ECONOMICS 581: LECTURE NOTES

CHAPTER 8: Nonlinear Programming: The Saddle Point Approach and the Kuhn-Tucker Conditions

W. Erwin Diewert March 30, 2009.

1. Introduction

The material to be presented in this chapter is useful in a wide variety of applied economic problems.

The main theoretical results are presented in sections 2 and 6. Section 2 presents the saddle point approach to concave programming problems that was developed by the economist Hirofumi Uzawa (1958) and the mathematician Samuel Karlin (1959; 200-205). Their approach is very general and does not assume any differentiability assumptions about the objective and constraint functions.[1] However, it is necessary that the constraint functions be written as inequalities rather than as equalities (as in classical constrained optimization). The restriction that the constraints be expressed in terms of inequalities is not a severe restriction for the applicability of this saddle point approach to economic problems since most real life economic problems are consistent with inequality constraints. In section 6, differentiability assumptions are added to the regularity conditions on objective and constraint functions and the famous Kuhn-Tucker (1951) conditions emerge in this section.

Section 3 presents a few examples to illustrate the general theorems presented in section 2. In particular, a geometric interpretation of the Slater (1950) constraint qualification condition is presented.

Section 4 indicates how the saddle point approach to constrained optimization problems is related to the classical Lagrange multiplier method which is used to solve equality constrained optimization problems.

Section 5 shows how the Karlin Uzawa saddle point theorem can be applied to a real economic problem. We show how the dimensionality of many economic problems can be reduced dramatically if we combine the saddle point approach with economic duality theory. This same idea will be used repeatedly in the following chapter.

2. The Karlin Uzawa Saddle Point Theorem

Suppose that we want to maximize some function of N variables, $g(x_1,...,x_N)$, subject to M inequality constraints of the following form:

(1) $f_1(x_1,...,x_N) \geq 0$; $f_2(x_1,...,x_N) \geq 0$; ... ; $f_M(x_1,...,x_N) \geq 0$.

[1] The main tool that they use is the supporting and separating hyperplane theorems developed by Minkowski (1911) and Fenchel (1953).

In addition, we restrict the column vector $x \equiv [x_1,...,x_N]^T$ to belong to a closed convex set of feasible x's, X. In this section, we will assume that the objective function, g(x) and all M of the constraint functions, $f_m(x)$, m = 1,2,...,M, are *concave functions* defined over the convex set X. The notation can be simplified if we define F(x) as the column vector $[f_1(x),f_2(x),...,f_M(x)]^T$. Then our basic *concave programming problem* can be written in a succinct manner as follows:

(2) $\max_x \{g(x) : F(x) \geq 0_M ; x \in X\}$.

A large number of economic problems can be written as the concave programming problem (2), including *linear programming problems* where the functions g and F are all linear and the set X is defined by simple inequalities on the components of the x vector.

Students of economics are generally well acquainted with the Lagrange multiplier technique in classical equality constrained maximization problems where differentiability of the objective and constraint functions is assumed. The student may also know that the optimal Lagrange multipliers always have interesting economic interpretations as the marginal change in the objective function due to the relaxation of the corresponding equality constraint by a marginal unit.[2] However, it is not as well known that we can associate similar optimal multipliers with the inequality constrained concave programming problem, (2) above, without assuming any differentiability restrictions. Furthermore, these optimal Lagrange multipliers can frequently be identified as prices in various economic models. The main results along these lines that we derive below are due to Uzawa (1958) and Karlin (1959; 201-203).

We start off by defining the *Lagrangian*, L(x,u), that corresponds to the nonlinear programming problem that is defined by (2) above:

(3) $L(x,u) \equiv g(x) + u^TF(x) = g(x) + \sum_{m=1}^M u_mf_m(x)$

where $u^T \equiv [u_1,u_2,...,u_M]$ is a vector of variables (Lagrange multipliers or dual prices). A *saddle point of the Lagrangian* is an $x^0 \in X$ and $u^0 \geq 0_M$ that satisfies the following inequalities:

(4) $L(x,u^0) \leq L(x^0,u^0) \leq L(x^0,u)$ for all $x \in X$ and $u \geq 0_M$.

The first inequality in (4) says that $L(x,u^0) = g(x) + u^{0T}F(x)$, regarded as a function of x, has an *unconstrained maximum* at $x = x^0$ over all x that belong to the set X. The second inequality in (4) says that $L(x^0,u) = g(x^0) + u^TF(x^0)$, regarded as a function of u, has an *unconstrained minimum* at $u = u^0$ over the set of all nonnegative u. Hence, we see that the Lagrangian has the curvature of a saddle at the point x^0,u^0, curving downward in the x direction(s) and curving upward in the u direction(s).

[2] See Samuelson (1947; 65).

What is the significance of a saddle point? The following theorem tells us that if we happen to find a saddle point of the Lagrangian, (x^0,u^0) say, then the x^0 part of the saddle point solves the nonlinear programming problem (2) above.

Theorem 1; Uzawa (1958): Suppose g(x) and $f_1(x), \ldots, f_M(x)$ are functions defined over a set X and consider the constrained maximization problem defined by (2) above. Suppose (x^0,u^0) is a saddle point of the corresponding Lagrangian; i.e., $x^0 \in X$ and $u^0 \geq 0_M$ satisfy the inequalities in (4). Then x^0 solves the nonlinear programming problem (2).

Proof: Using the definition of L(x,u), the second set of inequalities in (4) is equivalent to:

(5) $g(x^0) + u^{0T}F(x^0) \leq g(x^0) + u^TF(x^0)$ for all $u \geq 0_M$.

Now subtract $g(x^0)$ from each side of (5) and we obtain the following inequalities:

(6) $u^TF(x^0) \geq u^{0T}F(x^0)$ for all $u \geq 0_M$.

If any component of $F(x^0)$ were negative, say $f_m(x^0)$ were negative, then by taking u_m large enough, we would contradict the inequality (6) and so we deduce that

(7) $F(x^0) \geq 0_M$.

Since $u^0 \geq 0_M$ and $F(x^0) \geq 0_M$, we deduce that $u^{0T}F(x^0) \geq 0$. But now set $u = 0_M$ and from (6), we find that $u^{0T}F(x^0) \leq 0$. Hence

(8) $u^{0T}F(x^0) = 0$.

Now look at the first set of inequalities in (4), which are equivalent to:

(9) $g(x) + u^{0T}F(x) \leq g(x^0) + u^{0T}F(x^0)$ for all $x \in X$.

Using (8), (9) becomes:

(10) $g(x) + u^{0T}F(x) \leq g(x^0)$ for all $x \in X$.

Since $u^0 \geq 0_M$, for any $x \in X$ such that $F(x) \geq 0_M$, we have

(11) $g(x) \leq g(x) + u^{0T}F(x)$ for all $x \in X$ such that $F(x) \geq 0_M$
$\leq g(x^0)$ using (10).

But the inequalities in (11) show that x^0 solves (2). Q.E.D.

Note that we did not require the concavity of $g, f_1, \ldots, f_M$ or the convexity of the domain of definition set, X, in order to prove the above result.

The above theorem tells us that if we can find a saddle point of the Lagrangian, then we have a solution to the nonlinear programming problem (2). The problem with this result is that it does not tell us if such saddlepoints exist. In general, they will *not* exist. Hence to get the existence of a saddlepoint, we need to impose some *regularity conditions* on the functions $g, f_1, ..., f_M$ and the domain of definition set X. We will impose *concavity* on the functions $g, f_1, ..., f_M$ and *convexity* on the domain of definition set X. However, these conditions are not quite sufficient to imply the existence of at least one saddle point for the Lagrangian that corresponds to the nonlinear programming problem (2). We require at least one additional restriction on (2): a *constraint qualification condition*.

Before we list possible constraint qualification conditions, we need to explain in general terms why one is required. Note that when we defined the nonlinear programming problem (2), we did not specify that the number of constraint functions M be less than the number of variables N. Hence, if M is bigger than N, there is the possibility that the constraints will be *inconsistent*; i.e., there is no $x \in X$ that satisfies all of the constraints $F(x) \geq 0_N$. To solve this problem of inconsistency, we introduce our first constraint qualification condition:

(12) *Feasibility Constraint Qualification Condition*: There exists $x^* \in X$ such that $F(x^*) \geq 0_M$.

Obviously, (12) is a minimal constraint qualification condition that must be satisfied in order for solutions to the nonlinear programming problem (2) to exist. However, it turns out to be not quite strong enough for our purposes.[3] Consider the following condition:

(13) *Slater Constraint Qualification Condition*: There exists $x^* \in X$ such that $F(x^*) >> 0_M$.

The meaning of (13) is that there exists an $x^* \in X$ such that *each* of the inequality constraints is strictly satisfied when evaluated at x^*; i.e., we have $f_1(x^*) > 0$; $f_2(x^*) > 0$; ... ; $f_M(x^*) > 0$. This constraint qualification condition is due to Slater (1950). In most economic contexts, it will not be restrictive to assume that the Slater constraint qualification condition holds.

A seemingly weaker constraint qualification condition that will suffice for our purposes is due to Karlin (1959; 201):

(14) *Karlin Constraint Qualification Condition*: For each nonnegative vector u which is not zero (so that $u > 0_M$), there exists $x^* \in X$ such that $u^T F(x^*) > 0$.

Note that x^* can depend on u. Obviously, (13) implies (14); i.e., if the Slater constraint qualification condition holds, then so does the Karlin.[4]

---

[3] In Figure 2 below, we will explain why we need a stronger constraint qualification condition than (12) to get the existence of a saddlepoint.
[4] For additional material on constraint qualification conditions, see Arrow, Hurwicz and Uzawa (1961).

Now we are ready to state Theorem 2, which provides sufficient conditions for the nonlinear programming problem (2) to have a saddle point, given that an optimal solution to (2) exists.[5]

Theorem 2; Karlin (1959; 201-203): Let X be a closed convex set and let $g, f_1,...,f_M$ be concave functions defined over X. Assume also that the Karlin constraint qualification condition (14) is satisfied. Let x^0 solve the nonlinear programming problem (2). Then there exists at least one $u^0 \geq 0_M$ such that (x^0,u^0) is a saddle point of the Lagrangian L(x,y) defined by (3) above.

Proof: Let x^0 be a solution to (2). Then the following conditions are satisfied:

(15) $x^0 \in X$;
(16) $f_m(x^0) \geq 0$ for m = 1,...,M ;
(17) $g(x^0) \geq g(x)$ for all $x \in X$ such that $f_m(x) \geq 0$ for m = 1,...,M .

We now define two sets, A and B, in M+1 dimensional space. We will eventually show that these two sets are convex and disjoint and hence can be separated by a hyperplane. Define A and B as follows:

(18) $A \equiv \{(y_0,y_1,...,y_M) : y_0 \leq g(x) ; y_1 \leq f_1(x) ; ... ; y_M \leq f_M(x) \text{ for some } x \in X\}$;
(19) $B \equiv \{(y_0,y_1,...,y_M) : y_0 > g(x^0) ; y_1 > 0 ; ... ; y_M > 0\}$.

It is easy to see that B is a convex set since it is the interior of an orthant in M+1 dimensional space. To show that A is convex, let $y' \in A$, $y'' \in A$ and let $0 < \lambda < 1$. Since $y' \in A$ and $y'' \in A$, we have the existence of $x' \in X$ and $x'' \in X$ such that

(20) $y_0' \leq g(x') ; y_1' \leq f_1(x') ; ... ; y_M' \leq f_M(x')$ and
(21) $y_0'' \leq g(x'') ; y_1'' \leq f_1(x'') ; ... ; y_M'' \leq f_M(x'')$.

Using the concavity of g, we have

(22) $g(\lambda x' + (1-\lambda)x'') \geq \lambda g(x') + (1-\lambda)g(x'')$
$\geq \lambda y_0' + (1-\lambda) y_0''$ using (20), (21), $\lambda > 0$ and $1-\lambda > 0$.

For m = 1,2,...,M, using the concavity of f_m, we have

(23) $f_m(\lambda x' + (1-\lambda)x'') \geq \lambda f_m(x') + (1-\lambda)f_m(x'')$
$\geq \lambda y_m' + (1-\lambda) y_m''$ using (20), (21), $\lambda > 0$ and $1-\lambda > 0$.

But the inequalities in (22) and (23) show that $\lambda y' + (1-\lambda)y''$ belongs to A. Hence A is a convex set.

[5] Sufficient conditions for the existence of a solution to (2) are: X is a closed and bounded set, g is continuous and the functions $f_1,...,f_M$ are continuous from above (so that the sets $S^m \equiv \{x: f_m(x) \geq 0; x \in X\}$ are closed.

To show that the sets A and B are disjoint, we provisionally assume that they are not disjoint. Hence *suppose* that there exists a y^* such that $y^*\in A$ and $y^*\in B$. If $y^*\in A$, then there exists an $x^*\in X$ such that

(24) $y_0^* \leq g(x^*)$; $y_1^* \leq f_1(x^*)$; ... ; $y_M^* \leq f_M(x^*)$

where $y^{*T} \equiv [y_0^*, y_1^*, ..., y_M^*]$. If $y^*\in B$ as well, then the components of y^* satisfy the following inequalities:

(25) $y_0^* > g(x^0)$; $y_1^* > 0$; ... ; $y_M^* > 0$.

Putting the inequalities (24) and (25) together and eliminating the components of y^* means we have the existence of $x^*\in X$ such that

(26) $g(x^*) > g(x^0)$, $f_1(x^*) > 0$; ... ; $f_M(x^*) > 0$.

But the inequalities (26) contradict the optimality of x^0. Hence our *supposition* is false and the sets A and B have no points in common. Recall the following theorem from the convexity and concavity chapter:

Separating hyperplane theorem between two disjoint convex sets; Fenchel (1953; 48-49): Let A and B be two nonempty, convex sets in R^{M+1} that have no points in common; i.e., $A\cap B = \varnothing$ (the empty set). Assume that at least one of the two sets A or B has a nonempty interior. Then there exists a hyperplane that separates A and B; i.e., there exists a nonzero vector c and a scalar α such that

(27) $c^T y \leq \alpha \leq c^T z$ for all $y\in A$ and $z\in B$.

Suppose that some component of c were negative. Then since the set B is unbounded from above in each component of z, we could choose the component of z that corresponds to the negative component of c large enough to violate the second set of inequalities in (27). Hence, the components of c must be nonnegative.

We will set $c^T \equiv [v_0, v_1, ..., v_M] \equiv [v_0, v^T]$ and rewrite (27) as follows:

(28) $v_0 y_0 + v^T y \leq v_0 z_0 + v^T z$ for all $[y_0, y^T]^T \in A$ and all $[z_0, z^T]^T \in B$.

Now let $[z_0, z^T]^T \in B$. We can choose a sequence of z's, say z^n, such that $[z_0, z^{nT}]^T \in B$ and

(29) $\lim_{n\to\infty} z^n = 0_M$.

Using this sequence of z's in (28) and taking the limit as n tends to infinity leads to the following inequalities:

(30) $v_0 y_0 + v^T y \leq v_0 z_0$ for all $[y_0, y^T]^T \in A$.

Suppose $v_0 = 0$. Then (30) becomes:

(31) $v^T y \leq 0$ for all $[y_0, y^T]^T \in A$.

Using the definition of the set A, it can be seen that (31) is equivalent to the following inequalities:

(32) $v^T F(x) \leq 0$ for all $x \in X$.

But (32) contradicts the Karlin constraint qualification condition (14). Hence our *supposition* is false and v_0 is not equal to zero. Since we previously showed that v_0 must be nonnegative, this means that v_0 must be positive. Now define the M dimensional vector u^0 as follows:

(33) $u^0 \equiv v/v_0 \geq 0_M$.

where the inequalities follow since $v_0 > 0$ and $v \geq 0_M$. Now divide both sides of (30) through by v_0 and replace z_0 by its lower bound, $g(x^0)$. Using the definition of A, the resulting inequalities imply that the following inequalities are valid:

(34) $g(x) + u^{0T}F(x) \leq g(x^0)$ for all $x \in X$.

Since $x^0 \in X$, replacing x by x^0 in (34) implies the following inequality:

(35) $u^{0T}F(x^0) \leq 0$.

But from (33), $u^0 \geq 0_M$ and from the feasibility of the optimal solution x^0, $F(x^0) \geq 0_M$ as well and hence $u^{0T}F(x^0) \geq 0$. But this last inequality along with (35) implies

(36) $u^{0T}F(x^0) = 0$.

Using (36), we can rewrite (34) as follows:

(37) $g(x) + u^{0T}F(x) \leq g(x^0) + u^{0T}F(x^0)$ for all $x \in X$.

But (37) is equivalent to the first set of saddle point inequalities (4). Since $F(x^0) \geq 0_M$, for all $u \geq 0_M$, we have

(38) $g(x^0) \leq g(x^0) + u^T F(x^0)$ for all $u \geq 0_M$.

Using (36), we can rewrite (38) as follows:

(39) $g(x^0) + u^{0T}F(x^0) \leq g(x^0) + u^T F(x^0)$ for all $u \geq 0_M$.

But (39) is equivalent to the second set of inequalities in the definition of a saddle point, (4). Thus (37) and (39) imply that (x^0,u^0) is a saddle point of the Lagrangian. Q.E.D.

In the following section, we illustrate the above algebra with some simple examples and indicate what happens when the constraint qualification condition fails.

Problem

1. Show that the Slater and Karlin constraint qualification conditions are equivalent. Hint: It is immediately evident that the Slater constraint qualification condition implies that the Karlin condition holds. Now assume that the Slater condition does not hold and show that the Karlin condition does not hold. To accomplish this last task, define the following two sets in M dimensional space:

(a) $A^* \equiv \{(y_1,...,y_M) : y_1 \le f_1(x) ; ... ; y_M \le f_M(x)$ for some $x \in X\}$;
(b) $B^* \equiv \{(y_1,...,y_M) : y_1 > 0 ; ... ; y_M > 0\}$.

Show that A^* and B^* are convex sets and if the Slater condition does not hold, then these sets are disjoint and hence can be separated by a hyperplane; i.e., there will exist $p \ne 0_N$ and α such that

(c) $p^Tz \le \alpha \le p^Ty$ for all $z \in A^*$ and all $y \in B^*$.

Use (c) and the structure of B^* to show that $p > 0_N$. Then use (c) and the structure of B^* to show that $\alpha \le 0$. Thus

(d) $p^Tz \le \alpha \le 0$ for all $z \in A^*$.

Condition (d) along with $p > 0_N$ means that the Karlin constraint qualification condition is not satisfied, which completes the proof.

3. Some Examples

For our first example, we consider the following simple one output, one variable input cost minimization problem where the target output level is at least one unit of output and the variable input price is 1:

(40) $\min_x \{1x : f_1(x) \equiv x^{1/2} - 1 \ge 0 ; x \ge 0\}$.

Thus the production function is $y = x^{1/2}$ and we convert the constraint $x^{1/2} \ge 1$ into the standard format $f_1(x) \ge 0$ by subtracting 1 from both sides of the production function constraint. The cost minimization problem (40) is still not quite in standard format; we need to convert the minimization problem into a maximization problem. The equivalent standard format inequality constrained maximization problem is:

(41) $\max_x \{g(x) : f_1(x) \ge 0 ; x \in X\}$

where

(42) $g(x) \equiv -x$; $f_1(x) \equiv x^{1/2} - 1$; $X \equiv \{x : x \geq 0\}$.[6]

Define the corresponding Lagrangian as follows:

(43) $L(x,u) \equiv g(x) + uf_1(x) = -x + u[x^{1/2} - 1]$.

The simplest way to find a saddle point of the Lagrangian is to differentiate L(x,u) with respect to x and u, set the resulting partial derivatives equal to 0, find an (x^0,u^0) solution to the resulting two equations if possible and then check whether the resulting solution satisfies $x^0 \in X$ and $u^0 \geq 0$. If so, then we have found a saddle point, provided that g and f_1 are concave, which they are in this example. The resulting first order conditions are:

(44) $\partial L(x,u)/\partial x = -1 + u(1/2)x^{-1/2} = 0$;
(45) $\partial L(x,u)/\partial u = x^{1/2} - 1 = 0$.

It is easy to verify that $(x^0,u^0) = (1,2)$ solves (44) and (45).

Problems

2. Prove that $x^0 = 1$ solves

(a) $\max_x \{L(x,u^0) ; x \in X\}$.

3. Prove that the solution set to:

(b) $\min_u \{L(x^0,u) : u \geq 0\}$

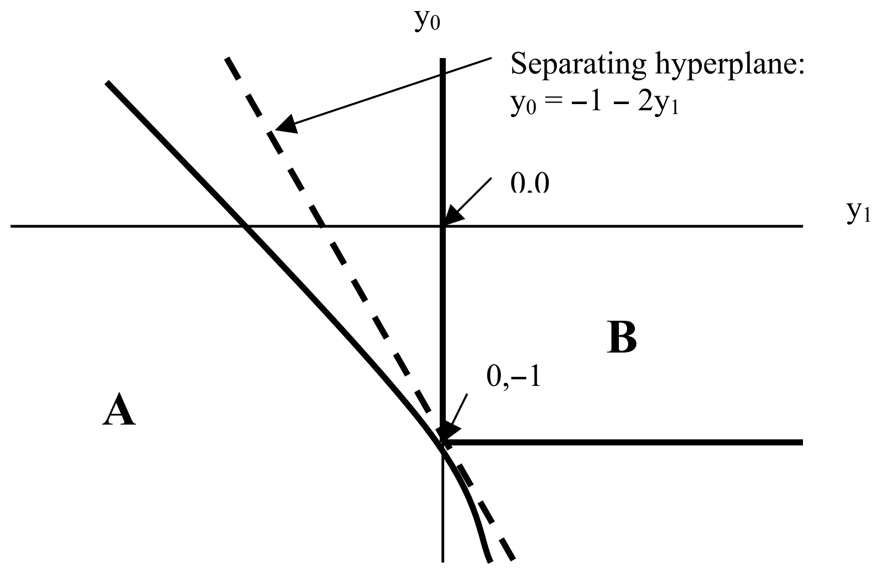
is the set $U \equiv \{u : u \geq 0\}$. Note: Problems 2 and 3 show that $(x^0,u^0) = (1,2)$ is a saddle point of the Lagrangian L(x,u) defined by (43) above.

In this simple example, the Slater and Karlin constraint qualification conditions are equivalent because there is only one constraint. To show that these conditions are satisfied, we need an $x^* \in X$ such that $f_1(x) > 0$. Take x^* to be 4. Then $f_1(x^*) = f_1(4) = 4^{1/2} - 1 = 2 - 1 = 1 > 0$ so the constraint qualification condition is satisfied.

The sets A and B defined by (18) and (19) above are graphed in Figure 1 below.

Figure 1: An Example where the Slater Constraint Qualification Condition Holds

[6] The second derivative of g(x) is 0 and the second derivative of $f_1(x)$ is $-(1/4)x^{-3/2}$, which is negative for $x > 0$. Hence both g and f_1 are concave over X.



In Figure 1, the dashed line separates the interiors of the sets A and B. The following saddle point inequalities are satisfied:

(46) $y_0 + 2y_1 \leq -1 \leq z_0 + 2z_1$ for all $(y_0,y_1)\in A$ and all $(z_0,z_1)\in B$.

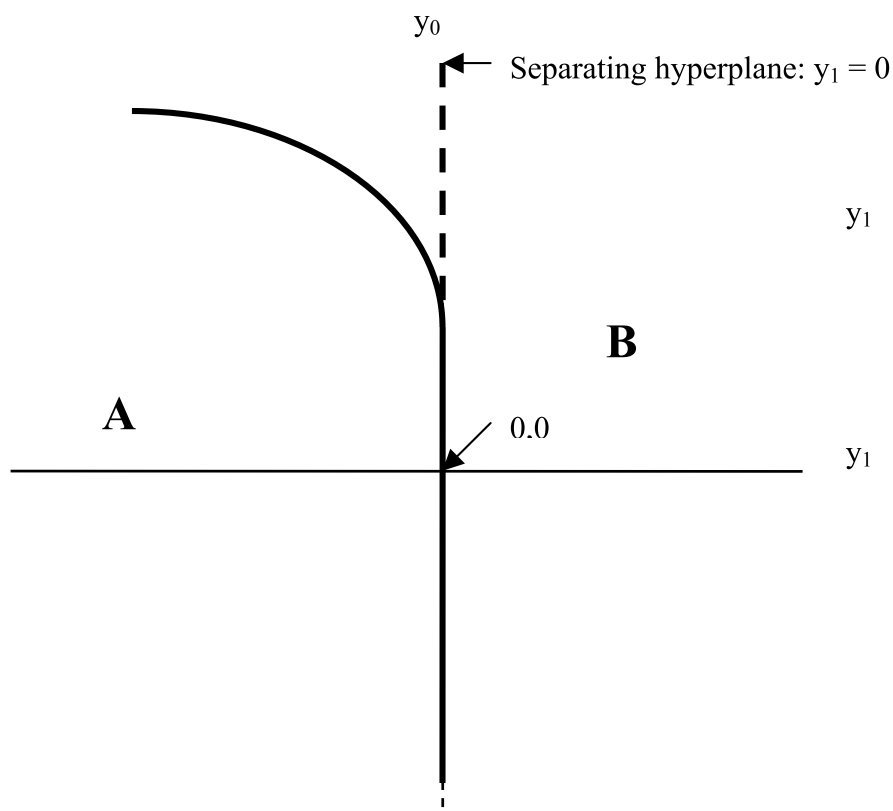
Our second example is due to Slater (1950). This example shows what happens when the feasibility constraint qualification condition (12) is satisfied but the Slater condition (13) is not satisfied. The inequality constrained maximization problem is again (41) but now g, f_1 and X are defined as follows:

(47) $g(x) \equiv x$; $f_1(x) \equiv -(x-1)^2$; $X \equiv \{x : x \geq 0\}$.[7]

For this problem, there is only one feasible x point; namely $x^0 = 1$, which solves the problem (41). The sets A and B defined by (18) and (19) for this example are graphed below in Figure 2. There is a separating hyperplane for the sets A and B but it turns out to be parallel to the y_0 axis; i.e., since the Slater or Karlin constraint qualification condition for this example are not satisfied, we find that v_0 is equal to 0 and the existence of a saddle point for this example fails.

Figure 2: An Example where the Slater Constraint Qualification Condition Fails

[7] The second derivative of g(x) is 0 and the second derivative of $f_1(x)$ is −1, which is negative. Hence both g and f_1 are concave over X.



4. Concave Programming and the Classical First Order Necessary Conditions

Theorem 1 in section 2 can be very useful in certain situations. In this section, we will again study the concave programming problem defined by (2) in section 2:

(48) $\max_x \{g(x) : F(x) \geq 0_M ; x \in X\}$.

As in section 2, we will assume that the objective function, $g(x)$ and all M of the constraint functions, $f_m(x)$, m = 1,2,...,M, are concave functions defined over the closed convex set X, which has a nonempty interior. In addition, we will assume that the objective function, $g(x)$ and all M of the constraint functions, $f_m(x)$, m = 1,2,...,M, are *differentiable* over X.

Suppose we treat the inequality constrained programming problem (48) as a classical equality constrained maximization problem. Then we could form the Lagrangian, $L(x,u)$, as in definition (3) above:

(49) $L(x,u) \equiv g(x) + u^TF(x) = g(x) + \sum_{m=1}^M u_mf_m(x)$.

Given some u^0, we could try to use classical unconstrained optimization theory to maximize $L(x,u^0)$ with respect to x. If a solution to this unconstrained maximization problem, x^0 say, lies in the interior of X, then the following first order necessary conditions must be satisfied:

(50) $\nabla_x L(x^0,u^0) \equiv \nabla_x g(x^0) + \sum_{m=1}^{M} u_m^0 \nabla_x f_m(x^0) = 0_N$.

Similarly, given some x^0, we could try to use classical unconstrained optimization theory to minimize $L(x^0,u)$ with respect to u. If a solution to this unconstrained minimization problem, u^0 say, is strictly positive, then the following first order necessary conditions must be satisfied:

(51) $\nabla_u L(x^0,u^0) \equiv F(x^0) = 0_M$.

Now suppose we solve the N+M equations in (50) and (51) and obtain a solution (x^0,u^0). Suppose that this solution satisfies the following additional conditions:

(52) $x^0 \in \text{Int } X$;
(53) $u^0 \gg 0_M$.

It can be shown (see the problems below) that under our concavity assumptions on g and $f_1,\ldots,f_M$, if (x^0,u^0) satisfies (50)-(53), then (x^0,u^0) is a saddle point of the Lagrangian. Hence x^0 solves (48) by Theorem 1.

Problems

4. Show that (50) and (52) are *sufficient* for x^0 to solve the following maximization problem:

(a) $\max_x \{L(x,u^0) : x \in X\}$.

Hint: Use the third characterization of concavity.

5. Show that (51) and (53) are *sufficient* for u^0 to solve the following minimization problem:

(a) $\min_u \{L(x^0,u) : u \geq 0_M\}$.

Hint: Use the third characterization of convexity. Note: Our concavity assumptions are a substitute for checking the rather complicated second order conditions that occur in equality constrained classical constrained maximization.

In section 6, we show how weaker sufficient conditions for the existence of a saddle point can be obtained. However, before presenting this material, the following section presents an application of the Lagrangian technique to an economic problem.

5. Concave Programming and Duality Theory

In this section, we show how the saddle point approach to solving a constrained maximization problem can be combined with duality theory in order to simplify dramatically the underlying optimization problem.

Consider a very simple general equilibrium model with one household and a single aggregate production sector. Assume that there are N commodities in model and the consumer's preferences can be represented by a concave utility function, $u = f(x)$ for $x \in X$ where X is closed convex subset of R^N of feasible net consumption vectors.[8] Given a vector of positive prices p and a feasible utility level u, we can define the consumer's *expenditure function* e(u,p) in the usual way:[9]

(54) $e(u,p) \equiv \min_x \{p^Tx : f(x) \geq u \,;\, x \in X\}$.

The producer's production possibilities set Y is a closed convex set which contains 0_N. The producer's *profit function*, $\pi(p)$, is defined in the usual way as follows:[10]

(55) $\pi(p) \equiv \max_y \{p^Ty : y \in Y\}$.

We could view the problem of maximizing the consumer's utility subject to the constraints of technology as the following *central planning problem*:

(56) $\max_{x,y} \{f(x) : y - x \geq 0_N \,;\, y \in Y \,;\, x \in X\}$.

Thus we maximize the consumer's utility, f(x), subject to the supply equal to or greater than demand constraints, $y \geq x$, which can be written in standard format as $y - x \geq 0_N$. Note that the objective function f(x) is a concave function of (x,y) and since the N constraint functions, $y - x$, are linear in the components of x and y, they too are concave functions of (x,y). Thus (56) is a concave programming problem of the type studied in the previous sections.[11] The Slater constraint qualification condition for (56) is:

(57) There exist $x^* \in X$ and $y^* \in Y$ such that $y^* >> x^*$.

In order to make further progress, we assume that f is subject to *local nonsatiation*; i.e., if $x \in X$, then in every neighbourhood of x, there exists an $x' \in X$ such that $f(x')$ is greater than f(x). With this assumption, we can replace the objective function f(x) in (56) by the

[8] If there are no labour supplies in the model, then we can take X to be the nonnegative orthant, $\{x : x \geq 0_N\}$. If the household supplies various types of labour services, then we index those supplies with negative numbers and the set X is more complex. The range of f over the set X is also a convex set U.

[9] We need to assume that f is continuous from above (in addition to being concave) in order to be certain that the minimum in (54) exists.

[10] In order to ensure that a solution to the maximization problem (55) exists, assume that Y is a closed, convex and bounded set that contains 0_N.

[11] The convex set $S \equiv \{(x,y) \,;\, x \in X,\, y \in Y\}$ replaces the convex domain of definition set X used in the previous sections.

utility level u and add the constraint $f(x) \geq u$ to (56). The resulting programming problem will have the same solution as the solution to (56). Thus we have:[12]

$$
\begin{aligned}
(58)\ & \max_{x,y} \{f(x) : y - x \geq 0_N ; y\in Y ; x\in X\} \\
& = \max_{x,y,u} \{u : y - x \geq 0_N ; f(x) - u \geq 0 ; y\in Y ; x\in X ; u\in U\} \\
& = \min_p \max_{x,y,u} \{u + p^T[y - x] : f(x) - u \geq 0 ; y\in Y ; x\in X ; u\in U ; p \geq 0_N\} && \text{applying Theorem 2} \\
& = \min_p \max_{x,y,u} \{u + p^Ty - p^Tx : f(x) - u \geq 0 ; y\in Y ; x\in X ; u\in U ; p \geq 0_N\} \\
& = \min_p \max_{x,u} \{u + \pi(p) - p^Tx : f(x) - u \geq 0 ; x\in X ; u\in U ; p \geq 0_N\} && \text{using definition (55)} \\
& = \min_p \max_u \{u + \pi(p) - e(u,p) : u\in U ; p \geq 0_N\} && \text{using definition (54)}
\end{aligned}
$$

where the last equality follows from the fact that minimizing p^Tx is the same as maximizing $- p^Tx$ and then reversing the sign; i.e., we have

$$(59)\ \max_x \{- p^Tx : f(x) \geq u ; x\in X\} = - \min_x \{p^Tx : f(x) \geq u ; x\in X\} = - e(u,p).$$

Now assume that $\pi(p)$ and $e(u,p)$ are differentiable with respect to their variables. The first order necessary conditions (for interior solutions) for the minimization and maximization problems in the last line of (58) are the following N+1 equations:

$$(60)\ \nabla_p\, \pi(p^0) - \nabla_p\, e(u^0,p^0) = 0_N ;$$
$$(61)\qquad 1 - \partial e(u^0,p^0)/\partial u = 0 .$$

Equations (60) and (61) are N+1 equations in the N unknown equilibrium prices p^0 and the equilibrium utility level u^0. Using Hotelling's Lemma and Shephard's Lemma, we see that equations (60) are simply the N supply equals demand equations. Equation (61) can be viewed as a normalization on the absolute scale of prices: $\partial e(u^0,p^0)/\partial u$ can be viewed as a marginal price index, which is set equal to unity. Equation (61) can be replaced by any other convenient normalization of prices.

Using the analysis presented in the previous section, it can be seen that if a (u^0,p^0) solution to (60) and (61) exists and in addition:

$$(61)\ p^0 \gg 0_N \text{ and } u^0\in \text{Int } U,$$

then these conditions are sufficient to imply that (u^0,p^0) solves the central planning problem that is defined by (58).

Note also how the use of duality theory has reduced the dimensionality of the programming problem going from the first line of (58) to the last line. This type of dimensionality reduction always occurs when we combine the Karlin Uzawa saddlepoint

[12] U is the range of f(x) for $x\in X$.

theorem with duality theory.[13] For additional applications of the basic idea presented in this section, see Diewert (1975) (1978) (1983a) (1983b), Diewert, Turunen-Red and Woodland (1989) (1991), Diewert and Woodland (1977) and Woodland (1982).

The traditional approach to prove the existence of a general equilibrium (in a model with no distortions) is via a fixed point theorem.[14] The example presented in this section shows that if there is only one household in the general equilibrium model, then it is not necessary to use a fixed point algorithm to find a set of equilibrium prices: instead, a relatively simple convex programming problem can be solved. In more complicated general equilibrium models with many households, the duality techniques used in this section can still be used in order to reduce the dimensionality of the fixed point algorithm; see Negishi (1960) and Diewert (1973) for applications along these lines.

We now return to the programming problem studied in section 4, where we assumed differentiability. In the following section, we will relax the interiority assumptions (52) and (53) which were made in section 4.

6. Concave Programs and the Kuhn-Tucker Conditions

In this section, we will revisit the two Theorems presented in section 2 above but in the present section, we will make two changes:

- We assume the functions $g, f_1,...,f_M$ are all once continuously differentiable over the domain of definition set X; i.e., the first order partial derivatives of these functions exist and are continuous over X and
- We assume that X is the nonnegative orthant; i.e., $X \equiv \{x : x \geq 0_N\}$.

Under these extra assumptions, our basic nonlinear programming problem (2) now becomes:

(62) $\max_x \{g(x) : F(x) \geq 0_M ; x \geq 0_N\}$.

As in section 2, define the Lagrangian that corresponds to (62) as follows:

(63) $L(x,u) \equiv g(x) + u^TF(x) = g(x) + \sum_{m=1}^M u_mf_m(x)$.

A *saddle point of the Lagrangian* is an $x^0 \geq 0_N$ and $u^0 \geq 0_M$ that satisfies the following inequalities:

(64) $L(x,u^0) \leq L(x^0,u^0) \leq L(x^0,u)$ for all $x \geq 0_N$ and $u \geq 0_M$.

---

[13] With many production sectors instead of only one, the reduction in dimensionality is even more pronounced.
[14] See Debreu (1959).

Obviously, versions of Theorems 1 and 2 in section 2 can be proven for the new problem: all that has changed is that the general convex domain of definition set X has been replaced by $\Omega \equiv \{x : x \geq 0_N\}$, the nonnegative orthant in N dimensional space.

Recall that Theorems 1 and 2 in section 2 did not involve any derivatives of the objective or constraint functions in the nonlinear programming problem. We now consider some conditions involving derivatives that might be necessary or sufficient for (x^0,u^0) to be a saddle point of the Lagrangian L(x,u) defined by (63):

(65) $\nabla_x L(x^0,u^0) \leq 0_N$; $x^0 \geq 0_N$; $x^{0T}\nabla_x L(x^0,u^0) = 0$;
(66) $\nabla_u L(x^0,u^0) \geq 0_N$; $u^0 \geq 0_M$; $u^{0T}\nabla_u L(x^0,u^0) = 0$.

Conditions (65) and (66) are due to Kuhn and Tucker (1951) and are known as the *Kuhn Tucker Conditions*. They require a bit of elaboration.

Consider conditions (65). Since $\nabla_x L(x^0,u^0) \leq 0_N$ and $x^0 \geq 0_N$, it must be the case that $x^{0T}\nabla_x L(x^0,u^0)$ is equal to or less than zero. But the last condition in (65) says that the inner product $x^{0T}\nabla_x L(x^0,u^0)$ is equal to zero. This means each term in the sum of terms must be equal to zero; i.e., (65) implies that

(67) $x_n^0\ \partial L(x^0,u^0)/\partial x_n = 0$; $\qquad n = 1,2,...,N.$

In view of the inequality restrictions in (65), conditions (67) imply the following conditions:

(68) Either $x_n^0 > 0$ and $\partial L(x^0,u^0)/\partial x_n = 0$ *or* $x_n^0 = 0$ and $\partial L(x^0,u^0)/\partial x_n \leq 0$.

Now the meaning of conditions (65) become clear. Recall that we are trying to find a saddle point of the Lagrangian L(x,u) and hence we want to maximize $L(x,u^0)$ with respect to the components of x. Suppose that such a maximizing point x^0 exists. Then if x_n^0 is positive, we must be at the top of a hill in the x_n direction at the point x_n^0; i.e., the standard calculus arguments tell us that we must have $\partial L(x^0,u^0)/\partial x_n = 0$, which is the first case in (68). On the other hand, if x_n^0 is equal to 0, then the feasibility constraint $x_n \geq 0$ prevents us from moving to the left of $x_n^0 = 0$, so we cannot argue as before that we must have $\partial L(x^0,u^0)/\partial x_n = 0$. However, we can argue that we will *not* be at a maximum point if $\partial L(x^0,u^0)/\partial x_n > 0$, since in this case, we could increase x_n from its initial 0 level and increase $L(x^0,u^0)$.[15] Hence if x_n^0 equals 0 and we are at a maximum with respect to the components of the x vector, we must have $\partial L(x^0,u^0)/\partial x_n \leq 0$.

The analysis of conditions (66) proceeds in an analogous manner except we want to *minimize* $L(x^0,u)$ with respect to the components of u and so the old partial derivative conditions for a maximum, $\nabla_x L(x^0,u^0) \leq 0_N$, are replaced by the new partial derivative conditions for a minimum, $\nabla_u L(x^0,u^0) \geq 0_N$.

[15] Use the continuity of the first order partial derivative function and the Mean Value Theorem in calculus to prove this.

The above analysis allows us to prove the following theorem:

Theorem 3; Kuhn and Tucker (1951), Karlin (1959; 204): Suppose $g(x)$ and $f_1(x), \ldots, f_M(x)$ are once continuously differentiable functions defined over $\{x : x \geq 0_N\}$. Suppose that (x^0,u^0) is a saddle point of the Lagrangian defined by (63). Then (x^0,u^0) must satisfy the Kuhn-Tucker conditions (65) and (66).

Thus the Kuhn-Tucker conditions are *necessary conditions* for finding a saddle point of the Lagrangian $L(x,u)$ that corresponds to the constrained maximization problem (62). The following result shows that if we add *concavity* of the objective and constraint functions to our differentiability assumptions, then the Kuhn-Tucker conditions are also *sufficient* to imply that (x^0,u^0) is a saddle point of the Lagrangian.

Theorem 4; Kuhn and Tucker (1951), Karlin (1959; 204): Suppose $g(x)$ and $f_1(x), \ldots, f_M(x)$ are once continuously differentiable, concave functions defined over $\{x : x \geq 0_N\}$. Define the corresponding Lagrangian by (63). Suppose that (x^0,u^0) satisfies the Kuhn-Tucker conditions (65) and (66). Then (x^0,u^0) is a saddle point of the Lagrangian and hence x^0 is a solution to the concave programming problem (62).

Proof: Since $u^0 \geq 0_M$, using definition (63) and the concavity of $g, f_1,\ldots,f_M$, it can be verified that $L(x,u^0)$ is a concave (and differentiable) function of x. Hence by the third characterization of concavity, for all $x \geq 0_N$, we have:

$$
\begin{aligned}
(69)\quad L(x,u^0) &\leq L(x^0,u^0) + \nabla_x L(x^0,u^0)^T(x - x^0) \\
&= L(x^0,u^0) + \nabla_x L(x^0,u^0)^T x - \nabla_x L(x^0,u^0)^T x^0 \\
&= L(x^0,u^0) + \nabla_x L(x^0,u^0)^T x && \text{since } \nabla_x L(x^0,u^0)^T x^0 = 0 \text{ using (65)} \\
&\leq L(x^0,u^0) && \text{using } \nabla_x L(x^0,u^0) \leq 0_N \text{ and } x \geq 0_N.
\end{aligned}
$$

This establishes the first set of inequalities in (64). Using definition (63), we see that $L(x^0,u)$ is a linear function of u and hence is a convex function of u. Hence by the third characterization of convexity, for all $u \geq 0_M$, we have:

$$
\begin{aligned}
(70)\quad L(x^0,u) &\geq L(x^0,u^0) + \nabla_u L(x^0,u^0)^T(u - u^0) \\
&= L(x^0,u^0) + \nabla_u L(x^0,u^0)^T u - \nabla_u L(x^0,u^0)^T u^0 \\
&= L(x^0,u^0) + \nabla_u L(x^0,u^0)^T u && \text{since } \nabla_u L(x^0,u^0)^T u^0 = 0 \text{ using (66)} \\
&\geq L(x^0,u^0) && \text{using } \nabla_u L(x^0,u^0) \geq 0_N \text{ and } u \geq 0_M.
\end{aligned}
$$

This establishes the second set of inequalities in (64). Q.E.D.

Problem

6. Consider the constrained maximization problem (62). Suppose $g(x)$ and $f_1(x), \ldots, f_M(x)$ are once continuously differentiable, concave functions defined over $\{x : x \geq 0_N\}$.

Define the corresponding Lagrangian by (63). Suppose also that the constraints also satisfy the following Slater constraint qualification condition:

(a) There exists $x^* \geq 0_N$ such that $F(x^*) >> 0_M$.

Finally, suppose that x^0 solves (62). Show that there exists $u^0 \geq 0_M$ such that (x^0,u^0) satisfy the Kuhn-Tucker conditions (65) and (66). Hint: This problem is not difficult; you need only use the results that have already been proven. The point of this problem is that under the regularity conditions listed in the problem, the Kuhn-Tucker conditions are necessary and sufficient to find a solution to the concave programming problem (62).

References

Arrow, K.J., L. Hurwicz and H. Uzawa (1961), "Constraint Qualifications in Maximization Problems", *Naval Research Logistics Quarterly* 8, 175-191.

Debreu, G (1959), *A Theory of Value*, New York: John Wiley and Sons.

Diewert, W.E. (1973), "On a Theorem of Negishi", *Metroeconomica* 25, 119-135.

Diewert, W.E. (1975), "The Samuelson Nonsubstitution Theorem and the Computation of Equilibrium Prices", *Econometrica* 43, 57-64.

Diewert, W.E. (1978), "Optimal Tax Perturbations", *Journal of Public Economics* 10, 138-177.

Diewert, W.E. (1983a), "Cost-Benefit Analysis and Project Evaluation: A Comparison of Alternative Approaches", *Journal of Public Economics* 22, 265-302.

Diewert, W.E. (1983b), "The Measurement of Waste within the Production Sector of an Open Economy", *Scandinavian Journal of Economics* 85, 159-179.

Diewert, W.E., A.H. Turunen-Red and A.D. Woodland (1989), "Productivity and Pareto Improving Changes in Taxes and Tariffs", *Review of Economic Studies* 56, 199-216.

Diewert, W.E., A.H. Turunen-Red and A.D. Woodland (1991), "Tariff Reform in a Small Open Multi-Household Economy with Domestic Distortions and Nontraded Goods", *International Economic Review* 32, 937-957.

Diewert, W.E. and A.D. Woodland (1977), "Frank Knight's Theorem in Linear Programming Revisited", *Econometrica* 45, 375-398.

Fenchel, W. (1953), "Convex Cones, Sets and Functions", Lecture Notes at Princeton University, Department of Mathematics, Princeton, N.J.

Karlin, S. (1959), *Mathematical Methods and Theory in Games, Programming and Economics*, Volume 1, Reading MA: Addison-Wesley Publishing Co.

Kuhn, H.W. and A.W. Tucker (1951), "Nonlinear Programming", pp. 481-492 in *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley CA: University of California Press.

Minkowski, H. (1911), *Theorie der konvexen Körper,* Gesammelte Abhandlungen, Zweiter Band, Berlin.

Negishi, T. (1960), "Welfare Economics and the Existence of an Equilibrium for a Competitive Economy", *Metroeconomica* 12, 92-97.

Samuelson, P.A. (1947), *Foundations of Economic Analysis*, Cambridge MA: Harvard University Press.

Slater, M. (1950), "Lagrange Multipliers Revisited: A Contribution to Nonlinear Programming", Cowles Commission Discussion Paper, Mathematics, Number 403, November.

Uzawa, H. (1958), "The Kuhn-Tucker Theorem in Concave Programming", pp. 32-37 in *Studies in Linear and Nonlinear Programming*, K.H. Arrow, L. Hurwicz and H. Uzawa (eds.), Stanford CA: Stanford University Press.

Woodland, A.D. (1982), *International Trade and Resource Allocation*, Amsterdam: North-Holland.