ECONOMICS 581: LECTURE NOTES

CHAPTER 7: Project Evaluation and Willingness to Pay

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1. Introduction

In this chapter, we look at the effects of a government project on welfare. We draw heavily on the results developed in chapter 5.

In section 2, we introduce a project into the economy that is undertaken by the government and where the benefits and costs of the project are fairly straightforward to calculate in the sense that the project produces market outputs and uses market inputs so that market prices can be used to evaluate the project revenues and costs. We also consider only a single household economy and the government sector is rather simple in that there are no commodity tax distortions. In section 5, we remove this last restriction.

In section 3, we generalize the model presented in section 2 to the many household case. We continue to assume that there are no commodity tax distortions in the model.

Section 4 applies the model developed in section 3 to take a first look at a problem of great practical importance in the evaluation of projects. This is the problem associated with the determination of the *social discount rate*; i.e., if some benefits and costs occur in the future for a publicly funded project, then what is the appropriate *interest rate* that should be used to discount future benefits and costs?

In section 5, we generalize the single household model presented in section 2 to the case where variable commodities are taxed in addition to fixed factors or pure profits. We also provide an introduction to *shadow pricing rules* and present a rigorous derivation of Harberger's (1969; 82) *weighted average shadow pricing rule*.

Sections 6-10 return to the model presented in section 2 but in these final sections, we assume that the project also produces public goods. These are goods that are supplied to the population free of charge, such as roads and parks. The main additional difficulty is the determination of prices to value the benefits associated with the provision of these public goods. Thus in the final sections of this chapter, we revisit the simple model explained in section 2 but we now assume that the project produces a K dimensional vector of public goods, $c \equiv [c_1,c_2,\ldots,c_K]^T \geq 0_K$. These public goods (or environmental variables) could benefit consumers directly (so that the vector c appears in household utility functions) or indirectly by facilitating production (so that the vector c appears in the economy's production functions). We will deal with the case where c appears in a utility function in this sections 6-8 and then deal with the case where c appears in the economy's technology set in section 9.

Section 6 starts off our discussion of how to measure the benefits of public goods to a household. We first prove an impossibility result that shows it will not be possible to estimate completely general household preferences defined over different combinations of market and nonmarket goods and services. However, we will then go on and show that with a homotheticity restriction on preferences, it will be possible to use econometric techniques to estimate flexible functional forms for the consumer's utility function in this restricted class of preferences.

Section 7 is rather different than the rest of this chapter. This section addresses an econometric issue; namely, is it preferable to estimate preferences by estimating a utility function directly (and use Wold's Identity to generate a system of inverse demand functions) or is it preferable to estimate preferences indirectly via the estimation of a dual expenditure function (and use Shephard's Lemma to generate as system of Hicksian demand functions)? In the first set up, quantities demanded are the dependent variables and prices are the independent or predetermined variables. In the second set up, prices are the dependent variables and quantities are the predetermined variables. Section 7 points out that unless the correlation between prices and quantities is very high, we are likely to obtain very different estimates of elasticities using the two approaches. Note that from the viewpoint of theory, the two methods are completely equivalent. However, from the viewpoint of econometric estimation, the two methods are likely to generate very different sets of elasticities. We try to explain this econometric problem in some detail in section 7. In the end, we suggest that it is probably preferable to estimate preferences indirectly via the estimation of a dual expenditure function.

In section 6, we showed how a flexible functional form for a utility function (that satisfied the assumption of homotheticity in market goods) could be estimated using market data on prices and the market choices of the consumer. In section 8, we show how a flexible functional form for a dual expenditure function could be estimated using the same data.

In section 9, we study the problem of measuring the effects of the provision of additional amounts of public goods and services on production and we look at the problems involved in finding a flexible functional form to model these effects.

Finally, in section 10, we draw on sections 8 and 9 and we rework our simple project evaluation model presented in section 2 (where the project only involved market goods and services) but we extend our earlier model to the case where the project also produces additional supplies of public goods or nonmarket goods and services.

2. The Evaluation of Costs and Benefits for a Small Project when There are no Commodity Tax Distortions: The One Household Case

In this section, we draw on the results obtained in sections 4 and 8 of chapter 5. We consider a one household economy with a government sector but the government is able to raise enough revenue to pay for its production of government services by taxing fixed factors. Thus there are no commodity tax distortions in the model developed in this section. This assumption greatly simplifies the analysis, but it is a limitation on the results.

Assume that the household sector supplies some factors of production inelastically. Conditional on this vector of fixed inputs, the production sector has the technology set S, which is a closed, convex subset of R^{N+1}, which is bounded from above. The set S represents the set of outputs[1] that the economy can produce. Given a vector of positive output prices, $[p_0,p]$, where $p_0 > 0$ and $p \equiv [p_1,\ldots,p_N]^T >> 0_N$, define the *profit function*, $\pi(p_0,p)$, that corresponds to S as follows:

$$(1)\ \pi(p_0,p) \equiv \max_{y_0,y} \{p_0y_0 + p^Ty: (y_0,y)\in S\}.$$

Throughout this chapter, we will assume that profit functions are twice continuously differentiable with respect to their price variables. Thus we can apply Hotelling's (1932) Lemma and the supply functions of the production sector are given by the following first order partial derivatives:

$$(2)\ y_0(p_0,p) = \partial\pi(p_0,p)/\partial p_0 \ ;$$
$$y(p_0,p) = \nabla_p\pi(p_0,p) \equiv [\partial\pi(p_0,p)/\partial p_1,\ldots,\partial\pi(p_0,p)/\partial p_N]^T .$$

Consumer preferences over the N+1 variable commodities are represented by the continuous, nondecreasing and quasiconcave utility function, $f(x_0,x_1,\ldots,x_N) = f(x_0,x)$, where $x \equiv [x_1,\ldots,x_N]^T$. The cost or *expenditure function*, e, that corresponds to these preferences is defined for some utility level u in the range of f and for $[p_0,p]$, where $p_0 > 0$ and $p \equiv [p_1,\ldots,p_N]^T >> 0_N$, as follows:

$$(3)\ e(u,p_0,p) \equiv \min_{x_0,x} \{p_0x_0 + p^Tx: f(x_0,x) = u\}.$$

Throughout this chapter, we will assume that expenditure functions are twice continuously differentiable with respect to their variables. Thus we can apply Shephard's Lemma and the Hicksian demand functions of the household sector are given by the following first order partial derivatives:

$$(4)\ x_0(u,p_0,p) = \partial e(u,p_0,p)/\partial p_0 \ ;$$
$$x(u,p_0,p) = \nabla_p e(u,p_0,p) \equiv [\partial e(u,p_0,p)/\partial p_1,\ldots,\partial e(u,p_0,p)/\partial p_N]^T .$$

We assume that the consolidated government sector taxes only profits (or fixed factors) at the rate τ. The government spends these tax revenues on purchases of goods and services that produce various government services and by possibly transferring money back to the

[1] Variable demands for labour services can also be included as negative net outputs. Since the fixed inputs in the economy will not be varied, we have omitted them from the notation.

household sector. Denote the transfer[2] as T and the purchases by the government on numeraire and nonnumeraire goods and services by g_0 and $g \equiv [g_1,\ldots,g_N]^T$ respectively.

The government also introduces a *project* or expenditure program into the economy. This project is characterized by a vector of net outputs that the project generates, $[a_0,a_1,\ldots,a_N] = [a_0,a^T]$. If $a_n > 0$, then the project produces commodity n as an output, while if $a_n < 0$, then the project uses commodity n as an input. We model the transition from the preproject economy to the post project economy by having a scale variable s run from 0 to 1 as in chapter 5. Our goal in this section is to obtain a first order approximation to the change in welfare due to the introduction of the project and to determine how the costs and benefits of the project should be evaluated.

The consumer expenditure equals income equation in this model is equation (5) below and the demand equals supply equations for the numeraire and nonnumeraire commodities are given by (6) and (7) below:

(5) $$e(u(s),p_0,p(s)) = (1-\tau)\pi(p_0,p(s)) + T(s) ;$$
(6) $$\partial e(u(s),p_0,p(s))/\partial p_0 + g_0 = \partial \pi(p_0,p(s))/\partial p_0 + a_0 s ;$$
(7) $$\nabla_p e(u(s),p_0,p(s)) + g = \nabla_p \pi(p_0,p(s)) + as.$$

The endogenous variables in the N+2 equations (5)-(7) are the utility level u, the nonnumeraire prices p and the transfer T, which is N+2 variables, while the exogenous variable is the project scale s.

There is another (redundant) equation which could be added to (5)-(7) and that is the government budget constraint. We show how the government budget constraint can be derived. Multiply both sides of equation (6) by the numeraire price p_0, premultiply both sides of (7) by the vector of nonnumeraire prices transposed $p(s)^T$ and sum the resulting equations. Using the linear homogeneity properties of the expenditure and profit functions in their price variables, we obtain the following equation:

(8) $$e(u(s),p_0,p(s)) + p_0 g_0 + p(s)^T g = \pi(p_0,p(s)) + [p_0 a_0 + p(s)^T a]s.$$

Now use equations (5) and (8) to eliminate $e(u(s),p_0,p(s))$ from the two equations. The resulting equation is the government budget constraint, equation (9) below:

(9) $$T(s) + p_0 g_0 + p(s)^T g = \tau\pi(p_0,p(s)) + [p_0 a_0 + p(s)^T a]s.$$

Equation (9) says that *government expenditures* (transfer expenditures, T(s), plus expenditures on goods and services, $p_0 g_0 + p(s)^T g$, needed to finance public services)

[2] If T(0) is negative, then the government is imposing a lump sum tax on the household sector at the initial equilibrium.

equals *government revenues* from taxing profits (or primary inputs), $\tau\pi(p_0,p(s))$, plus net revenues from the project run at scale s, $[p_0a_0 + p(s)^Ta]s$.[3]

Now use equation (9) to obtain an expression for the transfer T(s) and substitute this expression into equation (5). The resulting equation is:

$$(10)\ e(u(s),p_0,p(s)) = \pi(p_0,p(s)) - [p_0g_0 + p(s)^Tg] + [p_0a_0 + p(s)^Ta]s.$$

Equations (10) and (7) are N+1 equations in the N+1 unknowns u(s) and p(s). We will differentiate these equations with respect to s in order to obtain expressions for the derivatives of utility u and nonnumeraire prices p with respect to the project scale s, $u'(s)$ and $p'(s)$.

Differentiating (10) with respect to s leads to the following equation:

$$(11)\ \{\partial e(u(s),p_0,p(s))/\partial u\}u'(s)$$
$$= -[\nabla_p e(u(s),p_0,p(s)) + g - \nabla_p\pi(p_0,p(s)) - as]^Tp'(s) + p_0a_0 + p(s)^Ta$$
$$= p_0a_0 + p(s)^Ta \qquad \text{using (7).}$$

Differentiating (7) with respect to s leads to the following equation:

$$(12)\ \nabla_{pu}^{2}e(u(s),p_0,p(s))u'(s) + [\nabla_{pp}^{2}e(u(s),p_0,p(s)) - \nabla_{pp}^{2}\pi(p_0,p(s))]p'(s) = a.$$

In order to further simplify equations (11) and (12) when s = 0, we impose *money metric utility scaling* on the household preferences[4] using the initial equilibrium prices, $p_0,p(0)$, as the reference prices. From Chapter 4, it can be shown that this scaling assumption imposes the following restrictions on the expenditure function and its first and second order partial derivatives:

$$(13)\ e(u(0),p_0,p(0)) = \pi(p_0,p(s)) - [p_0g_0 + p(s)^Tg]\ ;$$
$$(14)\ \partial e(u(0),p_0,p(0))/\partial u = 1\ ;$$
$$(15)\ p_0\partial^2 e(u(0),p_0,p(0))/\partial p_0\partial u + p(0)^T\nabla_{pu}^{2}e(u(0),p_0,p(0)) = 1\ ;$$
$$(16)\ \partial^2 e(u(0),p_0,p(0))/\partial u^2 = 0\ .$$

Now substitute (14) into (11) and evaluate the resulting equation along with (12) at s = 0. We obtain the following matrix equation:

$$(17)\ \begin{bmatrix} 1, & 0_N^T \\ b, & S \end{bmatrix} \begin{bmatrix} u'(0) \\ p'(0) \end{bmatrix} = \begin{bmatrix} p_0a_0 + p(0)^T a \\ a \end{bmatrix}$$

[3] If government enterprises are included in the government sector, then it could be the case that government expenditures on goods and services are negative. It could also be the case that project profits could be negative.

[4] As was mentioned in Chapter 4, this assumption is empirically harmless because it cannot be refuted by observable data.

where the vector $b \equiv \nabla_{pu}^{2}e(0)$ shows how the demand for the nonnumeraire commodities changes as utility increases[5] and $S \equiv \nabla_{pp}^{2}e(u(0),p_0,p(0)) - \nabla_{pp}^{2}\pi(p_0,p(s))$ is the economy's *aggregate substitution matrix* for nonnumeraire commodities. Since $e(u,p_0,p)$ is concave in p and $\pi(p_0,p)$ is convex in p, S is a symmetric negative semidefinite matrix. We shall assume that S is *negative definite* so that its determinant is not equal to zero; i.e., we assume that

(18) S is a negative definite symmetric matrix.

Since the determinant of the N+1 by N+1 matrix in the left hand side of (17) is equal to the determinant of S, |S|, it can be seen that assumption (18) means that we can apply the Implicit Function Theorem and obtain the *existence* of the solution functions, u(s) and p(s), to equations (7) and (10) for s close to 0.

Assumption (18) implies that S^{-1} exists and we can invert the matrix on the left hand side of (17) and obtain the following expressions for the derivatives, $u'(0)$ and $p'(0)$:

$$(19)\quad \begin{bmatrix} u'(0) \\ p'(0) \end{bmatrix} = \begin{bmatrix} 1 & 0_N^T \\ -S^{-1}b & S^{-1} \end{bmatrix} \begin{bmatrix} p_0a_0 + p(0)^T a \\ a \end{bmatrix} = \begin{bmatrix} p_0a_0 + p(0)^T a \\ -S^{-1}b[p_0a_0 + p(0)^T a] + S^{-1}a \end{bmatrix}.$$

Now we can use (19) to form the following *first order approximation to the change in welfare of the consumer* if the project is implemented at level s = 1:

$$(20)\quad u(1) - u(0) \approx u'(0)[1 - 0] = p_0a_0 + p(0)^T a.$$

Thus *to the accuracy of a first order Taylor series approximation, the increase in consumer utility is equal to the profitability of the project*, *valued at the market prices of the preproject economy*, $p_0a_0 + p(0)^T a$.

Equation (20) forms the underlying rational for most practical applications of cost benefit analysis. The effect on economic welfare of undertaking a (small) project is equal to *expected revenues* minus *expected costs* and hence is equal to the net expected profits of the project (evaluated at prevailing market prices), $p_0a_0 + p(0)^T a$. Of course, in many practical applications of cost benefit analysis, it is not an easy matter to determine the expected costs and revenues of the project.

Problem

1. Show that $u''(0) \le 0$. Hint: Recall section 4 in chapter 5 and note the similarity of the present model with the model in that section.

3. The Evaluation of Costs and Benefits for a Small Project when there are no Commodity Tax Distortions: The Many Household Case

[5] If nonnumeraire commodity n is normal, then $b_n > 0$ while if nonnumeraire commodity n is inferior, then $b_n < 0$.

We generalize the model in the previous section from the case of only one household to the case where there are H households.

We assume that household h owns a *share*, $\alpha_h \geq 0$, of the aggregate production sector.[6] These nonnegative shares sum to unity:

(21) $\sum_{h=1}^{H} \alpha_h = 1$.

We also assume that the government makes transfer of size $\beta_h T$ to household h where the transfer shares sum to unity:[7]

(22) $\sum_{h=1}^{H} \beta_h = 1$.

Equations (5)-(7) that characterized the economy in the previous section are now replaced by the following equations:

(23) $e^h(u_h(s),p_0,p(s)) = \alpha_h(1-\tau)\pi(p_0,p(s)) + \beta_h T(s)$; $h = 1,\ldots,H$;
(24) $\sum_{h=1}^{H} \partial e^h(u_h(s),p_0,p(s)/\partial p_0 + g_0 = \partial\pi(p_0,p(s))/\partial p_0 + a_0 s$;
(25) $\sum_{h=1}^{H} \nabla_p e^h(u_h(s),p_0,p(s)) + g = \nabla_p \pi(p_0,p(s)) + as$

where e^h is the household h expenditure function, u_h is the household h utility level and the other variables are defined as in the previous section. The endogenous variables in the H+N+1 equations (22)-24) are the H utility levels u_h, the N nonnumeraire prices p and the transfer T, which is H+N+1 variables, while the exogenous variable is the project scale s. Equations (23) equate each household's expenditures on goods and services (less variable factor supplies) to its corresponding after tax income, $\alpha_h(1-\tau)\pi(p_0,p(s))$, plus its transfer income, $\beta_h T(s)$. Equations (24) and (25) equate the aggregate household demand for each commodity to the available supply for the numeraire and nonnumeraire commodities respectively. When s = 0, equations (23)-(25) are the preproject general equilibrium equations, and when s = 1, equations (23)-(25) are the post project equilibrium equations.

Multiply both sides of equation (24) by the numeraire price p_0, premultiply both sides of (25) by the vector of nonnumeraire prices transposed $p(s)^T$ and sum the resulting equations. Using the linear homogeneity properties of the expenditure and profit functions in their price variables, we obtain the following equation:

(26) $\sum_{h=1}^{H} e^h(u_h,p_0,p(s)) + p_0 g_0 + p(s)^T g = \pi(p_0,p(s)) + [p_0 a_0 + p(s)^T a]s$.

Now sum the H equations in (23). Using (21) and (22), we obtain the following equation:

[6] Alternatively, if there is only one fixed factor and constant returns to scale in production, the α_h can be interpreted as household h's share of the fixed factor.
[7] It is not necessary that these β_h shares be nonnegative; they need only sum to 1.

(27) $\sum_{h=1}^{H} e^h(u_h(s),p_0,p(s)) = (1 - \tau)\pi(p_0,p(s)) + T(s)$

Now use equations (26) and (27) to eliminate $\sum_{h=1}^{H} e^h(u_h(s),p_0,p(s))$ from the two equations. The resulting equation is the *government budget constraint*, equation (28) below:

(28) $T(s) + p_0g_0 + p(s)^Tg = \tau\pi(p_0,p(s)) + [p_0a_0 + p(s)^Ta]s.$

Equation (28) says that *government expenditures* (total transfer expenditures, T(s), plus expenditures on goods and services, $p_0g_0 + p(s)^Tg$, needed to finance public services) equals *government revenues* from taxing profits (or primary inputs), $\tau\pi(p_0,p(s))$, plus net revenues from the project run at scale s, $[p_0a_0 + p(s)^Ta]s$.

An equation that we will use shortly results when we use (28) to solve for T(s) and substitute this expression into (27). After this substitution, equation (27) becomes the following equation:

(29) $\sum_{h=1}^{H} e^h(u_h(s),p_0,p(s)) = \pi(p_0,p(s)) - [p_0g_0 + p(s)^Tg] + [p_0a_0 + p(s)^Ta]s.$

The final set of equations that we need results from using (28) to replace T(s) in equations (23). The new version of equations (23) becomes the following equations:

$$(30)\ e^h(u_h(s),p_0,p(s)) = \alpha_h(1 - \tau)\pi(p_0,p(s)) + \beta_h\{\tau\pi(p_0,p(s)) + [p_0a_0 + p(s)^Ta]s - [p_0g_0 + p(s)^Tg]\}\ ; \qquad h = 1,\ldots,H.$$

Now drop equation (24) and differentiate the H equations (30) and the N equations (25) with respect to s. Doing the differentiation and evaluating the derivatives at s = 0 lead to the following system of H + N equations:

$$(31)\ \begin{bmatrix} \partial e^1/\partial u_1 & \ldots & 0 & \nabla_p e^{1^T} - [\alpha_1(1-\tau)+\tau\beta_1]\nabla_p\pi^T + \beta_1 g^T \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots, & \partial e^H/\partial u_H & \nabla_p e^{H^T} - [\alpha_H(1-\tau)+\tau\beta_H]\nabla_p\pi^T + \beta_H g^T \\ \nabla^2_{pu}e^1 & \ldots\ldots & \nabla^2_{pu}e^H & \sum_{h=1}^{H}\nabla^2_{pp}e^h(0) - \nabla^2_{pp}\pi(0) \end{bmatrix} \begin{bmatrix} u'(0) \\ p'(0) \end{bmatrix} = \begin{bmatrix} \beta_1[p_0a_0 + p(0)^Ta] \\ \ldots \\ \beta_H[p_0a_0 + p(0)^Ta] \\ a \end{bmatrix}$$

where $e^h(0) \equiv e^h(u(0),p_0,p(0))$ for h = 1,…,H, $\pi(0) \equiv \pi(p_0,p(0))$, $u'(0) \equiv [u_1'(0),\ldots,u_H'(0)]^T$ and $p'(0) \equiv [p_1'(0),\ldots,p_N'(0)]^T$. Now define the following matrices using the initial equilibrium data:

(32) $S \equiv \sum_{h=1}^{H} \nabla_{pp}^{2}e^{h}[u_h(0),p_0,p(0)] - \nabla_{pp}^{2}\pi[p_0,p(0)] \equiv \sum_{h=1}^{H} S^{h} - S_P$;
(33) $B \equiv [b^{1},\ldots,b^{H}] \equiv [\nabla_{pu}^{2}e^{1}[u_1(0),p_0,p(0)],\ldots,\nabla_{pu}^{2}e^{1}[u_1(0),p_0,p(0)]]$;
(34) $C = [c^{1},\ldots,c^{H}]$

where c^h is defined as follows:

(35) $c^{h} \equiv \nabla_{p}e^{h}(u_h(0),p_0,p(0)) + \beta_h g - [\alpha_h(1-\tau) + \beta_h\tau]\,\nabla_p\pi(p_0,p(0))$; $h = 1,\ldots,H$.

S is the economy's negative semidefinite *aggregate substitution matrix* and it is equal to the sum of the individual household negative semidefinite substitution matrices, $\sum_{h=1}^{H} S^{h}$, minus the positive semidefinite production substitution matrix, S_P, at the period 0 equilibrium. B is the economy's period 0 N by H *matrix of household income effects*; i.e., b^h tells us how household h's demand for nonnumeraire commodities changes as real income or utility increases a marginal amount. Finally, C is an N by H matrix of *household net demand vectors for nonnumeraire commodities*. To explain the meaning of this matrix, consider column h, which is equal to:

(36) $c^{h} \equiv \nabla_{p}e^{h}(u_h(0),p_0,p(0)) - \alpha_h(1-\tau)\nabla_p\pi(p_0,p(0)) - \beta_h[\tau\nabla_p\pi(p_0,p(0)) - g]$; $h = 1,\ldots,H$
$= x^{h}(0) - \alpha_h(1-\tau)y(0) - \beta_h[\tau\, y(0) - g]$.

Thus column h of C, c^h, is equal to the commodity demand vector for household h for nonnumeraire commodities at the period 0 equilibrium, $x^h(0)$, less household h's share of period 0 after tax nonnumeraire production, $\alpha_h(1-\tau)y(0)$, less household h's share of the government's implied vector of net supplies of nonnumeraire goods and services, $\beta_h[\tau\, y(0) - \beta g]$.[8]

Note that the columns of C sum to 0_N:

$$\begin{aligned}
(37)\ C1_N &= \textstyle\sum_{h=1}^{H} c^{h} \\
&= \textstyle\sum_{h=1}^{H} \{x^{h}(0) - \alpha_h(1-\tau)y(0) - \beta_h[\tau\, y(0) - g]\} \\
&= \textstyle\sum_{h=1}^{H} x^{h}(0) - \sum_{h=1}^{H} \alpha_h(1-\tau)y(0) - \sum_{h=1}^{H} \beta_h[\tau\, y(0) - g] \\
&= \textstyle\sum_{h=1}^{H} x^{h}(0) - (1-\tau)y(0) - [\tau\, y(0) - g] && \text{using (21) and (22)} \\
&= \textstyle\sum_{h=1}^{H} x^{h}(0) - y(0) + g \\
&= 0_N && \text{using (24) when } s = 0.
\end{aligned}$$

As in the previous section, we impose *money metric utility scaling* on each household's preferences using the initial equilibrium prices, $p_0,p(0)$, as the reference prices. Now substitute the money metric utility scaling equation (14) into (31) for each household and using definitions (32)-(34), (31) becomes the following matrix equation:

[8] Through its taxation of fixed factors, the government can be thought of as collecting $\tau y(0)$ from producers and then contributing this amount of production to the market. But offsetting this supply of the government is its vector of nonnumeraire commodity demands g, which is required to produce government services. Thus the government's implicit net supply of nonnumeraire commodities is $\tau y(0) - g$ and household h's share of this vector is $\beta_h[\tau y(0) - g]$.

$$(38)\begin{bmatrix} I_H & C^T \\ B & S \end{bmatrix}\begin{bmatrix} u'(0) \\ p'(0) \end{bmatrix} = \begin{bmatrix} \beta_1[p_0a_0 + p(0)^T a] \\ \cdots \\ \beta_H[p_0a_0 + p(0)^T a] \\ a \end{bmatrix} = \begin{bmatrix} \beta[p_0a_0 + p(0)^T a \\ a \end{bmatrix}$$

where $\beta^T \equiv [\beta_1,\ldots,\beta_H]$. If we subtract B times the first H rows in (38) from the last N rows in (38), we obtain the equivalent system of equations:

$$(39)\begin{bmatrix} I_H & C^T \\ 0_{NxH} & S - BC^T \end{bmatrix}\begin{bmatrix} u'(0) \\ p'(0) \end{bmatrix} = \begin{bmatrix} \beta[p_0a_0 + p(0)^T a \\ a - B\beta[p_0a_0 + p(0)^T a] \end{bmatrix}.$$

Thus it can be seen that the determinant of the first matrix on the left hand side of (38) is not equal to zero if and only if the determinant of $S-BC^T$ is not equal to zero. In order to apply the Implicit Function theorem and get the existence of the solution functions u(t) and p(t) to the system of equations (98)-(100) around t = 0, we assume that[9]

(40) $| S-BC^T| \neq 0$.

Using row operations with partitioned matrices, it can be shown that the formula for the inverse of the H+N by H+N matrix on the left hand side of (38) is given by:

$$(41)\ \begin{bmatrix} I_H & C^T \\ B & S \end{bmatrix}^{-1} = \begin{bmatrix} I_H + C^T\left[S - BC^T\right]^{-1}B; & -C^T\left[S - BC^T\right]^{-1} \\ -\left[S - BC^T\right]^{-1}B & \left[S - BC^T\right]^{-1} \end{bmatrix}.$$

Obviously, (38) and (41) enable us to calculate the vectors of utility and price derivatives around the preproject economy, $u'(0)$ and $p'(0)$. However, the resulting formulae are not very transparent so we will use a different strategy in what follows.

Recall equation (29) above. Differentiate this equation with respect to s and evaluate the resulting derivatives at s = 0. We obtain the following equation:

$$\begin{aligned}(42)\ &\textstyle\sum_{h=1}^{H} [\partial e^h(u_h(0),p_0,p(0))/\partial u_h]u_h'(0) \\ &= -\textstyle\sum_{h=1}^{H} \nabla_p e^h(u_h(0),p_0,p(0))^T p'(0) + \nabla_p\pi(p_0,p(0))^T p'(0) - g^T p'(0) + [p_0a_0 + p(0)^T a] \\ &= [-\textstyle\sum_{h=1}^{H} \nabla_p e^h(u_h(0),p_0,p(0))^T + \nabla_p\pi(p_0,p(0))^T - g^T]p'(0) + [p_0a_0 + p(0)^T a] \\ &= 0_N^T p'(0) + [p_0a_0 + p(0)^T a] \qquad\qquad \text{using (25) for s = 0.}\end{aligned}$$

Using the money metric utility scaling assumptions (14) for each household, (42) becomes:

(43) $\sum_{h=1}^{H} u_h'(0) = p_0a_0 + p(0)^T a$

[9] If the aggregate substitution matrix S is negative definite, then only by chance will we have $| S-BC^T| = 0$.

which is a counterpart to (19) in the one household economy.

Define the *money metric utilitarian social welfare function* for the economy[10], using the prices of the period 0 economy as reference prices, as:

(44) $W(s) \equiv \sum_{h=1}^{H} u_h(s)$.

Thus aggregate welfare is simply defined as the sum of the money metric welfare measures for each household in the economy. Note that

(45) $W'(0) = \sum_{h=1}^{H} u_h'(0)$.

Now we can use (43) and (45) to form the following *first order approximation to the change in money metric utilitarian welfare assuming the project is undertaken*:

(46) $W(1) - W(0) \approx W'(0)[1 - 0] = p_0a_0 + p(s)^Ta$.

Thus *to the accuracy of a first order Taylor series approximation, the increase in money metric utilitarian welfare is equal to the value of revenues less costs that the project is expected to make, where the market prices prevailing in the preproject economy are used to calculate these revenues and costs.* Thus to the accuracy of a first order approximation, the profitability of the project at market prices translates dollar for dollar into an increase in utilitarian social welfare.

Problems

2. Assuming that condition (40) holds, determine necessary and sufficient conditions for $p'(0) = 0_N$. Hint: Look at equations (39).

3. Of course, not all households may gain if the project is undertaken even if project profits are large and positive. Develop some analytic expressions in order to determine $u_h'(0)$ for each h.

4. Show that $W''(0) \leq 0$. Hint: Use the techniques used in section 7 of chapter 5.

4. A First Approach to the Determination of the Social Discount Rate

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[10] With money metric scaling at the initial prices, the social welfare function defined by (444) is equivalent to $\sum_{h=1}^{H} e^h(u_h,p_0,p(0))$ where $u^h \equiv f^h(x_0^h,x^h)$ for h = 1,…,H and f^h is household h's primal utility function. This is a money metric utilitarian social welfare function and it is not independent of the reference prices, $p_0,p(0)$. The corresponding primal utilitarian social welfare function is defined as $\sum_{h=1}^{H} f^h(x_0^h,x^h)$ an if it is used in applications, it is usually cardinalized by using ray scaling; i.e., choose a reference consumption vector, $[x_0^*,x^*]$, (perhaps equal to the aggregate household consumption vector at the initial equilibrium), and then impose ray scaling on each utility function using this reference vector; i.e., set $f^h(x_0^*,x^*) = 1$ with $f^h(\lambda x_0^*,\lambda x^*) = \lambda$ for all $\lambda \geq 0$ for h = 1,…,H. Of course this ray scaled primal utilitarian social welfare function is not independent of the reference scaling vector, $[x_0^*,x^*]$.

A problem of great practical importance in the evaluation of projects is the determination of the *social discount rate*; i.e., if some benefits and costs occur in the future for a publicly funded project, then what is the appropriate *interest rate* that should be used to discount future benefits and costs?

As a first approach to this problem, we could apply the model developed in the previous section and adapt it to the intertemporal context.[11] Thus suppose we take a T period horizon with N+1 variable commodities in each period. Thus let $[p_0^t,p^t]$ denote the N+1 dimensional vector of prices for the N+1 variable commodities in period t for t = 1,2,…,T. The analysis in the previous section suggests that we simply use these T(N+1) intertemporal market prices in order to evaluate the project.

In order to see the connection of the model outlined in the previous paragraph with interest rates,[12] let us set the price of the numeraire commodity in period 1 equal to 1; i.e., set

(47) $p_0^1 = 1.$

Now consider the sequence of prices for the numeraire commodity in periods 2,3,…,T. These are the prices p_0^t for t = 2,3,…,T. We can use these prices in order to define the *sequence of one period interest rates*, $r_1, r_2,\ldots,r_{T-1}$:

(48) $p_0^1 = 1$;
$p_0^2 = 1/(1+r_1)$;
$p_0^3 = 1/(1+r_1)(1+r_2)$;
…
$p_0^T = 1/(1+r_1)(1+r_2)(1+r_3) \ldots (1+r_{T-1}).$

[11] For additional material on intertemporal production and consumption models, see Hicks (1946; 191-236 and 325-328).

[12] Böhm-Bawerk (1891; 24-72) looked at the factors that determined interest rates while Fisher (1930; 246-265) presented a clear algebraic and geometric description of the same theory. Böhm-Bawerk (1891; 285-286), Fisher (1897; 522) and Hicks (1946; 141-142) explained how the present price of a good purchased now for delivery next period is equal to the spot price of the good next period divided by one plus the current period interest rate. Hicks (1946; 136) generalized the simple one spot commodity and multiple time period model of Fisher (1930) into a general model with man commodities and many time periods (his "Futures Economy" where all commodities can be bought and sold on forward markets) and Debreu (1959) provided a rigorous proof of the existence of equilibrium in such a model. But Hicks (1946; 19-127), following along the lines pioneered by Walras (1954; 267-306), developed another model of intertemporal equilibrium that had an interest rate built into it: the temporary equilibrium model. This second Hicksian model uses the same basic building blocks as the futures model, except that instead of assuming the existence of futures markets, Hicks assumed the existence of current period spot markets for commodities and financial capital and the existence of definite expectations about future period spot prices (which could depend on current period spot prices) for all consumers and producers in the economy. In this model, these expected future period spot prices were used by producers and consumers in their intertemporal profit and utility maximization plans. For a more recent update of the temporary equilibrium model, see Diewert (1977). The algebra in equations (48)-(50) is also explained in Pollak (1989; 72-73).

We can also convert the period t prices for nonnumeraire commodities, p_n^t, into prices that are relative to the price of the numeraire commodity for that period, say ρ_n^t:

$$(49)\ \rho_n^t \equiv p_n^t/p_0^t\,; \qquad t = 1,\ldots,T\,;\ n = 1,\ldots,N.$$

Thus using (49), we have:

$$(50)\ p_n^t = p_0^t\,\rho_n^t \qquad t = 2,3,\ldots,T\,;\ n = 1,\ldots,N$$
$$= \rho_n^t/(1+r_1)(1+r_2)(1+r_3)\ \ldots\ (1+r_{t-1}) \qquad \text{using (48).}$$

Thus the intertemporal prices p_n^t that appear in the intertemporal model that is a counterpart to the model of the previous section can be written as interest rate discounted prices relative to the numeraire commodity in period t.

The above argument suggests that the appropriate *interest rate* that should be used to discount future benefits and costs in project analysis is a *representative market rate of interest*. This argument will suffice as a first approximation to the social discount rate but there are several problems with the above argument. We list some of these problems:

- The argument rests on the utilitarian social welfare function, which many economists would regard as being inappropriate.
- There is a finite T period horizon in our model. What happens if we take an infinite horizon?
- The model rests on the existence of a complete set of futures markets that will correctly equate supply equal to demand in future periods at the beginning of period 1. Alternatively, economic agents have to be able to accurately predict what future period equilibrium prices will be.
- The argument assumes that all households that are present at the beginning of period 1 are also present in future periods. How can we deal with deaths and births in the model?
- We have assumed that there are no commodity tax distortions in the model presented in the previous section and the results are not robust to this assumption.
- We have abstracted from risk and uncertainty. This too is a severe limitation of our analysis. Risk means that there will not be a single market rate of interest.

However, we cannot solve all difficult economic problems in one chapter. The argument given in this section is only a first approach to this problem of choosing the social discount rate.

5. The Evaluation of Costs and Benefits for a Small Project when There are Commodity Tax Distortions: The One Household Case

In this section, we generalize the model presented in section 2 to the case where variable commodities are taxed in addition to fixed factors or pure profits. We also provide an introduction to *shadow pricing rules* and present a rigorous derivation of Harberger's (1969; 82) *weighted average shadow pricing rule*.

We assume that the consolidated government sector taxes variable commodities at the tax rates τ_0 for the numeraire commodity and $\tau \equiv [\tau_1,\ldots,\tau_N]^T$ for the vector of nonnumeraire commodities. The government also taxes variable profits (or the underlying fixed factors) at the rate τ_π. The government spends these tax revenues on purchases of goods and services that produce various government services and by possibly transferring money back to the household sector. Denote the transfer as T[13] and the purchases by the government on numeraire and nonnumeraire goods and services by g_0 and $g \equiv [g_1,\ldots,g_N]^T$ respectively.

The counterparts to equations (5)-(7) in section 2 are the following equations:

(51) $$e(u(s),p_0+\tau_0,p(s)+\tau)) = (1-\tau_\pi)\pi(p_0,p(s)) + T(s)\ ;$$
(52) $$\partial e(u(s),p_0+\tau_0,p(s)+\tau)/\partial p_0 + g_0 = \partial\pi(p_0,p(s))/\partial p_0 + a_0 s\ ;$$
(53) $$\nabla_p e(u(s),p_0+\tau_0,p(s)+\tau) + g = \nabla_p\pi(p_0,p(s)) + as.$$

The endogenous variables in the N+2 equations (51)-(53) are the utility level u, the nonnumeraire prices p and the transfer T, which is N+2 variables, while the exogenous variable is the project scale s.

In order to obtain the government budget constraint, multiply both sides of equation (52) by the numeraire price p_0, premultiply both sides of (53) by the vector of nonnumeraire prices transposed $p(s)^T$ and sum the resulting equations. Using the linear homogeneity properties of the expenditure and profit functions in their price variables, we obtain the following equation:

(54) $$e(u(s),p_0+\tau_0,p(s)+\tau) + p_0g_0 + p(s)^Tg$$
$$= \pi(p_0,p(s)) + [p_0a_0 + p(s)^Ta]s + \tau_0\partial e(u(s),p_0+\tau_0,p(s)+\tau)/\partial p_0 + \tau^T\nabla_p e(u(s),p_0+\tau_0,p(s)+\tau).$$

Now use equations (51) and (54) to eliminate $e(u(s),p_0,p(s))$ from the two equations. The resulting equation is the government budget constraint, equation (55) below:

(55) $$T(s) + p_0g_0 + p(s)^Tg =$$
$$\tau_\pi\pi(p_0,p(s)) + \tau_0\partial e(u(s),p_0+\tau_0,p(s)+\tau)/\partial p_0 + \tau^T\nabla_p e(u(s),p_0+\tau_0,p(s)+\tau) + [p_0a_0 + p(s)^Ta]s.$$

Equation (55) says that *government expenditures* (transfer expenditures, T(s), plus expenditures on goods and services, $p_0g_0 + p(s)^Tg$, needed to finance public services) equals *government revenues* from taxing profits (or primary inputs), $\tau_\pi\pi(p_0,p(s))$, plus commodity tax revenues, $\tau_0\partial e(u(s),p_0+\tau_0,p(s)+\tau)/\partial p_0 + \tau^T\nabla_p e(u(s),p_0+\tau_0,p(s)+\tau)$), plus net revenues from the project run at scale s, $[p_0a_0 + p(s)^Ta]s$.

Now use equation (55) to obtain an expression for the transfer T(s) and substitute this expression into equation (51). The resulting equation is:

[13] If T(0) is negative, then the government is imposing a lump sum tax on the household sector at the initial equilibrium.

(56) $e(u(s),p_0+\tau_0,p(s)+\tau)$
$= \pi(p_0,p(s)) - [p_0g_0 + p(s)^Tg] + [p_0a_0 + p(s)^Ta]s$
$+ \tau_0\partial e(u(s),p_0+\tau_0,p(s)+\tau)/\partial p_0 + \tau^T\nabla_p e(u(s),p_0+\tau_0,p(s)+\tau)$
$= \pi(p_0,p(s)) - [p_0g_0 + p(s)^Tg] + [p_0a_0 + p(s)^Ta]s$
$+ \tau_0[\partial\pi(p_0,p(s))/\partial p_0 + a_0s - g_0] + \tau^T[\nabla_p\pi(p_0,p(s)) + as - g]$ using (52) and (53).

Equations (56) and (53) are N+1 equations in the N+1 unknowns u(s) and p(s). We will differentiate these equations with respect to s in order to obtain expressions for the derivatives of utility u and nonnumeraire prices p with respect to the project scale s, $u'(s)$ and $p'(s)$.

Differentiating (56) with respect to s leads to the following equation:

(57) $\{\partial e(u(s),p_0+\tau_0,p(s)+\tau)/\partial u\}u'(s) + [-\tau_0\nabla^2_{p_0p}\pi(p_0,p(s)) - \tau^T\nabla_{pp}{}^2\pi(p_0,p(s))]p'(s)$
$= -[\nabla_p e(u(s),p_0+\tau_0,p(s)+\tau) + g - \nabla_p\pi(p_0,p(s)) - as]^Tp'(s) + (p_0+\tau_0)a_0 + [p(s)+\tau]^Ta$
$= (p_0+\tau_0)a_0 + [p(s)+\tau]^Ta$ using (53).

Differentiating (53) with respect to s leads to the following equation:

(58) $\nabla_{pu}{}^2e(u(s),p_0+\tau_0,p(s)+\tau)u'(s) + [\nabla_{pp}{}^2e(u(s),p_0+\tau_0,p(s)+\tau) - \nabla_{pp}{}^2\pi(p_0,p(s))]p'(s) = a.$

In order to further simplify equations (57) when s = 0, we impose *money metric utility scaling* on the household preferences using the initial consumer equilibrium prices, $p_0+\tau_0,p(0)+\tau$, as the reference prices. This assumption implies:

(59) $\partial e(u(0),p_0+\tau_0,p(0)+\tau)/\partial u = 1.$

Now substitute (59) into (57) and evaluate the resulting equation along with (58) at s = 0. We obtain the following matrix equation:

(60) $$\begin{bmatrix} 1, & d^T \\ b, & S \end{bmatrix} \begin{bmatrix} u'(0) \\ p'(0) \end{bmatrix} = \begin{bmatrix} [p_0+\tau_0]a_0 + [p(0)+\tau]^T a \\ a \end{bmatrix}$$

where d, b and S are defined as follows:[14]

(61) $d^T \equiv -\tau_0\nabla^2_{p_0p}\pi(p_0,p(0)) - \tau^T\nabla_{pp}{}^2\pi(p_0,p(0))$;
(62) $b \equiv \nabla_{pu}{}^2\, e(u(0),p_0+\tau_0,p(0)+\tau)$;
(63) $S \equiv \nabla_{pp}{}^2e(u(0),p_0+\tau_0,p(0)+\tau) - \nabla_{pp}{}^2\pi(p_0,p(0),0).$

---

[14] Using the homogeneity properties of the profit function in prices, it can be shown that if the commodity taxes are all proportional to the preproject prices, so that $[\tau_0,\tau^T] = \lambda[p_0,p(0)^T]$, then $d = 0_N$. Since labour should normally be included as a variable commodity, this condition is unlikely to be satisfied in those economies.

As usual, S is the economy's aggregate substitution matrix at the initial equilibrium, b is the vector of income effects on the demand for nonnumeraire commodities at the initial equilibrium and $d^T p'(0)$ *represents the government's increased commodity tax revenues from taxing purchasers due to the effects of changes in the prices of nonnumeraire commodities*.

As usual, we need some conditions that will guarantee the existence of the inverse of the square matrix in (60). It can be seen that the inverse exists if and only if the following condition holds:

(64) $|S - bd^T| \neq 0$.

Using assumption (64), we can use elementary row operations and find the following formula for the inverse:

$$(65) \quad \begin{bmatrix} 1 & d^T \\ b & S \end{bmatrix}^{-1} = \begin{bmatrix} 1 + d^T\left[S - bd^T\right]^{-1} b; & -d^T\left[S - bd^T\right]^{-1} \\ -\left[S - bd^T\right]^{-1} b & \left[S - bd^T\right]^{-1} \end{bmatrix}.$$

Using (65), it can be seen that (60) is equivalent to the following matrix equation:

$$(66) \quad \begin{bmatrix} u'(0) \\ p'(0) \end{bmatrix} = \begin{bmatrix} 1 + d^T\left[S - bd^T\right]^{-1} b; & -d^T\left[S - bd^T\right]^{-1} \\ -\left[S - bd^T\right]^{-1} b & \left[S - bd^T\right]^{-1} \end{bmatrix} \begin{bmatrix} [p_0 + \tau_0]a_0 + [p(0) + \tau]^T a \\ a \end{bmatrix}.$$

Define the value of the project, v, as the benefits less the costs or the profits that the project is expected to generate at the preproject prices:

(67) $v \equiv (p_0 + \tau_0)a_0 + [p(0) + \tau]^T a$.

Substituting (67) into (66) leads to the following expressions for $u'(0)$ and $p'(0)$:

(68) $u'(0) = v + d^T[S - bd^T]^{-1}\{bv - a\}$;
(69) $p'(0) = -[S - bd^T]^{-1}\{bv - a\}$.

Substituting (69) into (68) leads to the following expression for $u'(0)$:[15]

---

[15] At first glance, (70) seems to indicate that we do not need local information about consumer preferences in order to evaluate $u'(0)$ since it appears that we need only know the preproject prices that consumers face so that we can calculate v and information about commodity taxes and producer price elasticities of net supply around the preproject equilibrium. But we also need to know what the induced first order change in producer prices is, $p'(0)$, and from (69), it can be seen that $p'(0)$ cannot in general be evaluated without knowing b and S, which means we require local information about consumer preferences as well.

$$\begin{aligned}(70)\ u'(0) &= v - d^Tp'(0)\\ &= v + \tau_0 \nabla^2_{p_0 p}\pi(p_0,p(0))p'(0) + \tau^T\nabla_{pp}{}^2\pi(p_0,p(0))p'(0) \quad \text{using definition (61) for d}\\ &= v + \tau_0\nabla_p y_0(p_0,p(0))p'(0) + \tau^T\nabla_p y(p_0,p(0))p'(0)\\ &= v + R'(0)\end{aligned}$$

where commodity tax revenue generated by production, R(s), is defined as follows:

$$(71)\ R(s) \equiv \tau_0 y_0(p_0,p(s)) + \tau^T y(p_0,p(s)).$$

Thus (70) says that the rate of change of welfare due to the introduction of the project is equal to the profitability of the project at preproject prices, v, plus the first order change in commodity tax revenue, $R'(0)$, that is induced by the first order changes in producer prices, $p'(0)$, which in turn is induced by the general equilibrium effects of the introduction of the project.

Typically, $p'(0)$ will be small; i.e., the project will not generate very big changes in nonnumeraire prices. These small changes in prices will induce small changes in the net supplies of the N+1 variable commodities and then these small changes will induce small changes in commodity tax revenues. Thus, typically, $R'(0)$ will be small.

Unfortunately, v, the profitability of the project can also be very small for a typical government project. If this is the case, then the effects of the project on commodity tax revenues becomes important. If labour is a variable commodity and it is heavily taxed (as is the case in most OECD countries), or if there are substantial value added taxes or general sales taxes on production, then the tax effects could well be decisive. This is somewhat unfortunate, since we will require accurate estimates of elasticities of supply and demand in order to calculate approximations to these tax effects and usually, accurate estimates of elasticities are not available.

It is of interest to look a bit more closely at equation (69), which defines $p'(0)$. The term bv can be interpreted as a first order approximation to the income effects on consumer demand for the nonnumeraire commodities that the project generates; i.e., the project creates (at constant prices) an extra amount of income equal to v and this induces an income generated increase in demand equal to bv. On the other side of the markets for nonnumeraire commodities, the project creates a net increase in supply equal to a. Thus $bv - a$ is an approximate *demand less supply imbalances vector*. To close these demand less supply gaps, prices will have to change a sufficient amount in order to induce consumers and producers to adjust their demands and supplies to overcome these gaps. The required amount of price change turns out to equal $p'(0) = -[S-bd^T]^{-1}\{bv - a\}$. If N = 1, the required amount of price change is

$$(72)\ p_1'(0) = -[S_{11} - b_1d_1]^{-1}\{b_1v - a_1\}.$$

S_{11}, which equals $\partial^2 e(u(0),p_0+\tau_0,p(0)+\tau)/\partial p_1^2 - \partial\pi(p_0,p(0),0)/\partial p_1^2$, must be negative or zero. Usually, it will be negative and if b_1d_1 is positive or not too negative, then $S_{11} - b_1d_1$ will be negative as well. In this case, it can be seen that if the demand effect is greater than the supply effect so that $b_1v - a_1 > 0$, then $p_1'(0) > 0$ as well; i.e., the price of commodity 1 increases to induce consumers to consume less and to induce producers to produce more, thus closing the gap. On the other hand, if the demand effect is less than the supply effect so that $b_1v - a_1 < 0$, then $p_1'(0) < 0$ as well.

There is yet another way of looking at formula (68) for the utility derivative, $u'(0)$, but we first need a theorem from matrix algebra. We can apply a theorem due to Bartlett (1951; 107) but attributed to Sherman and Morrison (1949) (1950) and obtain a formula (74) for the inverse of $(S - bd^T)$ provided that the following two conditions are satisfied:

$$(73)\ |S| \neq 0\ ;\ 1 - d^TS^{-1}b \neq 0.$$

The Bartlett formula is:

$$(74)\ [S- bd^T]^{-1} = S^{-1} + (1 - d^TS^{-1}b)^{-1}S^{-1}bd^TS^{-1}.$$

Now substitute (74) into (68):

$$\begin{aligned}(75)\ u'(0) &= v + d^T[S-bd^T]^{-1}\{bv - a\}\\ &= v + d^T[S^{-1} + (1 - d^TS^{-1}b)^{-1}S^{-1}bd^TS^{-1}]\{bv - a\}\\ &= v + (1 - d^TS^{-1}b)^{-1}[d^TS^{-1} - d^TS^{-1}bd^TS^{-1} + d^TS^{-1}bd^TS^{-1}]\{bv - a\}\\ &= v + (1 - d^TS^{-1}b)^{-1}[d^TS^{-1}]\{bv - a\}\\ &= (1 - d^TS^{-1}b)^{-1}\{v - d^TS^{-1}bv + d^TS^{-1}bv - d^TS^{-1}a\}\\ &= (1 - d^TS^{-1}b)^{-1}\{v - d^TS^{-1}a\}\\ &= (1 - d^TS^{-1}b)^{-1}\{(p_0 + \tau_0)a_0 + [p(0) + \tau]^Ta - d^TS^{-1}a\} \qquad \text{using definition (67)}\\ &= (1 - d^TS^{-1}b)^{-1}\{(p_0 + \tau_0)a_0 + [p(0) + \tau - S^{-1}d]^Ta\}.\end{aligned}$$

Equation (75) shows that to the accuracy of a first order approximation,[16] project inputs and outputs should be valued at the price $p_0 + \tau_0$ for the numeraire good and the price vector $p(0) + \tau - S^{-1}d$ for the nonnumeraire commodities. Thus these prices are appropriate *project evaluation prices* or *shadow prices* for pricing out the costs and revenues of small projects, taking into account the commodity tax distortions in the economy:

$$(76)\ p_0^* \equiv p_0 + \tau_0\ ;\ p^* \equiv p(0) + \tau - S^{-1}d$$

where d and S are defined by (61) and (63) respectively. Thus if a small project is profitable when evaluated at these shadow prices, it should be undertaken, since it will lead to a welfare increase.

[16] There is also a factor of proportionality $(1 - d^TS^{-1}b)^{-1}$ in the last equation of (75) that we will discuss shortly.

The factor of proportionality in (75), $(1 - d^T S^{-1} b)^{-1}$, will usually be positive in most real life economies.[17] Why is this? If the government introduces a project that produces a positive amount a_0 of the numeraire commodity and nothing else and uses no resources,[18] then $a = 0_N$ and formula (75) collapses to:

(77) $u'(0) = (1 - d^T S^{-1} b)^{-1}(p_0 + \tau_0)a_0$.

Since a_0 is positive as is $p_0 + \tau_0$, we see that $u'(0)$ is positive if and only if $1 - d^T S^{-1} b$ is greater than zero. If $1 - d^T S^{-1} b$ is negative, this is a sign that the structure of commodity taxes in the economy is far from optimal and we assume that the government ensures that this far from optimal situation will not occur.[19]

If the numeraire commodity is not taxed so that $\tau_0 = 0$,[20] then (75) simplifies as follows:

$$
\begin{aligned}
(78)\ u'(0) &= (1 - d^T S^{-1} b)^{-1}\{p_0 a_0 + [p(0) + \tau - d^T S^{-1}]^T a\} \\
&= (1 - d^T S^{-1} b)^{-1}\{p_0 a_0 + [p(0)^T + \tau^T + \tau^T \nabla_{pp}{}^2 \pi(p_0,p(0))\, S^{-1}]a\} \quad \text{using (61) with } \tau_0 = 0 \\
&= (1 - d^T S^{-1} b)^{-1}\{p_0 a_0 + [p(0)^T + \tau^T + \tau^T S_P S^{-1}]a\} \\
&\qquad \text{defining the producer substitution matrix } S_P \equiv \nabla_{pp}{}^2 \pi(p_0,p(0)) \\
&= (1 - d^T S^{-1} b)^{-1}\{p_0 a_0 + [p(0)^T + \tau^T + \tau^T S_P[S_C - S_P]^{-1}]a\} \quad \text{using definition (63) and} \\
&\qquad \text{defining the consumer substitution matrix as } S_C \equiv \nabla_{pp}{}^2 e(u(0),p_0+\tau_0,p(0)+\tau) \\
&= (1 - d^T S^{-1} b)^{-1}\{p_0 a_0 + p(0)^T + \tau^T[S_C - S_P][S_C - S_P]^{-1} + \tau^T S_P[S_C - S_P]^{-1}]a\} \\
&= (1 - d^T S^{-1} b)^{-1}\{p_0 a_0 + p(0)^T + \tau^T S_C[S_C - S_P]^{-1}a\}.
\end{aligned}
$$

Thus in the case where the numeraire good is not taxed, the appropriate *project evaluation prices* or *shadow prices* for pricing out the outputs and inputs of small projects, taking into account the commodity tax distortions in the economy are defined as follows:

(79) $p_0^{**} \equiv p_0$; $p^{**} \equiv p(0) + [S_C - S_P]^{-1} S_C\, \tau$.

If we further assume that all of the nonnumeraire commodities are *only* substitutable with the numeraire commodity, both in production and consumption, then the N by N substitution matrices in (79) become diagonal, with negative elements on the main diagonals and (79) simplifies to *Harberger's* (1969; 82) *weighted average shadow pricing rule*:[21]

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[17] If it is negative, then we want the project to be unprofitable when evaluated at the shadow prices defined by (76).
[18] See Bruce and Harris (1982; 768-769) for a similar assumption.
[19] Recall that $d = 0_N$ if the commodity taxes are proportional. In order to obtain a negative $1 - d^T S^{-1} b$, we need large and nonproportional commodity taxes and very special income effects b.
[20] This assumption is not as restrictive as it might seem since if any commodity is untaxed, we can always choose it to be the numeraire commodity.
[21] Result was rigorously derived by Bruce and Harris (1982; 770). Note that the coefficients $\sigma_{nn}{}^C/[\sigma_{nn}{}^C - \sigma_{nn}{}^P]$ are nonnegative numbers between 0 and 1.

(80) $p_n^{***} \equiv p_n(0) + \{\sigma_{nn}^{C}/[\sigma_{nn}^{C} - \sigma_{nn}^{P}]\}\tau_n$; $n = 1,..,N$

where σ_{nn}^{C} is the (negative) nth element on the main diagonal of the consumer substitution matrix S_C and σ_{nn}^{P} is the (positive) nth element on the main diagonal of the producer substitution matrix S_P. Thus each of the nonnumeraire shadow prices defined by (80) is equal to a weighted average of the preproject producer price $p_n(0)$ and the preproject consumer price $p_n(0) + \tau_n$. This is a very pretty result but the assumptions are very far from being empirically valid. Looking at the more general formula (79), it can be seen that Harberger's rule is not likely to be satisfied in more general economies.[22]

As a final exercise in this section, we rewrite the basic comparative statics matrix equation (60) in elasticity and share format. First we note that the first equation in (56), evaluated at s = 0 can be rewritten as follows:

(81) $u(0) = Y(0) - G(0) + R(0)$

where

(82) $u(0) \equiv e(u(0),p_0+\tau_0,p(0)+\tau)$;
(83) $Y(0) \equiv \pi(p_0,p(0))$;
(84) $G(0) \equiv p_0g_0 + p(0)^Tg$;
(85) $R(0) \equiv \tau_0\partial e(u(0),p_0+\tau_0,p(0)+\tau)/\partial p_0 + \tau^T\nabla_p e(u(0),p_0+\tau_0,p(0)+\tau)$.

Note that u(0) is equal to preproject consumption expenditures evaluated at consumer prices, Y(0) is equal to the value of preproject private sector production evaluated at producer prices, G(0) is equal to the preproject value of government expenditures on goods and services valued at producer prices and R(0) is equal to preproject commodity tax revenues.

We need to define several sets of "shares" or expenditures relative to the value of consumption expenditures. Thus we first define the ratio of the net revenue generated by the project by commodity n relative to the value of initial consumption expenditures, s_n^a, as follows:

(86) $s_0^a \equiv (p_0 + \tau_0)a_0/u(0)$; $s_n^a \equiv (p_n(0) + \tau_n)a_n/u(0)$ for $n = 1,\ldots,N$.

Denote the vector of nonnumeraire project expenditure ratios by $s^a \equiv [s_1^a,\ldots,s_N^a]^T$. Next we define the ratio of preproject commodity tax revenue for commodity n relative to the value of initial consumption expenditures, s_n^t, as follows:

(87) $s_0^t \equiv \tau_0x_0(0)/u(0)$; $s_n^t \equiv \tau_nx_n(0)/u(0)$ for $n = 1,\ldots,N$.

[22] Note that when N = 1, (79) and (80) are the same; i.e., when N = 1, Harberger's diagonality assumptions are no longer restrictive.

Denote the vector of nonnumeraire commodity tax revenue ratios by $s^t \equiv [s_1^t,\ldots,s_N^t]^T$. Next we define the ratio of preproject consumer expenditures on commodity n relative to the value of initial consumption expenditures, s_n^C, as follows:

$$(88)\ s_0^C \equiv (p_0 + \tau_0)x_0(0)/u(0)\ ;\ s_n^c \equiv (p_n(0) + \tau_n)x_n(0)/u(0) \text{ for } n = 1,\ldots,N.$$

Denote the vector of nonnumeraire consumer commodity expenditure shares by $s^C \equiv [s_1^C,\ldots,s_N^C]^T$. Finally, we define the ratio of preproject producer net revenues (valued at consumer prices) on commodity n relative to the value of initial consumption expenditures, s_n^P, as follows:

$$(89)\ s_0^P \equiv (p_0 + \tau_0)y_0(0)/u(0)\ ;\ s_n^P \equiv (p_n(0) + \tau_n)y_n(0)/u(0) \text{ for } n = 1,\ldots,N.$$

Denote the vector of nonnumeraire producer commodity expenditure ratios by $s^P \equiv [s_1^P,\ldots,s_N^P]^T$.

We also need to define various producer and consumer elasticities, evaluated at the preproject equilibrium. Thus define the consumer's income elasticity of demand for commodity n, η_n, for nonnumeraire commodities as follows:

$$\begin{aligned}(90)\ \eta_n &\equiv [u(0)/x_n(u(0),p_0+\tau_0,p(0)+\tau)]\partial x_n(u(0),p_0+\tau_0,p(0)+\tau)/\partial u \\ &= [u(0)/x_n(0)]\partial^2 e(u(0),p_0+\tau_0,p(0)+\tau)/\partial p_n\partial u \\ &= [u(0)/x_n(0)]b_n\ ; \qquad n = 1,\ldots,N.\end{aligned}$$

Denote the column vector of nonnumeraire income elasticities of demand by $\eta \equiv [\eta_1,\ldots,\eta_N]^T$.

Define the N by N matrix of producer elasticities of net supply with respect to nonnumeraire prices as follows:

$$(91)\ \Sigma_{pp}^P \equiv \hat{y}(0)^{-1}\nabla_{pp}^2\pi(p_0,p(0))\,\hat{p}(0) = \hat{y}(0)^{-1}\nabla_p y(p_0,p(0))\,\hat{p}(0)$$

where $\hat{y}(0)$ and $\hat{p}(0)$ are N by N diagonal matrices with the elements of the vectors y(0) and p(0) running down the main diagonals. We also need to define a row vector of elasticities of net supply for the numeraire commodity with respect to changes in nonnumeraire prices:

$$(92)\ \Sigma_{p_0p}^P \equiv y_0(0)^{-1}\nabla_{p_0p}^2\pi(p_0,p(0))\,\hat{p}(0) = y_0(0)^{-1}\nabla_p y_0(p_0,p(0))\,\hat{p}(0).$$

Finally, define the N by N matrix of consumer elasticities of net demand with respect to nonnumeraire prices as follows:

$$(93)\ \Sigma_{pp}^C \equiv \hat{x}(0)^{-1}\nabla_{pp}^2 e(p_0+\tau_0,p(0)+\tau)\,\hat{p}(0) = \hat{x}(0)^{-1}\nabla_p x(p_0+\tau_0,p(0)+\tau)\,\hat{p}(0)$$

where $\hat{x}(0)$ and $\hat{p}(0)$ are N by N diagonal matrices with the elements of the vectors x(0) and p(0) running down the main diagonals.

Using definitions (86) to (93), it can be shown that the fundamental matrix equation (60) can be rewritten in share and elasticity format as follows:

$$(94)\quad \begin{bmatrix} 1, & -[s_0^t\Sigma_{p_0p}^P + \hat{s}^{tT}\Sigma_{pp}^P] \\ \hat{s}^C\eta, & \hat{s}^C\Sigma_{pp}^C(\hat{p}(0)+\hat{\tau})^{-1}\hat{p}(0) - \hat{s}^P\Sigma_{pp}^P \end{bmatrix} \begin{bmatrix} d\ln u(0)/ds \\ d\ln p(0)/ds \end{bmatrix} = \begin{bmatrix} s_0^a + \sum_{n=1}^N s_n^a \\ s^a \end{bmatrix}.$$

The logarithmic derivative dlnu(0)/ds can be interpreted as (a first order approximation to) the percentage change in utility or real consumption that is generated by the project while the components of the logarithmic price derivative vector dlnp(0)/ds can be interpreted as (a first order approximation to) the percentage change in the producer prices that are generated by the project. Obviously, formula (80) may be used to invert the square matrix in (94) with a suitable redefinition of the elements of b, d and S. We do not obtain any new theoretical results using (94). However, (94) might be useful in empirical applications when we have information about consumer and producer elasticities (as opposed to information about the second derivatives of the expenditure and profit functions, which will not usually be available).

There is a great deal of controversy about methods that attempt to determine the benefits of a project. The straightforward general equilibrium approach that we have taken in this chapter has its origins in the work of Mead (1955) and has been further developed by Harberger (1971) (1978), Boadway (1975) (1976) (1978) , Starrett (1979), Hammond (1980), Bruce and Harris (1982), Diewert (1983) and many others.

Diamond and Mirrlees (1971) (1976) and Little and Mirrlees (1974) argue that one can simply use producer prices as shadow prices. On the other hand, Hammond (1980) has argued that one requires only local (elasticity) information about consumer preferences to determine the correct shadow prices. The model presented in this section indicates that *local information about elasticities is required on both preferences and technology when there are nonproportional commodity taxes*.

We note that we have followed the example of Diamond and McFadden (1974) in using expenditure functions rather than direct or indirect utility functions in order to represent consumer preferences.

Problems

5. Let A be an N by N matrix that has an inverse and let b and c be N dimensional vectors. Suppose that:

(i) $1 + c^TA^{-1}b \neq 0$.

Show that the following formula due to Bartlett (1951; 107) is valid:[23]

(ii) $[A + bc^T]^{-1} = A^{-1} - (1 + c^TA^{-1}b)^{-1}A^{-1}bc^TA^{-1}$.

6. Let A be an N by N matrix, let B and C be N by K dimensional matrices where $K \le N$, and let D be a K by K matrix. Suppose that:

(i) A^{-1} exists;
(ii) D^{-1} exists; and
(iii) $[D + C^TA^{-1}B]^{-1}$ exists.

Show that the following formula due to Duncan (1944) and Guttman (1946;342) is valid:[24]

(iv) $[A + BD^{-1}C^T]^{-1} = A^{-1} - A^{-1}B[D + C^TA^{-1}B]^{-1}C^TA^{-1}$.

7. Let A be an N by N matrix, let B and C be N by K dimensional matrices where $K \le N$, and let D be a K by K matrix. Suppose that:

(i) A^{-1} exists;
(ii) D^{-1} exists; and
(iii) $[D - C^TA^{-1}B]^{-1}$ exists.

Show that the following formula is valid:

(iv) $[A - BD^{-1}C^T]^{-1} = A^{-1} + A^{-1}B[D - C^TA^{-1}B]^{-1}C^TA^{-1}$.

Hint: Apply the previous problem, replacing B by −B. Thus (iv) is a completely equivalent version of the Duncan Guttman formula for updating an inverse matrix.

8. Define the N+K by N+K partitioned matrix E as follows:

(i) $E \equiv \begin{bmatrix} A & B \\ C^T & D \end{bmatrix}$.

[23] Sherman and Morrison obtained special cases of this result. In Sherman and Morrison (1949; 621), they obtained the special case where either b or c were unit vectors (some typographical errors were corrected in Hager (1989; 223)). Sherman and Morrison (1950; 124) obtained yet a further special case where b and c were both unit vectors but one of them was multiplied by a number; this corresponds to the case where only one element of the original matrix changes.

[24] This formula is usually attributed to Woodbury (1950) in the mathematics literature; see Hager (1989; 222-223) for the history of this result. Plackett (1950) also independently obtained this result. Note that the Duncan, Guttman, Plackett and Woodbury result contains the Bartlett, Sherman and Morrison result as a special case: set K = 1, let D = 1, let B = b and let C = c.

Make assumptions (i)-(iii) in problem 7. (a) Show that a formula for E^{-1} is the following one:

(ii) $E^{-1} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$

where the E_{ij} are defined as follows:

(iii) $E_{11} \equiv A^{-1} + A^{-1}B[D - C^TA^{-1}B]^{-1}C^TA^{-1}$;
$E_{12} \equiv - A^{-1}B[D - C^TA^{-1}B]^{-1}$;
$E_{21} \equiv - [D - C^TA^{-1}B]^{-1}C^TA^{-1}$;
$E_{22} \equiv [D - C^TA^{-1}B]^{-1}$.

Note that this formula results if we use block matrix row operations, starting off by premultiplying the first set of rows in E by A^{-1}. (b) Show that another formula for E^{-1} is the following one:

(iv) $E_{11} \equiv [A - BD^{-1}C^T]^{-1}$
$E_{12} \equiv -[A - BD^{-1}C^T]^{-1}BD^{-1}$;
$E_{21} \equiv - D^{-1}C^T[A - BD^{-1}C^T]^{-1}$;
$E_{22} \equiv D^{-1} + D^{-1}C^T[A - BD^{-1}C^T]^{-1}BD^{-1}$.

Note that this formula results if we use block matrix operations, starting off by premultiplying the second set of rows in E by D^{-1}. (c) There is one detail that should be checked: show that assumptions (i)-(iii) in problem 7 are sufficient to imply the existence of the following inverse matrix: $[A - BD^{-1}C^T]^{-1}$. Finally, note that if we equate the formulae for E_{11} in (iii) and (iv) above, we obtain the Duncan Guttman result that was in problem 7 above.[25]

6. Evaluating the Benefits of Public Goods: A Primal Approach to Econometric Estimation

In many cases, the government produces *public goods*. These are commodities that are generally consumed free of charge by the national population. Examples of such goods are national defence, product safety, provision of parks, protection (police) services, judicial services, roads, educational and health services and many other goods and services. The problem with these goods is that there are usually no prices to guide investment decisions.

In the final sections of this chapter, we revisit the simple model explained in section 2 but we now assume that the project produces an incremental addition, $b \equiv [b_1,b_2,\ldots,b_K]^T$, to the initial consumption of public goods, $c \equiv [c_1,c_2,\ldots,c_K]^T$. These public goods (or environmental variables) could benefit consumers directly (so that the vector c appears in

[25] In fact, this is how Guttman (1946; 342) obtained his identity.

household utility functions) or indirectly by facilitating production (so that the vector c appears in the economy's production functions). We will deal with the case where c appears in a utility function in this section and then deal with the case where c appears in the economy's technology set in section 9.[26]

We assume that the consumer has preferences defined over alternative bundles of market goods and services and public goods. These preferences over the N variable market commodities, $[x_1,\ldots,x_N]^T \equiv x$ and the K public goods, $c \equiv [c_1,c_2,\ldots,c_K]^T$ are represented by the continuous, nondecreasing[27] and quasiconcave in x utility function, f(x,c). We also assume that f is twice continuously differentiable over its domain of definition. We assume that the domain of definition for f is the set $S \equiv \{(x,c): x \geq 0_N ; c \in R^K\}$.[28] Note that we do not assume any regularity properties for f(x,c) with respect to its c variables (except for differentiability).

When we attempt to use econometric techniques in order to estimate preferences where the consumer has joint preferences over both market goods and services x (which can be purchased at the prices p) and nonmarket goods and services (or the environmental or public good variables) c (which are provided free of charge), we find that *we can no longer identify completely these preferences* over combinations of x and c using observable data.[29] The problem can be explained as follows. Suppose the consumer faces the vector of market prices $p >> 0_N$ and has "income" $Y > 0$ to spend on market commodities. The consumer's utility maximization problem will be the following one:

(95) $\max_x \{f(x,c) : p^Tx = Y ; x \geq 0_N\}$

where the consumer takes the vector of nonmarket supplies c as a given vector of constants in solving (95). Now let g(c) be an arbitrary function of c and suppose we add this arbitrary function to f(x,c) and ask the same consumer to solve the resulting utility maximization problem:

(96) $\max_x \{f(x,c) + g(c) : p^Tx = Y ; x \geq 0_N\} = g(c) + \max_x \{f(x,c) : p^Tx = Y ; x \geq 0_N \}$.

By comparing (95) and (96), it can be seen if x^* solves the original utility maximization problem (95), then it will also solve the new utility maximization problem (96) and vice versa! Thus we have an *impossibility theorem*: *it is not possible to completely recover*

[26] It may be the case that different components of c appear in the utility function and in the production function. However, this case can be regarded as a special case where the "wrong" components of c that appear in say the utility function simply do not affect the utility function; i.e., u = f(x,c) is simply constant with respect to variations in some of the c components.

[27] We will make the somewhat stronger assumption that for every $(x,c)\in S$, we have $\nabla_x f(x,c) > 0_N$.

[2828] In real life applications, the domain of definition of f will be a subset of S. In particular, the domain of definition for c is unlikely to be all of R^K. The restriction that the market consumption vector x be nonnegative means that if there is disutility from supplying labour, we need to convert this disutility into nonnegative leisure to apply our present model. We leave the details to the reader.

[29] In Chapter 4, we also encountered a mild version of this problem in that we had to make some cardinalizing normalizations on our representations of preferences in order to identify the remaining parameters of the utility function or the dual cost or expenditure function.

the consumer's preferences over market and nonmarket goods and services by using observed market demand functions. This is a rather disappointing result!

But if we cannot completely recover the consumer's preferences if they are completely unrestricted, then perhaps if we make some "reasonable" restrictions on their preferences, it will be possible to recover these restricted preferences. This conjecture turns out to be true as we shall show below.

The a priori restriction that we will impose upon preferences is that they be *homothetic in market goods and services, conditional on any vector of nonmarket goods and services* This assumption is equivalent to the following *linear homogeneity assumption* on f:

(97) $f(\lambda x,c) = \lambda f(x,c)$ for all $\lambda > 0$; $x \geq 0_N$; $c \in R^K$.

As usual, Euler's Theorems on homogeneous functions mean that the linear homogeneity assumption (97) implies the following restrictions on the first and second order partial derivatives of f at any point in the domain of definition of f:

(98) $\nabla_x f(x,c)^T x = f(x,c)$;
(99) $\nabla_{xx}^2 f(x,c)x = 0_N$;
(100) $\nabla_{cx}^2 f(x,c)x = \nabla_c f(x,c)$.

In what follows, we will attempt to find a flexible functional form for f(x,c) in the class of homothetic in market commodities utility functions, where f satisfies (97). Once we have found such a function, we will use Wold's (1944; 69-71) Identity to obtain the consumer's system of inverse demand functions and see if the parameters in our suggested flexible functional form can be identified.[30]

Neglecting the restrictions (98)-(100), a fully flexible functional form, f(x,c), for preferences over market and nonmarket commodities would require 1 + (N+K) + (N+K+1)(N+K)/2 parameters. There is 1 restriction on the derivatives of f in (98), N restrictions in (99) and K restrictions in (100). Thus the total number of independent parameters we will need in a flexible functional form for a homothetic in market goods and services utility function is:

(101) $1 + (N+K) + (N+K+1)(N+K)/2 - [1+N+K] = N(N+1)/2 + NK + K(K+1)/2$.

We will attempt to determine whether the following functional form is flexible in our restricted class of functions:

(102) $f(x,c) \equiv a^T x + (1/2)(\alpha^T x)^{-1} x^T A x + x^T B c + (1/2)(\alpha^T x) c^T C c$

where α is a positive vector of predetermined constants, a is an unknown N dimensional parameter vector, A is an N by N matrix of unknown parameters, B is an N by K matrix

[30] Recall problem 19 in Chapter 4.

of unknown parameters and C is a K by K matrix of unknown parameters. The vector α and the matrices A and C are assumed to satisfy the following restrictions:

(103) $\alpha^T x^* = 1$;
(104) $A = A^T$;
(105) $C = C^T$;
(106) $Ax^* = 0_N$

where $x^* >> 0_N$ is some point that we have chosen to form the basis for our second order approximation to an arbitrary utility function, $f^*(x,c)$ and c^* is the corresponding c point of approximation. There are N free parameters in the a vector, N(N−1)/2 free parameters in the A matrix taking into account the restrictions in (104) and (106), NK free parameters in the B matrix and K(K+1)/2 free parameters in the C matrix and this sums up to the total number of parameters on the right hand side of (101) and hence the normalized quadratic functional form[31] defined by (102)-(106) has just enough parameters to be a parsimonious (homothetic in x) flexible functional form. We will show below that this functional form is indeed a flexible one.

We first calculate the vectors of first order partial derivatives of the f defined by (102):

(107) $\nabla_x f(x,c) = a + (\alpha^T x)^{-1}Ax - (1/2)(\alpha^T x)^{-2}x^T Ax\alpha + Bc + (1/2)c^T Cc\alpha$;
(108) $\nabla_c f(x,c) = B^T x + (\alpha^T x)Cc$.

The matrices of second order partial derivatives may be computed as follows:

(109) $\nabla_{xx}^2 f(x,c) = (\alpha^T x)^{-1}A - (\alpha^T x)^{-2}\alpha x^T A - (\alpha^T x)^{-2}Ax\alpha^T + (\alpha^T x)^{-3}x^T Ax\alpha\alpha^T$;
(110) $\nabla_{xc}^2 f(x,c) = B + \alpha c^T C$;
(111) $\nabla_{cc}^2 f(x,c) = (\alpha^T x)C$.

Let $f^*(x,c)$ be an arbitrary twice continuously differentiable utility function that is homothetic in market goods and services. Pick an arbitrary point x^*,c^* as our point of approximation. Since $f^*(x,c)$ is linearly homogeneous in x, the derivatives of f^* evaluated at (x^*,c^*) will satisfy the following counterparts to the restrictions (98)-(100):

(112) $\nabla_x f^*(x^*,c^*)^T x^* = f^*(x^*,c^*)$;
(113) $\nabla_{xx}^2 f^*(x^*,c^*)x^* = 0_N$;
(114) $\nabla_{cx}^2 f^*(x^*,c^*)x^* = \nabla_c f^*(x^*,c^*)$.

We now want to pick a vector a and matrices A, B and C, satisfying the restrictions (104)-(106), such that $f(x^*,c^*)$ equals $f^*(x^*,c^*)$ and all of the first and second order partial derivatives of f evaluated at (x^*,c^*) equal the corresponding first and second order partial derivatives of f^* evaluated at (x^*,c^*).

[31] Diewert and Wales (1987) (1988a) (1988b) introduced the normalized quadratic functional form into the literature on flexible functional forms. Using a result in Diewert and Wales (1987; 66), it can be shown that f(x,c) defined by (102) is globally concave provided that the A matrix is negative semidefinite.

Equating $\nabla_{cc}^2 f(x^*,c^*)$ to $\nabla_{cc}^2 f^*(x^*,c^*)$ and using (111) and (103) leads to the following equation:

$$(115)\ C = \nabla_{cc}^2 f^*(x^*,c^*).$$

Obviously, (115) determines the matrix C. Note that C will be a symmetric matrix since Young's Theorem implies that $\nabla_{cc}^2 f^*(x^*,c^*)$ is a symmetric K by K matrix.

Equating $\nabla_{xc}^2 f(x^*,c^*)$ to $\nabla_{xc}^2 f^*(x^*,c^*)$ and using (110) leads to the following equation:

$$(116)\ B + \alpha c^T C = \nabla_{xc}^2 f^*(x^*,c^*).$$

Since α is predetermined and C has been determined by (115), we can use (116) to determine the matrix B.

Now equate $\nabla_{xx}^2 f(x^*,c^*)$ to $\nabla_{xx}^2 f^*(x^*,c^*)$. Assuming for the moment that A satisfies the restrictions $Ax^* = 0_N$, then using (109) and (103) leads to the following equation:

$$(117)\ A = \nabla_{xx}^2 f^*(x^*,c^*).$$

Use (117) to define the N by N matrix A. It will be symmetric because Young's Theorem implies that $\nabla_{xx}^2 f^*(x^*,c^*)$ is a symmetric matrix. Recalling (113) above, Euler's Theorem on homogeneous functions implies that $\nabla_{xx}^2 f^*(x^*,c^*)x^*$ equals 0_N and hence the A defined by (117) will satisfy the restrictions $Ax^* = 0_N$.

Finally, equate $\nabla_x f(x^*,c^*)$ to $\nabla_x f^*(x^*,c^*)$. Using (107), (103) and (106), this equation simplifies to the following one:

$$(118)\ a + Bc^* + (1/2)c^{*T}Cc^*\alpha = \nabla_x f^*(x^*,c^*).$$

Since B and C have already been determined, (118) can be solved for the vector a. Now all of the parameters in the f(x,c) defined by (102) have been determined. To complete the proof of the flexibility of this functional form, we need only show that the following equations are also satisfied:

$$(119)\quad f(x^*,c^*) = f^*(x^*,c^*)\ ;$$
$$(120)\ \nabla_x f(x^*,c^*) = \nabla_x f^*(x^*,c^*)\ .$$

But it can be verified that (119) and (120) are satisfied using our already established results on the equality of the first and second order partial derivatives of f and f^* evaluated at (x^*,c^*) plus equations (98), (100), (112) and (114). This completes the proof of the flexibility of the f defined by (102)-(106).

We now ask whether the parameters of the homothetic in market goods normalized quadratic utility function defined by (102) can be estimated using observable data on the choices of a consumer who has these preferences. Suppose that we have data x^t, c^t, p^t on such a consumer for period t = 1,...,T. The consumer's period t "income" or expenditure on market goods and services can be defined as follows:

$$(121)\quad Y^t \equiv p^{tT}x^t\,; \qquad t = 1,\ldots,T.$$

Using the first order necessary conditions for the consumer's utility maximization problem in period t leads to the following relationships between the consumer's period t chosen market consumption vector, x^t and the prices p^t and "income" Y^t that the consumer takes as given:

$$\begin{aligned}(122)\quad p^t/Y^t &= \nabla_x f(x^t,c^t)/x^{tT}\nabla_x f(x^t,c^t) && \text{using Wold's (1944; 69-71) Identity}\\ &= \nabla_x f(x^t,c^t)/f(x^t,c^t) && \text{using the linear homogeneity of f in x; see (98)}\\ &= [a + (\alpha^T x^t)^{-1}Ax^t - (1/2)(\alpha^T x^t)^{-2}x^{tT}Ax^t\alpha + Bc^t + (1/2)c^{tT}Cc^t\alpha]\\ &\quad /[a^T x^t + (1/2)(\alpha^T x^t)^{-1}x^{tT}Ax^t + x^{tT}Bc^t + (1/2)(\alpha^T x^t)c^{tT}Cc^t]\,; && t = 1,\ldots,T.\end{aligned}$$

Note that the market and nonmarket consumption vectors, x^t and c^t respectively, are regarded as predetermined variables in equations (122) while the normalized price vectors for market goods and services, p^t/Y^t, are regarded as the dependent variables in these equations. If the x^t grow substantially over the T periods in the sample of observations, then the dependent variables p^t/Y^t may exhibit substantial downward trends, leading to heteroskedasticity problems. To control this heteroskedasticity problem, premultiply both sides of equation t in (122) by a diagonal matrix, $\hat{x}^t$, which has the elements of the market consumption vector for period t, x^t, running down the main diagonal. On the left hand side of the resulting equation, we will obtain the consumer's period t expenditure share on market goods vector, $s^t \equiv [s_1^t,s_2^t,\ldots,s_N^t]^T$ where $s_n^t \equiv p_n^t x_n^t/p^{tT}x^t = p_n^t x_n^t/Y^t$ for $n = 1,\ldots,N$; $t = 1,\ldots,T$. The resulting system of transformed equations (122) becomes the following system of equations:

$$\begin{aligned}(123)\quad s^t = \hat{x}^t\,&[a + (\alpha^T x^t)^{-1}Ax^t - (1/2)(\alpha^T x^t)^{-2}x^{tT}Ax^t\alpha + Bc^t + (1/2)c^{tT}Cc^t\alpha]\\ &/[a^T x^t + (1/2)(\alpha^T x^t)^{-1}x^{tT}Ax^t + x^{tT}Bc^t + (1/2)(\alpha^T x^t)c^{tT}Cc^t]\,; \qquad t = 1,\ldots,T.\end{aligned}$$

Looking at the right hand sides of (122) or (123), it can be seen that because these right hand sides are homogeneous of degree zero in the parameter vector and matrices a, A, B and C, not all of these parameters can be identified. However, this is normal when estimating consumer preferences; we will require at least one additional normalization in order to cardinalize utility. A natural normalization is to set utility equal to "income" Y^* at the point of approximation x^*,c^*; i.e., we set $f(x^*,c^*)$ equal to Y^* which leads to the following additional restriction on the parameters of f:[32]

$$\begin{aligned}(124)\quad Y^* &= a^T x^* + (1/2)(\alpha^T x^*)^{-1}x^{*T}Ax^* + x^{*T}Bc^* + (1/2)(\alpha^T x^*)c^{*T}Cc^*\\ &= a^T x^* + x^{*T}Bc^* + (1/2)c^{*T}Cc^* && \text{using (103) and (106).}\end{aligned}$$

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[32] Typically, (x^*,c^*) will be chosen to be one of our sample observations, (x^t,c^t), and the corresponding income Y^* will be Y^t.

We can further simplify the normalization (124) if we choose a new origin for measuring nonmarket production c so we choose that origin to be c^*. Now convert to the new system of measuring nonmarket production as deviations around c^*. We abuse our notation and continue to use the old symbol c for the new units. Hence our point of approximation in the new units is

(125) $c^* = 0_K$.

Substituting (125) into (124) leads to the following new normalization on the elements of a:

(126) $a^T x^* = Y^*$.

Now we can use (126) to solve for say a_N in terms of Y^* and the other a_n and x^* as follows:

(127) $a_N = [Y^* - \sum_{n=1}^{N-1} a_n x_n^*]/x_N^*$.

Now substitute (127) into (123), drop the last share equation, and we obtain a nice system of N−1 share equations that can be used to estimate the parameters of the f defined by (102). We note that if the matrices B and C happen to be equal to $0_{N\times K}$ and $0_{K\times K}$ respectively so that all elements of both matrices are zero, then the system of estimating equations reduces to a conventional system of estimating equations for a system of inverse demand equations with no nonmarket goods and services.

Looking at equations (122) or (123), it can be seen that the matrix C is unlikely to play a significant role in these systems of inverse demand functions, since the variation that C generates in the inverse demand is proportional to the vector $c^{tT}Cc^t\alpha$ in period t for (122), which is more or less the same across all market goods. Thus it may be preferable to simply set C equal to $0_{K\times K}$ and generalize the model by allowing splines in the linear terms in c defined by Bc^t. However, further discussion of these technical details can best be left to an actual application. At this point, we tentatively conclude that we have exhibited, in theory, a "practical" method for estimating preferences (without imposing too many unwarranted restrictions on elasticities of substitution a priori) where the consumer has preferences over both market and nonmarket goods and services.

7. Econometric Issues

There are some econometric issues that emerge from the previous section that require further discussion. We have simply assumed that once we have derived our theoretical exact mathematical model, we can simply add some error terms to our exact equations and proceed to the econometric estimation of the unknown parameters in the model. To a certain extent, this straightforward procedure can be justified from the statistical point of view, provided that the dependent variables do not appear on the right hand side of any equation as a predetermined variable; i.e., we can simply condition on the right hand side variables and find the conditional distribution of the left hand side variables. In the present section, we have chosen to condition on quantities and we set up an econometric model which will enable us to obtain the conditional distribution of prices. However, in the following section, we will choose to condition on prices and we will set up an econometric model that will enable us to obtain the conditional distribution of quantities. Both models will generate estimates for elasticities of substitution but in general, the estimates for these elasticities will be different. This raises the difficult issue: which set of estimates best represents the "truth"?

To show that it can sometimes matter a great deal whether we condition on prices or quantities in order to obtain elasticity estimates, we will illustrate the econometric problems by considering a simple two variable problem. Thus suppose we have two variables, y_1 and y_2 say, where we think of one variable being a price and the other being a quantity and we suppose that $y = [y_1,y_2]^T$ has a bivariate normal distribution[33] $N(\mu, \Sigma)$ where $\mu \equiv [\mu_1,\mu_2]^T$ is the mean vector for y and Σ is the variance covariance matrix for y defined as

$$(128)\ \Sigma \equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} .$$

We assume that σ_{12} is not equal to zero and that $\sum$ is positive definite so that

$$(129)\ \sigma_{12} \neq 0\ ;\ \sigma_{11} > 0\ ;\ \sigma_{22} > 0 \text{ and}$$
$$(130)\ \sigma_{11}\ \sigma_{22} > \sigma_{12}^2.$$

Anderson (1958; 28) shows that the conditional distribution of y_1 given y_2 is normal with mean equal to $\mu_1 + \sigma_{12}(\sigma_{22})^{-1}[y_2 - \mu_2]$ and variance equal to $\sigma_{11} - \sigma_{12}^2/\sigma_{22}$. Thus the average rate of change of y_1 with respect to a change in y_2 using this conditional regression line based on regressing y_1 on y_2 will be

$$(131)\ \beta_{12} \equiv \sigma_{12}/\sigma_{22} .$$

However, again using the results in Anderson, we can show that the conditional distribution of y_2 given y_1 is normal with mean equal to $\mu_2 + \sigma_{12}(\sigma_{11})^{-1}[y_1 - \mu_1]$ and variance equal to $\sigma_{22} - \sigma_{12}^2/\sigma_{11}$. Thus the average rate of change of y_2 with respect to a change in y_1 using this conditional regression line will be

$$(132)\ \delta \equiv \sigma_{12}/\sigma_{11} .$$

If we take the reciprocal of δ, we obtain the following alternative estimate of the average rate of change of y_1 with respect to a change in y_2 based on regressing y_2 on y_1:

$$(133)\ \beta_{21} \equiv \sigma_{11}/\sigma_{12} .$$

[33] For material on multivariate normal distributions and their properties, see Anderson (1958; 11-30).

Now the question is: how do the two estimates of the slope $\partial y_1/\partial y_2$ defined by (131) and (133) compare? We compare the ratio of β_{12} to β_{21} to answer this question:

(134) $\beta_{12}/\beta_{21} = [\sigma_{12}/\sigma_{22}]/[\sigma_{11}/\sigma_{12}]$ using definitions (131) and (133)
$= \sigma_{12}^2/\sigma_{11}\sigma_{22}$
< 1 using (130).

Thus

(135) if $\sigma_{12} > 0$, then $0 < \beta_{12} < \beta_{21}$ and
(136) if $\sigma_{12} < 0$, then $0 > \beta_{12} > \beta_{21}$.

Note that in both cases, β_{21} will be bigger in magnitude than β_{12}. Thus there is a systematic inequality between our two (equally valid) estimates for the slope $\partial y_1/\partial y_2$. This is a very unsatisfactory state of affairs! There is scope for consultants and analysts to strategically choose to regress y_1 on y_2[34] or regress y_2 on y_1[35] in order to obtain an estimate for the average slope that best suits their client's agenda.

Under what conditions will the two estimates for the slope be close? The two estimates will converge to each other as $\sigma_{12}^2/\sigma_{11}\sigma_{22}$ approaches unity. Note that $\sigma_{12}^2/\sigma_{11}\sigma_{22}$ is the square of the correlation coefficient between y_1 and y_2. Thus as y_1 and y_2 are more closely correlated (either positive or negative correlation will work), then the closer will be our two estimates for the average slope $\partial y_1/\partial y_2$. Unfortunately, in many applications, the two variables will not be highly correlated and hence the difference between β_{12} and β_{21} can be enormous.

We pursue the above example of a joint distribution of dependent and predetermined variables a bit further. We now indicate how estimates for μ and the variance covariance matrix Σ can be obtained. Thus suppose that we have T independent observations $y^t \equiv [y_1^t, y_2^t]^T$ for t = 1,2,...,T and each observation is a drawing from the bivariate normal distribution $N(\mu,\Sigma)$. Anderson (1958; 48) shows that the maximum likelihood estimators for the components of μ and Σ are given by the following expressions:

(137) $\mu^* \equiv (1/T) \sum_{t=1}^T y^t$;

(138) $\Sigma^* \equiv (1/T) \sum_{t=1}^T [y^t - y^*][y^t - y^*]^T \equiv \begin{bmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{12}^* & \sigma_{22}^* \end{bmatrix}$.

We can substitute these maximum likelihood estimates for σ_{11}, σ_{12} and σ_{22} (denote them by σ_{11}^*, σ_{12}^* and σ_{22}^*) into (131) and (133) in order to obtain the following estimators for β_{12} and β_{21} (we assume that $\sigma_{12}^* \neq 0$):

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[34] This will lead to an estimate of $\partial y_1/\partial y_2$ that is relatively *small* in magnitude.
[35] This will lead to an estimate of $\partial y_2/\partial y_1$ that is relatively *large* in magnitude.

(139) $\beta_{12}^* \equiv \sigma_{12}^*/\sigma_{22}^*$;
(140) $\beta_{21}^* \equiv \sigma_{11}^*/\sigma_{12}^*$.

From (138), it can be seen that Σ^* is a sum of T positive semidefinite matrices of rank 1 and hence Σ^* is positive semidefinite. It will be positive definite if at least two of the vectors $y^1, ..., y^T$ are linearly independent, an assumption that we will make. Hence the determinantal conditions for positive definiteness will imply that

(141) $\sigma_{11}^* \sigma_{22}^* > (\sigma_{12}^*)^2$.

Hence we can deduce that our estimates for β_{12} and β_{21} defined by (139) and (140) will satisfy the following inequalities, which are sample counterparts to the population inequalities (135) and (136) that we obtained earlier:

(142) If $\sigma_{12}^* > 0$, then $0 < \beta_{12}^* < \beta_{21}^*$ and
(143) if $\sigma_{12}^* < 0$, then $0 > \beta_{12}^* > \beta_{21}^*$.

Thus there is a systematic inequality between our two (equally valid) estimates for the slope $\partial y_1/\partial y_2$, β_{12}^* and β_{21}^*, which is again a very unsatisfactory state of affairs!

The inequalities (142) and (143) have their counterparts in ordinary least squares regression analysis, where we first regress y_1^t on y_2^t and then run the reverse regression. Thus consider the following ordinary least squares regression model where we regress y_1^t on y_2^t:

(144) $y_1^t = \alpha + \beta y_2^t + \varepsilon^t$; $\qquad t = 1,...,T$

where α and β are scalar parameters to be estimated and the ε^t are independently distributed normal random variables with means 0 and constant variances. Letting y^1 and y^2 be T dimensional vectors of the y_1^t and y_2^t respectively, then the least squares estimators for α and β, which we denote by α^* and β^* respectively, are defined as follows:

(145) $[\alpha^*, \beta^*]^T \equiv \{[1_T,y^2]^T[1_T,y^2]\}^{-1}[1_T,y^2]^T y^1$.

Now consider the following reverse ordinary least squares regression model where we regress y_2^t on y_1^t:

(146) $y_2^t = \gamma + \delta y_1^t + \eta^t$; $\qquad t = 1,...,T$

where γ and δ are scalar parameters to be estimated and the η^t are independently distributed normal random variables with means 0 and constant variances. Then the least squares estimators for γ and δ, which we denote by γ^* and δ^* respectively, are defined as follows:

(147) $[\gamma^*, \delta^*]^T \equiv \{[1_T,y^1]^T[1_T,y^1]\}^{-1}[1_T,y^1]^T y^2$.

Recall the maximum likelihood estimators β_{12}^* and β_{21}^* defined by (139) and (140) respectively. It can be shown (see Problem 9 below) that the least squares estimator β^* defined in the direct regression results (145) and the least squares estimator δ^* defined in the reverse regression results (147) are numerically related to β_{12}^* and β_{21}^* as follows:

(148) $\beta_{12}^* = \beta^*$;
(149) $\beta_{21}^* = 1/\delta^*$.

Since $1/\delta^*$ is the reverse regression estimator for $\partial y_1/\partial y_2$ while β^* is the direct regression estimator for $\partial y_1/\partial y_2$, we see again that there will be the same systematic inequality between these two estimators that we found for β_{12}^* and β_{21}^*; i.e., the counterparts to (142) and (143) above are now:

(150) If $\beta^* > 0$, then $0 < \beta^* < 1/\delta^*$ and
(151) if $\beta^* < 0$, then $0 > \beta^* > 1/\delta^*$.

The inequalities (150)-(151) relating the direct and reverse OLS regression estimates for a slope parameter are reasonably well known in the literature; e.g., see Kendall and Stuart (1967; 380) or Bartelsman (1995). However, there is no consensus in the literature on how to deal with this problem.[36] But in some instances, we can make an a priori choice between the two regressions; i.e., if we are interested in forecasting y_1^t, then the direct regression of y_1^t on y_2^t, (144), is the appropriate one to choose whereas if we are interested in forecasting y_2^t, then the reverse regression of y_2^t on y_1^t, (146), is the appropriate one to choose. In our present context, we are interested in forecasting utility or welfare as the vector of public goods c^t changes and as the vector of after tax commodity prices p^t changes. Since utility u^t depends on the quantity vectors x^t (and c^t), we should probably use the approach to preference estimation that will be developed in the following section rather than use the approach developed in the prior section; i.e., we

[36] The use of instrumental variables is not a solution to this estimation problem. Let $x^1 \equiv y^1 - \mu_1^* 1_T$ and $x^2 \equiv y^2 - \mu_2^* 1_T$; i.e., the x^i are the corresponding y^i vectors less their sample means for i = 1,2. Then using (137)-(140), it can be shown that $\beta_{12}^* = \beta^* = x^{2T}x^1/x^{2T}x^2$ and $\beta_{21}^* = 1/\delta^* = x^{1T}x^1/x^{1T}x^2$. Let z be a T dimensional vector of instruments. The instrumental variable estimator for β is $\beta_z \equiv z^T x^1/z^T x^2$. If z is numerically close to being proportional to the x^2 vector, then β_z will be numerically close to $x^{2T}x^1/x^{2T}x^2 \equiv \beta_{12}^*$, the x^1 on x^2 direct regression estimator for β. However, if z is numerically close to being proportional to the x^1 vector, then β_z will be numerically close to $x^{1T}x^1/x^{1T}x^2 \equiv \beta_{21}^*$, the x^2 on x^1 reverse regression estimator for β. More generally, let x^1 and x^2 be linearly independent. Since an arbitrary z vector can be represented as a linear combination of x^1 and x^2 and T−2 vectors that are orthogonal to both x^1 and x^2, it can be verified that the instrumental variables estimator for β using this arbitrary z could lie anywhere between plus and minus infinity. If $z = ax^1 + bx^2 + u$ where u is orthogonal to x^1 and x^2 and if we restrict the weights a and b to be nonnegative, then the resulting instrumental variable estimator using this z could lie anywhere between β_{12}^* and β_{21}^*. Since different researchers will choose a wide variety of instrument vectors z, it can be seen that the resulting estimates for β will not be reproducible across different econometricians who pick different instrument vectors (unless β_{12}^* is close to β_{21}^* which will happen only if x^1 is close to being proportional to x^2).

should treat prices p^t, income Y^t and public goods production c^t as predetermined and x^t as endogenous.[37]

The problems associated with converting exact relationships between variables into stochastic relationships (and the related problem of whether to regress y_1^t on y_2^t or y_2^t on y_3^t) have a long history in economics; see Frisch (1934; 47-70), Allen (1939; 191-194), and Haavelmo (1943; 1). Haavelmo explains the basic problem as follows:

"Observable economic variables do not satisfy exact relationships (except, perhaps, some trivial identities). Therefore, if we start out with such a theoretical scheme, we have—for the purpose of application—to add some stochastical elements , to bridge the gap between the theory and the facts. One much discussed way of doing this is to adopt the convention that the observable variables considered are each made up of two parts, viz., a *systematic* part which, by assumption, satisfies the exact relation considered, and an error part, or 'disturbance' of a stochastical nature." Trygve Haavelmo (1944; 52-53).

Thus this early econometric research proceeded as follows: an exact theoretical model was derived but then it was recognized that the model equations unlikely to hold exactly but they may be approximately true. In this approximate case, it is natural to assume that one or more of the variables or parameters in these equations is a stochastic variable. Once we determine which variables are stochastic, econometric estimation can proceed. The estimation is particularly easy if only a single stochastic variable can be isolated on the left hand side of each equation. However, when the early econometricians tried to implement this strategy, they soon ran into problems. For example, when Marshak and Andrews (1944; 156) attempted to estimate aggregate production functions for the U.S., they had an exact production function model, say output in period t is equal to a function f of input in period t. They then introduced productivity shocks into their exact production function equations and obtained a stochastic model. But they were then faced with a problem: does the random part of the productivity shock get absorbed into aggregate output growth (in which case the direct regression of output growth on input growth is the "right" regression model to use) or does it get absorbed into aggregate input growth (in which case the reverse regression of input growth on output growth is the "right" regression model to use) or does the productivity shock get spread between input and output growth? In the latter case, neither regression model is the "correct" one.

In addition to the above "shock" allocation problem, Marshak and Andrews (1944; 156) recognized that there could be other sources of error that could be introduced into their exact system of equations including errors in profit maximization, random technological coefficients, various measurement errors and aggregation errors.[38]

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[37] This advice neglects the problem of possible measurement errors in p^t, Y^t and c^t.

[38] "Similarly, unless the revenue functions and the outlay functions of all firms in all years coincide, some or all of the β's and b's (or B's) in the above equations will be considered random (and accordingly supplied with the f subscript). In addition, not all entrepreneurs may have the same urge, ability or luck to choose the most profitable combinations of production factors: even if entrepreneur A be technically efficient as B, he may have smaller '*economic efficiency*': the combination of resources which A will choose may bring his profit not as close to the possible maximum as the combination chosen by B. Accordingly, the equations (1.11), (1.12) [or their equivalents (1.13), (1.14); and, under perfect competition, also (1.15), (1.16) or (1.17), (1.18)] will not be satisfied exactly. Another cause of their

Eventually, many researchers realized that the problem of adding stochastic error terms to exact equations could be combined with a model that recognized that not all variables in the exact equations are measured without error. This led to the intensive study of errors in variables models. Perhaps the best exposition of these types of models can be found in Chapter 29, "Functional and Structural Relationships" in Kendall and Stuart (1967; 375-418). But the overall conclusions reached by this literature were not overly encouraging. Returning to our simple two variable regression model that was studied at the beginning of this section, the errors in variable literature suggests that we can only make progress if we can form some a priori judgements about the relative magnitude of the "error" variances in our two variables. Armed with this a priori information, we can obtain sensible estimates for the parameter of interest, β in equations (144). However, if the error variance in the y_1^t variable is zero, then the resulting estimator for β turns out to be β_{21}^{*}, the y_2^t on y_1^t regression estimator, while if error variance in the y_2^t variable is zero, then the resulting estimator for β turns out to be β_{12}^{*}, the y_1^t on y_2^t regression estimator. For cases where both variables have positive variances, as the variance ratio changes, we can obtain any estimate for β that lies between the direct and reverse regression estimators for β; see Kendall and Stuart (1967; 380-381 and 387).

At first glance, these results from the errors in variables literature are disappointing since we will often not have a very clear idea of the relative variances of the errors in the variables that appear in our exact model. Thus it seems that we could justify any estimate for β that lies between the estimates generated by the direct and reverse regressions; i.e., we are back to the conditional regression models that we started out with in this section. It seems clear that the "truth" lies somewhere between the estimates generated by these two conditional regressions but the range of uncertainty can be large.

However, at a second glance, the errors in variables literature tells us a bit more than we were able to deduce from our first conditional regression approach. In many situations, we will be able to form an opinion that the "error" variance (relative to our exact model) in say y_1^t is much bigger than the corresponding "error" variance in y_2^t. Under these conditions, the y_1^t on y_2^t regression results are likely to be more accurate than the y_2^t on y_1^t regression results. For example, in the production function estimation context, output growth rates are typically much more variable than input growth rates and thus regressing output growth rates on input growth rates is more likely to give more accurate estimates of production function parameters than the reverse regression. The logic behind this conclusion is as follows: the theoretical model of production suggests that output growth should be proportional to input growth plus productivity shocks. Thus the variance of output growth rates should be equal to the sum of the variance of input growth rates plus the variance of the productivity shocks and hence the direct regression of output growth on input growth is the "right" one.

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failure to be satisfied exactly may be the absence of perfect competition among the buyers of the product or among the sellers of productive resources." Jacob Marshak and William H. Andrews Jr. (1944; 156).

In our present context, quantities demanded by households exhibit much more variability than the corresponding variability in prices. Hence it is likely that the "error" variances (relative to our exact model) in quantities demanded are much bigger than the corresponding "error" variances in the corresponding prices and so we will obtain more accurate estimates of the consumer's structural utility function parameters using a dual representation of preferences and regressing quantities demanded on prices rather than vice versa. Thus in the following section, we pursue a dual approach to the estimation of preferences when the consumer has preferences over both market and nonmarket goods and services.

There are some additional tentative conclusions that we can draw from the discussion in this section:

- If the fit in our estimating equations is good (i.e., the R^2 is high), then the problem of choosing the "right" conditional regression diminishes since both the initial direct regression and a reverse regression will give much the same answer. Thus fit matters, contrary to what is said in many econometric textbooks.
- Errors in the predetermined variables will tend to lead to coefficient estimates that are smaller in magnitude than the true structural parameters. The bigger is the error variance, the bigger the bias will be.
- Instrumental variable techniques will not cure the problems generated by endogenous predetermined variables nor will they cure the problems caused by predetermined variables that are measured with error: different choices for the instrumental variables will generally lead to different parameter estimates[39] so that in general, instrumental variable techniques are not reproducible across analysts (i.e., different analysts will choose different instruments, leading to different parameter estimates).

Problems

9. Show that equations (148) and (149) are valid.

10. Verify that the assertions made in footnote 36 are true.

8. Evaluating the Benefits of Public Goods: A Dual Approach to the Econometric Estimation of Household Preferences

Recall the model presented in section 6 above, where we assumed that the consumer has preferences defined over alternative bundles of market goods and services and public goods. These preferences over the N variable market commodities, $[x_1,\ldots,x_N]^T \equiv x$ and the K public goods, $c \equiv [c_1,c_2,\ldots,c_K]^T$ public goods were represented by the continuous, nondecreasing and quasiconcave in x utility function, f(x,c). We also assumed that f is twice continuously differentiable over its domain of definition and we assumed that the domain of definition for f is the set $S \equiv \{(x,c): x \geq 0_N ; c \in R^K\}$.

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[39] For a good example of this phenomenon, see Burnside (1996).

As in section 6, we will assume (97); i.e., that the consumer's preferences are homothetic in market goods and services, conditional on any vector of nonmarket goods and services.

We define the (*conditional*) *expenditure function* that is dual to the utility function f(x,c) in the following way: for $p >> 0_N$ and c and u such that there exists an x such that f(x,c) = u with $(x,c) \in S$, define E as follows:

(152) $E(u,p,c) \equiv \min_x \{p^T x : f(x,c) = u\}$.

Holding c fixed, it can be seen that E(u,p,c), regarded as a function of u and p will have the same properties as a cost or expenditure function (recall the analysis in Chapter 4). In particular, E(u,p,c) will be linearly homogeneous and concave in p over the set $S^* \equiv \{p : p >> 0_N\}$.

Recall that for $(x,c) \in S$, we assumed that

(153) $\nabla_x f(x,c) > 0_N$.

Hence if $(x,c) \in S$ and $x >> 0_N$, we have:

(154) $0 < x^T \nabla_x f(x,c)$ using $x >> 0_N$ and (153)
$= f(x,c)$

where the last equality follows from our linear homogeneity assumption (97) and Euler's Theorem on homogeneous functions. But now the linear homogeneity of f(x,c) in x and the inequality (154) implies that for each fixed c, the range of f(x,c) as x varies over the nonnegative orthant is the set $R \equiv \{u: 0 \leq u \leq +\infty\}$. In particular, u = 1 belongs to this range set R. Thus we can now define the *unit* (utility) *expenditure function* as follows:

(155) $e(p,c) \equiv \min_x \{p^T x : f(x,c) = 1\}$
$= E(1,p,c)$

where the last equality follows using definition (152).

We now show that for each u > 0, $p >> 0_N$ and c such that there exists an x such that f(x,c) = u with $(x,c) \in S$, we have the following relationship between E and e:

(156) $E(u,p,c) \equiv \min_x \{p^T x : f(x,c) = u\}$
$= \min_x \{p^T x : u^{-1} f(x,c) = 1\}$
$= \min_x \{p^T x : f(u^{-1}x,c) = 1\}$ using the linear homogeneity property (97) on f
$= \min_x \{up^T x/u : f(u^{-1}x,c) = 1\}$
$= u \min_z \{p^T z : f(z,c) = 1\}$ letting z = x/u
$= ue(p,c)$ using definition (155).

Thus under the homothetic in market goods assumption (97) on preferences, the expenditure function E(u,p,c) that is dual to f simplifies into ue(p,c) where e(p,c) is the consumer's unit (utility) expenditure function. This restriction on E simplifies our search for a flexible functional form for E since the decomposition (156) tells us that we need only find a flexible functional form for e(p,c).

Since e(p,c) is linearly homogeneous in p, Euler's Theorems on homogeneous functions imply the following restrictions on the first and second order partial derivatives of e:

(157) $\nabla_p e(p,c)^T p = e(p,c)$;
(158) $\nabla_{pp}{}^2 e(p,c)p = 0_N$;
(159) $\nabla_{cp}{}^2 e(p,c)p = \nabla_c e(p,c)$.

In what follows, we will attempt to find a flexible functional form for e(p,c). Once we have found such a function, we will use Shephard's Lemma to obtain the consumer's system of Hicksian demand functions and then substitute the indirect utility function into these Hicksian demand functions in order to obtain market demand functions.

Neglecting the restrictions (157)-(159), a fully flexible functional form for the unit expenditure function, e(p,c), would require 1 + (N+K) + (N+K+1)(N+K)/2 parameters. There is 1 restriction on the derivatives of e in (157), N restrictions in (158) and K restrictions in (159). Thus the total number of independent parameters we will need in a flexible functional form for a unit expenditure function that is dual to a homothetic in market goods and services utility function is:

(160) $1 + (N+K) + (N+K+1)(N+K)/2 - [1+N+K] = N(N+1)/2 + NK + K(K+1)/2$.

We will attempt to determine whether the following functional form is flexible in our restricted class of functions:

(161) $e(p,c) \equiv a^Tp + (1/2)(\alpha^Tp)^{-1}p^TAp + p^TBc + (1/2)(\alpha^Tp)c^TCc$

where α is a positive vector of predetermined constants, a is an unknown N dimensional parameter vector, A is an N by N matrix of unknown parameters, B is an N by K matrix of unknown parameters and C is a K by K matrix of unknown parameters. The vector α and the matrices A and C are assumed to satisfy the following restrictions:

(162) $\alpha^Tp^* = 1$;
(163) $A = A^T$;
(164) $C = C^T$;
(165) $Ap^* = 0_N$

where $p^* \gg 0_N$ is some point that we have chosen to form the basis for our second order approximation to an arbitrary unit expenditure function, $e^*(p,c)$ and c^* is the corresponding c point of approximation. There are N free parameters in the a vector, N(N−1)/2 free parameters in the A matrix taking into account the restrictions in (163) and

(165), NK free parameters in the B matrix and K(K+1)/2 free parameters in the C matrix and this sums up to the total number of parameters on the right hand side of (160) and hence the normalized quadratic functional form defined by (161)-(165) has just enough parameters to be a parsimonious flexible functional form. We will show below that this functional form is indeed a flexible one. Note that our analysis in this section largely parallels our analysis presented in section 6 above.

We first calculate the vectors of first order partial derivatives of the e defined by (161):

(166) $\nabla_p e(p,c) = a + (\alpha^T p)^{-1}Ap - (1/2)(\alpha^T p)^{-2}p^T Ap\alpha + Bc + (1/2)c^T Cc\alpha$;
(167) $\nabla_c e(p,c) = B^T p + (\alpha^T p)Cc$.

The matrices of second order partial derivatives may be computed as follows:[40]

(168) $\nabla_{pp}{}^2 e(p,c) = (\alpha^T p)^{-1}A - (\alpha^T p)^{-2}\alpha p^T A - (\alpha^T p)^{-2}Ap\alpha^T + (\alpha^T p)^{-3}p^T Ap\alpha\alpha^T$;
(169) $\nabla_{pc}{}^2 e(p,c) = B + \alpha c^T C$;
(170) $\nabla_{cc}{}^2 e(p,c) = (\alpha^T p)C$.

Let $e^*(p,c)$ be an arbitrary twice continuously differentiable unit expenditure function. Pick an arbitrary point p^*,c^* as our point of approximation. Since $e^*(p,c)$ is linearly homogeneous in p, the derivatives of e^* evaluated at (p^*,c^*) will satisfy the following counterparts to the restrictions (157)-(159):

(171) $\nabla_p e^*(p^*,c^*)^T p^* = e^*(p^*,c^*)$;
(172) $\nabla_{pp}{}^2 e^*(p^*,c^*)p^* = 0_N$;
(173) $\nabla_{cp}{}^2 e^*(p^*,c^*)p^* = \nabla_c e^*(p^*,c^*)$.

We now want to pick a vector a and matrices A, B and C, satisfying the restrictions (163)-(165), such that $e(p^*,c^*)$ equals $e^*(p^*,c^*)$ and all of the first and second order partial derivatives of e evaluated at (p^*,c^*) equal the corresponding first and second order partial derivatives of e^* evaluated at (p^*,c^*). Recalling equations (115)-(120) in section 6 above, it can be seen that that the same proof of flexibility will work in the present context. Thus the normalized quadratic unit expenditure function defined by (161)-(165) is indeed a parsimonious flexible functional form.

The *indirect utility function*, g(Y,p,c), that corresponds to the normalized quadratic expenditure function defined by (161) can be obtained by equating ue(p,c) to "income" Y and solving for u. The resulting indirect utility function is defined as follows:

(174) $g(Y,p,c) \equiv Y/[a^T p + (1/2)(\alpha^T p)^{-1}p^T Ap + p^T Bc + (1/2)(\alpha^T p)c^T Cc]$.

[40] Using the proof of Theorem 10 in Diewert and Wales (1987; 66), it can be shown that a necessary and sufficient condition for global concavity in p for each c of e(p,c) defined by (161)-(165) is that the matrix A be negative semidefinite. The techniques used by Diewert and Wales (1967; 52) to impose concavity on A can be used in the present context as well.

Using Shephard's (1953; 11) Lemma and the decomposition (156), the consumer's system of conditional Hicksian demand functions, x(u,p,c), is equal to $u\nabla_p e(p,c)$:

$$(175)\ x(u,p,c) = u\nabla_p e(p,c)$$
$$= u\{a + (\alpha^T p)^{-1}Ap - (1/2)(\alpha^T p)^{-2}p^T Ap\alpha + Bc + (1/2)c^T Cc\alpha\} \quad \text{using (166).}$$

Now we may use the indirect utility function defined by the right hand side of (174) to replace u on the right hand side of (175) and we obtain the consumer's system of market demand functions. Suppose that we have data x^t, c^t, p^t on the consumer for period t = 1,...,T. The consumer's period t "income" or expenditure on market goods and services Y^t is again defined $p^{tT}x^t$ for t = 1,...,T. Thus we obtain the following system of *market demands* for the consumer:

$$(176)\ x^t = Y^t[a + (\alpha^T p^t)^{-1}Ap^t - (1/2)(\alpha^T p^t)^{-2}p^{tT}Ap^t\alpha + Bc^t + (1/2)c^{tT}Cc^t\alpha]$$
$$/[a^T p^t + (1/2)(\alpha^T p^t)^{-1}p^{tT}Ap^t + p^{tT}Bc^t + (1/2)(\alpha^T p^t)c^{tT}Cc^t]\ ; \qquad t = 1,...,T.$$

Note that the vector of market prices p^t, the vector of nonmarket consumption c^t and income Y^t are regarded as predetermined variables in equations (176) while the market consumption vector x^t is regarded as the vector of dependent variables in these equations. If the x^t grow substantially over the T periods in the sample of observations, then there will be a heteroskedasticity problem. To control this heteroskedasticity problem, divide both sides of equation t in (176) by Y^t and premultiply both sides of equation t in (176) by a diagonal matrix, $\hat{p}^t$, which has the elements of the market price vector for period t, p^t, running down the main diagonal. On the left hand side of the resulting equation, we will obtain the consumer's period t expenditure share on market goods vector, s^t, and the resulting system of transformed equations (176) becomes the following system of equations:

$$(177)\ s^t = \hat{p}^t[a + (\alpha^T p^t)^{-1}Ap^t - (1/2)(\alpha^T p^t)^{-2}p^{tT}Ap^t\alpha + Bc^t + (1/2)c^{tT}Cc^t\alpha]$$
$$/\ [a^T p^t + (1/2)(\alpha^T p^t)^{-1}p^{tT}Ap^t + p^{tT}Bc^t + (1/2)(\alpha^T p^t)c^{tT}Cc^t]\ ; \qquad t = 1,...,T.$$

Looking at the right hand sides of (176) or (177), it can be seen that because these right hand sides are homogeneous of degree zero in the parameter vector and matrices a, A, B and C, not all of these parameters can be identified. Thus we will require at least one additional normalization in order to cardinalize utility. A natural normalization is to set utility equal to "income" Y^* at the point of approximation p^*,c^*; i.e., we set u^* equal to Y^*. This turns out to be equivalent to *money metric utility scaling* at the reference prices p^*. Since

$$(178)\ Y^* = E(u^*,p^*,c^*) = u^*e(p^*,c^*)$$

and u^* equals Y^*, we see that we must impose the following normalization on e:

$$(179)\ 1 = e(p^*,c^*)$$

$= a^T p^* + (1/2)(\alpha^T p^*)^{-1} p^{*T} A p^* + p^{*T} B c^* + (1/2)(\alpha^T p^*) c^{*T} C c^*$

$= a^T p^* + p^{*T} B c^* + (1/2) c^{*T} C c^*$ using (162) and (165).

We can further simplify the normalization (179) if we choose a new origin for measuring nonmarket production c so we choose that origin to be c^*. Now convert to the new system of measuring nonmarket production as deviations around c^*. We abuse our notation and continue to use the old symbol c for the new units. Hence our point of approximation in the new units is

(180) $c^* = 0_K$.

Substituting (180) into (179) leads to the following new normalization on the elements of a:

(181) $a^T p^* = Y^*$.

Now we can use (181) to solve for say a_N in terms of Y^* and the other a_n and p^* as follows:

(182) $a_N = [Y^* - \sum_{n=1}^{N-1} a_n p_n^*]/p_N^*$.

Now substitute (182) into (177), drop the last share equation, and we obtain a nice system of N−1 share equations that can be used to estimate the parameters of the e defined by (161). We note that if the matrices B and C happen to be equal to $0_{N\times K}$ and $0_{K\times K}$ respectively so that all elements of both matrices are zero, then the system of estimating equations reduces to a conventional system of estimating equations[41] for a system of demand equations with no nonmarket goods and services.

An advantage of the present set up is that we can apply this demand model to aggregate data under the assumptions that:

- All households have the same preferences;
- All households in the aggregate face the same nonmarket vectors c^t in each period t and
- All households face the same price vector p^t for market goods and services in each period t.

Make these assumptions and suppose that each household h has preferences that are dual to the homothetic in market goods normalized quadratic unit expenditure function defined by (161) for h = 1,...,H. Letting x^{ht} denote the demand vector for household h in period t and Y^{ht} be the corresponding income, we can obtain the following counterpart to (176) for each household h:

(183) $x^{ht} = Y^{ht}[a + (\alpha^T p^t)^{-1} A p^t - (1/2)(\alpha^T p^t)^{-2} p^{tT} A p^t \alpha + B c^t + (1/2) c^{tT} C c^t \alpha]$

[41] In fact, under these conditions, we obtain the demand system that corresponds to the normalized quadratic expenditure function pioneered by Diewert and Wales (1987) (1988a) (1988b).

$$/[a^T p^t + (1/2)(\alpha^T p^t)^{-1} p^{tT} A p^t + p^{tT} B c^t + (1/2)(\alpha^T p^t) c^{tT} C c^t] ; \qquad t = 1,...,T.$$

Now define the period t aggregate demand vector x^t and aggregate income Y^t in the obvious way:

$$(184)\ x^t \equiv \sum_{h=1}^H x^{ht} ;\ Y^t \equiv \sum_{h=1}^H Y^{ht} ; \qquad t = 1,...,T.$$

Substituting (183) into (184), we find that the aggregate consumption vectors x^t and the aggregate income scalars Y^t satisfy equations (176), which were originally derived for just a single household.

There is one problem with the use of homothetic preferences that must be mentioned. In Chapter 4, we showed that homothetic preferences led to unitary income elasticities of demand. However, in sections 9 and 10 of Chapter 4, we showed how the origin of the consumption space could be shifted so that homothetic preferences relative to this shifted origin were consistent with an arbitrary set of income elasticities. The same technique can be used in the present context.[42]

Our conclusion is that it is possible to adapt the usual econometric techniques for estimating consumer preferences to a situation where the consumer has preferences over both market and nonmarket goods and services.

The government may supply goods and services to the production sector of the economy as well as to the household sector. In the following section, we adapt the techniques developed in this section to model the provision of public goods to the production sector.

9. Evaluating the Benefits of Public Goods: A Dual Approach to the Econometric Estimation of Production Functions

As in section 6, we will assume that a government project produces a K dimensional vector of public goods, $c \equiv [c_1,c_2,\ldots,c_K]^T \geq 0_K$, but in the present section, we will examine the effects of these public goods on production rather than consumption.

It turns out that the identification problem that was analyzed in section 6[43] is not a problem in the production context because, in principle, we can observe outputs whereas in the consumer case, we cannot directly observe utility. Thus consider momentarily a simple one output, N input production function f, where the public goods vector c affects the production function. Thus we have:

$$(185)\ y = f(x,c)$$

[42] It is possible to aggregate over households in this translated origin framework as well; see Gorman (1953).

[43] Recall the analysis around (95) and (96).

where y is output, x is a vector of inputs and c is a vector of public goods. If we now think about adding an arbitrary function of c, say g(c), to the production function f(x,c), we would obtain the following new production function:

(186) $y = f(x,c) + g(c)$.

It can be seen that the new public goods contribution to production, g(c), can be identified because we can observe directly the effects of this contribution to output y. In the consumer context, utility u replaces output y and f is a utility function rather than a production function. Comparing equations (185) and (186) with our earlier consumer theory equations (95) and (96), it can be seen that in the production context, there is no problem in identifying the contribution of public goods to production, since output y is directly observable, in contrast to the consumer theory situation where we cannot directly observe utility.

We now study a more general model of production. Thus let y be an N dimensional vector of variable outputs and inputs (outputs are indexed with a plus sign and inputs with a minus sign as usual), let x be an M dimensional vector of nonnegative fixed inputs and let c be a vector of public goods that affect production. We assume that there is a production possibilities set S that characterizes the technology; recall section 11 in Chapter 4. However, in the present context, we assume that this technology set is affected by the production of public goods. Thus S becomes S(c).

Let $p \gg 0_N$ be a strictly positive vector of variable net output prices that the economy faces during a production period. Then conditional on a given vector of fixed inputs x, we assume that the economy solves the following *variable profit maximization problem*:

(187) $\max_y \{p^T y : (y,x) \in S(c)\} \equiv \pi(p,x,c)$.

Some regularity conditions on the technology set S(c) are required in order to ensure that the maximum in (187) exists. A simple set of sufficient conditions are:

(188) S(c) is a closed set in R^{N+M};
(189) For each $x \geq 0_M$, the set of y such that $(y,x) \in S(c)$ is not empty and is bounded from above; i.e., for each $x \geq 0_M$ and y such that $(y,x) \in S(c)$, there exists a number b(x) such that $y \leq b(x)1_N$.

Condition (189) means that for each vector of fixed inputs, $x \geq 0_M$, the amount of each variable net output that can be produced by the technology is bounded from above, which is not a restrictive condition.

Note that (187) serves to define the economy's *variable profit function*,[44] $\pi(p,x,c)$; i.e., $\pi(p,x,c)$ is equal to the optimized objective function in (187) and is regarded as a function

[44] This concept is due to Hicks (1946; 319) and Samuelson (1953-54), who determined many of its properties using primal optimization techniques. For more general approaches to this function using

of the net output prices for variable commodities that the firm faces, p, as well as a function of the vector of fixed inputs, x, that the firm has at its disposal and it also depends on the vector of public goods c that the government is supplying to the economy. As was done in Chapter 4, following McFadden (1966) (1978), Gorman (1968) and Diewert (1973), we can show that $\pi(p,x,c)$ is linearly homogeneous and convex in p for each fixed x and c. We can also show that Hotelling's Lemma (1932; 594) also holds in the present context; i.e., the economy's system of profit maximizing net supply functions, conditional on the vector of fixed inputs x and the vector of public goods c, y(p,x,c), can be obtained by differentiating the variable profit function with respect to the components of p, provided that $\pi(p,x,c)$ is differentiable with respect to the components of p:

(190) $y(p,x,c) = \nabla_p\pi(p,x,c)$.

In some cases, the government may be providing free inputs to the private production sector so that these components of the c vector behave in the same manner as a purchased fixed input or a purchased variable input. However, in other cases, the public goods can act in a manner that is similar to technical progress; i.e., the components of c simply *shift* the technology in some way. Since the shifting interpretation of the c variables can be specialized to the case where the public goods are direct substitutes for private inputs, we will work with the more general model where the c variables simply shift the technology set S(c) in some way (and hence the c variables also shift variable profits $\pi(p,x,c)$ in some way).

In the following section, we will hold the vector of fixed inputs constant and hence, we can replace $\pi(p,x,c)$ by $\pi(p,c)$. Thus in the present section, we will attempt to find a flexible functional form for $\pi(p,c)$. Note that $\pi(p,c)$ is linearly homogeneous and convex in p for each fixed c.

Since $\pi(p,c)$ is linearly homogeneous in p, Euler's Theorems on homogeneous functions imply the following restrictions on the first and second order partial derivatives of π, assuming that $\pi(p,c)$ is twice continuously differentiable:

(191) $\nabla_p \pi(p,c)^T p = \pi(p,c)$;
(192) $\nabla_{pp}^2 \pi(p,c)p = 0_N$;
(193) $\nabla_{cp}^2 \pi(p,c)p = \nabla_c\pi(p,c)$.

In what follows, we will attempt to find a flexible functional form for $\pi(p,c)$. But notice that (191)-(193) are exact counterparts to the restrictions (157)-(159) on the partial derivatives of e(p,c) in the previous section. Thus we can simply adapt the proofs in the previous section in order to obtain a flexible functional form for π. The following *normalized quadratic profit function with public goods* will do the job:

(194) $\pi(p,c) \equiv a^Tp + (1/2)(\alpha^Tp)^{-1}p^TAp + p^TBc + (1/2)(\alpha^Tp)c^TCc$

duality theory, see Gorman (1968), McFadden (1966) (1978) and Diewert (1973). McFadden used the term "conditional profit function" while Diewert used the term "variable profit function".

where α is a positive vector of predetermined constants, a is an unknown N dimensional parameter vector, A is an N by N matrix of unknown parameters, B is an N by K matrix of unknown parameters and C is a K by K matrix of unknown parameters. The vector α and the matrices A and C are assumed to satisfy the following restrictions:

(195) $\alpha^T p^* = 1$;
(196) $A = A^T$;
(197) $C = C^T$;
(198) $Ap^* = 0_N$.

The only difference between the present model and the model in the previous section is that we now require the matrix A to be positive semidefinite in order to ensure the global convexity of $\pi(p,c)$ in p, whereas in the previous section, we required A to be negative semidefinite.

The system of variable profit maximizing net supply functions can be obtained by using (190), Hotelling's Lemma:

(199) $y(p,c) = a + (\alpha^T p)^{-1}Ap - (1/2)(\alpha^T p)^{-2}p^T Ap\alpha + Bc + (1/2)c^T Cc\alpha$.

Equations (199) are linear in the unknown parameters and can be used as econometric estimating equations.[45]

Now we are ready to analyze the effects of a government project that has externalities in both consumption and production.

10. The Introduction of a Project that has Externalities in Consumption and Production

In this section, we revisit the simple model explained in section 2 but we now assume that the project produces at unit scale, a K dimensional vector of public goods, $b \equiv [b_1,b_2,\ldots,b_K] \geq 0_K$.

As in sections 6 and 8, we assume that the consumer has preferences defined over alternative bundles of market goods and services and public goods. These preferences over the N+1 variable market commodities, $x_0,x_1,\ldots,x_N$, and the public goods are represented by the continuous, nondecreasing and quasiconcave utility function, $f(x_0,x_1,\ldots,x_N;c) = f(x_0,x,c)$, where $x \equiv [x_1,\ldots,x_N]^T$. In this section, we will not assume that

[45] If it is necessary to impose positive semidefiniteness on A using the techniques explained in Diewert and Wales (1987; 52), the estimating equations become nonlinear. If there is a single "fixed" input k say that varies from period to period but is held fixed at the beginning of the each period and there is overall constant returns to scale in production over variable outputs and inputs y and the "fixed" input k, then we can obtain a system of econometric estimating equations by dividing the left hand side of (199) by k; i.e., we obtain the following period t estimating equation: $y^t/k^t = a + (\alpha^T p^t)^{-1}Ap^t - (1/2)(\alpha^T p^t)^{-2}p^{tT}Ap^t\alpha + Bc^t + (1/2)c^{tT}Cc^t\alpha$.

these preferences are necessarily homothetic in the vector of private goods and services, (x_0,x), so we will deal with a general situation in this section.[46] The dual *expenditure function*, E, that corresponds to these preferences is defined for $u > 0$ and $p_0 > 0$ and $p \equiv [p_1,\ldots,p_N]^T >> 0_N$, as follows:

$$(200)\ E(u,p_0,p,c) \equiv \min_{x\text{'s}} \{p_0x_0 + p^Tx: f(x_0,x,c) = u\}.$$

Shephard's Lemma is still applicable in this situation and the Hicksian conditional demand functions of the household sector are given by the following first order partial derivatives:

$$(201)\ x_0(u,p_0,p,c) = \partial E(u,p_0,p,c)/\partial p_0\ ;$$
$$x(u,p_0,p,c) = \nabla_p E(u,p_0,p,c) \equiv [\partial E(u,p_0,p,c)/\partial p_1,\ldots,\partial E(u,p_0,p,c)/\partial p_N]^T .$$

We need to work out an economic interpretation for the vector of partial derivatives of the conditional expenditure function $E(u,p_0,p,c)$ with respect to the components of the c vector. To do this, suppose for the moment that the consumer faces the positive prices $w >> 0_K$ for the vector of public goods so that provision of public goods is no longer free. Under these circumstances, the consumer will want to minimize the overall cost of achieving any given utility level and thus he or she will want to solve the following expenditure minimization problem:

$$(202)\ E^\circ(u,p_0,p,w) \equiv \min_{x\text{'s},c} \{p_0x_0 + p^Tx + w^Tc: f(x_0,x,c) = u\}$$
$$= \min_c \{E(u,p_0,p,c) + w^Tc\}$$

where we have minimized first with respect to x_0 and x, conditioning on c, and used definition (200) to derive the second line in (202). If $E(u,p_0,p,c)$ is differentiable with respect to c, and $c^* >> 0_K$ is the solution to the second minimization problem in (202), then the following first order necessary conditions for an interior minimum must hold:

$$(203)\ w = -\nabla_c E(u,p_0,p,c^*) \equiv w(u,p_0,p,c^*).$$

Thus $-\nabla_c E(u,p_0,p,c^*)$ is a vector of consumer *willingness to pay functions for marginal units of the public goods*.[47]

To see this more clearly, differentiate $E(u,p_0,p,c)$ with respect to the kth element of c. The resulting derivative, $\partial E(u,p_0,p,c)/\partial c_k$, gives us the reduction in income or expenditure on market goods that the consumer will require to maintain his or her standard of living at the utility level u that is due to an additional unit of c_k. Thus $-\partial E(u,p_0,p,c)/\partial c_k \equiv w_k(u,p_0,p,c)$ is the value to the consumer of the extra unit of c_k.

[46] Of course, in order to uniquely identify the preferences that are dual to E, we will have to make at least one extra assumption such as the homotheticity in market goods assumption that we have made in previous sections.
[47] This derivation of willingness to pay functions in an expenditure function framework is due to Diewert (1986; 173) but precursors of this result may be found in Samuelson (1953-54; 10) and Diewert (1974; 140).

Turning now to the production side of our model, we assume that the technology of the market sector of the economy can be represented by a variable profit function, $\pi(p_0,p,c)$ of the type studied in the previous section. Formally, π is defined as follows:

$$(204)\ \pi(p_0,p,c) \equiv \max_{y\text{'s}} \{p_0y_0 + p^Ty : (y_0,y)\in S(c)\}$$

where S(c) is the market sector's production possibilities set, conditional on the provision of the public goods vector c.

We need to work out an economic interpretation for the vector of partial derivatives of the variable profit function $\pi(p_0,p,c)$ with respect to the components of the c vector. To do this, suppose for the moment that the production sector faces the positive prices $W >> 0_K$ for the vector of public goods so that provision of public goods is no longer free. Under these circumstances, the producer will want to maximize variable profits less the cost of public goods provision and thus the producer will want to solve the following short run profit maximization problem:

$$(205)\ \pi^\circ(p_0,p,W) \equiv \max_{y\text{'s},c} \{p_0y_0 + p^Ty - W^Tc: (y_0,y)\in S(c)\}$$
$$= \max_c \{\pi(p_0,p,c) - W^Tc\}$$

where we have maximized first with respect to y_0 and y, conditioning on c, and used definition (204) in order to derive the second line in (205). If $\pi(p_0,p,c)$ is differentiable with respect to c, and $c^* >> 0_K$ is the solution to the second maximization problem in (205), then the following first order necessary conditions for an interior minimum must hold:

$$(206)\ W^* = \nabla_c\pi(p_0,p,c^*) \equiv W(u,p_0,p,c^*).$$

Thus $\nabla_c\pi(p_0,p,c)$ is a vector of producer *willingness to pay functions for marginal units of the public goods*.[48]

We now introduce a government project into the economy along the lines of the model presented in section 2. We assume that the consolidated government sector taxes only profits (or fixed factors) at the rate τ. The government spends these tax revenues on purchases of goods and services that produce various government services and by possibly transferring money back to the household sector. Denote the transfer as T and the purchases by the government on numeraire and nonnumeraire goods and services by g_0 and $g \equiv [g_1,\ldots,g_N]^T$ respectively.

The government introduces a *project* or expenditure program into the economy. This project is characterized by a vector of net outputs of market goods and services that the project generates, $[a_0,a_1,\ldots,a_N] = [a_0,a^T]$ at unit scale *plus* an incremental vector of public

[48] This derivation of producer willingness to pay functions using profit functions is due to Diewert (1986; 150).

goods, b, at unit scale. We model the transition from the preproject economy to the post project economy by having a scale variable s run from 0 to 1 as in section 2. Our goal in this section is to obtain a first order approximation to the change in welfare due to the introduction of the project and to determine how the costs and benefits of the project should be evaluated.

The consumer expenditure equals income equation in this model is equation (207) below and the demand equals supply equations for the numeraire and nonnumeraire commodities are given by (208) and (209) below:

$$(207)\quad E(u(s),p_0,p(s),c(0)+bs) = (1-\tau)\pi(p_0,p(s),c(0)+bs) + T(s)\ ;$$
$$(208)\quad \partial E(u(s),p_0,p(s),c(0)+bs)/\partial p_0 + g_0 = \partial\pi(p_0,p(s),c(0)+bs)/\partial p_0 + a_0 s\ ;$$
$$(209)\quad \nabla_p E(u(s),p_0,p(s),c(0)+bs) + g = \nabla_p\pi(p_0,p(s),c(0)+bs) + as.$$

The endogenous variables in the N+2 equations (207)-(209) are the utility level u, the nonnumeraire prices p and the transfer T, which is N+2 variables, while the exogenous variable is the project scale s.

Obviously, we can repeat the various steps that were undertaken in section 2. The counterpart to equation (7) is (209) above and the counterpart to equation (10) is:

$$(210)\quad E(u(s),p_0,p(s),c(0)+bs) = \pi(p_0,p(s),c(0)+bs) - [p_0g_0 + p(s)^Tg] + [p_0a_0 + p(s)^Ta]s.$$

Equations (209) and (210) are N+1 equations in the N+1 unknowns u(s) and p(s). We differentiate these equations with respect to s in order to obtain expressions for the derivatives of utility u and nonnumeraire prices p with respect to the project scale s, $u'(s)$ and $p'(s)$.

Differentiating (210) with respect to s leads to the following equation:

$$\begin{aligned}(211)\quad &\{\partial E(u(s),p_0,p(s),c(0)+bs)/\partial u\}u'(s)\\ &= -[\nabla_p E(u(s),p_0,p(s),c(0)+bs) + g - \nabla_p\pi(p_0,p(s),c(0)+bs) - as]^Tp'(s)\\ &\quad + p_0a_0 + p(s)^Ta - \nabla_c E(u(s),p_0,p(s),c(0)+bs)^Tb + \nabla_c\pi(p_0,p(s),c(0)+bs)^Tb\\ &= p_0a_0 + p(s)^Ta - \nabla_c E(u(s),p_0,p(s),c(0)+bs)^Tb + \nabla_c\pi(p_0,p(s),c(0)+bs)^Tb \quad \text{using (209)}\\ &= p_0a_0 + p(s)^Ta + w(u(s),p_0,p(s),c(0)+bs)^Tb + W(p_0,p(s),c(0)+bs)^Tb\end{aligned}$$

where the consumer and producer willingness to pay for public goods price vectors are $w(u(s),p_0,p(s),c(0)+bs)$ and $W(p_0,p(s),c(0)+bs)$ defined by (203) and (206) respectively.

Differentiating (209) with respect to s leads to the following equation:

$$\begin{aligned}(212)\quad &\nabla_{pu}^2E(u(s),p_0,p(s),c(0)+bs)u'(s) + [\nabla_{pp}^2E(u(s),p_0,p(s),c(0)+bs)\\ &\qquad - \nabla_{pp}^2\pi(p_0,p(s),c(0)+bs)]p'(s)\\ &= a - \nabla_{pc}^2E(u(s),p_0,p(s),c(0)+bs)b + \nabla_{pc}^2\pi(p_0,p(s),c(0)+bs)b\end{aligned}$$

In order to further simplify equations (211) and (212) when s = 0, we impose *local money metric utility scaling* on the household preferences using the initial equilibrium prices and quantities, p_0,p(0),c(0) as the reference prices and quantities. This scaling assumption imposes the following restriction on the first and second derivatives of the expenditure function with respect to utility:[49]

(213) $\partial E(u(0),p_0,p(0),c(0))/\partial u = 1$;
(214) $\partial^2 E(u(0),p_0,p(0),c(0))/\partial u^2 = 0$.

Now evaluate (211) and (212) at s = 0. Using (213), we obtain the following matrix equation:

$$(215)\quad \begin{bmatrix} 1, & 0_N^T \\ b, & S \end{bmatrix} \begin{bmatrix} u'(0) \\ p'(0) \end{bmatrix} = \begin{bmatrix} p_0 a_0 + p(0)^T a + w(0)^T b + W(0)^T b \\ a - \nabla_{pc}^2 E(0) b + \nabla_{pc}^2 \pi(0) b \end{bmatrix}$$

where $w(0) \equiv w(u(0),p_0,p(0),c(0)) = -\nabla_c E(u(0),p_0,p(0),c(0))$ is the consumer's willingness to pay for additional units of the public goods at the initial preproject equilibrium, $W(0) \equiv W(p_0,p(0),c(0))$ is the producer's willingness to pay for additional units of the public goods at the initial preproject equilibrium, the vector $b \equiv \nabla_{pu}^2 E(0)$ shows how the demand for the nonnumeraire commodities changes as utility increases and $S \equiv \nabla_{pp}^2 E(u(0),p_0,p(0),c(0)) - \nabla_{pp}^2 \pi(p_0,p(0),c(0))$ is the economy's *aggregate substitution matrix* for nonnumeraire commodities. Since $E(u,p_0,p,c)$ is concave in p and $\pi(p_0,p,c)$ is convex in p, S is a symmetric negative semidefinite matrix. We shall assume that S is *negative definite* so that its determinant is not equal to zero; i.e., we assume that

(216) S is a negative definite symmetric matrix.

Since the determinant of the N+1 by N+1 matrix in the left hand side of (215) is equal to the determinant of S, |S|, it can be seen that assumption (216) means that we can apply the Implicit Function Theorem and obtain the *existence* of the solution functions, u(s) and p(s), to equations (211) and (212) for s close to 0.

Assumption (216) implies that S^{-1} exists and we can invert the matrix on the left hand side of (215) and obtain the following expressions for the derivatives, u′(0) and p′(0):

$$(217)\quad \begin{bmatrix} u'(0) \\ p'(0) \end{bmatrix} = \begin{bmatrix} 1 & 0_N^T \\ -S^{-1} b & S^{-1} \end{bmatrix} \begin{bmatrix} p_0 a_0 + p(0)^T a + w(0)^T b + W(0)^T b \\ a - \nabla_{pc}^2 E(0) b + \nabla_{pc}^2 \pi(0) b \end{bmatrix}$$

[49] Note that with an appropriate choice of the reference point, assumptions (213) and (214) will be satisfied using the homothetic in market goods normalized quadratic expenditure function that was defined in section 8.

$$= \begin{bmatrix} p_0 a_0 + p(0)^T a + w(0)^T b + W(0)^T b \\ -S^{-1}b[p_0 a_0 + p(0)^T a + w(0)^T b + W(0)^T b] + S^{-1}[a - \nabla^2_{pc}E(0)b + \nabla^2_{pc}\pi(0)b] \end{bmatrix}.$$

Now we can use (217) to form the following *first order approximation to the change in welfare of the consumer* if the project is implemented at level s = 1:

(218) $u(1) - u(0) \approx u'(0)[1 - 0] = p_0a_0 + p(0)^Ta + w(0)^Tb + W(0)^Tb.$

Thus *to the accuracy of a first order Taylor series approximation, the increase in consumer utility is equal to the profitability of the project, valued at the market prices of the preproject economy*, $p_0a_0 + p(0)^Ta$, *plus the value to the consumer and producer of the additional public goods produced by the project*, $[w(0) + W(0)]^Tb$.

Formula (218) is a common sense extension of our earlier formula (20) in section 2. As we noted before, in many practical applications of cost benefit analysis, it is not an easy matter to determine the expected costs and revenues of the project but in virtually every application, it is extremely difficult to obtain estimates for the consumer's and producer's willingness to pay for units of public goods; i.e., it is extremely difficult to obtain estimates for w(0) and W(0). However, the new econometric techniques developed in sections 8 and 9 above should be helpful in this regard.

Problem

11. Work out a second order Taylor series approximation to the change in welfare that is a counterpart to (218) above. Hint: Do not forget to use assumption (214).

References

Allen, R.G.D. (1939), "The Assumptions of Linear Regression", *Economica N.S.* 6, 191-201.

Anderson, T.W. (1958), *An Introduction to Multivariate Statistical Analysis*, New York: John Wiley and Sons.

Bartlett, M.S. (1951), "An Inverse Matrix Adjustment Arising in Discriminant Analysis". *Annals of Mathematical Statistics* 22, 107-111.

Bartelsman, E. J. (1995), "Of Empty Boxes: Returns to Scale Revisited," *Economics Letters* 49, 59-67.

Boadway, R.W. (1975), "Cost Benefit Rules in General Equilibrium", *Review of Economic Studies* 42, 361-374.

Boadway, R.W. (1976), "Integrating Equity and Efficiency in Applied Welfare Economics", *Quarterly Journal of Economics* 90, 541-556.

Boadway, R.W. (1978), "A Note on the Treatment of Foreign Exchange in Project Evaluation", *Economica* 45, 391-399.

Böhm-Bawerk, E.V. (1891), *The Positive Theory of Capital*, translated by W. Smart, New York: G.E. Stechert.

Bruce, N. and R. Harris (1982), "Cost Benefit Criteria and the Compensation Principle in Evaluating Small Projects", *Journal of Political Economy* 90, 755-776.

Burnside, C. (1996), "Production Function Regressions, Returns to Scale, and Externalities," *Journal of Monetary Economics* 37, 177-201.

Debreu, G. (1959), *Theory of Value*, New York: John Wiley and Sons.

Diamond, P.A. and D. McFadden (1974), "Some Uses of the Expenditure Function in Public Finance", *Journal of Public Economics* 3, 3-21.

Diamond, P.A. and J.A. Mirrlees (1971), "Optimal Taxation and Public Production, Parts I and II", *American Economic Review* 61, 8-27 and 261-278.

Diamond, P.A. and J.A. Mirrlees (1976), "Private Constant Returns and Public Shadow Prices", *Review of Economic Studies* 43, 41-47.

Diewert, W.E. (1973), "Functional Forms for Profit and Transformation Functions", *Journal of Economic Theory* 6, 284-316.

Diewert, W.E. (1974), "Applications of Duality Theory", pp. 106-171 in *Frontiers of Quantitative Economics*, Volume 2, M.D. Intriligator and D.A. Kendrick (eds.), Amsterdam: North-Holland.

Diewert, W.E. (1977), "Walras' Theory of Capital Formation and the Existence of a Temporary Equilibrium', pp. 73-126 in *Equilibrium and Disequilibrium in Economic Theory*, G. Schwödiauer (ed.), Dordrecht: D. Reidel.

Diewert, W.E. (1983), "Cost Benefit Analysis and Project Evaluation: A Comparison of Alternative Approaches", *Journal of Public Economics* 22, 265-302.

Diewert, W.E. (1986), *The Measurement of the Economic Benefits of Infrastructure Services*, Lecture Notes in Economics and Mathematical Systems 278, Berlin: Springer-Verlag.

Diewert, W.E. and T.J. Wales (1987), "Flexible Functional Forms and Global Curvature Conditions", *Econometrica* 55, 43-68.

Diewert, W.E. and T.J. Wales (1988a), "Normalized Quadratic Systems of Consumer Demand Functions", *Journal of Business and Economic Statistics* 6, 303-12.

Diewert, W.E. and T.J. Wales (1988b), "A Normalized Quadratic Semiflexible Functional Form", *Journal of Econometrics* 37, 327-42.

Duncan, W.J. (1944), "Some Devices for the Solution of Large Sets of Simultaneous Equations", *Philosophy Magazine*, Series 7, 35, 660-670.

Fisher, I. (1897), "The Role of Capital in Economic Theory", *Economic Journal* 7, 511-537.

Fisher, I. (1930), *The Theory of Interest*, New York: Macmillan.

Frisch, R. (1934), *Statistical Confluence Analysis by Means of Complete Regression Systems*, Oslo: University Institute of Economics.

Gorman, W.M. (1953), "Community Preference Fields", *Econometrica* 21, 63-80.

Gorman, W.M. (1968), "Measuring the Quantities of Fixed Factors", pp. 141-172 in *Value, Capital and Growth: Papers in Honour of Sir John Hicks*, J.N. Wolfe (ed.), Chicago: Aldine.

Guttman, L. (1946), "Enlargement Methods for Computing the Inverse Matrix", *Annals of Mathematical Statistics* 17, 336-343.

Hager, W.W. (1989), "Updating the Inverse of a Matrix", *SIAM Review* 31:2, 221-239.

Hammond, P.J. (1980), "Cost Benefit as a Planning Procedure", pp. 221-250 in Contemporary Economic Analysis, Volume 2, London: Croom Helm.

Harberger, A.C. (1969), "Discussion: Professor Arrow on the Social Discount Rate", pp. 76-88 in G.G. Somers and W.D. Wood (eds.), *Cost Benefit Analysis of Manpower Policies*, Kingston, Canada: Industrial Relations Centre at Queen's University.

Harberger, A.C. (1971), "Three Basic Postulates for Applied Welfare Economics: An Interpretive Essay", *Journal of Economic Literature* 9, 785-797.

Harberger, A.C. (1978), "On the Use of Distributional Weights in Social Cost Benefit Analysis", *Journal of Political Economy* 86, S87-S120.

Haavelmo, T. (1943), "The Statistical Implications of a System of Simultaneous Equations", *Econometrica* 11, 1-12.

Haavelmo, T. (1944), "The Probability Approach in Econometrics", *Econometrica* 12, *Supplement*, 1-115.

Hicks, J.R. (1946), *Value and Capital*, Second Edition, Oxford: Clarendon Press.

Hotelling, H. (1932), "Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions", *Journal of Political Economy* 40, 577-616.

Kendall, M.G. and A.S. Stuart (1967), *The Advanced Theory of Statistics: Volume 2: Inference and Relationship*, Second Edition, New York: Hafner Publishing Co.

Little, I.M.D. and J.A. Mirrlees ((1974), *Project Appraisal and Planning for Developing Countries*, London: Heinemann.

Marshak, J. and W.H. Andrews, Jr. (1944), "Random Simultaneous Equations and the Theory of Production", *Econometrica* 12, 143-205.

McFadden, D. (1966), "Cost, Revenue and Profit Functions: A Cursory Review", IBER Working Paper 86, University of California at Berkeley, March.

McFadden, D. (1978), "Cost, Revenue and Profit Functions", pp. 3-109 in *Production Economics: A Dual Approach to Theory and Applications*, Volume 1, M. Fuss and D. McFadden (eds.), Amsterdam: North-Holland.

Meade, J.E. (1955), *Trade and Welfare*, London: Oxford University Press.

Mirrlees, J.A. (1976), "Optimal Tax Theory: A Synthesis", *Journal of Public Economics* 6, 327-358.

Plackett, R.L. (1950), "Some Theorems in Least Squares", *Biometrika* 37, 149-157.

Pollak, R.A. (1969), "Conditional Demand Functions and Consumption Theory", *Quarterly Journal of Economics* 83, 60-78.

Pollak, R.A. (1989), *The Theory of the Cost of Living Index*, Oxford: Oxford University Press.

Samuelson, P.A. (1953-54), "Prices of Factors and Goods in General Equilibrium", *Review of Economic Studies* 21, 1-20.

Shephard, R.W. (1953), *Cost and Production Functions*, Princeton: Princeton University Press.

Sherman, J. and W.J. Morrison (1949), "Adjustment of an Inverse Matrix Corresponding to Changes in the Elements of a Given Column or a Given Row of the Original Matrix", *Annals of Mathematical Statistics* 20, 621.

Sherman, J. and W.J. Morrison (1950), "Adjustment of an Inverse Matrix Corresponding to a Change in One Element of a Given Matrix", *Annals of Mathematical Statistics* 21, 124-127.

Starrett, D. (1979), "Second Best Welfare Economics in the Mixed Economy", *Journal of Public Economics* 12, 329-349.

Walras, L. (1954), *Elements of Pure Economics*, W. Jaffé (translator), New York: Richard D. Irwin (translation of the Edition Définitive (1926) of the *Eléments d'économie politique pure*).

Wold, H. (1944), "A Synthesis of Pure Demand Analysis; Part 3", *Skandinanvisk Aktuarietidskrift* 27, 69-120.

Woodbury, M. (1950), *Inverting Modified Matrices*, Memorandum Report 42, Statistical Research Group, Princeton University, Princeton N.J.