ECONOMICS 581: LECTURE NOTES

CHAPTER 6: Consumer Surplus and the Measurement of Ex Post Welfare Change

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1. Introduction

In this chapter, we will consider how to measure the welfare change of a consumer or household, assuming cost minimizing behavior on the part of the consumer and assuming that we can observe the choices made by the consumer during the two periods under consideration. Thus we attempt to use observed price and quantity data in order to measure ex post welfare change. In the previous chapter, we also attempted to measure welfare change for households but instead of assuming that we had ex post information for two equilibrium situations, we assumed that we had information about prices and quantities at an initial equilibrium plus information on elasticities of demand and supply around this initial equilibrium. The methodology developed in chapter 5 is useful when one attempts to analyze *potential* policy changes around an initial equilibrium whereas the analysis to be presented in the present chapter is useful when we attempt to determine *ex post* whether the welfare of a consumer group has increased or decreased going from one period to another.

The problem of how exactly to measure welfare change and utility and the closely related topic of consumer surplus analysis has challenged economists since the first paper by Dupuit (1844) on the concept of consumer surplus.[1] In section 2, we provide an introduction to consumer surplus analysis and in section 8 of this chapter, we return to this subject and we will present some recent more rigorous justifications for consumer surplus type measures of welfare change.

In chapter 4, we introduced money metric measures of utility. They will play a large role in the present chapter, starting in section 2 below, since they are rigorous cardinal measures of utility or welfare. Of course, there is a practical problem associated with their use in empirical applications: these measures are not directly observable. In sections 3 and 4, we apply some first and second order Taylor series approximations to these money metric measures of utility change in order to derive some empirically useful measures, using techniques pioneered by Hicks (1941-42; 129-133) (1946; 332), Weitzman (1988) and Diewert (1992).

In sections 5-7, we consider some alternative techniques that have been used in index number theory in order to measure welfare change. As we shall see in section 8, it turns out that these index number methods have a connection with the consumer surplus type measures of welfare change that will be studied in section 2.

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[1] For a good review of many aspects of the consumer surplus literature and an excellent list of references on this topic, see Ekelund and Hébert (1985).

In sections 5-8, we will assume that the consumer's preferences are *homothetic*; i.e., the preferences can be represented by a linearly homogeneous utility function. In sections 9 and 10, we again draw on index number theory techniques in an attempt to relax this restriction.

Our overall conclusion in this chapter is that there are fairly satisfactory measures for measuring ex post welfare change that are easy to implement empirically.

2. The Marshall Bennet Consumer Surplus Measure of Welfare Change

Suppose that a household has the utility function $f(x)$ defined for $x \geq 0_N$ and the corresponding dual expenditure or cost function is

$$(1)\ e(u,p) \equiv \min_x \{p^Tx: f(x) \geq u\}$$

where $p \gg 0_N$ and u belongs to the range of f. We assume that we observe the household's price and quantity vectors, p^t and x^t for $t = 0,1$ and that the household minimizes the cost of achieving its utility level u^t in each period t so that we have:

$$(2)\ p^{tT}x^t = e(u^t,p^t) = e(f(x^t),p^t)\ ; \qquad t = 0,1.$$

Our goal is to obtain an approximation to the household's welfare change going from period 0 to period 1 using only observable data.

Consider the period 0 equilibrium. As prices change from p^0 to p^1, we could ask: what is the change in income that the consumer will require in order to be able to purchase the initial consumption vector x^0. This amount of income is obviously equal to the following *Laspeyres type measure of compensation for the price change*:[2]

$$(3)\ C_L(p^0,p^1,x^0,x^1) \equiv (p^1 - p^0)^Tx^0 = p^{1T}x^0 - p^{0T}x^0 = \textstyle\sum_{n=1}^N (p_n^{\,1} - p_n^{\,0})x_n^{\,0}\ .$$

The reason for the term *Laspeyres* is that the above measure is analogous to the Laspeyres (1871) price index, P_L, which is the measure of price change defined as follows:

$$(4)\ P_L(p^0,p^1,x^0,x^1) \equiv p^{1T}x^0/p^{0T}x^0\ .$$

[2] Diewert (2005; 319) defined this measure of price change as the *Laspeyres indicator of price change*. Hicks (1939; 41) (1946; 330) was clearly aware of this measure of price change and established versions of the inequality (6). This measure of price change has also been introduced in the accounting literature in the context of variance analysis; see Mensah (1982), Marcinko and Petri (1984) and Darrough (1986). Variance analysis in accounting decomposes a value difference into a price change component and a quantity or efficiency change component. Barlev and Callen (1986) contrast the ratio approach to the measurement of quantity change (the economist's traditional index number measurement approach) with the difference approach (the accountant's variance analysis approach).

Comparing (3) and (4), it can be seen that C_L and P_L are both *measures of pure price change* going from period 0 to 1 but C_L uses a *difference* in the cost of the same base period basket of goods and services x^0, using the prices of periods 0 and 1, but P_L uses a *ratio* of the same costs.

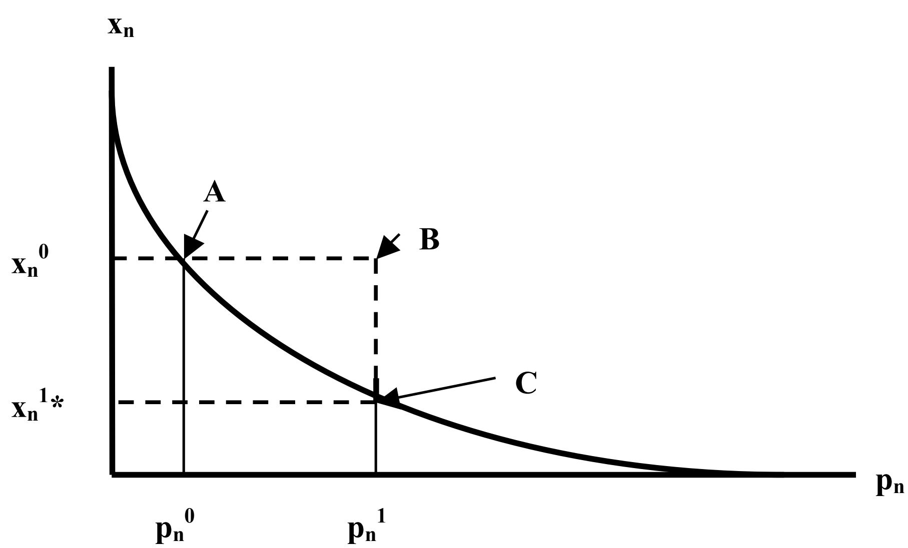
Recall from chapter 4 that the *consumer's system of Hicksian demand functions* is defined as follows, assuming that e(u,p) is differentiable with respect to its price variables:

$$(5)\ x(u,p) \equiv \nabla_p e(u,p).$$

Now consider the nth Hicksian demand function, $x_n(u^0,p_1^0,\ldots,p_{n-1}^0,p_n,\ p_{n+1}^0,\ldots,\ p_N^0)$, where utility is held constant at the period 0 level u^0 and all prices except the nth are held fixed at their period 0 levels. This Hicksian demand function for commodity n is graphed in Figure 1 below.

The nth term in the Laspeyres measure of compensation for the price change going from period 0 to 1, $(p_n^1 - p_n^0)x_n^0$, can be represented by the rectangle $p_n^0ABp_n^1$ in Figure 1. The amount x_n^0 is equal to the initial period 0 demand for commodity n; i.e., $x_n^0 = x_n(u^0,p_1^0,\ldots,p_{n-1}^0,p_n^0,\ p_{n+1}^0,\ldots,\ p_N^0)$ while the point x_n^{1*} is equal to $x_n(u^0,p_1^0,\ldots,p_{n-1}^0,p_n^1,\ p_{n+1}^0,\ldots,\ p_N^0)$. However, $(p_n^1 - p_n^0)x_n^0$ overstates the amount of money that is required to keep the consumer at the initial welfare level u^0 since it does not take into account the fact that the consumer will tend to consume less of commodity n as its price increases; i.e., *consumer substitution effects are not taken into account*.

Figure 1: The Period 0 Hicksian Demand Function for Commodity n



To see that the Laspeyres measure of compensation for the price change going from period 0 to 1, C_L defined by (3), is an *overestimate* of the amount of compensation required to keep the consumer at the same period 0 of utility, we note that *the analytically correct measure of the required compensation* for this amount of money is:[3]

$$\begin{aligned}
(6)\ e(u^0,p^1) - e(u^0,p^0) &= e(u^0,p^1) - p^{0T}x^0 && \text{using (2) for } t = 0 \\
&= \min_x \{p^{1T}x : f(x) \geq f(x^0)\} - p^{0T}x^0 \\
&\leq p^{1T}x^0 - p^{0T}x^0 && \text{since } x^0 \text{ is feasible for the min problem} \\
&= (p^1 - p^0)^T x^0 \\
&\equiv C_L(p^0,p^1,x^0,x^1) && \text{using definition (3).}
\end{aligned}$$

If consumer preferences are of the Leontief or no substitution variety or if the period 0 price vector p^0 is proportional to the period 1 price vector, then the inequality in (6) becomes an equality and C_L will be the correct amount of compensation that is required to compensate the consumer for the price change and keep the household at the period 0 level of welfare. However, in general, C_L will be too large.

Now we can repeat the above analysis but from the perspective of the period 1 level of welfare. We ask what is the amount of income that the consumer will require in order to be able to purchase the period 1 consumption vector x^1 (and hence maintain period 1's standard of living) but at the prices of period 0. This amount of income is obviously equal to $p^{0T}x^1$. Thus the extra amount that we must *add* to the period 1 income, $p^{1T}x^1$, to enable the household to purchase the period 1 consumption vector at the prices of period 0 is $p^{0T}x^1 - p^{1T}x^1$. In order to make this compensation measure comparable to the Laspeyres measure of compensation for the price change, we change the sign of $p^{0T}x^1 - p^{1T}x^1$. Thus we define the following *Paasche type measure of compensation for the price change*:

$$(7)\ C_P(p^0,p^1,x^0,x^1) \equiv (p^1 - p^0)^T x^1 = p^{1T}x^1 - p^{0T}x^1 = \textstyle\sum_{n=1}^N (p_n^1 - p_n^0)x_n^1 .$$

The reason for the term Paasche is that the above measure is analogous to the Paasche (1874) price index, P_P, which is the measure of price change defined as follows:

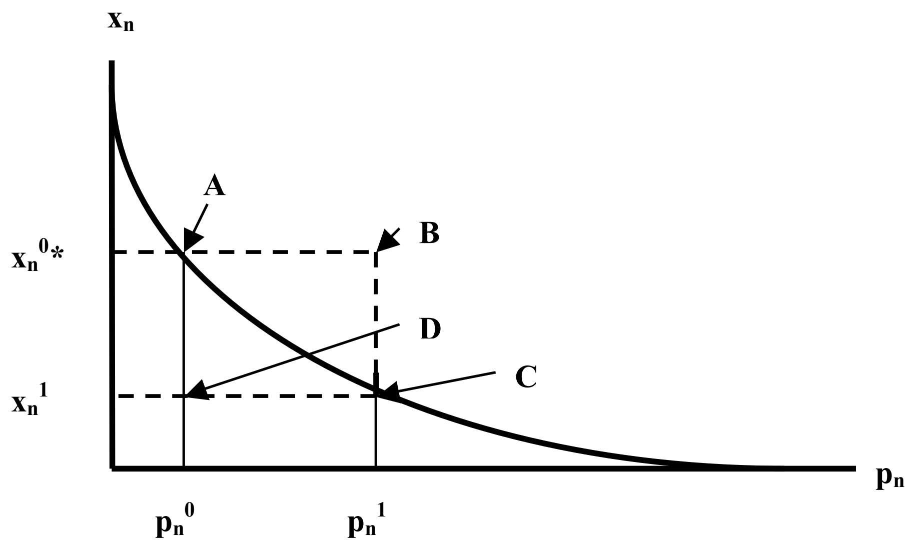
$$(8)\ P_P(p^0,p^1,x^0,x^1) \equiv p^{1T}x^1/p^{0T}x^1 .$$

[3] This measure of compensation (a measure of price change) was originally defined by Hicks (1939; 40-41) in words as follows: "As we have seen, the best way of looking at consumer's surplus is to regard it as a means of expressing, in terms of money income, the gain which accrues to the consumer as a result of a fall in price. Or better, it is the *compensating variation* in income, whose loss would just offset the fall in price and leave the consumer no better off than before." However, later, Hicks (1941-2; 127-128), following Henderson (1940-41; 120) defined (geometrically) the compensating variation as $e(u^1,p^1) - e(u^0,p^1)$ and the equivalent variation as $e(u^1,p^0) - e(u^0,p^0)$, which are measures of welfare (or quantity) change, and it is the latter terminology that we will follow in this chapter. The confusion between the two concepts is perhaps due to the fact that in his early definitions, Hicks assumed that money income was the same in the two situations being compared; i.e., he assumed $e(u^0,p^0) = e(u^1,p^1)$. Under these conditions, the compensating measure of price change, $e(u^0,p^1) - e(u^0,p^0)$ is equal to $e(u^0,p^1) - e(u^1,p^1)$, the negative of the compensating measure of quantity or utility change. In the index number literature, $e(u^0,p^1)/e(u^0,p^0)$ is known as the Laspeyres Konüs (1924) (1939; 17) *true cost of living index* or price index; see Pollak (1983) or Diewert (1983).

Comparing (7) and (8), it can be seen that C_P and P_P are both *measures of pure price change* going from period 0 to 1 but C_P uses a *difference* in the cost of the same base period basket of goods and services x^1, using the prices of periods 0 and 1, but P_P uses a *ratio* of the same costs.

Now consider the nth Hicksian demand function, $x_n(u^1,p_1^1,\ldots,p_{n-1}^1,p_n, p_{n+1}^1,\ldots, p_N^1)$, where utility is held constant at the period 1 level u^1 and all prices except the nth are held fixed at their period 1 levels. This Hicksian demand function for commodity n is graphed in Figure 1 below.[4]

Figure 2: The Period 1 Hicksian Demand Function for Commodity n



The nth term in the Paasche measure of compensation for the price change going from period 0 to 1, $(p_n^1 - p_n^0)x_n^1$, can be represented by the rectangle $p_n^0DCp_n^1$ in Figure 2. The amount x_n^1 is equal to the period 1 demand for commodity n; i.e., $x_n^1 = x_n(u^1,p_1^1,\ldots,p_{n-1}^1,p_n^1,p_{n+1}^1,\ldots, p_N^1)$, while the point x_n^0* is equal to $x_n(u^1,p_1^1,\ldots,p_{n-1}^1,p_n^0, p_{n+1}^1,\ldots, p_N^1)$. However, $(p_n^1 - p_n^0)x_n^1$ *understates* the amount of money that is required to keep the consumer at the period 1 welfare level u^1 since it does not take into account the fact that the consumer will consume more of commodity n as its price decreases; i.e., consumer substitution effects are not taken into account.

To see that the Paasche measure of compensation for the price change going from period 0 to 1, C_P defined by (7), is an *underestimate* of the amount of compensation required to

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[4] Note that this Hicksian demand function differs from that illustrated in Figure 1 for two reasons: (1) it holds constant a different utility level, u^0 for the Hicksian demand function $x_n(u^0,p_1^0,\ldots,p_{n-1}^0,p_n, p_{n+1}^0,\ldots, p_N^0)$ and u^1 for the Hicksian demand function $x_n(u^1,p_1^1,\ldots,p_{n-1}^1,p_n, p_{n+1}^1,\ldots, p_N^1)$ and (2) it holds constant different prices for commodities 1,…n−1,n+1,…,N.

keep the consumer at the period 1 level of utility when facing the prices of period 0, we note that the *analytically correct measure of the required compensation* for this amount of money is:

$$
\begin{aligned}
(9)\ e(u^1,p^1) - e(u^1,p^0) &= p^{1T}x^1 - e(u^1,p^0) && \text{using (2) for } t = 1 \\
&= p^{1T}x^1 - \min_x \{p^{0T}x : f(x) \geq u^1 \equiv f(x^1)\} \\
&\geq p^{1T}x^1 - p^{0T}x^1 && \text{since } x^1 \text{ is feasible for the min problem[5]} \\
&= (p^1 - p^0)^T x^1 \\
&\equiv C_P(p^0,p^1,x^0,x^1) && \text{using definition (7).}
\end{aligned}
$$

Again, if consumer preferences are of the Leontief or no substitution variety or if the period 0 price vector p^0 is proportional to the period 1 price vector, then the inequality in (9) becomes an equality and C_P will be the correct amount of compensation that is required to compensate the consumer for the price change in order to keep the household at the period 1 level of welfare. However, in general, C_P will be too small.

Recall the analytically correct measure of compensation for the price change that keeps the consumer at the period 0 level of welfare, (6) above. We now show that this measure is related to certain areas under Hicksian demand curves. Starting off with definition (6) and assuming that the expenditure function is once differentiable with respect to commodity prices, we have:

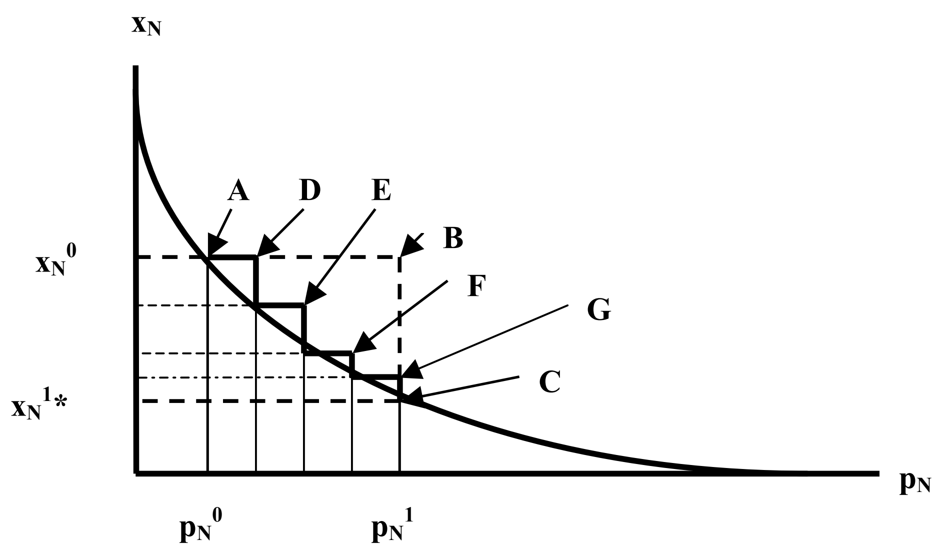
$$
\begin{aligned}
(10)\ e(u^0,p^1) - e(u^0,p^0) &= e(u^0,p_1^1,p_2^1,\ldots,p_N^1) - e(u^0,p_1^0,p_2^1,\ldots,p_N^1) \\
&\quad + e(u^0,p_1^0,p_2^1,p_3^1,\ldots,p_N^1) - e(u^0,p_1^0,p_2^0,p_3^1,\ldots,p_N^1) \\
&\quad + \ldots \\
&\quad + e(u^0,p_1^0,p_2^0,\ldots, p_{N-1}^0,p_N^1) - e(u^0,p_1^0,p_2^0,\ldots, p_{N-1}^0,p_N^0) \\
&= \int_{p_1^0}^{p_1^1} [\partial e(u^0,p_1,p_2^1, \ldots,p_N^1)/\partial p_1]dp_1 \\
&\quad + \int_{p_2^0}^{p_2^1} [\partial e(u^0,p_1^0,p_2,p_3^1\ldots,p_N^1)/\partial p_2]dp_2 \\
&\quad + \ldots \\
&\quad + \int_{p_N^0}^{p_N^1} [\partial e(u^0,p_1^0,p_2^0,\ldots, p_{N-1}^0,p_N)/\partial p_N]dp_N \\
&= \int_{p_1^0}^{p_1^1} x_1(u^0,p_1,p_2^1, \ldots,p_N^1)dp_1 \\
&\quad + \int_{p_2^0}^{p_2^1} x_2(u^0,p_1^0,p_2,p_3^1\ldots,p_N^1)dp_2 \\
&\quad + \ldots \\
&\quad + \int_{p_N^0}^{p_N^1} x_N(u^0,p_1^0,p_2^0,\ldots, p_{N-1}^0,p_N)dp_N
\end{aligned}
$$

[5] Hence $\min_x \{p^{0T}x: f(x) \geq f(x^1)\} \leq p^{0T}x^1$ but $-\min_x \{p^{0T}x: f(x) \geq f(x^1)\} \geq -p^{0T}x^1$.

where the x_n are various Hicksian demand functions. The N integrals in (10) can be interpreted as areas under the various Hicksian demand functions $x_n(u^0,p_1^0,p_2^0,\ldots,p_{n-1}^0,p_n,p_{n+1}^1\ldots,p_N^1)$ where only one price p_n in the nth integral varies between the period 0 level, p_n^0, and the period 1 level, p_n^1.

The last integral on the right hand side of (10) is illustrated in Figure 3 below. The amount x_N^0 is equal to the initial period 0 demand for commodity 1; i.e., $x_N^0 = x_N(u^0,p_1^0,p_2^0,\ldots,p_{N-1}^0,p_N^0)$ while the point x_N^{1*} is equal to $x_N(u^0,p_1^0,p_2^0,\ldots,p_{N-1}^0,p_N^1)$.[6] Obviously the integral of $x_N(u^0,p_1^0,p_2^0,\ldots,p_{N-1}^0,p_N)$ with respect to p_N between the limits p_N^0 and p_N^1 is equal to the area under the Nth Hicksian demand curve, regarded as a function of p_N, in Figure 3. A discrete approximation to this area is equal to the staircase area bounded by p_N^0ADEFGp_n^1. Note that the formula for the analytically correct measure of price change, $e(u^0,p^1) - e(u^0,p^0)$, given by (10) is not particularly useful unless we have estimated the consumer's system of Hicksian demand functions.

Figure 3: The Period 0 Hicksian Demand Function for Commodity N



It is possible to make a very rough approximation to the areas under the N Hicksian demand curves that are in (10). Referring again to Figure 3, it can be seen that an approximation to the area p_N^0ACp_N^1 is $(1/2)[x_N^0 + x_N^{1}*][p_N^1 - p_N^0]$ and this area can be further approximated by $(1/2)[x_N^0 + x_N^1][p_N^1 - p_N^0]$. If we approximate the other integrals in (10) by similar approximations, we obtain the following the following *Bennet* (1920; 456-457) *measure of compensation for the price change*:

[6] Note that $x_N(u^0,p_1^0,\ldots p_{N-1}^0,p_N^1)$ is not observable although it is sometimes approximated by the observable consumption of commodity N in period 1, x_N^1, particularly in the case where only the price of commodity N changes going from 0 to 1. Thus formula (10) is not an empirically useful formula; i.e., the N integrals in (10) cannot be evaluated unless empirical estimates of the consumer's Hicksian demand functions are available but if such estimates are available, then we also have empirical estimates of the consumer's expenditure function and we can readily evaluate $e(u^0,p^1) - e(u^0,p^0)$ without performing any integration.

(11) $C_B(p^0,p^1,x^0,x^1) \equiv (1/2)[x^0 + x^1]^T(p^1 - p^0)$.

It can be seen that the Bennet measure of compensation for the price changes is equal to the arithmetic average of the Laspeyres and Paasche measures of compensation.

We now convert the Laspeyres and Paasche measures of price change into measures of welfare change.

A measure of welfare change going from period 0 to 1 is the household's actual income change, $p^{1T}x^1 - p^{0T}x^0$, minus the Laspeyres measure of the amount of money required to compensate the household for the price changes going from period 0 to 1. Thus define the following *Paasche variation measure of welfare change* for the consumer[7] as follows:

$$\begin{aligned}(12)\ V_P(p^0,p^1,x^0,x^1) &\equiv p^{1T}x^1 - p^{0T}x^0 - C_L(p^0,p^1,x^0,x^1) \\ &= p^{1T}x^1 - p^{0T}x^0 - [p^{1T}x^0 - p^{0T}x^0] && \text{using definition (3) for } C_L \\ &= p^{1T}x^1 - p^{1T}x^0 \\ &= p^{1T}(x^1 - x^0).\end{aligned}$$

An alternative measure of welfare change going from period 0 to 1 is the household's actual income change, $p^{1T}x^1 - p^{0T}x^0$, minus the Paasche measure of the welfare cost of the price changes going from period 0 to 1. Thus define the following *Laspeyres variation measure of welfare change* for the consumer[8] as follows:

$$\begin{aligned}(13)\ V_L(p^0,p^1,x^0,x^1) &\equiv p^{1T}x^1 - p^{0T}x^0 - C_P(p^0,p^1,x^0,x^1) \\ &= p^{1T}x^1 - p^{0T}x^0 - [p^{1T}x^1 - p^{0T}x^1] && \text{using definition (7) for } C_P \\ &= p^{0T}x^1 - p^{0T}x^0 \\ &= p^{0T}(x^1 - x^0).\end{aligned}$$

The Paasche and Laspeyres variation measures of ex post welfare change, V_P and V_L defined by (12) and (13), are not analytically exact measures since they are based on the

[7] Hicks (1941-2; 127-128) defined this variation but it also had been considered earlier as a measure of efficiency or aggregate quantity change in the business management and industrial engineering literature. It is a bit confusing to define (11) as the Paasche variation V_P since it matches up with the Laspeyres measure of price change, C_L. However, we are following the conventions used in index number theory where the Paasche quantity index is defined as $Q_P(p^0,p^1,x^0,x^1) \equiv p^{1T}x^1/p^{1T}x^0$. Thus (11) is the difference counterpart to the Paasche ratio definition of quantity change used in index number theory. Diewert (2005; 319) called the measure of quantity change (12) the *Paasche indicator of quantity change*.

[8] Hicks (1941-2; 127-128) defined this variation but it also had been considered earlier as a measure of aggregate quantity change or efficiency change in the early industrial engineering literature by Harrison (1918; 393). Again, it is a bit confusing to define (13) as the Laspeyres variation V_L since it matches up with the Paasche measure of price change, C_P. However, we are following the conventions used in index number theory where the Laspeyres quantity index is defined as $Q_L(p^0,p^1,x^0,x^1) \equiv p^{0T}x^1/p^{0T}x^0$. Thus (13) is the difference counterpart to the Laspeyres ratio definition of quantity change used in index number theory. Diewert (2005; 319) called the measure of quantity change (13) the *Laspeyres indicator of quantity change*.

assumption of no consumer substitution effects.[9] Analytically correct measures of welfare change, using money metric utility scaling, are the *equivalent variation* V_E and the *compensating variation* V_C defined as follows:[10]

(14) $V_E(u^0,u^1,p^0) \equiv e(f(x^1),p^0) - e(f(x^0),p^0) = e(u^1,p^0) - e(u^0,p^0)$;
(15) $V_C(u^0,u^1,p^1) \equiv e(f(x^1),p^1) - e(f(x^0),p^1) = e(u^1,p^1) - e(u^0,p^1)$.

Thus the equivalent variation is equal to money metric utility change using the period 0 prices p^0 as reference prices while the compensating variation is equal to money metric utility change using the period 1 prices p^1 as reference prices.[11]

The equivalent variation V_E defined by (14) is related to the Laspeyres variation V_L defined by (13) as follows:

$$
\begin{aligned}
(16)\ V_L(p^0,p^1,x^0,x^1) &= p^{0T}x^1 - p^{0T}x^0 && \text{using definition (13)}\\
&= p^{0T}x^1 - e(u^0,p^0) && \text{using (2) for } t = 0\\
&\geq \min_x \{p^{0T}x;\ f(x) \geq f(x^1)\} - e(u^0,p^0) && \text{since } x^1 \text{ is feasible for the}\\
& && \text{the minimization problem}\\
&= e(u^1,p^0) - e(u^0,p^0) && \text{since } u^1 \equiv f(x^1)\\
&\equiv V_E(u^0,u^1,p^0) && \text{using definition (14).}
\end{aligned}
$$

Thus the observable Laspeyres variation is an upper bound to the unobservable equivalent variation.

The compensating variation V_C defined by (15) is related to the Paasche variation V_P defined by (12) as follows:

$$
\begin{aligned}
(17)\ V_P(p^0,p^1,x^0,x^1) &= p^{1T}x^1 - p^{1T}x^0 && \text{using definition (12)}\\
&= e(u^1,p^1) - p^{1T}x^0 && \text{using (2) for } t = 1\\
&\leq e(u^1,p^1) - \min_x \{p^{0T}x;\ f(x) \geq f(x^0)\} && \text{since } x^0 \text{ is feasible for the}\\
&\text{the minimization problem and hence } p^{1T}x^0 \geq \min_x \{p^{0T}x;\ f(x) \geq f(x^0)\}\\
&= e(u^1,p^1) - e(u^0,p^1) && \text{since } u^0 \equiv f(x^0)\\
&\equiv V_C(u^0,u^1,p^0) && \text{using definition (15).}
\end{aligned}
$$

Thus the observable Paasche variation is an lower bound to the unobservable compensating variation.[12]

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[9] However these measures do have the advantage that they can be evaluated using observable data on household choices for the two periods under consideration.
[10] Henderson (1940-41; 120) introduced these variations in the N = 2 case and Hicks (1941-42) introduced them in the general case. The terms compensating and equivalent variation are due to Hicks (1940-41;110) (1941-42; 128) but as we indicated above, there is some ambiguity as to the exact meaning that Hicks had in mind.
[11] Note that the hypothetical expenditure $e(u^1,p^0)$ in (14) is not observable while the hypothetical expenditure $e(u^0,p^1)$ in (15) is not observable.

Since the Laspeyres variation is an overestimate of a theoretically correct measure of welfare change and the Paasche variation is an underestimate of another theoretically correct measure of welfare change, this suggests taking a average of these two observable variations in order to obtain a less biased measure of welfare change. This leads us to the *Bennet* (1920; 457) *measure of welfare change*:

$$(18)\ V_B(p^0,p^1,x^0,x^1) = (1/2)V_L(p^0,p^1,x^0,x^1) + (1/2)V_P(p^0,p^1,x^0,x^1)$$
$$= (1/2)p^{1T}(x^1 - x^0) + (1/2)p^{0T}(x^1 - x^0)$$
$$\equiv (1/2)[p^1 + p^0]^T(x^1 - x^0)$$
$$= \sum_{n=1}^N (1/2)(p_n^0 + p_n^1)(x_n^1 - x_n^0).$$

By reversing the role of prices and quantities, Bennet also defined a companion *indicator of price change* as follows:[13]

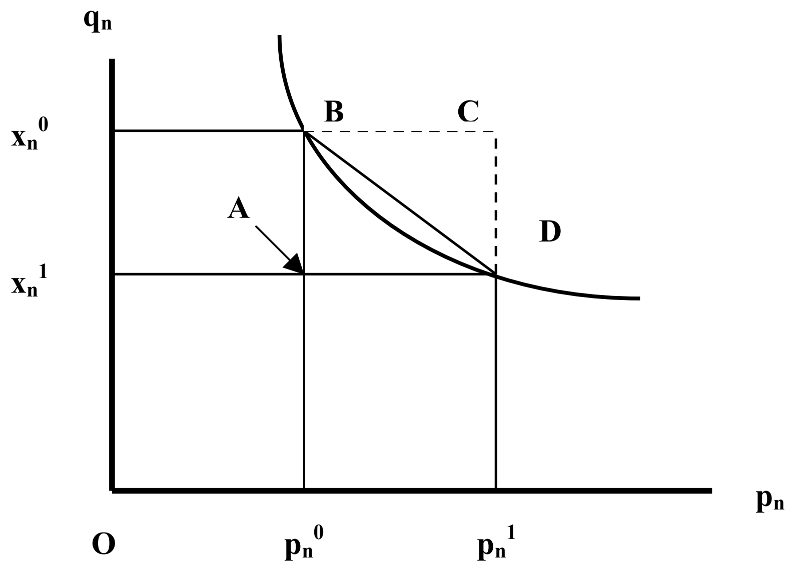
$$(19)\ I_B(p^0,p^1,x^0,x^1) \equiv (1/2)[x^1 + x^0]^T(p^1 - p^0)$$
$$= \sum_{n=1}^N (1/2)(x_n^0 + x_n^1)(p_n^1 - p_n^0).$$

Bennet (1920; 456-457) justified his measure of welfare change as a linear approximation to the area under an inverse demand curve and his price indicator as a linear approximation to an area under a demand curve. Hence, Bennet was following the partial equilibrium consumer surplus approach of Dupuit (1844) and Marshall (1898; 203). The geometry of the Bennet measures of price and quantity change for an increase in the price of commodity n is illustrated in Figure 4 for the case where commodity n increases from p_n^0 to a higher price p_n^1.

[12] Counterparts to the inequalities (16) and (17) in the index number context were first obtained by Konüs (1924) (1939; 17). Hicks (1941-42; 129) obtained the inequalities (16) and (17), except that he reversed the names for the Paasche and Laspeyres variations as compared to our terminology.

[13] Note that I_B is equal to C_B defined earlier by (11).

Figure 4: The Bennet Measures of Price and Welfare Change



The Bennet measures of price and quantity change can be given a simple interpretation that does not depend on economic theory. Obviously, the rectangle $p_n^0ADp_n^1$ represents a change in value going from period 0 to 1 that can be attributed to price change since the area of this rectangle is equal to the minimum of x_n^1 and x_n^0 times the difference in the prices. However, this estimate is a lower bound to the amount of money that can be attributed to price change since it excludes the rectangle ABCD, which represents the change in expenditures due to the combined effects of price change and quantity change; i.e., the area of this rectangle is equal to the absolute value of $(p_n^1-p_n^0)(x_n^1-x_n^0)$. The Bennet measure of price change takes one half of this amount and attributes it to price change while the other half is attributed to quantity change. Thus the Bennet measure of price change is equal to the area of the trapezoid $p_n^0BDp_n^1$. Similarly, the rectangle $x_n^1ABx_n^0$ represents a change in value going from period 0 to 1 that can be attributed to quantity change since the area of this rectangle is equal to the minimum of p_n^1 and p_n^0 times the difference in the quantities. Again, this estimate is a lower bound to the amount of money that can be attributed to quantity change since it excludes the rectangle ABCD, which represents the change in expenditures due to the combined effects of price change and quantity change. The Bennet measure of quantity change takes one half of this amount and attributes it to quantity change. Thus the Bennet measure of quantity change is equal to the area of the trapezoid $x_n^1DBx_n^0$. The amount of money spent on commodity n during period 0 is $p_n^0x_n^0$ which is the area of the rectangle $Op_n^0Bx_n^0$ and the amount of money spent on commodity n during period 1is $p_n^1x_n^1$which is the area of the rectangle $Op_n^1Dx_n^1$. The difference in these areas, $p_n^1x_n^1 - p_n^0x_n^0$, is equal to the Bennet measure of price change, $x_n^1(p_n^1-p_n^0) - (1/2)(p_n^1-p_n^0)(x_n^1-x_n^0)$, (this is the area of the trapezoid $p_n^0BDp_n^1$ indexed with a positive sign) plus the Bennet measure of quantity change, $p_n^1(x_n^1-x_n^0) + (1/2)(p_n^1-p_n^0)(x_n^1-x_n^0)$, (this is the area of the trapezoid $x_n^1DBx_n^0$ indexed with a negative sign).

The Bennet measures of price and quantity change can also be given an economic interpretation. In Figure 4, the curved line through B and D can be thought of as a partial equilibrium demand curve with the period 0 point, (p_n^0, x_n^0), and the period 1 point, (p_n^1, x_n^1), being on this curve. Now approximate this demand curve by the straight line through B and D. The area below this approximate partial equilibrium demand curve between the points p_n^0 and p_n^1 is the Bennet measure of price change for commodity n and the area to the left of this approximate partial equilibrium demand curve between the points x_n^0 and x_n^1 is the Bennet measure of quantity change for commodity n.[14]

Thus the Bennet measure of quantity change can be regarded as a linear approximation to Dupuit's (1969; 280) partial equilibrium measure of utility change or to Marshall's (1898; 200-206) consumer surplus measure of utility change.

The problem with these measures of welfare change is that they are partial equilibrium in nature: exactly what is being held constant when we draw the N partial equilibrium demand curves? And how can we be certain that the observed x^0 and x^1 are on these demand curves? Thus in the following sections of this chapter, we will return to the analytically correct measures of welfare change defined by the equivalent and compensating variations and see if we can obtain adequate approximations to these measures.

Problems

1. Show that the Bennet indicator of price change defined by (19) is equal to the arithmetic average of the Laspeyres and Paasche indicators of price change, $C_L(p^0,p^1,x^0,x^1)$ defined by (3) and $C_P(p^0,p^1,x^0,x^1)$ defined by (7).

2. Show that the Bennet indicator of price change defined by (19) plus the Bennet measure of welfare change defined by (18) add up to the actual income change of the consumer; i.e., show that

(i) $p^{1T}x^1 - p^{0T}x^0 = V_B(p^0,p^1,x^0,x^1) + I_B(p^0,p^1,x^0,x^1)$.

3. Show that the Bennet measure of welfare change defined by (18) is equal to the following expression:[15]

(i) $V_B(p^0,p^1,x^0,x^1) = p^{0T}(x^1-x^0) + (1/2)[p^1-p^0]^T[x^1-x^0]$.

[14] In Figure 4, we have drawn the demand curve as a function of price, which is consistent with normal mathematical conventions where the independent variable is always put on the horizontal axis. Dupuit (1969; 280-283) followed this convention but Marshall (1898; 203) reversed this convention and put quantity on the horizontal axis and economists have followed his example ever since. Thus using this Marshallian convention for demand curves, the Bennet measure of quantity change becomes the area under the (linearized) demand curve and the Bennet measure of price change becomes the area to the left of the (linearized) demand curve.

[15] This formula for measuring welfare change has been derived by Hotelling (1938; 253-254), Hicks (1941-42; 134) (1945-46; 73) and Harberger (1971; 788).

4.Consider the following *revealed preference* table:[16]

	$p^{0T}x^1 > p^{0T}x^0$	$p^{0T}x^1 = p^{0T}x^0$	$p^{0T}x^1 < p^{0T}x^0$
$p^{1T}x^0 > p^{1T}x^1$:	Zone of indeterminancy	x^0 revealed preferred to x^1	x^0 revealed preferred to x^1
$p^{1T}x^0 = p^{1T}x^1$:	x^1 revealed preferred to x^0	x^1 and x^0 revealed to be equivalent	x^0 revealed preferred to x^1
$p^{1T}x^0 < p^{1T}x^1$:	x^1 revealed preferred to x^0	x^1 revealed preferred to x^0	inconsistent preferences have been revealed

(a) Evaluate the Bennet measure of welfare change for each of the 9 cases in the above table and determine whether it correctly indicates the direction of welfare change.

(b) If the data are in the zone of indeterminacy, show that you can make the Bennet measure of welfare change positive or negative by scaling the prices in period 0. (This indicates a major problem with the Bennet indicator: it is not invariant to the scaling of prices in either period. Harberger (1971; 793) realized that this was a problem with the use of the Bennet measure and he suggested dividing prices in each period by the corresponding expenditure in each period and then using the resulting normalized prices in place of the original prices.)

3. Taylor Series Approximations to the Compensating and Equivalent Variations

Recall definitions (14) and (15), which defined the equivalent and compensating variation measures of welfare change. Let us start off with the definition of equivalent variation defined by (14) and approximate the unobservable term $e(u^1,p^0)$ by a certain first order approximation:

$$
\begin{aligned}
(20)\quad V_E(u^0,u^1,p^0) &\equiv e(u^1,p^0) - e(u^0,p^0) \\
&= -\, p^{0T}x^0 + e(u^1,p^0) && \text{using (2) for } t = 0 \\
&\cong -\, p^{0T}x^0 + e(u^1,p^1) + \nabla_p e(u^1,p^1)^T(p^0 - p^1) \\
&= -\, p^{0T}x^0 + p^{1T}x^1 + x^{1T}(p^0 - p^1) && \text{using Shephard's Lemma} \\
& && \text{and (2) for } t = 1 \\
&= p^{0T}[x^1 - x^0].
\end{aligned}
$$

Thus a first order approximation to the equivalent variation is the observable Laspeyres variation defined by (16), which we earlier showed is an upper bound to the equivalent variation.

[16] Revealed preference theory dates back to Pigou (1924; 53-60), Konüs (1939; 15) and Samuelson (1947; 146-149). This problem is based on results in Diewert (1976a) and Weymark and Vartia (1981).

In a similar manner, start off with the definition of compensating variation defined by (15) and approximate the unobservable term $e(u^0,p^1)$ by a certain first order approximation:

$$\begin{aligned}
(21)\ V_C(u^0,u^1,p^1) &\equiv e(u^1,p^1) - e(u^0,p^1) \\
&= p^{1T}x^1 - e(u^0,p^1) && \text{using (2) for } t = 1 \\
&\cong p^{1T}x^1 - e(u^0,p^0) - \nabla_p e(u^0,p^0)^T(p^1 - p^0) \\
&= p^{1T}x^1 - p^{0T}x^0 - x^{0T}(p^1 - p^0) && \text{using Shephard's Lemma and (2) for } t = 0 \\
&= p^{1T}[x^1 - x^0].
\end{aligned}$$

Thus a first order approximation to the compensating variation is the observable Paasche variation defined by (12), which we earlier showed is a lower bound to the compensating variation.[17]

Instead of using first order approximations in (20) and (21), we could have used second order approximations, which we now undertake:[18]

$$\begin{aligned}
(22)\ V_E(u^0,u^1,p^0) &\equiv e(u^1,p^0) - e(u^0,p^0) \\
&\cong - p^{0T}x^0 + e(u^1,p^1) + \nabla_p e(u^1,p^1)^T(p^0 - p^1) + (1/2)[p^0 - p^1]^T\nabla_{pp}{}^2 e(u^1,p^1)[p^0 - p^1] \\
&= - p^{0T}x^0 + p^{1T}x^1 + x^{1T}(p^0 - p^1) + (1/2)[p^0 - p^1]^T S^1[p^0 - p^1] \\
&\qquad \text{using Shephard's Lemma and (2) for } t = 1 \text{ and defining } S^1 \equiv \nabla_{pp}{}^2 e(u^1,p^1) \\
&= p^{0T}[x^1 - x^0] + (1/2)[p^1 - p^0]^T S^1[p^1 - p^0].
\end{aligned}$$

In a similar manner, start off with the definition of compensating variation defined by (15) and approximate the unobservable term $e(u^0,p^1)$ by a certain second order approximation:

$$\begin{aligned}
(23)\ V_C(u^0,u^1,p^1) &\equiv e(u^1,p^1) - e(u^0,p^1) \\
&\cong p^{1T}x^1 - e(u^0,p^0) - \nabla_p e(u^0,p^0)^T(p^1 - p^0) - (1/2)[p^1 - p^0]^T\nabla_{pp}{}^2 e(u^0,p^0)[p^1 - p^0] \\
&= p^{1T}x^1 - p^{0T}x^0 - x^{0T}(p^1 - p^0) - (1/2)[p^1 - p^0]^T S^0[p^1 - p^0] \\
&\qquad \text{using Shephard's Lemma and (2) for } t = 0 \text{ and defining } S^0 \equiv \nabla_{pp}{}^2 e(u^0,p^0) \\
&= p^{1T}[x^1 - x^0] - (1/2)[p^1 - p^0]^T S^0[p^1 - p^0].
\end{aligned}$$

Comparing (22) and (23), it is tempting to assume that the substitution terms in these expressions are equal; i.e., it is tempting to assume that

$$(24)\ (1/2)[p^1 - p^0]^T S^1[p^1 - p^0] = (1/2)[p^1 - p^0]^T S^0[p^1 - p^0].$$

[17] The linear approximations (20) and (21) are essentially due to Hicks (1941-42; 134).

[18] These approximations are also due to Hicks (1946; 331). In his earlier work, Hicks (1941-42; 133-134), Hicks did not distinguish between S^0 and S^1; i.e., he simply defined a generalized substitution terms equal to $[p^1 - p^0]^T\nabla_{pp}{}^2 e(u,p)[p^1 - p^0]$.

If (24) were true, then we could take an arithmetic average of the equivalent and compensating variations and obtain a second order approximation to this average by taking the average of the Paasche and Laspeyres variations; i.e., if (24) held, then:

(25) $(1/2)V_E(u^0,u^1,p^0) + (1/2)V_C(u^0,u^1,p^1)$
$\cong (1/2)\{p^{0T}[x^1 - x^0] + (1/2)[p^1 - p^0]^T S^1[p^1 - p^0] + p^{1T}[x^1 - x^0] - (1/2)[p^1 - p^0]^T S^0[p^1 - p^0]\}$
$= (1/2)\{p^{0T}[x^1 - x^0] + p^{1T}[x^1 - x^0]\}$ using (24)
$\equiv V_B(p^0,p^1,x^0,x^1)$ using definition (18).

Thus if (24) held, an arithmetic average of the equivalent and compensating variations could be approximated to the second order by the Bennet measure of welfare change.

Unfortunately, the equality (24) may be far from being satisfied. The problem is that the elements of each substitution matrix, S^0 and S^1, are homogeneous of degree minus 1 in p; i.e., the elements of $\nabla_{pp}{}^2 e(u^t,p^t)$ are homogeneous of degree -1 for each t; see problem 5 below. Thus by scaling prices in one of the two periods, we can make the left and right hand sides of (24) differ by arbitrarily large amounts. This again illustrates the weakness of the Bennet measure of welfare change.

Problem

5. Let f(x) be a twice continuously differentiable positive function of N variables, defined for $x >> 0_N$. Suppose also that f is positively homogeneous of degree one so that

(i) $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$ and all $x >> 0_N$.

(a) Show that $f_n(x) \equiv \partial f(x)/\partial x_n$ is (positively) homogeneous of degree 0 for n = 1,…,N; i.e., show that

(ii) $f_n(\lambda x) = f_n(x)$ for all $\lambda > 0$ and all $x >> 0_N$.

(b) Show that $f_{nm}(x) \equiv \partial^2 f(x)/\partial x_n \partial x_m$ is (positively) homogeneous of degree -1 for all n and m, i.e., show that

(iii) $f_{nm}(\lambda x) = \lambda^{-1} f_{nm}(x)$ for all $\lambda > 0$ and all $x >> 0_N$.

4. Weitzman's Approach to the Treatment of General Inflation between Periods

As we have seen in the previous two sections, a major problem with the Bennet measure of ex post welfare change is that it is not invariant to changes in general inflation that might take place between the two periods under consideration. Weitzman (1988) addressed this issue using some new techniques for the measurement of welfare change. Diewert (1992) reworked Weitzman's analysis using duality theory and we will review and extend their work in this section.

As usual, we assume that a household has the utility function $f(x)$ defined for $x \geq 0_N$ and the corresponding dual expenditure function, $e(u,p)$, is twice continuously differentiable with respect to its variables. We assume that we observe the household's price and quantity vectors, p^t and x^t for $t = 0,1$ and that the household minimizes the cost of achieving its utility level u^t in each period t so that we have:

$$(26)\ p^{tT}x^t = e(u^t,p^t) = e(f(x^t),p^t)\ ; \qquad t = 0,1.$$

We are going to form a second order approximation to the *equivalent variation*, $e(u^1,p^0) - e(u^0,p^0)$, which involves making a second order approximation to the unobservable term, $e(u^1,p^0)$. However, instead of making this approximation around the point (u^1,p^1) as we did in (22) above, we follow the example of Weitzman (1988) and make this approximation around the point $(u^1,\alpha p^1)$, where $\alpha > 0$ is a convenient scaling factor, which adjusts for the effects of inflation between periods 0 and 1 and will be chosen later. For later reference, we note that the fact that $e(u^1,p)$ is homogeneous of degree one in the components of p implies the following equations:

$$(27)\quad e(u^1,\alpha p^1) = \alpha e(u^1,p^1) = \alpha p^{1T}x^1 \qquad \text{using (26) for } t = 1;$$
$$(28)\quad \nabla_p e(u^1,\alpha p^1) = \nabla_p e(u^1,p^1) = x^1 \qquad \text{using Shephard's Lemma;}$$
$$(29)\ \nabla_{pp}{}^2 e(u^1,\alpha p^1) = \alpha^{-1}\nabla_{pp}{}^2 e(u^1,p^1) = \alpha^{-1}S^1\ ;$$
$$(30)\ \nabla_{pu}{}^2 e(u^1,\alpha p^1) = \nabla_{pu}{}^2 e(u^1,p^1).$$

Approximate $e(u^1,p^0)$ to the second order around the point $(u^1,\alpha p^1)$:

$$\begin{aligned}(31)\ e(u^1,p^0) &\cong e(u^1,\alpha p^1) + \nabla_p e(u^1,\alpha p^1)^T(p^0 - \alpha p^1) + (1/2)[p^0 - \alpha p^1]^T\nabla_{pp}{}^2 e(u^1,\alpha p^1)[p^0 - \alpha p^1] \\ &= \alpha p^{1T}x^1 + x^{1T}(p^0 - \alpha p^1) + (1/2)[p^0 - \alpha p^1]^T\nabla_{pp}{}^2 e(u^1,\alpha p^1)[p^0 - \alpha p^1] \\ &\qquad \text{using (27) and (28)} \\ &= p^{0T}x^1 + (1/2)[p^0 - \alpha p^1]^T\nabla_{pp}{}^2 e(u^1,\alpha p^1)[p^0 - \alpha p^1].\end{aligned}$$

Approximate $\nabla_p e(u^0,p^0)$ to the first order around the point $(u^1,\alpha p^1)$:

$$(32)\ \nabla_p e(u^0,p^0) \cong \nabla_p e(u^1,\alpha p^1) + \nabla_{pp}{}^2 e(u^1,\alpha p^1)(p^0 - \alpha p^1) + \nabla_{pu}{}^2 e(u^1,\alpha p^1)[u^0 - u^1].$$

Premultiply both sides of (32) by $[p^0 - \alpha p^1]^T$ and solve the resulting equation for the quadratic term in prices:[19]

$$\begin{aligned}(33)\ [p^0 - \alpha p^1]^T\nabla_{pp}{}^2 e(u^1,\alpha p^1)[p^0 - \alpha p^1] &\cong [p^0 - \alpha p^1]^T\nabla_p e(u^0,p^0) - [p^0 - \alpha p^1]^T\nabla_p e(u^1,\alpha p^1) \\ &\quad - [p^0 - \alpha p^1]^T\nabla_{pu}{}^2 e(u^1,\alpha p^1)[u^0 - u^1] \\ &= [p^0 - \alpha p^1]^T x^0 - [p^0 - \alpha p^1]^T x^1 - [p^0 - \alpha p^1]^T\nabla_{pu}{}^2 e(u^1,\alpha p^1)[u^0 - u^1] \qquad \text{using (28).}\end{aligned}$$

[19] The right hand side of (33) is a second order approximation to the left hand side of (33) around the point $(u^1,\alpha p^1)$ in the differences $u^0 - u^1$ and $p^0 - \alpha p^1$.

Weitzman's (1988; 550) strategy was to choose the inflation adjustment factor α to make the last term on the right hand side of (33) equal to 0.[20] However, in order to do this, we need to know the consumer's real income derivatives,

(34) $\nabla_{pu}{}^2 e(u^1,\alpha p^1) = \nabla_{pu}{}^2 e(u^1,p^1) = \partial x(u^1,p^1)/\partial u$

around the period 1 consumption vector x^1. However, *if we assume that the consumer's preferences are homothetic*, then we will be able to determine these income derivatives as we now show. If preferences are homothetic, then we can rescale the consumer's utility function f to be linearly homogeneous. In this case, the dual expenditure function has the following decomposition:

(35) $e(u,p) = ue(1,p)$.

Thus the consumer's demand vector x^1 for period 1 is equal to:

(36)
$$\begin{aligned} x^1 &= \nabla_p e(u^1,p^1) && \text{using Shephard's Lemma} \\ &= u^1\nabla_p e(1,p^1) && \text{using (35)} \\ &= u^1\nabla_p e(1,\alpha p^1) && \text{using (27).} \end{aligned}$$

Differentiating the right hand side of (36) with respect to u leads to the following equation:

(37)
$$\begin{aligned} \nabla_{pu}{}^2 e(u^1,p^1) &= \nabla_p e(1,p^1) && \text{using (36)} \\ &= x^1/u^1 && \text{using (36) again.} \end{aligned}$$

In what follows, we will assume homothetic preferences so that (37) holds. Now substitute (37) into the last term on the right hand side of (33) and choose α to make this last term equal to 0. This leads to the following equation for α:

(38)
$$\begin{aligned} [p^0 - \alpha p^1]^T[x^1/u^1][u^0 - u^1] &= 0 && \text{or} \\ [p^0 - \alpha p^1]^T x^1 &= 0 && \text{or} \end{aligned}$$

(39) $\alpha^* = 1/[p^{1T}x^1/p^{0T}x^1] = 1/P_P(p^0,p^1,x^0,x^1)$

where $P_P(p^0,p^1,x^0,x^1) \equiv p^{1T}x^1/p^{0T}x^1$ is the *Paasche price index* going from period 0 to 1. Thus assuming homothetic preferences and with α set equal to α^* defined by (39), we can drop the last term in equation (33). We now use the two second order approximation equations (31) and (33) in order to obtain a second order approximation to the equivalent variation.

Substituting our second order approximation (31) for $\alpha = \alpha^*$ into the definition of the equivalent variation leads to the following equations:

---

[20] Weitzman (1988; 549) used a mixture of primal and dual techniques to obtain his approximate measure of welfare change and so he did not follow this step exactly but the general idea is due to him.

$$
\begin{aligned}
(40)\ V_E(u^0,u^1,p^0) &\equiv e(u^1,p^0) - e(u^0,p^0) \\
&\cong e(u^1,\alpha^*p^1) + \nabla_p e(u^1,\alpha^*p^1)^T(p^0 - \alpha^*p^1) \\
&\quad + (1/2)[p^0 - \alpha^*p^1]^T\nabla_{pp}{}^2 e(u^1,\alpha^*p^1)[p^0 - \alpha^*p^1] - p^{0T}x^0 && \text{using (26) for } t = 0 \\
&= p^{0T}(x^1 - x^0) + (1/2)[p^0 - \alpha^*p^1]^T\nabla_{pp}{}^2 e(u^1,\alpha^*p^1)[p^0 - \alpha^*p^1] && \text{using (31)} \\
&\cong p^{0T}(x^1 - x^0) + (1/2)\{[p^0 - \alpha^*p^1]^T x^0 - [p^0 - \alpha^*p^1]^T x^1\} && \text{using (33) when } \alpha = \alpha^* \\
&= (1/2)[p^0 + \alpha^*p^1]^T[x^1 - x^0] \\
&\equiv V_B(p^0,\alpha^*p^1,x^0,x^1)
\end{aligned}
$$

where $V_B(p^0,\alpha^*p^1,x^0,x^1)$ is the Bennet indicator of welfare change defined by (18) above, except instead of using the period 1 price vector p^1 in the Bennet formula, we use the period 1 price vector deflated by the Paasche price index, $\alpha^*p^1 = p^1/[\, p^{1T}x^1/p^{0T}x^1]$. Thus a second order approximation to the equivalent variation is the following measure of welfare change first suggested by Diewert (1992; 571):[21]

$$
\begin{aligned}
(41)\ V_{DE}(p^0,p^1,x^0,x^1) &\equiv V_B(p^0,\alpha^*p^1,x^0,x^1) \\
&= (1/2)[p^0 + p^1/\{p^{1T}x^1/p^{0T}x^1\}]^T[x^1 - x^0].
\end{aligned}
$$

Weitzman (1988; 551) ended up suggesting the following measure as an approximation to the equivalent variation in the case where the consumer had homothetic preferences:

$$(42)\ V_{WE}(p^0,p^1,x^0,x^1) \equiv (1/2)[p^0 + p^1/\{p^{1T}x^0/p^{0T}x^0\}]^T[x^1 - x^0].$$

Comparing (42) with (41), it can be seen that the Weitzman measure of welfare change is also a Bennet type measure except that Weitzman suggested the Laspeyres price index as the deflator for the period 1 prices while Diewert suggested the Paasche price index as the deflator for the period 1 prices.

We conclude this section by trying to form a second order approximation to the *compensating variation*, $e(u^1,p^1) - e(u^0,p^1)$, which involves making a second order approximation to the unobservable term, $e(u^0,p^1)$. Obviously, we can repeat the above analysis with some obvious modifications. Instead of approximating $e(u^0,p^1)$ around the point (u^0,p^0) as we did in (23) above, we again follow the example of Weitzman (1988) and make this approximation around the point $(u^0,\beta p^0)$, where $\beta > 0$ is a convenient scaling factor, which adjusts for the effects of inflation between periods 0 and 1 and will be chosen later. The fact that $e(u^0,p)$ is homogeneous of degree one in the components of p implies the following equations:

$$
\begin{aligned}
&(43)\quad e(u^0,\beta p^0) = \beta e(u^0,p^0) && = \alpha p^{0T}x^0 && \text{using (26) for } t = 0; \\
&(44)\quad \nabla_p e(u^0,\beta p^0) = \nabla_p e(u^0,p^0) && = x^0 && \text{using Shephard's Lemma;}
\end{aligned}
$$

Approximate $e(u^0,p^1)$ to the second order around the point $(u^0,\beta p^0)$:

[21] Diewert (1992; 571) called this measure the Weitzman Paasche welfare change indicator.

$$(45)\ e(u^0,p^1) \cong e(u^0,\beta p^0) + \nabla_p e(u^0,\beta p^0)^T(p^1 - \beta p^0) + (1/2)[p^1 - \beta p^0]^T\nabla_{pp}{}^2 e(u^0,\beta p^0)[p^1 - \beta p^0]$$
$$= \beta p^{0T}x^0 + x^{0T}(p^1 - \beta p^0) + (1/2)[p^1 - \beta p^0]^T\nabla_{pp}{}^2 e(u^0,\beta p^0)[p^1 - \beta p^0] \quad \text{using (43) and (44)}$$
$$= p^{1T}x^0 + (1/2)[p^1 - \beta p^0]^T\nabla_{pp}{}^2 e(u^0,\beta p^0)[p^1 - \beta p^0].$$

Approximate $\nabla_p e(u^1,p^1)$ to the first order around the point $(u^0,\beta p^0)$:

$$(46)\ \nabla_p e(u^1,p^1) \cong \nabla_p e(u^0,\beta p^0) + \nabla_{pp}{}^2 e(u^0,\beta p^0)(p^1 - \beta p^0) + \nabla_{pu}{}^2 e(u^0,\beta p^0)[u^1 - u^0].$$

Premultiply both sides of (46) by $[p^1 - \beta p^0]^T$ and solve the resulting equation for the quadratic term in prices:

$$(47)\ [p^1 - \beta p^0]^T\nabla_{pp}{}^2 e(u^0,\beta p^0)[p^1 - \beta p^0] \cong [p^1 - \beta p^0]^T\nabla_p e(u^1,p^1) - [p^1 - \beta p^0]^T\nabla_p e(u^0,\beta p^0)$$
$$- [p^1 - \beta p^0]^T\nabla_{pu}{}^2 e(u^0,\beta p^0)[u^1 - u^0]$$
$$= [p^1 - \beta p^0]^T x^1 - [p^1 - \beta p^0]^T x^0 - [p^1 - \beta p^0]^T\nabla_{pu}{}^2 e(u^0,\beta p^0)[u^1 - u^0] \quad \text{using (44).}$$

Again, assuming that the consumer's preferences are homothetic, we can determine the income derivatives $\nabla_{pu}{}^2 e(u^0,\beta p^0) = \nabla_{pu}{}^2 e(u^0,p^0)$. Using the same techniques that we used to derive (37), we can obtain the following equations:

$$(48)\ \nabla_{pu}{}^2 e(u^0,\beta p^0) = \nabla_{pu}{}^2 e(u^0,p^0) = \nabla_p e(1,p^0) = x^0/u^0.$$

In what follows, we will assume homothetic preferences so that (48) holds. Now substitute (48) into the last term on the right hand side of (47) and choose β to make this last term equal to 0. This leads to the following equation for β:

$$(49)\ [p^1 - \beta p^0]^T[x^0/u^0][u^1 - u^0] = 0 \quad \text{or}$$
$$[p^1 - \beta p^0]^T x^0 = 0 \quad \text{or}$$

$$(50)\ \beta^* = [p^{1T}x^0/p^{0T}x^0] = P_L(p^0,p^1,x^0,x^1)$$

where $P_L(p^0,p^1,x^0,x^1) \equiv p^{1T}x^0/p^{0T}x^0$ is the *Laspeyres price index* going from period 0 to 1. Thus assuming homothetic preferences and with β set equal to β^* defined by (50), we can drop the last term in equation (47). We now use the two second order approximation equations (45) and (47) in order to obtain a second order approximation to the Compensating variation.

Substituting our second order approximation (45) for $\beta = \beta^*$ into the definition of the compensating variation leads to the following equations:

$$(51)\ V_C(u^0,u^1,p^1) \equiv e(u^1,p^1) - e(u^0,p^1)$$
$$\cong p^{1T}x^1 - \{ e(u^0,\beta^* p^0) + \nabla_p e(u^0,\beta^* p^0)^T(p^1 - \beta^* p^0)$$
$$+ (1/2)[p^1 - \beta^* p^0]^T\nabla_{pp}{}^2 e(u^0,\beta^* p^0)[p^1 - \beta^* p^0]\} \quad \text{using (26) for } t = 1$$
$$= p^{1T}x^1 - \{p^{1T}x^0 + (1/2)[p^1 - \beta^* p^0]^T\nabla_{pp}{}^2 e(u^0,\beta^* p^0)[p^1 - \beta^* p^0]\} \quad \text{using (45)}$$
$$= p^{1T}(x^1 - x^0) - (1/2)[p^1 - \beta^* p^0]^T\nabla_{pp}{}^2 e(u^0,\beta^* p^0)[p^1 - \beta^* p^0]\}$$

$$= p^{1T}(x^1 - x^0) - (1/2)\{[p^1 - \beta^* p^0]^T x^1 - [p^1 - \beta^* p^0]^T x^0\} \qquad \text{using (47) when } \beta = \beta^*$$
$$= (1/2)[\beta^* p^0 + p^1]^T[x^1 - x^0]$$
$$\equiv V_B(\beta^* p^0, p^1, x^0, x^1)$$

where $V_B(\beta^* p^0,p^1,x^0,x^1)$ is the Bennet indicator of welfare change defined by (18) above, except instead of using the period 0 price vector p^0 in the Bennet formula, we use the period 0 price vector times the Laspeyres price index, $\beta^* p^0 = [p^{1T}x^0/p^{0T}x^0]p^0$. Thus a second order approximation to the compensating variation is the following measure of welfare change:

$$(52)\ V_{DC}(p^0,p^1,x^0,x^1) \equiv V_B(\beta^* p^0,p^1,x^0,x^1)$$
$$= (1/2)[\{p^{1T}x^0/p^{0T}x^0\}p^0 + p^1]^T[x^1 - x^0].$$

Note that the Diewert approximation to the compensating variation, $V_{DC}(p^0,p^1,x^0,x^1)$ defined by (52), is related to the Weitzman (1988; 551) approximation to the equivalent variation as follows:

$$(53)\ V_{DC}(p^0,p^1,x^0,x^1) = (1/2)[\beta^* p^0 + p^1]^T[x^1 - x^0]$$
$$= \beta^*(1/2)[p^0 + p^1/\beta^*]^T[x^1 - x^0]$$
$$= \beta^* V_{WE}(p^0,p^1,x^0,x^1) \qquad \text{using definition (42)}$$
$$= P_L(p^0,p^1,x^0,x^1)V_{WE}(p^0,p^1,x^0,x^1).$$

Thus the Diewert approximation to the compensating variation, V_{DC}, is equal to the Laspeyres price index times the Weitzman approximation to the equivalent variation, V_{WE}.

The three measures of welfare change, V_{DE}, V_{DC} and V_{WE}, all show how the Bennet indicator of welfare change can be modified to deal adequately with the effects of inflation between the two periods being compared.

In the following section, we will suggest another second order approximation approach to the measurement of ex post welfare change, assuming that the household has homothetic preferences.

Problems

6. Recall that the Laspeyres and Paasche price indexes were defined as follows:

(i) $P_L(p^0,p^1,x^0,x^1) \equiv p^{1T}x^0/p^{0T}x^0$;
(ii) $P_P(p^0,p^1,x^0,x^1) \equiv p^{1T}x^1/p^{0T}x^1$.

It is obvious that these two indexes have the same value, 1, if they are evaluated at a point where the two price vectors are equal and where the two quantity vectors are equal, so that we have $p^0 = p^1 = p$ and $x^0 = x^1 = x$; i.e., we have:

(iii) $P_L(p,p,x,x) = P_P(p,p,x,x) = 1$.

(a) Show that $P_L(p^0,p^1,x^0,x^1)$ and $P_P(p^0,p^1,x^0,x^1)$ *approximate each other to the first order around an equal price and quantity point*; i.e., show that when $p^0 = p^1 = p$ and $x^0 = x^1 = x$, we have:

(iv) $\nabla_{p^0} P_L(p^0,p^1,x^0,x^1) = \nabla_{p^0} P_P(p^0,p^1,x^0,x^1)$;
(v) $\nabla_{p^1} P_L(p^0,p^1,x^0,x^1) = \nabla_{p^1} P_P(p^0,p^1,x^0,x^1)$;
(vi) $\nabla_{x^0} P_L(p^0,p^1,x^0,x^1) = \nabla_{x^0} P_P(p^0,p^1,x^0,x^1)$;
(vii) $\nabla_{x^1} P_L(p^0,p^1,x^0,x^1) = \nabla_{x^1} P_P(p^0,p^1,x^0,x^1)$.

7. Consider the Bennet, Weitzman and Diewert approximations to the equivalent variation as functions of p^0,p^1,x^0,x^1:

(i) $V_B(p^0,p^1,x^0,x^1) \equiv (1/2)[p^0 + p^1]^T[x^1 - x^0]$;
(ii) $V_{WE}(p^0,p^1,x^0,x^1) \equiv (1/2)[p^0 + \{P_L(p^0,p^1,x^0,x^1)\}^{-1}p^1]^T[x^1 - x^0]$;
(iii) $V_{DE}(p^0,p^1,x^0,x^1) \equiv (1/2)[p^0 + \{P_P(p^0,p^1,x^0,x^1)\}^{-1}p^1]^T[x^1 - x^0]$.

It is obvious that these three indicators of welfare change have the same value, 0, if they are evaluated at a point where the two price vectors are equal and where the two quantity vectors are equal, so that we have $p^0 = p^1 = p$ and $x^0 = x^1 = x$; i.e., we have:

(iv) $V_B(p,p,x,x) = V_{WE}(p,p,x,x) = V_{DE}(p,p,x,x) = 0$.

(a) Show that these three indicators of welfare change *approximate each other to the first order around an equal price and quantity point*.

(b) Show that V_{WE} and V_{DE} *approximate each other to the second order around an equal price and quantity point*. Hint: Define the difference f between these two indicators of welfare change as

(v) $f(p^0,p^1,x^0,x^1) \equiv V_{WE}(p^0,p^1,x^0,x^1) - V_{DE}(p^0,p^1,x^0,x^1)$
$= (1/2)[\{P_L(p^0,p^1,x^0,x^1)\}^{-1}p^1 - \{P_P(p^0,p^1,x^0,x^1)\}^{-1}p^1]^T[x^1 - x^0]$.

It is easy to see that $f(p,p,x,x) = 0$. Now show that

(vi) $\nabla f(p,p,x,x) = 0_{4N}$ and
(vii) $\nabla^2 f(p,p,x,x) = O_{4N\times 4N}$.

(c) Show that the Bennet indicator of welfare change, V_B, does *not* approximate V_{WE} and V_{DE} to the second order around an equal price and quantity point. Hint: Show that

(vii) $\nabla^2_{p^1x^1} V_B(p,p,x,x) \neq \nabla^2_{p^1x^1} V_{WE}(p,p,x,x)$.

This problem shows that if the amounts of price change and quantity change going from period 0 to 1 are very small, then the Bennet, Weitzman and Diewert indicators of welfare change will give much the same answer; i.e., the indicators will approximate each other to the first order. However, if there is a moderate amount of price or quantity change between the two periods, then the Bennet indicator will tend to diverge from the other two indicators. This problem again indicates the weakness of the Bennet indicator of welfare change if it is not adjusted for inflation between the two periods being considered.

8. Recall that the Diewert approximation to the compensating variation V_{DC} was defined by (52) as follows:

(i) $V_{DC}(p^0,p^1,x^0,x^1) \equiv V_B(\beta^* p^0,p^1,x^0,x^1) = (1/2)[\{p^{1T}x^0/p^{0T}x^0\}p^0 + p^1]^T[x^1 - x^0]$

where the scaling factor β^* for the period 0 price vector p^0 was chosen to be the Laspeyres price index $P_L \equiv p^{1T}x^0/p^{0T}x^0$. Suppose instead we chose the scaling factor β^* to be the Paasche price index $P_P \equiv p^{1T}x^1/p^{0T}x^1$. This leads to the following *Diewert Paasche approximation to the compensating variation*:[22]

(ii) $V_{DCP}(p^0,p^1,x^0,x^1) \equiv (1/2)[\{p^{1T}x^1/p^{0T}x^1\}p^0 + p^1]^T[x^1 - x^0]$.

(a) Show that the three indicators of welfare change, V_B, V_{DCP} and V_{DCL}, *approximate each other to the first order around an equal price and quantity point*.

(b) Show that V_{DCP} and V_{DCL} *approximate each other to the second order around an equal price and quantity point*.

(c) Show that the Bennet indicator of welfare change, V_B, does *not* approximate V_{DCP} and V_{DCL} to the second order around an equal price and quantity point.

5. The Fisher Quantity Index Approach to the Measurement of Welfare Change

Suppose that the consumer's utility function, f(x), is positive, nondecreasing, concave and linearly homogeneous for $x \geq 0_N$. Then as noted in the previous section, the dual expenditure function e satisfies for $u > 0$ and $p >> 0_N$:

(54) $e(u,p) = ue(1,p) = uc(p)$

where $c(p) \equiv e(1,p)$ is the *consumer's unit cost* (or expenditure) *function*. Assume that the consumer minimizes the cost of achieving his or her utility level in periods 0 and 1 and the observed price and quantity vectors for period t are $p^t >> 0_N$ and $x^t > 0_N$ for $t = 0,1$. Then using (54), the observed expenditures in each period satisfy the following equations:

(55) $p^{tT}x^t = e(u^t,p^t) = e(f(x^t),p^t) = f(x^t)c(p^t)$; $t = 0,1$.

---

[22] We could redefine $V_{DC}(p^0,p^1,x^0,x^1)$ defined by (52) as $V_{DCL}(p^0,p^1,x^0,x^1)$, the Diewert Laspeyres approximation to the compensating variation.

Using (55), the equivalent and compensating variations take on the following forms when preferences are linearly homogeneous:

$$(56)\ V_E(u^0,u^1,p^0) \equiv e(f(x^1),p^0) - e(f(x^0),p^0) = c(p^0)[f(x^1) - f(x^0)] ;$$
$$(57)\ V_C(u^0,u^1,p^1) \equiv e(f(x^1),p^1) - e(f(x^0),p^1) = c(p^1)[f(x^1) - f(x^0)].$$

With homogeneous preferences and a differentiable expenditure function, Shephard's Lemma implies the consumer's period t demand vector is equal to:

$$(58)\ x^t = \nabla_p e(u^t,p^t) = f(x^t)\nabla_p c(p^t) ; \qquad t = 0,1.$$

Now divide the left hand side of equation t in (58) by the left hand side of (55) for equation t and the right hand side of equation t in (58) by the right hand side of (55) for equation t and we obtain the following equations:

$$(59)\ x^t/p^{tT}x^t = f(x^t)\nabla_p c(p^t)/f(x^t)c(p^t) = \nabla_p c(p^t)/c(p^t) ; \qquad t = 0,1.$$

Now suppose that the consumer's unit cost function has the following functional form:

$$(60)\ c(p) \equiv [p^T Bp]^{1/2} \qquad B = B^T$$

where $B \equiv [b_{ij}]$ is an N by N symmetric matrix of parameters. Diewert (1976b; 130) showed that the c(p) defined by (60) is a *flexible functional form*; i.e., it can provide a second order approximation to an arbitrary twice continuously differentiable unit cost function. If we differentiate c(p) with respect to the components of p, we find:

$$(61)\ \nabla_p c(p) = Bp/c(p).$$

Substituting (61) into (59) gives us the following equations:

$$(62)\ x^t/p^{tT}x^t = Bp^t/[c(p^t)]^2 ; \qquad t = 0,1.$$

Irving Fisher (1922) defined his *ideal price index* P_F as the geometric mean of the Paasche and Laspeyres price indexes defined earlier; i.e., define

$$\begin{aligned}(63)\ P_F(p^0,p^1,x^0,x^1) &\equiv [P_L(p^0,p^1,x^0,x^1)P_P(p^0,p^1,x^0,x^1)]^{1/2} \\ &= [\{p^{1T}x^0/p^{0T}x^0\}\{p^{1T}x^1/p^{0T}x^1\}]^{1/2} \\ &= [[\{p^{1T}x^0/p^{0T}x^0\}/\{p^{0T}x^1/p^{1T}x^1\}]^{1/2} \\ &= [\{p^{1T}Bp^0/[c(p^0)]^2\}/\{p^{0T}Bp^1/[c(p^1)]^2\}]^{1/2} && \text{using (62)} \\ &= [c(p^1)^2/c(p^0)^2]^{1/2} && \text{using } B = B^T \\ &= c(p^1)/c(p^0).\end{aligned}$$

Thus if preferences are homothetic and the consumer's unit cost function c(p) is defined by the flexible functional form (60), then the cost function ratio, $c(p^1)/c(p^0)$,[23] is exactly equal to the Fisher ideal price index, $P_F(p^0,p^1,x^0,x^1)$, which can be calculated using observable price and quantity data pertaining to periods 0 a 1 for the consumer.[24]

The Fisher (1922) ideal quantity index Q_F can be defined as the value ratio for the two periods divided by the Fisher ideal price index; i.e., we have:

$$
\begin{aligned}
(64)\ Q_F(p^0,p^1,x^0,x^1) &\equiv [p^{1T}x^1/p^{0T}x^0]/P_F(p^0,p^1,x^0,x^1) \\
&= [f(x^1)c(p^1)/f(x^0)c(p^0)]/P_F(p^0,p^1,x^0,x^1) && \text{using (55)} \\
&= [f(x^1)c(p^1)/f(x^0)c(p^0)]/c(p^1)/c(p^0) && \text{using (63)} \\
&= f(x^1)/f(x^0).
\end{aligned}
$$

Thus if preferences are homothetic and the consumer's unit cost function c(p) is defined by the flexible functional form (60), then the consumer's utility ratio, $f(x^1)/f(x^0)$, is exactly equal to the Fisher ideal quantity index, $Q_F(p^0,p^1,x^0,x^1)$, which can be calculated using observable price and quantity data pertaining to periods 0 and 1 for the consumer.

Now we are ready to calculate the consumer's equivalent and compensating variations under the assumption that preferences are homothetic and the consumer's unit cost function is defined by (60). Using (56), the equivalent variation may be written as follows:

$$
\begin{aligned}
(65)\ V_E(u^0,u^1,p^0) &= c(p^0)[f(x^1) - f(x^0)] \\
&= c(p^0)f(x^0)[\{f(x^1)/f(x^0)\} - 1] \\
&= p^{0T}x^0[\{f(x^1)/f(x^0)\} - 1] && \text{using (55) for } t = 0 \\
&= p^{0T}x^0[Q_F(p^0,p^1,x^0,x^1) - 1] && \text{using (64).}
\end{aligned}
$$

Using (57), the compensating variation may be written as follows:

$$
\begin{aligned}
(66)\ V_C(u^0,u^1,p^1) &= c(p^1)[f(x^1) - f(x^0)] \\
&= c(p^1)f(x^1)[1 - \{f(x^1)/f(x^0)\}^{-1}] \\
&= p^{1T}x^1[1 - \{f(x^1)/f(x^0)\}^{-1}] && \text{using (55) for } t = 1 \\
&= p^{1T}x^1[1 - Q_F(p^0,p^1,x^0,x^1)^{-1}] && \text{using (64).}
\end{aligned}
$$

Thus under assumption (60), we have exact representations for the equivalent and compensating variations, which can be calculated using only the observed price and quantity data pertaining to the observed choices of the consumer for the two periods under consideration.

Problems

[23] Note that this ratio is not directly observable unless we have econometrically estimated the consumer's preferences.

[24] This result was derived in Diewert (1976; 134). Diewert called a price index number formula *superlative* if it is equal to the ratio of the unit cost functions as in (63) where the unit cost function is a flexible functional form. Thus the Fisher ideal price index is a superlative index number formula.

9. Define the Laspeyres and Paasche quantity indexes as follows:

(i) $Q_L(p^0,p^1,x^0,x^1) \equiv p^{0T}x^1/p^{0T}x^0$;
(ii) $Q_P(p^0,p^1,x^0,x^1) \equiv p^{1T}x^1/p^{1T}x^0$.

Show that the Fisher quantity index defined by the first line in (64) is equal to the geometric mean of the Paasche and Laspeyres quantity indexes; i.e., show that

(iii) $Q_F(p^0,p^1,x^0,x^1) = [Q_L(p^0,p^1,x^0,x^1)Q_P(p^0,p^1,x^0,x^1)]^{1/2}$.

10. Suppose the matrix B in (60) is equal to bb^T where b is an N dimensional positive column vector; i.e., $b >> 0_N$.

(a) Derive the system of Hicksian demand functions for this special case of the preferences defined by (60).

(b) Show that under our present assumptions on B, $x^0 \equiv \nabla_p e(u^0,p^0)$ and $x^1 \equiv \nabla_p e(u^1,p^1)$ are proportional to each other and hence we have

(i) $Q_F(p^0,p^1,x^0,x^1) = Q_L(p^0,p^1,x^0,x^1) = Q_P(p^0,p^1,x^0,x^1)$

where Q_F, Q_L and Q_P are the Fisher, Laspeyres and Paasche quantity indexes defined in the previous problem.

6. Normalized Quadratic Preferences

The previous section showed that there are some close connections between index number theory and finding exact measures for the equivalent and compensating variations. In the present section, we exhibit some additional connections between the two topics.

Suppose that a consumer has preferences that can be represented by the following *normalized quadratic expenditure function*:[25]

(67) $e(u,p) \equiv [p^Tb + (1/2)p^TAp/\alpha^Tp]u$; $\quad u > 0$; $p >> 0_N$; $\alpha > 0_N$; $A = A^T$;
(68) A is negative semidefinite[26] ;

where u is the consumer's utility level, p is the positive vector of commodity prices that the consumer faces and the vectors b and α are regarded as parameter vectors and the

[25] This function was introduced in the producer context by Diewert and Wales (1987; 53) and applied by Diewert and Wales (1992) and Diewert and Lawrence (2002) in this context and by Diewert and Wales (1988a) (1988b) (1993) in the consumer context. The advantages of this flexible functional form are explained in Diewert and Wales (1993).
[26] Diewert and Wales (1987; 66) show that this condition is necessary and sufficient for e(u,p) to be concave in p.

symmetric matrix A is a matrix of (unknown to us) parameters. In the following section, we will assume that α is known to us[27] and in fact, we will set it equal to the consumer's period 0 or period 1 (observable) consumption vector.

Let $p^* \gg 0_N$ be a reference commodity price vector. In addition to the restrictions in (67) and (68), we will impose the following restrictions on e:

(69) $p^{*T}b = 1$;
(70) $Ap^* = 0_N$.

It can be seen that the effect of the restrictions (69) and (70) on e is to impose *money metric utility scaling* on e at the reference prices p*; i.e., with the restrictions (69) and (70), we have:

(71) $e(u,p^*) = u \quad$ for all $u > 0$.

Now suppose that we are given another expenditure function that corresponds to homothetic preferences, say $e^*(u,p)$. Let $u^* > 0$ and suppose that $e^*(u,p)$ is twice continuously differentiable at (u^*,p^*). Then it can be shown[28] that for any choice of α such that $p^{*T}\alpha > 0$, we can find a parameter vector b and symmetric parameter matrix A which satisfy (69) and (70) such that the following equalities are satisfied:

(72) $\quad e(u^*,p^*) = e^*(u^*,p^*)$;
(73) $\quad \nabla_p e(u^*,p^*) = \nabla_p e^*(u^*,p^*)$;
(74) $\nabla_{pp}^2 e(u^*,p^*) = \nabla_{pp}^2 e^*(u^*,p^*)$.

Thus the normalized quadratic expenditure function e(u,p) defined by (67)-(70) can approximate an arbitrary twice continuously differentiable expenditure function e*(u,p) that corresponds to homothetic preferences and has the following money metric utility scaling property at the reference prices p*:

(75) $e^*(u,p^*) = u \quad$ for all $u > 0$;

i.e., the normalized quadratic expenditure function is another example of a *flexible functional form*, in the class of homothetic preferences, just as was the Fisher expenditure function, uc(p), where c(p) was defined by (60) in the previous section.[29]

Suppose that a consumer has preferences that can be represented by a normalized quadratic expenditure function, e(u,p), that is defined by (67)-(70) where we know the parameter vector α.[30] Suppose further that we can observe the consumer's quantity

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[27] Note that p^TAp is homogeneous of degree 2 in prices but we know that expenditure functions must be homogeneous of degree one in prices. Hence, the homogeneous of degree one term $p^T\alpha$ is simply used to deflate p^TAp into a linearly homogeneous function.
[28] See problem 11 below.
[29] The unit cost function that corresponds to (67) is $c(p) = e(1,p) = [p^Tb + (1/2)p^TAp/\alpha^Tp]$.
[30] In the following section, we will assume that α equals either x^0 or x^1.

choices x^0 and x^1 for two periods where the corresponding price vectors were p^0 and p^1 respectively.[31] Let $u^0 > 0$ and $u^1 > 0$ be the utility levels that the consumer attains for the two periods. Then differentiating the e(u,p) defined by (67) with respect to the components of p leads to the following expressions for x^0 and x^1 using Shephard's Lemma:

(76) $x^0 = [b + (\alpha^T p^0)^{-1} A p^0 - (1/2)(\alpha^T p^0)^{-2} p^{0T} A p^0 \alpha] u^0$;
(77) $x^1 = [b + (\alpha^T p^1)^{-1} A p^1 - (1/2)(\alpha^T p^1)^{-2} p^{1T} A p^1 \alpha] u^1$.

We are now going to derive an exact index number formula that will enable us to calculate the utility ratio u^1/u^0 using just the observable price and quantity data for the two situations, p^0,p^1,x^0,x^1, and the parameter vector α (which is assumed to be known to us). Thus we are going to derive a formula that is similar to the Fisher exact formula (63) derived in the previous section, except that this time, the exact formula will be for a quantity index rather than for a price index.

Premultiply both sides of (76) and (77) by the transpose of the price vector $(\alpha^T p^0)p^1 + (\alpha^T p^1)p^0$. After some simplification, we obtain the following formulae:

(78) $[(\alpha^T p^0)p^1 + (\alpha^T p^1)p^0]^T x^0 = \{[(\alpha^T p^0)p^1 + (\alpha^T p^1)p^0]^T b + p^{1T} A p^0\} u^0$;
(79) $[(\alpha^T p^0)p^1 + (\alpha^T p^1)p^0]^T x^1 = \{[(\alpha^T p^0)p^1 + (\alpha^T p^1)p^0]^T b + p^{0T} A p^1\} u^1$.

Since A is symmetric, $p^{1T} A p^0 = [p^{1T} A p^0]^T = p^{0T} A^T p^1 = p^{0T} A p^1$, and hence, we have:[32]

(80) $u^1/u^0 = [(\alpha^T p^0)p^1 + (\alpha^T p^1)p^0]^T x^1 / [(\alpha^T p^0)p^1 + (\alpha^T p^1)p^0]^T x^0 \equiv Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$

where $Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$ is the *normalized quadratic quantity index*.[33] Thus if we know α, $Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$ can be calculated using only observable price and quantity data pertaining to the two situations being considered and (80) tells us that this quantity index is equal to the utility ratio u^1/u^0, which is equal to $f(x^1)/f(x^0)$ where f is the linearly homogeneous utility function that is dual to the expenditure function defined by (67)-(70). Thus $Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$ is a *superlative index number formula* since $Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$ is exactly equal to the utility ratio $f(x^1)/f(x^0)$ where f is dual to a flexible functional form for a unit cost function.[34]

The price index $P_{NQ}(p^0,p^1,x^0,x^1;\alpha)$ that corresponds to the normalized quadratic quantity index defined by (80), $Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$, is defined as follows:

(81) $P_{NQ}(p^0,p^1,x^0,x^1;\alpha) \equiv p^{1T} x^1 / p^{0T} x^0 Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$;

[31] We suppose that $x^t > 0_N$ and $p^t >> 0_N$ for t = 0,1.
[32] This result was obtained by Diewert (1992; 576).
[33] Diewert (1992; 576) introduced this index to the economics literature.
[34] Diewert (1976b) introduced this definition for a superlative index number formula.

i.e., the corresponding price index $P_{NQ}(p^0,p^1,x^0,x^1;\alpha)$ is defined as the value ratio $p^{1T}x^1/p^{0T}x^0$ divided by the quantity index, $Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$.

Using the result (80), we can readily obtain *exact formulae for the equivalent and compensating variations*, $V_E(u^0,u^1,p^0)$ and $V_C(u^0,u^1,p^0)$, in the same manner that we derived (65) and (66) in the previous section; i.e., we have

$$
\begin{aligned}
(82)\quad V_E(u^0,u^1,p^0) &= c(p^0)[f(x^1) - f(x^0)] \\
&= c(p^0)f(x^0)[\{f(x^1)/f(x^0)\} - 1] \\
&= p^{0T}x^0[\{u^1/u^0\} - 1] && \text{using (55) for } t = 0 \\
&= p^{0T}x^0[Q_{NQ}(p^0,p^1,x^0,x^1;\alpha) - 1] && \text{using (80).}
\end{aligned}
$$

Similarly, the compensating variation may be written as follows:

$$
\begin{aligned}
(83)\quad V_C(u^0,u^1,p^1) &= c(p^1)[f(x^1) - f(x^0)] \\
&= c(p^1)f(x^1)[1 - \{f(x^1)/f(x^0)\}^{-1}] \\
&= p^{1T}x^1[1 - \{u^1/u^0\}^{-1}] && \text{using (55) for } t = 1 \\
&= p^{1T}x^1[1 - Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)^{-1}] && \text{using (80).}
\end{aligned}
$$

In the following section, we will specialize the formulae (81)-(83) to the cases where α equals x^0 or x^1.

Problems

11. Suppose $c^*(p)$ is a linearly homogeneous unit cost function that is twice continuously differentiable at $p = p^* >> 0_N$. (a) Show that the derivatives of c^* satisfy the following 1 + N equations:

(i) $p^{*T}\nabla_p c^*(p^*) = c^*(p^*)$;
(ii) $\nabla_{pp}^2 c^*(p^*)p^* = 0_N$.

(b) Define $c(p) \equiv e(1,p)$ where $e(u,p)$ is the normalized quadratic expenditure function defined by (67)-(70). Find a vector b and symmetric matrix A which satisfy (69) and (70) such that the following equations are satisfied:

(iii) $c(p^*) = c^*(p^*)$;
(iv) $\nabla_p c(p^*) = \nabla_p c^*(p^*)$;
(v) $\nabla_{pp}^2 c(p^*) = \nabla_{pp}^2 c^*(p^*)$.

We assume that the c^* function satisfies the conditions derived in part (a) of the problem plus the additional restriction (which will impose money metric utility scaling on the expenditure function $e^*(u,p) \equiv uc^*(p)$ at the reference prices p^*):

(vi) $c^*(p^*) = 1$.

In order to simplify the computations, we also assume that the (known) vector of parameters α is scaled so that

(vii) $p^{*T}\alpha = 1$.

This problem shows that the normalized quadratic expenditure function defined by (67)-(70) is indeed a flexible functional form.

12. Recall the formula for the equivalent variation given by (82), which uses the formula for the normalized quadratic quantity index, $Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$. Obtain the following formula for the equivalent variation that uses the normalized quadratic price index $P_{NQ}(p^0,p^1,x^0,x^1;\alpha)$ defined by (81):

(i) $V_E(u^0,u^1,p^0) = [p^{1T}x^1 / P_{NQ}(p^0,p^1,x^0,x^1;\alpha)] - p^{0T}x^0$.

13. Recall the formula for the compensating variation given by (83), which uses the formula for the normalized quadratic quantity index, $Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$. Obtain the following formula for the compensating variation that uses the normalized quadratic price index $P_{NQ}(p^0,p^1,x^0,x^1;\alpha)$ defined by (81):

(i) $V_C(u^0,u^1,p^0) = p^{1T}x^1 - P_{NQ}(p^0,p^1,x^0,x^1;\alpha)p^{0T}x^0$.

14. Recall (65) and (66) in the previous section that defined the equivalent and compensating variations when the consumer had the preferences which were dual to the unit cost function defined by (60). These formulae for the two variations can be written in terms of the Fisher ideal quantity index $Q_F(p^0,p^1,x^0,x^1)$ as follows:

(i) $V_E(u^0,u^1,p^0) = p^{0T}x^0[Q_F(p^0,p^1,x^0,x^1) - 1]$;
(ii) $V_C(u^0,u^1,p^0) = p^{1T}x^1[1 - Q_F(p^0,p^1,x^0,x^1)^{-1}]$.

Rewrite (i) and (ii) above in terms of the Fisher ideal price index, $P_F(p^0,p^1,x^0,x^1)$. Hint: Use the fact that $Q_F(p^0,p^1,x^0,x^1)$ and $P_F(p^0,p^1,x^0,x^1)$ satisfy the first line of (64).

7. Normalized Quadratic Preferences: Special Cases

In this section, we specify that the vector of parameters α that occurred in the definition of the normalized quadratic expenditure function defined by (67)-(70) in the previous section be either x^0 or x^1.

Case 1: $\alpha = x^0$:

Replacing α by x^0 in (80) leads to the following special case for the normalized quadratic quantity index:

$$\begin{aligned}(84)\ Q_{NQ}(p^0,p^1,x^0,x^1;x^0) &= [(x^{0T}p^0)p^1 + (x^{0T}p^1)p^0]^Tx^1/[(x^{0T}p^0)p^1 + (x^{0T}p^1)p^0]^Tx^0 \\ &= [(x^{0T}p^0)p^{1T}x^1 + (x^{0T}p^1)p^{0T}x^1]/2(p^{0T}x^0)(p^{1T}x^0)\end{aligned}$$

$$= (1/2)[p^{1T}x^1/p^{1T}x^0] + (1/2)[p^{0T}x^1/p^{0T}x^0]$$
$$= (1/2)Q_P(p^0,p^1,x^0,x^1) + (1/2)Q_L(p^0,p^1,x^0,x^1)$$

where Q_L and Q_P are the Laspeyres and Paasche quantity indexes defined in problem 9 above. Thus when the parameter vector α is equal to x^0, the normalized quadratic quantity index reduces to the arithmetic average of the Paasche and Laspeyres quantity indexes.

If we substitute (84) into (82) and (83) above, we find that when α is equal to x^0, the equivalent and compensating variations have the following exact decompositions (85) and (87):

$$\begin{aligned}(85)\ V_E(u^0,u^1,p^0) &= p^{0T}x^0[Q_{NQ}(p^0,p^1,x^0,x^1;x^0) - 1] && \text{using (82)}\\ &= p^{0T}x^0\{(1/2)[p^{1T}x^1/p^{1T}x^0] + (1/2)[p^{0T}x^1/p^{0T}x^0]\} - p^{0T}x^0 && \text{using (84)}\\ &= (1/2)[(p^1/P_L) + p^0]^Tx^1 - p^{0T}x^0\end{aligned}$$

where the *Laspeyres price index* P_L is defined as

$$(86)\ P_L \equiv P_L(p^0,p^1,x^0,x^1) \equiv p^{1T}x^0/p^{0T}x^0.$$

Similarly, the compensating variation may be written as follows:

$$\begin{aligned}(87)\ V_C(u^0,u^1,p^1) &= p^{1T}x^1[1 - Q_{NQ}(p^0,p^1,x^0,x^1;x^0)^{-1}] && \text{using (83)}\\ &= p^{1T}x^1 - p^{1T}x^1/Q_{NQ}(p^0,p^1,x^0,x^1;x^0)\\ &= p^{1T}x^1 - p^{1T}x^1/\{(1/2)[p^{1T}x^1/p^{1T}x^0] + (1/2)[p^{0T}x^1/p^{0T}x^0]\} && \text{using (84)}\\ &= p^{1T}x^1 - \{(1/2)[p^{1T}x^0]^{-1} + (1/2)[p^{0T}x^0/P_P]^{-1}\}^{-1}\end{aligned}$$

where the *Paasche price index* P_P is defined as

$$(88)\ P_P \equiv P_P(p^0,p^1,x^0,x^1) \equiv p^{1T}x^1/p^{0T}x^1.$$

The formula for the compensating variation is a bit difficult to interpret but the formula for the equivalent variation, (85), is fairly simple: the first set of terms on the right hand side of (85) is $e(u^1,p^0)$ and under our assumptions, we have

$$(89)\ e(u^1,p^0) = (1/2)[(p^1/P_L) + p^0]^Tx^1\ .$$

Thus to calculate $e(u^1,p^0)$, we need only take the inner product of the period 1 quantity vector x^1 with the arithmetic average of the period 0 price vector p^0 and the period 1 price vector deflated by the Laspeyres price index, p^1/P_L, which is a very simple formula.

The price index $P_{NQ}(p^0,p^1,x^0,x^1;x^0)$ which corresponds to the normalized quadratic quantity index defined by (84), $Q_{NQ}(p^0,p^1,x^0,x^1;x^0)$, can be defined as follows using (81):

$$\begin{aligned}(90)\ P_{NQ}(p^0,p^1,x^0,x^1;x^0) &\equiv p^{1T}x^1/p^{0T}x^0Q_{NQ}(p^0,p^1,x^0,x^1;x^0)\\ &= p^{1T}x^1/p^{0T}x^0\{(1/2)[p^{1T}x^1/p^{1T}x^0] + (1/2)[p^{0T}x^1/p^{0T}x^0]\}\end{aligned}$$

$$= \{(1/2)[p^{0T}x^0/p^{1T}x^0] + (1/2)[p^{0T}x^1/p^{1T}x^1]\}^{-1}$$
$$= \{(1/2)[p^{1T}x^0/p^{0T}x^0]^{-1} + (1/2)[p^{1T}x^1/p^{0T}x^1]^{-1}\}^{-1}$$
$$= \{(1/2)[P_L]^{-1} + (1/2)[P_P]^{-1}\}^{-1}.$$

Thus the price index $P_{NQ}(p^0,p^1,x^0,x^1;x^0)$ which matches up with the normalized quadratic quantity index $Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$ when we choose α equal to x^0 is the *harmonic mean of the Paasche and Laspeyres price indexes*. Recall that the corresponding quantity index, $Q_{NQ}(p^0,p^1,x^0,x^1;x^0)$, is the *arithmetic mean of the Paasche and Laspeyres quantity indexes*.

Case 2: $\alpha = x^1$:

Replacing α by x^1 in (80) leads to the following special case for the normalized quadratic quantity index:

$$(91)\ Q_{NQ}(p^0,p^1,x^0,x^1;x^1) = [(x^{1T}p^0)p^1 + (x^{1T}p^1)p^0]^Tx^1/[(x^{1T}p^0)p^1 + (x^{1T}p^1)p^0]^Tx^0$$
$$= 2p^{1T}x^1p^{0T}x^1/\{p^{0T}x^1p^{1T}x^0 + p^{1T}x^1p^{0T}x^0\}$$
$$= \{(1/2)[p^{1T}x^0/p^{1T}x^1] + (1/2)[p^{0T}x^0/p^{0T}x^1]\}^{-1}$$
$$= \{(1/2)[p^{1T}x^1/p^{1T}x^0]^{-1} + (1/2)[p^{0T}x^1/p^{0T}x^0]^{-1}\}^{-1}$$
$$= \{(1/2)Q_P(p^0,p^1,x^0,x^1)^{-1} + (1/2)Q_L(p^0,p^1,x^0,x^1)^{-1}\}^{-1}$$

where Q_L and Q_P are the Laspeyres and Paasche quantity indexes defined in problem 9 above. Thus when the parameter vector α is equal to x^1, the normalized quadratic quantity index reduces to the harmonic average of the Paasche and Laspeyres quantity indexes.

If we substitute (91) into (82) and (83) above, we find that when α is equal to x^1, the equivalent and compensating variations have the following exact decompositions (92) and (93):

$$(92)\ V_E(u^0,u^1,p^0) = p^{0T}x^0[Q_{NQ}(p^0,p^1,x^0,x^1;x^1) - 1] \qquad \text{using (82)}$$
$$= p^{0T}x^0\{(1/2)[p^{1T}x^1/p^{1T}x^0]^{-1} + (1/2)[p^{0T}x^1/p^{0T}x^0]^{-1}\}^{-1} - p^{0T}x^0 \qquad \text{using (91)}$$
$$= \{(1/2)[\ p^{0T}x^0\,p^{1T}x^1/p^{1T}x^0]^{-1} + (1/2)[p^{0T}x^1]^{-1}\}^{-1} - p^{0T}x^0$$
$$= \{(1/2)[(p^1/P_L)^Tx^1]^{-1} + (1/2)[p^{0T}x^1]^{-1}\}^{-1} - p^{0T}x^0$$

where the Laspeyres price index P_L is defined by (86). Similarly, the compensating variation may be written as follows:

$$(93)\ V_C(u^0,u^1,p^1) = p^{1T}x^1[1 - Q_{NQ}(p^0,p^1,x^0,x^1;x^1)^{-1}] \qquad \text{using (83)}$$
$$= p^{1T}x^1 - p^{1T}x^1/Q_{NQ}(p^0,p^1,x^0,x^1;x^1)$$
$$= p^{1T}x^1 - p^{1T}x^1\{(1/2)[p^{1T}x^0/p^{1T}x^1] + (1/2)[p^{0T}x^0/p^{0T}x^1]\} \qquad \text{using (91)}$$
$$= p^{1T}x^1 - \{(1/2)[p^{1T}x^0] + (1/2)[P_P\,p^{0T}x^0]\}$$
$$= p^{1T}x^1 - [(1/2)p^1 + (1/2)P_P\,p^0]^Tx^0$$

where the Paasche price index P_P is defined by (88).

The formula (92) for the equivalent variation is somewhat complex but the formula for the compensating variation, (93), is fairly simple under our present assumptions: the first set of terms in the last line of (93) is $e(u^1,p^1) = p^{1T}x^1$ and the second set of terms is equal to $-e(u^0,p^1)$. Thus under our present assumptions, we have

(94) $e(u^0,p^1) = [(1/2)p^1 + (1/2)P_P\, p^0]^T x^0$

Thus to calculate $e(u^0,p^1)$, we need only take the inner product of the period 0 quantity vector x^0 with the arithmetic average of the period 1 price vector p^1 and the period 0 price vector scaled up by the Paasche price index, $p^0 P_P$, which is a very simple formula.

The price index $P_{NQ}(p^0,p^1,x^0,x^1;x^1)$ which corresponds to the normalized quadratic quantity index defined by (91), $Q_{NQ}(p^0,p^1,x^0,x^1;x^1)$, can be defined as follows using (81):

(95) $$\begin{aligned} P_{NQ}(p^0,p^1,x^0,x^1;x^1) &\equiv p^{1T}x^1/p^{0T}x^0 Q_{NQ}(p^0,p^1,x^0,x^1;x^1) \\ &= \{p^{1T}x^1/p^{0T}x^0\}\{(1/2)[p^{1T}x^0/p^{1T}x^1] + (1/2)[p^{0T}x^0/p^{0T}x^1]\} \\ &= (1/2)[p^{1T}x^0/p^{0T}x^0] + (1/2)[p^{1T}x^1/p^{0T}x^1] \\ &= (1/2)P_L + (1/2)P_P. \end{aligned}$$

Thus the price index $P_{NQ}(p^0,p^1,x^0,x^1;x^1)$ which matches up with the normalized quadratic quantity index $Q_{NQ}(p^0,p^1,x^0,x^1;\alpha)$ when we choose α equal to x^1 is the *arithmetic mean of the Paasche and Laspeyres price indexes*.[35] Recall that the corresponding quantity index, $Q_{NQ}(p^0,p^1,x^0,x^1;x^1)$, is the *harmonic mean of the Paasche and Laspeyres quantity indexes*.

The results in this section and the two previous sections show that we have three different methods that we could use to estimate a consumer's quantity index or utility ratio, $u^1/u^0 = f(x^1)/f(x^0)$, and the corresponding equivalent and compensating variations:

- Assume that the consumer's unit cost function $c(p)$ has the homogeneous quadratic functional form defined by (60). In this case, the utility ratio is given by $u^1/u^0 = f(x^1)/f(x^0) = [Q_LQ_P]^{1/2}$, the geometric mean of the Laspeyres and Paasche quantity indexes, where f is the utility function which is dual to the given unit cost function $c(p)$. In this case, the consumer's true cost of living index is the unit cost function ratio, $c(p^1)/c(p^0) = P_F \equiv [P_LP_P]^{1/2}$, which is the geometric mean of the Laspeyres and Paasche price indexes (which in turn is known as the Fisher (1922) ideal price index).
- Assume that the consumer's unit cost function $c(p)$ has the normalized quadratic functional form defined by (67)-(70) with α defined to be x^0. In this case, the utility ratio is given by $u^1/u^0 = f(x^1)/f(x^0) = (1/2)Q_L + (1/2)Q_P$, the arithmetic mean of the Laspeyres and Paasche quantity indexes, where f is the utility function which is dual to the given unit cost function $c(p)$. In this case, the consumer's

[35] This price index was suggested in the economics literature by Sidgwick (1883; 68) and Bowley (1901; 227).

true cost of living index is the unit cost function ratio, $c(p^1)/c(p^0) = \{(1/2)[P_L]^{-1} + (1/2)[P_P]^{-1}\}^{-1}$, which is exactly equal to the harmonic mean of the Laspeyres and Paasche price indexes.

- Assume that the consumer's unit cost function $c(p)$ has the normalized quadratic functional form defined by (67)-(70) with α defined to be x^1. In this case, the utility ratio is given by $u^1/u^0 = f(x^1)/f(x^0) = \{(1/2)[Q_L]^{-1} + (1/2)[Q_P]^{-1}\}^{-1}$, the harmonic mean of the Laspeyres and Paasche quantity indexes, where f is the utility function which is dual to the given unit cost function $c(p)$. In this case, the consumer's true cost of living index is the unit cost function ratio, $c(p^1)/c(p^0) = (1/2)P_L + (1/2)P_P$, which is exactly equal to the arithmetic mean of the Laspeyres and Paasche price indexes.

Each of the above three functional forms for the unit cost function can approximate an arbitrary unit cost function to the second order around any given point so that the resulting expenditure functions can approximate an arbitrary expenditure function to the second order in the class of homothetic preference type expenditure functions. Thus if we want to use the above theory in empirical applications, it would seem at this point, that each of the above 3 methods for calculating equivalent or compensating variations is equally good from an a priori theoretical point of view. Fortunately, in most cases, it will not matter very much which of the 3 methods is used since it can be shown that the harmonic, geometric and arithmetic means of the Laspeyres and Paasche quantity indexes, regarded as functions of the prices and quantities pertaining to the two periods, p^0,p^1,x^0,x^1, approximate each other to the second order around an equal price (i.e., $p^0 = p^1$) and equal quantity (i.e., $x^0 = x^1$) point; i.e., recall problems 6 and 7 above where we proved similar properties for various approximations to the equivalent variation.[36]

Problems

15. Show that the following inequalities are always satisfied:

(i) $[(1/2)Q_L^{-1} + (1/2)Q_P^{-1}]^{-1} \leq Q_F \leq (1/2)Q_L + (1/2)Q_P$;
(ii) $[(1/2)P_L^{-1} + (1/2)P_P^{-1}]^{-1} \leq P_F \leq (1/2)P_L + (1/2)P_P$

where Q_L, Q_P and Q_F are the Laspeyres, Paasche and Fisher quantity indexes respectively and P_L, P_P and P_F are the Laspeyres, Paasche and Fisher price indexes respectively.

16. Show that $[(1/2)Q_L^{-1} + (1/2)Q_P^{-1}]^{-1}$, Q_F and $(1/2)Q_L + (1/2)Q_P$ approximate each other to the second order around an equal price and quantity point.

8. Consumer Surplus Analysis Revisited

[36] This approximation result was first established by Diewert (1978; 897). The technique of proof is simple (but very tedious): just do the differentiation and verify that the derivatives coincide when evaluated at an equal price and quantity point.

Recall the result (85) in the previous section, i.e., if the parameter vector α in definition (67) is equal to x^0, then the normalized quadratic quantity index reduces to the arithmetic average of the Paasche and Laspeyres quantity indexes and the equivalent variation is exactly equal to the right hand side of the following equation:

(96) $V_E(u^0,u^1,p^0) = (1/2)[(p^1/P_L) + p^0]^T x^1 - p^{0T}x^0$

where P_L is the Laspeyres price index, $p^{1T}x^0/p^{0T}x^0$. Note that

(97) $(1/2)(p^1/P_L)^T x^0 = (1/2)p^{1T}x^0/P_L = (1/2)p^{1T}x^0/[p^{1T}x^0/p^{0T}x^0] = (1/2)p^{0T}x^0$.

We now substitute (97) into (96) and we obtain the following expression for the equivalent variation $V_E(u^0,u^1,p^0)$ when the consumer has normalized quadratic preferences with α equal to x^0:

$$
\begin{aligned}
(98)\ V_E(u^0,u^1,p^0) &= (1/2)[(p^1/P_L) + p^0]^T x^1 - [(1/2)p^0 + (1/2)p^0]^T x^0 \\
&= [(1/2)(p^1/P_L) + (1/2)p^0]^T x^1 - [(1/2)(p^1/P_L) + (1/2)p^0]^T x^0 \qquad \text{using (97)} \\
&= [(1/2)(p^1/P_L) + (1/2)p^0]^T [x^1 - x^0] \\
&= V_{WE}(p^0,p^1,x^0,x^1)
\end{aligned}
$$

where $V_{WE}(p^0,p^1,x^0,x^1)$ is the Weitzman (1988; 551) approximation to the equivalent variation defined earlier by (42). *Thus we have provided another strong justification for the Weitzman approximation to the equivalent variation*; namely, it is an exact (global) measure of welfare change for a consumer with homothetic preferences that can be represented by normalized quadratic preferences, which in turn can approximate arbitrary homothetic preferences to the second order around any chosen point. Of course, (98) is also a form of the Bennet measure of welfare change (except that the period 1 price vector p^1 is deflated by the Laspeyres price index P_L), which in turn is a consumer surplus type measure of welfare change.

The above analysis has a counterpart for the compensating variation. Recall the result (93) in the previous section, i.e., if the parameter vector α in definition (67) is equal to x^1, then the normalized quadratic quantity index reduces to the harmonic average of the Paasche and Laspeyres quantity indexes and the compensating variation is exactly equal to the right hand side of the following equation:

(99) $V_C(u^0,u^1,p^1) = p^{1T}x^1 - [(1/2)p^1 + (1/2)P_P\, p^0]^T x^0$

where P_P is the Paasche price index, $p^{1T}x^1/p^{0T}x^1$. Note that

(100) $(1/2)(P_P\, p^0)^T x^1 = (1/2)[p^{1T}x^1/p^{0T}x^1]p^{0T}x^1 = (1/2)p^{1T}x^1$.

We now substitute (100) into (99) and we obtain the following expression for the compensating variation $V_C(u^0,u^1,p^0)$ when the consumer has normalized quadratic preferences with α equal to x^1:

$$\begin{aligned}(101)\ V_C(u^0,u^1,p^0) &= [(1/2)p^1 + (1/2)p^1]^T x^1 - [(1/2)p^1 + (1/2)P_P\, p^0]^T x^0 \\ &= [(1/2)p^1 + (1/2)P_P\, p^0]^T x^1 - [(1/2)p^1 + (1/2)P_P\, p^0]^T x^0 \qquad \text{using (100)} \\ &= [(1/2)p^1 + (1/2)P_P\, p^0]^T [x^1 - x^0] \\ &= V_{DCP}(p^0,p^1,x^0,x^1)\end{aligned}$$

where $V_{DCP}(p^0,p^1,x^0,x^1)$ is the Diewert Paasche approximation to the compensating variation defined earlier in problem 8. *Thus we have provided another strong justification for this Diewert approximation to the compensating variation*; namely, it is an exact (global) measure of welfare change for a consumer with homothetic preferences that can be represented by normalized quadratic preferences, which in turn can approximate arbitrary homothetic preferences to the second order around any chosen point. As was the case with formula (98), (101) is also a form of the Bennet measure of welfare change (except that the period 1 price vector p^0 is inflated by the Paasche price index P_P), which in turn is a consumer surplus type measure of welfare change.

Our conclusion is that Bennet type measures of welfare change are adequate measures of welfare change for a large class of preferences: all we need to do is to scale the prices of one period by the appropriate index number estimate of price change between the two periods.

However, there is one big weakness with all of the index number methods for estimating welfare change that we have considered in sections 5-8: they all rely on the assumption of homothetic preferences. In the following two sections, we consider another index number method which allows us to approximate the general case where preferences could be nonhomothetic.

9. The Measurement of Price Change in the Nonhomothetic Case

Before we derive our main result, we require a preliminary result. Suppose the function of N variables, $f(z_1,\ldots,z_N) \equiv f(z)$, is quadratic; i.e.,

$$(102)\quad f(z_1,\ldots,z_N) \equiv a_0 + \textstyle\sum_{i=1}^N a_i\, z_i + (1/2) \sum_{i=1}^N \sum_{k=1}^N a_{ik}\, z_i z_k \ ;\ a_{ik} = a_{ki} \text{ for all i and k,}$$

where the a_i and the a_{ik} are constants. Let $f_i(z)$ denote the first order partial derivative of f evaluated at z with respect to the ith component of z, z_i. Let $f_{ik}(z)$ denote the second order partial derivative of f with respect to z_i and z_k. Then it is well known that the second order Taylor series approximation to a quadratic function is *exact*; i.e., if f is defined by (102) above, then for any two points, z^0 and z^1, we have

$$(103)\quad f(z^1) - f(z^0) = \textstyle\sum_{i=1}^N f_i(z^0)[z_i^1 - z_i^0] + (1/2) \sum_{i=1}^N \sum_{k=1}^N f_{ik}(z^0)[z_i^1 - z_i^0][z_k^1 - z_k^0].$$

It is less well known that *an average of two first order Taylor series approximations* to a quadratic function is also *exact*; i.e., if f is defined by (102) above, then for any two points, z^0 and z^1, we have[37]

(104) $f(z^1) - f(z^0) = (1/2)\sum_{i=1}^N [f_i(z^0) + f_i(z^1)][z_i^1 - z_i^0]$.

Diewert (1976; 118) and Lau (1979) showed that equation (104) characterized a quadratic function and called the equation the *quadratic approximation lemma*. We will be more brief and refer to (104) as the *quadratic identity*.

We now suppose that the consumer's *expenditure function*, e(u,p), has the following *translog functional form*:[38]

$$(105)\quad \ln e(u,p) \equiv a_0 + \sum_{i=1}^N a_i \ln p_i + (1/2) \sum_{i=1}^N \sum_{k=1}^N a_{ik} \ln p_i \ln p_k + b_0 \ln u + \sum_{i=1}^N b_i \ln p_i \ln u + (1/2)\, b_{00} [\ln u]^2$$

where ln is the natural logarithm function and the parameters a_i, a_{ik}, and b_i satisfy the following restrictions:

(106) $a_{ik} = a_{ki}$; $\quad i,k = 1,\ldots,N$;
(107) $\sum_{i=1}^N a_i = 1$;
(108) $\sum_{i=1}^N b_i = 0$;
(109) $\sum_{k=1}^N a_{ik} = 0$; $\quad i = 1,\ldots,N$.

The parameter restrictions (106)-(109) ensure that e(u,p) defined by (105) is linearly homogeneous in p, a property that a cost or expenditure function must have. It can be shown that the translog expenditure function defined by (105)-(109) can provide a second order Taylor series approximation to an arbitrary expenditure function.[39]

We assume that the consumer has preferences that correspond to the translog expenditure function and that the consumer engages in cost minimizing behavior during periods 0 and 1. Let p^0 and p^1 be the period 0 and 1 observed price vectors and let x^0 and x^1 be the period 0 and 1 observed quantity vectors. Thus we have:

(110) $e(u^0,p^0) = \sum_{i=1}^N p_i^0 x_i^0 = p^{0T}x^0$ and $e(u^1,p^1) = \sum_{i=1}^N p_i^1 x_i^1 = p^{1T}x^1$

where e is the translog expenditure function defined above. We can also apply Shephard's lemma:

---

[37] To prove that (103) and (104) are true, use (102) and substitute into the left hand sides of (103) and (104). Then calculate the partial derivatives of the quadratic function defined by (102) and substitute these derivatives into the right hand side of (103) and (104).
[38] Christensen, Jorgenson and Lau (1971) introduced this function into the economics literature.
[39] It can also be shown that if all of the $b_i = 0$ and $b_{00} = 0$, then $e(u,p) = ue(1,p) \equiv uc(p)$; i.e., with these additional restrictions on the parameters of the general translog cost function, we have homothetic preferences. Note that we also assume that utility u is scaled so that u is always positive.

(111) $x_i^t = \partial e(u^t,p^t)/\partial p_i$; $\qquad$ $i = 1,\ldots,N$; $t = 0,1$
$\qquad = [e(u^t,p^t)/p_i^t]\ \partial \ln e(u^t,p^t)/\partial \ln p_i$.

Now use (110) to replace $e(u^t,p^t)$ in (111). After some cross multiplication, equations (111) become the following system of equations:

(112) $p_i^t x_i^t / \sum_{k=1}^N p_k^1 x_k^1 \equiv s_i^t = \partial \ln e(u^t,p^t)/\partial \ln p_i$; $\qquad$ $i = 1,\ldots,n$; $t = 0,1$ or

(113) $s_i^t = a_i + \sum_{k=1}^N a_{ik} \ln p_k^t + b_i \ln u^t$; $\qquad$ $i = 1,\ldots,N$; $t = 0,1$

where s_i^t is the period t expenditure share on commodity i and (113) follows from (112) by differentiating (105) with respect to $\ln p_i$.

Define the geometric average of the period 0 and 1 utility levels as u*; i.e., define

(114) $u^* \equiv [u^0 u^1]^{1/2}$.

Now observe that the right hand side of the equation that defines the natural logarithm of the translog cost function, equation (105), is a quadratic function of the variables $z_i \equiv \ln p_i$ if we hold utility constant at the level u*. Hence we can apply the quadratic identity, (104), and get the following equation:

(115) $\ln e(u^*,p^1) - \ln e(u^*,p^0)$
$= (1/2)\sum_{i=1}^N [\partial \ln e(u^*,p^0)/\partial \ln p_i + \partial \ln e(u^*,p^1)/\partial \ln p_i\ ][\ln p_i^1 - \ln p_i^0]$
$= (1/2)\sum_{i=1}^N [a_i + \sum_{k=1}^N a_{ik} \ln p_k^0 + b_i \ln u^* + a_i + \sum_{k=1}^N a_{ik} \ln p_k^1 + b_i \ln u^*][\ln p_i^1 - \ln p_i^0]$
differentiating (105) at the points (u^*,p^0) and (u^*,p^1)
$= (1/2)\sum_{i=1}^N [a_i + \sum_{k=1}^N a_{ik} \ln p_k^0 + b_i \ln[u^0u^1]^{1/2} + a_i + \sum_{k=1}^N a_{ik} \ln p_k^1 + b_i \ln[u^0u^1]^{1/2}][\ln p_i^1 - \ln p_i^0]$
using definition (114) for u*
$= (1/2)\sum_{i=1}^N [a_i + \sum_{k=1}^N a_{ik} \ln p_k^0 + b_i \ln u^0 + a_i + \sum_{k=1}^N a_{ik} \ln p_k^1 + b_i \ln u^1][\ln p_i^1 - \ln p_i^0]$
rearranging terms
$= (1/2)\sum_{i=1}^N [\partial \ln e(u^0,p^0)/\partial \ln p_i + \partial \ln e(u^1,p^1)/\partial \ln p_i\ ][\ln p_i^1 - \ln p_i^0]$
differentiating (105) at the points (u^0,p^0) and (u^1,p^1)
$= (1/2)\sum_{i=1}^N [s_i^0 + s_i^1][\ln p_i^1 - \ln p_i^0]$ $\qquad$ using (112)
$= \sum_{i=1}^N (1/2)(s_i^0 + s_i^1) \ln(p_i^1/p_i^0)$
$\equiv \ln P_T(p^0,p^1,x^0,x^1)$.

The last line of (115) defines the *Törnqvist Theil price index*, P_T,[40] which can be calculated using observable price and quantity data pertaining to the consumer for periods 0 and 1. Exponentiating both sides of (115) yields the following equality between the true cost of living between periods 0 and 1, $e(u^*,p^1)/e(u^*,p^0)$, evaluated at the intermediate utility level u* and the observable Törnqvist Theil index P_T:[41]

(116) $e(u^*,p^1)/e(u^*,p^0) = P_T(p^0,p^1,x^0,x^1)$.

[40] See Törnqvist and Törnqvist (1937) and Theil (1967; 136-137).
[41] This result is due to Diewert (1976; 122).

Since the translog expenditure function which appears on the left hand side of (116) is a flexible functional form, the Törnqvist-Theil price index P_T is also a *superlative index*.

It is somewhat mysterious how a ratio of *unobservable* expenditure functions of the form appearing on the left hand side of the above equation can be *exactly* estimated by an *observable* index number formula but the key to this mystery is the assumption of expenditure minimizing behavior and the quadratic identity (104) along with the fact that derivatives of cost functions are equal to quantities, as specified by Shephard's (1953; 11) lemma.[42]

There is one additional important fact about the result (116) that must be mentioned: (116) holds without assuming that the consumer's preferences are homothetic; i.e., the translog expenditure function defined by (105)-(109) can approximate an arbitrary expenditure function to the second order; i.e., it is not necessary to assume homothetic preferences in order to derive the exact result (116). However, there is an important limitation on the result (116) as well: as we shall see in the next section, we would like to have empirically observable expressions for the measures of price change, $e(u^0,p^1)/e(u^0,p^0)$ (this is the consumer's true cost of living index using the base period indifference surface as the reference utility level that is being held constant), or $e(u^1,p^1)/e(u^1,p^0)$ (this is the consumer's true cost of living index using the current period indifference surface as the reference utility level that is being held constant), but all we have available is $e(u^*,p^1)/e(u^*,p^0)$, which is the consumer's true cost of living index using the geometric average of the base and current period utility levels as the reference utility level that is being held constant. Thus if u^0 and u^1 are very far apart and preferences are very nonhomothetic, then $e(u^*,p^1)/e(u^*,p^0)$ may not be a very close approximation to either $e(u^0,p^1)/e(u^0,p^0)$ or to $e(u^1,p^1)/e(u^1,p^0)$.

Problem

17. (a) How many independent parameters does the translog expenditure function defined by (105)-(109) have? (b) How many independent parameters does an arbitrary twice continuously differentiable expenditure function $e^*(u,p)$ have to have in order to be flexible? (c) What additional restrictions on the parameters of the translog are required in order to impose money metric utility scaling on $e(u,p)$ at the reference prices $p^* = 1_N$? (d) What additional restrictions (additional to those made in part (c) above) on the translog expenditure function are required in order to impose the homotheticity restriction, $e(u,p^*) = ue(1,p^*)$ with $p^* = 1_N$?

10. Approximations to the Equivalent and Compensating Variations when Preferences are Nonhomothetic

Suppose that the consumer has preferences that are dual to the translog expenditure function $e(u,p)$ defined by (105). Suppose that u^0 and u^1 are fairly close to each other.

[42] All known superlative index number formula involve some form of quadratic preferences; see Diewert (2002).

Then the geometric mean $u^* \equiv [u^0u^1]^{1/2}$ will be close to u^0 and u^1. Under these conditions, using the continuity of the expenditure function e(u,p) in u, the exact equality (116) will translate into the following approximate equalities:

$$
\begin{aligned}
&(117)\ P_T(p^0,p^1,x^0,x^1) = e(u^*,p^1)/e(u^*,p^0) \\
&(118)\qquad\qquad\qquad\quad \approx e(u^0,p^1)/e(u^0,p^0) \\
&(119)\qquad\qquad\qquad\quad \approx e(u^1,p^1)/e(u^1,p^0).
\end{aligned}
$$

Starting with the definition (14) for the equivalent variation, we can rewrite it as follows:

$$
\begin{aligned}
(120)\ V_E(u^0,u^1,p^0) &\equiv e(u^1,p^0) - e(u^0,p^0) \\
&= e(u^1,p^1)/[e(u^1,p^1)/e(u^1,p^0)] - e(u^0,p^0) \\
&= p^{1T}x^1/[e(u^1,p^1)/e(u^1,p^0)] - p^{0T}x^0 \\
&\approx p^{1T}x^1/P_T(p^0,p^1,x^0,x^1) - p^{0T}x^0
\end{aligned}
$$

where the last line follows using the approximation (119). Similarly, starting with the definition (15) for the compensating variation, we can rewrite it as follows:

$$
\begin{aligned}
(121)\ V_C(u^0,u^1,p^0) &\equiv e(u^1,p^1) - e(u^0,p^1) \\
&= e(u^1,p^1) - [e(u^0,p^1)/e(u^0,p^0)]e(u^0,p^0) \\
&= p^{1T}x^1 - [e(u^0,p^1)/e(u^0,p^0)]p^{0T}x^0 \\
&\approx p^{1T}x^1 - P_T(p^0,p^1,x^0,x^1)p^{0T}x^0
\end{aligned}
$$

where the last line follows using the approximation (118).

The expressions on the right hand sides of (120) and (121) can be calculated using observable data. These expressions will be *exactly* equal to the equivalent and compensating variations for a consumer who has preferences that are dual to the translog expenditure function defined by (105)-(109), provided that the consumer's preferences are homothetic.[43] In the general case, (120) and (121) will provide adequate approximations to the equivalent and compensating variations provided that:

- u^0 and u^1 are "reasonably" close to each other *or*
- the consumer's preferences can be adequately approximated by homothetic preferences.

Thus unless the utility changes and price changes are very large going from period 0 to period 1, we can expect the expressions for the equivalent and compensating variations on the right hand side of (120) and (121) to provide adequate approximations to the "true" equivalent and compensating variations.

Problems

[43] In the case of homothetic preferences, e(u,p) equals ue(1,p) and hence $e(u^*,p^1)/e(u^*,p^0) = e(u^0,p^1)/e(u^0,p^0) = e(u^1,p^1)/e(u^1,p^0)$ and the approximate equalities (118) and (119) become exact equalities when the consumer has homothetic translog preferences.

18. (a) Show that the Fisher price index $P_F(p^0,p^1,x^0,x^1)$ defined by the first equality in (63) and the Törnqvist Theil price index $P_T(p^0,p^1,x^0,x^1)$ defined by the last equality in (115) *approximate each other to the first order around an equal price and quantity point*; i.e., show that when $p^0 = p^1 = p$ and $x^0 = x^1 = x$, we have:

(i) $\nabla_{p^0} P_F(p^0,p^1,x^0,x^1) = \nabla_{p^0} P_T(p^0,p^1,x^0,x^1)$;

(ii) $\nabla_{p^1} P_F(p^0,p^1,x^0,x^1) = \nabla_{p^1} P_T(p^0,p^1,x^0,x^1)$;

(iii) $\nabla_{x^0} P_F(p^0,p^1,x^0,x^1) = \nabla_{x^0} P_T(p^0,p^1,x^0,x^1)$;

(iv) $\nabla_{x^1} P_F(p^0,p^1,x^0,x^1) = \nabla_{x^1} P_T(p^0,p^1,x^0,x^1)$.

(b) Show that $P_F(p^0,p^1,x^0,x^1)$ and $P_T(p^0,p^1,x^0,x^1)$ *approximate each other to the second order around an equal price and quantity point*; i.e., show that when $p^0 = p^1 = p$ and $x^0 = x^1 = x$, we have:[44]

(v) $\nabla^2 P_F(p^0,p^1,x^0,x^1) = \nabla^2 P_T(p^0,p^1,x^0,x^1)$.

19. In section 5, we obtained the following formulae for the equivalent and compensating variations, using the Fisher quantity index:

(i) $V_E(u^0,u^1,p^0) = p^{0T}x^0[Q_F(p^0,p^1,x^0,x^1) - 1] \equiv V_{EF}(p^0,p^1,x^0,x^1)$;
(ii) $V_C(u^0,u^1,p^1) = p^{1T}x^1[1 - Q_F(p^0,p^1,x^0,x^1)^{-1}] \equiv V_{CF}(p^0,p^1,x^0,x^1)$

where we have defined the Fisher version of the equivalent variation, $V_{EF}(p^0,p^1,x^0,x^1)$, as the observable function of prices and quantities pertaining to the two periods, $p^{0T}x^0[Q_F(p^0,p^1,x^0,x^1) - 1]$ and we have defined the Fisher version of the compensating variation, $V_{CF}(p^0,p^1,x^0,x^1)$, as the observable function of prices and quantities pertaining to the two periods, $p^{1T}x^1[1 - Q_F(p^0,p^1,x^0,x^1)^{-1}]$. Now make similar definitions for the Törnqvist Theil versions of the equivalent and compensating variations:

(iii) $V_E(u^0,u^1,p^0) \approx p^{1T}x^1/P_T(p^0,p^1,x^0,x^1) - p^{0T}x^0 \equiv V_{ET}(p^0,p^1,x^0,x^1)$;
(iv) $V_C(u^0,u^1,p^0) \approx p^{1T}x^1 - P_T(p^0,p^1,x^0,x^1)p^{0T}x^0 \equiv V_{CT}(p^0,p^1,x^0,x^1)$.

(a) Show that $V_{EF}(p^0,p^1,x^0,x^1)$ approximates $V_{ET}(p^0,p^1,x^0,x^1)$ to the second order around an equal price and quantity point.

(b) Show that $V_{CF}(p^0,p^1,x^0,x^1)$ approximates $V_{CT}(p^0,p^1,x^0,x^1)$ to the second order around an equal price and quantity point.

This problem shows that the Fisher index approach to estimating equivalent and compensating variations will give numerical results that are very close to those given by the Törnqvist Theil approach.

---

[44] Diewert (1978; 888) established this result.

20. Recall the Weitzman approximation to the equivalent variation, $V_{WE}(p^0,p^1,x^0,x^1)$, defined by (98) above and the Diewert Paasche approximation to the compensating variation, $V_{DCP}(p^0,p^1,x^0,x^1)$, defined by (101):

(i) $V_{WE}(p^0,p^1,x^0,x^1) = [(1/2)(p^1/P_L) + (1/2)p^0]^T[x^1 - x^0]$;
(ii) $V_{DCP}(p^0,p^1,x^0,x^1) = [(1/2)p^1 + (1/2)P_P\, p^0]^T[x^1 - x^0]$.

(a) Show that $V_{EF}(p^0,p^1,x^0,x^1)$ approximates $V_{WE}(p^0,p^1,x^0,x^1)$ to the second order around an equal price and quantity point.

(b) Show that $V_{CF}(p^0,p^1,x^0,x^1)$ approximates $V_{DPC}(p^0,p^1,x^0,x^1)$ to the second order around an equal price and quantity point.

This problem shows that the Fisher index approach to estimating equivalent and compensating variations will give numerical results that are very close to those given by the consumer surplus approaches of Weitzman and Diewert.

References

Barlev, B. and J.L. Callen (1986), "Total Factor Productivity and Cost Variances: Survey and Analysis", *Journal of Accounting Literature* 5, 35-56.

Bennet, T.L. (1920), "The Theory of Measurement of Changes in Cost of Living", *Journal of the Royal Statistics Society* 83, 455-462.

Bowley, A.L. (1901), *Elements of Statistics*, Westminster: P.S. King and Son.

Darrough, M.N. (1986), "Variance Analysis: A Unifying Cost Function Approach", *Contemporary Accounting Research* 5, 199-221.

Diewert, W.E. (1976a), "Harberger's Welfare Indicator and Revealed Preference Theory", *The American Economic Review* 66, 143-152.

Diewert, W.E. (1976b), "Exact and Superlative Index Numbers", *Journal of Econometrics* 4, 114-145.

Diewert, W.E. (1978), "Superlative Index Number and Consistency in Aggregation", *Econometrica* 46, 883-900.

Diewert, W.E. (1983a), "The Theory of the Cost of Living Index and the Measurement of Welfare Change", pp. 163-233 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada, reprinted as pp. 79-147 in *Price Level Measurement*, W.E. Diewert (ed.), Amsterdam: North-Holland, 1990.

Diewert, W.E. (1992), “Exact and Superlative Welfare Change Indicators”, *Economic Inquiry* 30, 565-582.

Diewert, W.E. (2002), “The Quadratic Approximation Lemma and Decompositions of Superlative Indexes”, *Journal of Economic and Social Measurement* 28, 63-88.

Diewert, W.E. (2005), “Index Number Theory Using Differences Instead of Ratios”, *The American Journal of Economics and Sociology* 64:1, 311-360.

Diewert, W.E. and D. Lawrence (2002), “The Deadweight Costs of Capital Taxation in Australia”, pp. 103-167 in *Efficiency in the Public Sector*, Kevin J. Fox (ed.), Boston: Kluwer Academic Publishers.

Diewert, W.E. and T.J. Wales (1987), “Flexible Functional Forms and Global Curvature Conditions”, *Econometrica* 55, 43-68.

Diewert, W.E. and T.J. Wales (1988a), “Normalized Quadratic Systems of Consumer Demand Functions”, *Journal of Business and Economic Statistics* 6, 303-12.

Diewert, W.E. and T.J. Wales (1988b), “A Normalized Quadratic Semiflexible Functional Form”, *Journal of Econometrics* 37, 327-42.

Diewert, W.E. and T.J. Wales (1992), “Quadratic Spline Models For Producer’s Supply and Demand Functions”, *International Economic Review* 33, 705-722.

Diewert, W.E. and T.J. Wales (1993), “Linear and Quadratic Spline Models for Consumer Demand Functions”, *Canadian Journal of Economics* 26, 77-106.

Dupuit, J. (1844), “De la mesure de l’utilité des travaux publics”, *Annales de Ponts et Chaussées*, Second Series, Volume 8. Translated as: Dupuit, J. (1952), “On the Measurement of Utility of Public Works”, *International Economic Papers* 2, 83-110. English translation reprinted as pages 255-283 in K. J. Arrow and T. Scitovsky (1969), *Readings in Welfare Economics*, Homewood, Illinois: Richard D. Irwin.

Ekelund, R.B. and R.F. Hébert (1985), “Consumer Surplus: The First Hundred Years”, *History of Political Economy* 17:3, 419-454.

Fisher, I. (1922), *The Making of Index Numbers*, Houghton-Mifflin, Boston.

Harberger, A.C. (1971), “Three Basic Postulates for Applied Welfare Economics: An Interpretive Essay”, *The Journal of Economic Literature* 9, 785-797.

Harrison, G.C. (1918), “Cost Accounting to Aid Production”, Part I, II and III, *Industrial Management* 56; 273-282, 391-398 and 456-463.

Henderson, A. (1940-41), “Consumer’s Surplus and the Compensating Variation”, *The Review of Economic Studies* 8, 117-121.

Hicks, J.R. (1939), *Value and Capital*, Oxford: Clarendon Press.

Hicks, J.R. (1940-41), “The Rehabilitation of Consumers’ Surplus”, *The Review of Economic Studies* 8, 106-116.

Hicks, J.R. (1941-42), “Consumers’ Surplus and Index Numbers”, *The Review of Economic Studies* 9, 126-137.

Hicks, J.R. (1945-46), “The Generalized Theory of Consumers’ Surplus”, *The Review of Economic Studies* 13, 68-74.

Hicks, J.R. (1946), *Value and Capital*, Second Edition, Oxford: Clarendon Press.

Hotelling, H. (1938), “The General Welfare in Relation to Problems of Taxation and of Railway and Utility Rates”, *Econometrica* 6, 242-269.

Konüs, A.A. (1924), “The Problem of the True Index of the Cost of Living”, translated in *Econometrica* 7, (1939), 10-29.

Laspeyres, E. (1871), “Die Berechnung einer mittleren Waarenpreissteigerung”, *Jahrbücher für Nationalökonomie und Statistik* 16, 296-314.

Lau, L.J. (1979), “On Exact Index Numbers”, *Review of Economics and Statistics* 61, 73-82.

Marcinko, D. and E. Petri (1984), “Use of the Production Function in Calculation of Standard Cost Variances-An Extension”, *The Accounting Review* 59, 488-495.

Marshall, A. (1890), *Principles of Economics*, London: The Macmillan Co.

Marshall, A. (1898), *Principles of Economics*, Fourth Edition, Cambridge: Cambridge University Press.

Mensah, Y.M. (1982), “A Dynamic Approach to the Evaluation of Input-Variable Cost Center Performance”, *The Accounting Review* 57, 681-700.

Paasche, H. (1874), “Über die Preisentwicklung der letzten Jahre nach den Hamburger Borsennotirungen”, *Jahrbücher für Nationalökonomie und Statistik* 12, 168-178.

Pigou, A.C. (1924), *The Economics of Welfare*, Second Edition, London: Macmillan.

Pollak, R.A. (1983), “The Theory of the Cost-of-Living Index”, pp. 87-161 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics

Canada; reprinted as pp. 3-52 in R.A. Pollak, *The Theory of the Cost-of-Living Index*, Oxford: Oxford University Press, 1989; also reprinted as pp. 5-77 in *Price Level Measurement*, W.E. Diewert (ed.), Amsterdam: North-Holland, 1990.

Samuelson, P.A. (1947), *Foundations of Economic Analysis*, Cambridge MA: Harvard University Press.

Shephard, R.W. (1953), *Cost and Production Functions*, Princeton: Princeton University Press.

Sidgwick, H. (1883), *The Principles of Political Economy*, London: Macmillan.

Theil, H. (1967), *Economics and Information Theory*, Amsterdam: North-Holland Publishing.

Törnqvist, L. and E. Törnqvist (1937), Vilket är förhällandet mellan finska markens och svenska kronans köpkraft?", *Ekonomiska Samfundets Tidskrift* 39, 1-39 reprinted as pp. 121-160 in *Collected Scientific Papers of Leo Törnqvist*}, Helsinki: The Research Institute of the Finnish Economy, 1981.

Weitzman, M.L. (1988), "Consumer's Surplus as an Exact Approximation when Prices are Appropriately Deflated", *The Quarterly Journal of Economics* 102, 543-553.

Weymark, J. and Y.O. Vartia (1981), "Four Revealed Preference Tables", *Scandinavian Journal of Economics* 83, 408-418.