homothetic preferences assumptions made in Proposition 5, we know that $u'(T)$ will be a positive fraction and hence if t is small, $g_0'(T)$ will be a negative fraction which is approximately equal in magnitude to $u'(T)$.

What can be said about the term $t^T \nabla_{ww}{}^2 \pi(p,w) w'(T)$ which appears in (111)? This term is a first order approximation to the change in domestic commodity tax revenue that is induced by the small increase in the transfer T and normally, we would expect this term to be *positive*; i.e., the increase in the transfer T induces the consumer to spend more on domestic goods and services which in turn drives up domestic prices w(T) which induces producers to divert production from exports to domestic production (which incidentally decreases g_0) which in turn increases domestic commodity tax revenues for the government (which allows the government to finance the increased level of transfers). For the sake of definiteness, we make this presumption that $t^T \nabla_{ww}{}^2 \pi(p,w) w'(T)$ is positive a formal assumption:

$$(112)\ t^T \nabla_{ww}{}^2 \pi(p,w) w'(T) > 0.$$

The positivity assumption (112) along with the assumption that $u'(T) > 0$ and equation (111) imply that $g_0'(T)$ is equal to $-\,u'(T)$ plus the positive term $t^T \nabla_{ww}{}^2 \pi(p,w) w'(T)$ and hence if this positive term is not too big, then $g_0'(T)$ will be negative but its *magnitude will be less than the (positive) rate of utility increase*, $u'(T)$. Thus if we value money metric utility increases at the same valuation as we value decreases in government spending on foreign currency, then equation (111) and assumption (112) imply that *increasing the transfer* T and *decreasing government spending* on foreign exchange g_0 will lead to an overall "general" welfare increase; i.e., under assumption (112), we have the following result using (111):

$$(113)\ u'(T) + g_0'(T) = t^T\nabla_{ww}{}^2\,\pi(p,w)w'(T) > 0.$$

However, increasing the transfer has the effect of *decreasing* government spending which is *not* the experiment that we have in mind when we calculate marginal excess burdens of tax increases. In the marginal excess burden literature, we increase a tax and work out what extra spending the tax increase facilitates and weigh this *benefit* of the tax increase against its *cost* in terms of the household loss of utility over private goods and services. The counterpart to this mental exercise in the present context is that we *reduce* the transfer and this will facilitate *increased government spending on foreign exchange* (the benefit) and we offset this benefit with the *loss of household utility* over privately delivered goods and services (the cost). Thus in the present context, we measure the benefit of a decrease in the transfer T as $-g_0'(T)$ (and this rate of change in g_0 will be positive under normal conditions) and the cost will be the absolute value of the induced fall in household utility, which turns out to be the positive derivative $u'(T)$. Thus the utility cost per unit of extra government revenue raised will be the following ratio:

$$\begin{aligned}(114)\ u'(T)/[-\,g_0'(T)] &= u'(T)/[u'(T) - t^T\nabla_{ww}{}^2\,\pi(p,w)w'(T)] && \text{using (111)}\\ &> 1\end{aligned}$$

where the inequality follows using the inequality in assumption (113) along with the assumption that t is small enough so that the following inequality is satisfied:

$$(115)\ u'(T) > t^T\nabla_{ww}{}^2\,\pi(p,w)w'(T).$$

The marginal excess burden associated with a decrease in the transfer (so that more government spending can be accommodated) is the excess of the ratio $u'(T)/[-\,g_0'(T)]$ above 1; i.e., define the MEB(T) as follows:

$$(116)\ \text{MEB}(T) \equiv \{u'(T)/[-\,g_0'(T)]\} - 1.$$

The bigger MEB(T) is, the more "expensive" is the financing of increased government spending in terms of the loss of consumer welfare over privately delivered goods and services. In the following section, we will work out comparable marginal excess burdens for changes in the domestic commodity taxes t.

Problems

7. If $t = 0_N$, show that MEB(T) = 0. Thus if there are no domestic commodity tax distortions, then the marginal excess burden of a decrease in the transfer is zero.

8. Assume that there is only one domestic commodity so that $N = 1$. Recall definitions (91) above. Assume that the following restrictions hold:

(a) $c_1 > 0$; (b) $b_1 > 0$; (c) $S_{ww} \equiv s > 0$; (d) $0 < t_1 < 1/b_1$.

(i) Show that the consumer substitution matrix $\sum_{ww}$ equals 0 when $N = 1$.
(ii) Obtain specific formulae for $u'(T)$, $w_1'(T)$ and $g_0'(T)$ and sign these derivatives if possible.
(iii) Obtain a specific formula for MEB(T) in terms of the underlying parameters of the model in this $N = 1$ case. Try to determine whether the MEB will increase, decrease or remain the same (or we cannot say) as the producer domestic commodity substitution parameter s increases or as the domestic tax rate t_1 increases.

Note that problem 8 shows that the straightforward assumptions (a) to (d) above are sufficient to get the existence of the inverse of the two by two matrix in (90) and to sign the key derivatives in our model in this $N = 1$ case.

7. Marginal Excess Burdens: The Case of Changes in Domestic Commodity Taxes

In this section, we work out the marginal excess burdens of changes in the domestic commodity taxes $t_1, \ldots, t_N$ that are present in the commodity tax vector $t \equiv [t_1,\ldots,t_N]^T$. The basic equations that define our model are still equations (84)-(86) which are listed in the previous section. As in the previous section, we note that government spending on internationally traded commodities, g_0, appears only in equation (84) and so equations (85) and (86) can be differentiated with respect to a commodity tax t_n in order to obtain formulae for the derivative of utility with respect to t_n, $u'(t_n)$, and the vector of derivatives of producer prices for domestic commodities with respect to t_n, $[\partial w_1(t_n)/\partial t_n, \ldots, \partial w_N(t_n)/\partial t_n]^T \equiv \nabla_{t_n} w(t_n)$. These derivatives can be rewritten in the following compact form:

$$(117) \quad \begin{bmatrix} 1 & c^T \\ b & S \end{bmatrix} \begin{bmatrix} \nabla_t u(t) \\ \nabla_t w(t) \end{bmatrix} = \begin{bmatrix} -c^T \\ -\Sigma_{ww} \end{bmatrix}$$

where $c \equiv [c_1,\ldots,c_N]^T$, b, $\sum_{ww}$ and $S \equiv \sum_{ww} - S_{ww}$ are defined as in (91) above and $\nabla_t u(t)$ is the one by N vector of derivatives of u(t) with respect to the N components of t and $\nabla_t w(t)$ is the N by N matrix of derivatives of w(t) with respect to the components of t, which has i,j element equal to $\partial w_i(t)/\partial t_j$. In order to get the existence of the inverse for the matrix on the left hand side of (117), we can make assumption (96) or assumptions (98) and (99) as in the previous section, since the same matrix appeared in that section.

Using assumption (96), we can use formula (97) for the inverse of the matrix on the left hand side of (117). Using this formula, we have the following expressions for the derivatives of utility and domestic prices with respect to changes in domestic tax rates:

$$(118)\quad \begin{bmatrix} \nabla_t u(t) \\ \nabla_t w(t) \end{bmatrix} = \begin{bmatrix} 1 + c^T[S - bc^T]^{-1}b, & -c^T[S - bc^T]^{-1} \\ -[S - bc^T]^{-1}b, & [S - bc^T]^{-1} \end{bmatrix} \begin{bmatrix} -c^T \\ -\Sigma_{ww} \end{bmatrix}$$

$$= \begin{bmatrix} -c^T - c^T[S - bc^T]^{-1}[bc^T - \Sigma_{ww}] \\ -[S - bc^T]^{-1}[-bc^T + \Sigma_{ww}] \end{bmatrix}$$

$$= \begin{bmatrix} c^T[S - bc^T]^{-1}S_{ww} \\ -I_N - [S - bc^T]^{-1}S_{ww} \end{bmatrix}$$

where the last equality follows using the fact that S equals $\sum_{ww} - S_{ww}$ and $[S - bc^T]^{-1}$ times $[\sum_{ww} - bc^T - S_{ww}]$ equals the N by N identity matrix, I_N. Picking out the nth column on both sides of (118) leads to the following expressions for the derivatives of u and w with respect to t_n, where e_n denotes a (column) unit vector of dimension N with a 1 in position n:

(119) $u'(t_n) = c^T[S - bc^T]^{-1}S_{ww}e_n$; $\qquad n = 1,2,...,N$

(120) $w'(t_n) = -e_n - [S - bc^T]^{-1}S_{ww}e_n$; $\qquad n = 1,2,...,N.$

There is a presumption that an increase in commodity tax t_n will lead to lower producer prices for domestic commodities and lower utility for the household. The following problem 10 shows that this presumption is true if we have only one domestic consumer good.

Problems

9. Show that under assumption (96), we have the following equalities:

(a) $u'(t_n) + c^Tw'(t_n) = -c_n$ for n = 1,2,...,N.

10. Suppose there is only one domestic commodity so that N = 1. Assume that $c_1 > 0$, $b_1 > 0$ and $S_{ww} = s > 0$. Prove that:

(i) $u'(t_1) < 0$ and (ii) $w_1'(t_1) < 0$.

Obviously, once we have expressions for $u'(t_n)$ and $w'(t_n)$, we could differentiate equation (84) in order to obtain an expression for $g_0'(t_n)$. However, as in the previous section, a more instructive formula for $g_0'(t_n)$ emerges if we differentiate the government budget constraint (88) with respect to t_n:

(121) $g_0'(t_n) + g^Tw'(t_n) + g_n = \nabla_w\pi(p,w)^Tw'(t_n) + t^T\nabla_{ww}^2\,\pi(p,w)w'(t_n) + \partial\pi(p,w)/\partial w_n.$

where g_n is the nth component of the governments domestic requirements vector g and $\partial\pi(p,w)/\partial w_n$ is equal to y_n, the nth component of the initial supply vector for domestic goods and services. Using demand equals supply equations (86) (which implies $\nabla_w\pi(p,w)$ equals c minus g) leads to the following simplification of equation (121):

$$(122)\ g_0'(t_n) = c_n + c^T w'(t_n) + t^T\nabla_{ww}^2 \pi(p,w)w'(t_n) ; \qquad n = 1,...,N$$
$$(123)\qquad = -\ u'(t_n) + t^T\nabla_{ww}^2 \pi(p,w)w'(t_n)$$

where (123) follows from (122) using the results in problem 9.

Note that the term $t^T\nabla_{ww}^2 \pi(p,w)w'(t_n)$ which appears in (122) and (123) represents *the first order effects of the change in commodity tax revenue* due to a marginal increase in t_n but excluding the main first order effect of the increase, which is equal to y_n.[18] In general, we expect this induced tax revenue change term to be *negative* since the components of the t vector will be predominantly positive, the producer substitution matrix $\nabla_{ww}^2 \pi(p,w)$ is positive semidefinite and hence will have positive (or zero) elements along its main diagonal and a predominance of positive elements and the vector of derivatives will have at least one element that is negative and relatively large in magnitude. Problem 11 below shows that this induced change in commodity tax revenue term will in fact be negative if the number of domestic commodities N is equal to one.

Equations (123) are the key results in this section. If the domestic commodity tax rates t are small, then (123) shows that $u'(t_n)$ and $g_0'(t_n)$ will be approximately equal in magnitude but of opposite signs. We expect the effect on utility of an increase in a commodity tax rate t_n to be negative (so we expect $u'(t_n)$ to be negative) and the effect on government expenditures on foreign exchange to be positive (so we expect $g_0'(t_n)$ to be positive). In addition, we expect commodity tax change term $t^T\nabla_{ww}^2 \pi(p,w)w'(t_n)$ to be negative. Suppose that all of these expectations are true so that we have the following inequalities:

$$(124)\ u'(t_n) < 0 ;\ g_0'(t_n) > 0 ;\ t^T\nabla_{ww}^2 \pi(p,w)w'(t_n) < 0.$$

Thus the *cost* of the marginal increase in the tax t_n is the loss of utility $-\ u'(t_n) > 0$ (which we have converted into a positive number) while the *benefit* of the tax increase is the amount of extra spending on foreign goods and services by the government that the tax increase facilitates, $g_0'(t_n)$. The ratio of the utility loss to the foreign exchange benefit is $-\ u'(t_n)/g_0'(t_n)$ and assumptions (124) plus the identity (123) implies that the benefit $g_0'(t_n)$ is less than the cost $-\ u'(t_n)$ and hence the ratio $-\ u'(t_n)/g_0'(t_n)$ is greater than one. The excess of this cost benefit ratio over one is defined to be the *marginal excess burden of an increase in* t_n, $MEB(t_n)$:

$$(125)\ MEB(t_n) \equiv [-\ u'(t_n)/g_0'(t_n)] - 1 ; \qquad n = 1,...,N.$$

[18] Define commodity tax revenue as a function of t_n as $\Gamma(t_n) = t^T\nabla_w\pi(p,w(t_n))$ and so the derivative of this tax revenue with respect t t_n is $\Gamma'(t_n) = t^T\nabla_{ww}^2 \pi(p,w(t_n))\ w'(t_n) + \partial\pi(p,w)/\partial w_n = t^T\nabla_{ww}^2 \pi(p,w(t_n))\ w'(t_n) + y_n$. Thus the term $t^T\nabla_{ww}^2 \pi(p,w(t_n))\ w'(t_n)$ is the first order change in commodity tax revenue but excluding the term y_n.

If assumptions (124) hold, then as we have seen, the marginal excess burden $MEB(t_n)$ is greater than zero.

The commodity tax marginal excess burdens $MEB(t_n)$ defined by (125) can be compared to the transfer marginal excess burden MEB(T) defined earlier by (116). If the government wants to increase expenditures on public goods or other government services, it should increase the commodity tax t_n or decrease the household transfer T that has the lowest marginal excess burden. Conversely, if the government is able to decrease government expenditures, then it should reduce the commodity tax t_n that has the highest marginal excess burden.

The problem with marginal excess burden analysis is that it depends critically on having a good knowledge of consumer preferences and producer production possibility sets or more narrowly on the various elasticities of demand and supply that pertain to the tax distorted equilibrium. But it is not easy to estimate econometrically these elasticities!

Problems

11. Suppose there is only one domestic commodity so that N = 1. Assume that $c_1 > 0$, $b_1 > 0$, $t_1 < 1/b_1$ and $S_{ww} = s > 0$. Prove that $w'(t_1) < 0$ and that the three inequalities in (124) hold for n = 1.

12. Extend the model in this section to allow for tax, subsidy and tariff distortions on international trade transactions; i.e., assume that domestic producers face the tax distorted price vector $p+\tau$ instead of the international price vector p. Try to develop some interesting results along the lines that we have established in this section and the previous one.

13. Extend the model in this section to allow for a profits tax. Try to develop some interesting results along the lines that we have established in this section and the previous one.

14. Extend the model in this section to allow for an arbitrary number of households. One way of doing this is to retain equation (84) and replace equations (85) and (86) by the following equations:

(a) $e^h(u^h,w+t) = \alpha_h T$; $\qquad h = 1,\ldots,H$;
(b) $\sum_{h=1}^H \nabla_w e^h(u^h,w+t) + g = \nabla_w \pi(p,w)$

where the nonnegative constants α_h sum to unity. We now have the problem of weighing the utility changes of the different households in order to come up with an overall social welfare function. Try defining social welfare U as the following *utilitarian social welfare function*:

(c) $U(u^1,\ldots.u^H,w+t) \equiv \sum_{h=1}^H u^h$.

Now change a commodity tax t_n (or the overall level of transfers T) and work out formulae for the derivatives $U'(t_n)$ and $g_0'(t_n)$ and then go on to define marginal excess burdens.

References

Allais, M. (1943), *A la recherche d'une discipline économique*, Tome 1, Paris: Imprimerie Nationale.

Allais, M. (1973), "La theorie général des surplus et l'apport fondamental de Vilfredo Pareto", *Revue d'Economie Politique* 83, 1044-1097.

Allais, M. (1977), "Theories of General Economic Equilibrium and Maximum Efficiency", pp. 129-201 in *Equilibrium and Disequilibrium in Economic Theory*, E. Schwödiauer (ed.), Dordrecht, Holland: D. Reidel Publishing.

Boiteux, M. (1951), "Le 'revenu distruable' et les pertes économiques", *Econometrica* 19, 112-133.

Debreu, G. (1951), "The Coefficient of Resource Utilization", *Econometrica* 19, 273-292.

Debreu, G. (1954), "A Classical Tax-Subsidy Problem", *Econometrica* 22, 14-22.

Diewert, W.E. (1983), "The Measurement of Waste within the Production Sector of an Open Economy", *Scandinavian Journal of Economics* 85, 159-179.

Diewert, W.E., A.H. Turunen-Red and A.D. Woodland (1989), "Productivity and Pareto Improving Changes in Taxes and Tariffs", *Review of Economic Studies* 56, 199-216.

Diewert, W.E., A.H. Turunen-Red and A.D. Woodland (1991), "Tariff Reform in a Small Open Multi-Household Economy with Domestic Distortions and Nontraded Goods", *International Economic Review* 32, 937-957.

Diewert, W.E. and A.D. Woodland (1977), "Frank Knight's Theorem in Linear Programming Revisited", *Econometrica* 45, 375-398.

Farrell, M.J. (1957), "The Measurement of Productive Efficiency", *Journal of the Royal Statistical Society*, Series A, 120, 253-281.

Feenstra, R.C. (2004), *Advanced International Trade: Theory and Evidence*, Princeton N.J.: Princeton University Press.

Harberger, A.C. (1964), "The Measurement of Waste" *American Economic Review* 54, 58-78.

Hicks, J.R. (1940), "The Valuation of the Social Income", *Economica* 7, 105-124.

Hicks, J.R. (1941-42), "Consumers' Surplus and Index Numbers", *The Review of Economic Studies* 9, 126-137.

Hotelling, H. (1932), "Edgeworth's Taxation Paradox and the Nature of Supply and Demand Functions", *Journal of Political Economy* 40, 577-616.

Hotelling, H. (1938), "The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates", *Econometrica* 6, 242-269.

Karlin, S. (1959), *Mathematical Methods and Theory in Games, Programming and Economics*, Volume 1, Reading MA: Addison-Wesley Publishing Co.

Kohli, U. (1978), "A Gross National Product Function and the Derived Demand for Imports and Supply of Exports", *Canadian Journal of Economics* 11, 167-182.

Kohli, U. (1990), "Growth Accounting in the Open Economy: Parametric and Nonparametric Estimates", *Journal of Economic and Social Measurement* 16, 125-136.

Kohli, U. (1991), *Technology, Duality and Foreign Trade: The GNP Function Approach to Modeling Imports and Exports*, Ann Arbor: University of Michigan Press.

Kohli, U. (2003), "Growth Accounting in the Open Economy: International Comparisons", *International Review of Economics and Finance* 12, 417-435.

Slater, M. (1950), "Lagrange Multipliers Revisited: A Contribution to Nonlinear Programming", Cowles Commission Discussion Paper, Mathematics, Number 403, November.

Uzawa, H. (1958), "The Kuhn-Tucker Theorem in Concave Programming", pp. 32-37 in *Studies in Linear and Nonlinear Programming*, K.H. Arrow, L. Hurwicz and H. Uzawa (eds.), Stanford CA: Stanford University Press.

Woodland, A.D. (1982), *International Trade and Resource Allocation*, Amsterdam: North-Holland.