Lecture Notes on Cost Benefit Analysis January 1, 2011.

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Chapter 1: Introduction to Cost Benefit Analysis

1. Introduction

Cost benefit analysis is usually defined as the evaluation of the costs and benefits of undertaking specific *projects* or more broadly speaking, of undertaking alternative *economic policies*. Thus Sugden and Williams introduce their popular textbook on cost benefit analysis with the following paragraph:

"This book is about the appraisal of projects. A *project*, broadly defined, is a way of using resources; a decision between undertaking and not undertaking a project is a choice between alternative ways of using resources. *Project appraisal* is a process of investigation and reasoning designed to assist a decision-maker to reach an informed and rational choice." Robert Sugden and Alan Williams (1978; 3).

We will define cost benefit analysis as *the evaluation of the effects of alternative policy decisions in the context of an economic model*. Typically, the policy choices can be imbedded in a (small) general equilibrium model and so our general strategy in this course will be to formulate a model of consumer and producer behavior in the context of the policy problem at hand, introduce the policy change and attempt to determine how the producers and consumers in our model are affected by the variation in the policy instruments. Thus our approach to cost benefit analysis overlaps with several other areas of economics, including:

- Consumer theory including the measurement of benefits and welfare change;
- Producer theory including the measurement of various types of costs;
- National income accounting and financial accounting;
- Applied general equilibrium modeling;
- Public finance and the evaluation of alternative taxation and expenditure policies;
- Environmental economics;
- The economics of regulation and
- Econometrics (for estimating statistically preferences and technology).

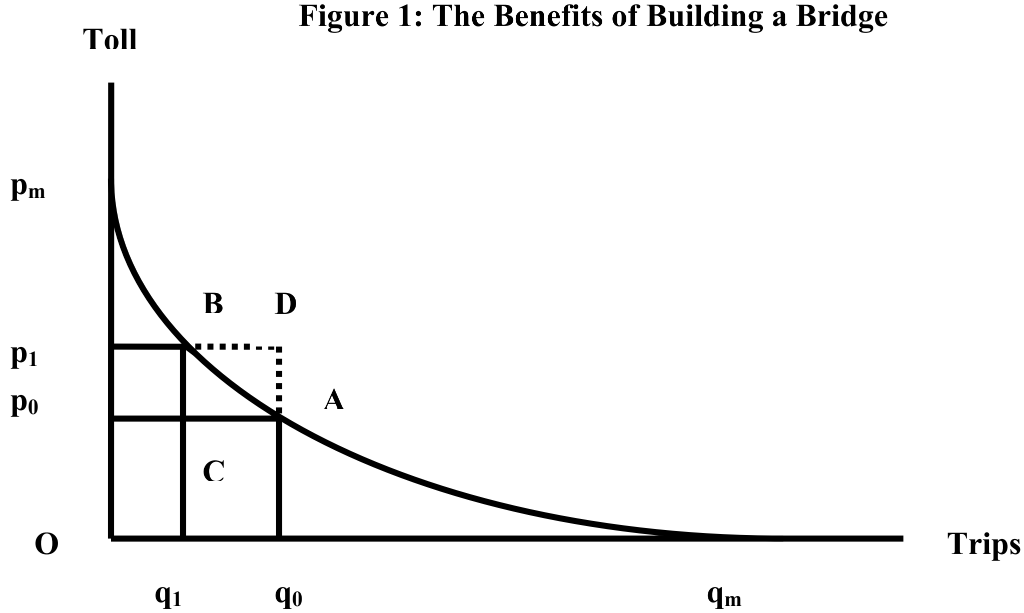
2. Dupuit's Bridge Example

To illustrate the types of questions that we will attempt to answer in the course, consider the following cost benefit model that dates back to the French engineer, Jules Dupuit (1952), originally published in 1844. Dupuit's basic model consisted of the following demand curve,[1] which related the number of trips across a bridge as a function of the toll that is charged per crossing:

[1] Dupuit (1969; 280-283) followed mathematical convention, which has the independent variable (price) on the horizontal axis and the dependent variable (quantity demanded) on the vertical axis. However, Figure 1 follows the modern economics convention, which puts price on the vertical axis and quantity demanded on the horizontal axis. This modern economics convention appears to be due to Marshall (1898; 203).

If the toll is set higher than the (maximum) price p_m, then no one uses the bridge. If the toll is set equal to 0, then there are q_m trips made across the bridge over the time period under consideration. If the toll is set equal to the price p_0, then users will make q_0 trips over the bridge.

Figure 1: The Benefits of Building a Bridge



The question we now ask is: what are the benefits to users of the bridge if the toll is decreased from the maximum price, p_m, to some other price, p_0? In order to answer this question, we will simplify the model by assuming that there is only a single consumer that uses the bridge and we will initially ask a slightly different question: how much has the consumer lost if the toll increases from p_0 to a slightly higher price p_1?[2]

When the crossing price is p_0, the consumer's preferred number of crossings is q_0. Thus when the price is p_0, the consumer prefers the point A in Figure 1 to the point C. Similarly, when the crossing price is p_1, the consumer's preferred number of crossings is q_1. Thus when the price is p_1, the consumer prefers the point B in Figure 1 to the point D.

Now consider the effects of moving the consumer from point A to point D; i.e., we increase the initial price p_0 to p_1, but force the consumer to continue consuming q_0 trips. The consumer's expenditure on bridge crossings will increase by the amount $(p_1-p_0)q_0$, which can be represented by the area p_0p_1DA in Figure 1. Thus if the consumer were

[2] The argument which follows is a modification of an argument due to Sugden and Williams (1978; 114-115) but many of the basic ideas date back to Dupuit and Marshall as we shall see.

given the amount of money $(p_1-p_0)q_0$, then he or she would be fully compensated for the price increase since the bundle of other goods and services could still be purchased by the consumer and the initial number of crossings q_0 could still be made. However, given normal substitution effects between bridge crossings and the consumption of other commodities, the consumer facing the higher price p_1 and given the additional amount of income $(p_1-p_0)q_0$ would not choose to stay at the point D: the consumer could attain a higher level of utility[3] by substituting away from bridge crossings and by using the extra income to purchase more of other goods and services. Thus the sum of money $(p_1-p_0)q_0$ represents an *upper bound to the welfare loss* of the consumer if the toll increases from p_0 to p_1.

Now consider the effects of moving the consumer from point B to point C; i.e., we begin at the equilibrium point B for the consumer where the price is p_1 and the number of crossings is q_1. Now decrease the initial price p_1 to p_0, but force the consumer to continue consuming q_1 trips. The consumer's expenditure on bridge crossings will decrease by the amount $(p_1-p_0)q_1$, which can be represented by the area p_0p_1BE in Figure 1. Thus if the amount of money $(p_1-p_0)q_1$ were taken away from the consumer, he or she could still attain the initial utility level that corresponded to the initial point B. However, after the toll has decreased to p_0 from p_1 and the amount of money $(p_1-p_0)q_1$ has been taken away from the consumer, the consumer will normally obtain an increased level of welfare (compared to the initial level attained at the point B) by increasing the number of crossings and decreasing expenditures on other commodities; i.e., the consumer will normally substitute towards the good whose price has decreased.[4] Thus the gain in

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[3] If the consumer's indifference curves between bridge crossings and other commodities were L shaped so that there were no substitution effects, then utility would not increase but it would remain the same.

[4] Dupuit and Marshall made similar arguments: "Consider the establishment of a water system in a town which, being situated at a high altitude, could previously procure water only at considerable trouble. Water then was so valuable that the supply of 1 hectoliter per day cost 50 francs, by annual subscription. It is obvious that each hectoliter consumed in these circumstances has a utility of 50 francs. With the installation of pumps this same quantity of water costs only 30 francs. What will happen? The inhabitant who was consuming 1 hectoliter will at first continue to do so and will derive a profit of 20 francs on this first hectoliter; but it is highly probable that the fall in price will induce him to increase his consumption; instead of using the water sparingly for personal purposes he will employ it also for less urgent and essential needs, the satisfaction of which is worth more than 30 francs to him—since that is the sacrifice he makes to obtain the water—but less than 50 francs, since at that price this consumption was foregone. Thus, of these two hectoliters supplied to the same individual by the public pumps, one has a utility greater than 50 francs, while the other has a utility of between 30 and 50 francs." Jules Dupuit (1969; 258).

"In order to give definiteness to our notions, let us consider the case of tea purchased for domestic consumption. Let us take the case of a man, who, if the price of tea were 20s a pound, would just be induced to buy one pound annually; who would just be induced to buy two pounds if the price were 14s, three pounds if the price were 10s, four pounds if the price were 6s, five pounds if the price were 4s, six pounds if the price were 3s, and who, the price being actually 2s, does purchase seven pounds. We have to investigate the consumer's surplus which he derives from his power of purchasing tea at 2s a pound. The fact that he would just be induced to purchase one pound if the price were 20s, proves that the total enjoyment or satisfaction which he derives from that pound is as great as that which he could obtain by spending 20s on other things. When the price falls to 14s, he could, if he chose, continue to buy only one pound. He would then get for 14s what was worth to him at least 20s; and he will obtain a surplus satisfaction worth to him at least 6s, or in other words a consumer's surplus of at least 6s. But in fact he buys a second pound of his own free choice, thus showing that he regards it as worth to him at least 14s. He obtains for 28 s what is worth to him at least 20s + 14s; i.e., 34s. His surplus satisfaction is at all events welfare to the consumer due to the price drop from p_1 to p_0 will normally[5] be *greater* than the sum of money $(p_1-p_0)q_1$. Thus the sum of money $(p_1-p_0)q_1$ represents a *lower bound to the welfare gain* to the consumer if the toll decreases from p_1 to p_0. Put the other way around, the sum of money $(p_1-p_0)q_1$ represents a *lower bound to the welfare loss* to the consumer if the toll increases from p_0 to p_1.

Thus we have two monetary estimates for the consumer's loss due to an increase in the toll from p_0 to p_1, where one estimate, $(p_1-p_0)q_0$, is an upper bound and another estimate, $(p_1-p_0)q_1$, is a lower bound to the loss of welfare. In order to obtain a final monetary measure of the loss in welfare to the consumer of a price increase from p_0 to p_1, it is natural to take the arithmetic average of the upper and lower bound, which leads to the following (approximate) measure of welfare loss due to the price change:

(1) $\Delta W_{CS} \equiv (1/2)[q_0 + q_1][p_1 - p_0]$.

From Figure 1, it can be seen that the measure of welfare loss due to the price increase from p_0 to p_1 can be represented by the trapezoidal area p_0p_1BA. Now if we broke up the increase in the toll from p_0 to p_1 into a series of smaller increases, it can be seen that in the limit, the overall measure of welfare loss of the price increase going from p_0 to p_1 can be represented by the curvilinear area p_0p_1BA; i.e., *it can be represented by the area to the left of the demand curve* between the prices p_0 and p_1. In the cost benefit literature, this is known as the *consumer surplus measure of welfare change*.[6]

Assuming that we can apply the same arguments as made above to an aggregate demand curve, it can be seen that a measure of the benefits of building a bridge and charging the toll p_0 can be represented by the area Ap_mp_0; i.e., by the area to the left of the aggregate demand curve and above the price p_0.

Now suppose that the toll bridge is privately built and that the maximum revenue that the operator of the bridge can extract out of the population occurs at the point q_0. This maximal revenue is $R_0 = p_0q_0$. Let the operator's costs per period be C_0. These costs consist of depreciation on the bridge, the opportunity costs of the financial capital tied up in the bridge and the operating and maintenance costs. If C_0 is less than R_0, then ignoring uncertainty, the bridge can be successfully built by a private operator and p_0 will be the profit maximizing toll for the operator.

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not diminished by buying it, but remains worth at least 6s to him. The total utility of the two pounds is worth at least 34 s, his consumer's surplus is at least 6s." Alfred Marshall (1898; 200-201).

[5] In the special case of L shaped indifference curves, the sum of money $(p_1-p_0)q_1$ will represent an exact monetary measure of welfare gain to the consumer of the price decrease instead of representing a lower bound to the gain.

[6] This terminology is due to Marshall (1898; 201). If b is the lowest price where the consumer demand for a commodity is 0 and a is an observed nonnegative price for the commodity, then Marshall (1898; 792) in his mathematical appendix defined the *total utility of the commodity* as follows: "If y be the price at which an amount x of a commodity can find purchasers in a given market, and y = f(x) be the equation to the demand curve, then the total utility of the commodity is measured by $\int_0^a f(x)dx$, where a is the amount consumed." To obtain the *consumer's surplus for the commodity*, the amount $p_a a$ is subtracted from the above integral where $p_a \equiv f(a)$.

However, now suppose that C_0 is slightly greater than the maximum period revenue R_0. In this case, the bridge will not be built. But if the government were to build the bridge and charge a zero toll, the total benefits to consumers would be the entire area to the left of the demand curve, Oq_mp_m, and this measure of benefits would easily exceed the operating cost measure, which was assumed to be slightly greater than p_0q_0 or the rectangular area Oq_0Ap_0. This is the type of argument used by Dupuit to justify government involvement in the building of public infrastructure.

Following the above argument to its logical conclusion, it appears that the government should build everything and produce everything and then sell its production at a zero price in order to maximize aggregate household welfare! Why is this not done in practice?

The basic problem is this: in order to build things or produce commodities, the government will require monetary resources in order to hire the inputs to do the production. These resources can only come from forced labour or from taxation. In either case, there will be a loss of efficiency (or in public finance terms, an excess burden) imposed by serfdom or taxation; i.e., government revenues and resources cannot be raised perfectly efficiently.

Thus the building of bridge (or more generally, the undertaking of any kind of a project) cannot be evaluated in isolation; we have to have some idea of what is going on in the rest of the economy. This is one reason why cost benefit analysis is so complicated; most policies have to be evaluated in some sort of a general equilibrium model.

The above bridge example, although extremely simple, already introduces one into the complexities of cost benefit analysis. The example raises many questions:

- The measure of welfare change was regarded as an approximation to the actual change in welfare; how accurate is this approximation?
- The consumer surplus welfare change measure (1) was derived for a single consumer; can it be generalized to many consumers? Can benefits be simply added up across households?
- The example dealt with a change only in a single market; what happens when the project is rather complex, involving many markets?
- The bridge is a durable input; i.e., it produces a stream of services over many periods. How should the initial capital cost of the bridge be allocated across the periods of its use?
- Our diagram of the demand curve applied to only one period of use. Over time, we could expect demand to *increase* (or in the case of many European countries and Japan, to *decrease* due to depopulation) and costs may also change over time due to increased real wage rates for workers and increased maintenance charges. How can these intertemporal complications be taken into account?

- We mentioned that the opportunity cost of the financial capital that was required to construct the bridge was a cost that should be taken into account. But how exactly should this cost be determined? Is it a nominal or real interest rate that should be considered and exactly which interest rate should be used?
- How should project costs and benefits be adjusted if future costs and benefits are uncertain?
- Are there methods for estimating the government's marginal cost of raising an additional dollar of tax revenue? And if so, how can these estimates of marginal excess burden be incorporated into cost benefit analysis?

In order to attempt to answer the above questions (and many others), our strategy in this course is the following one:

(i) We will study basic consumer and producer theory using duality theory. Duality theory is useful because it may be used to dramatically reduce the number of equations in a general equilibrium model.
(ii) Secondly, we will set up some simple (static) general equilibrium models and show how project evaluation "works" in this context.
(iii) We will look at methods for measuring discrete welfare change; i.e., methods for evaluating the welfare change between two observed equilibria.
(iv) Next, we will look at intertemporal general equilibrium models.
(v) Tax distorted general equilibrium models and shadow pricing models is our next topic along with measuring the excess burdens of taxation.
(vi) Our final set of topics is concerned with modeling risk and uncertainty. This material will be applied to the discussion of what discount rate to use in project evaluation.

This course is much more theoretical than the "average" course in cost benefit analysis but our goal is to give students an adequate toolkit so that they may use the techniques discussed in the course in order to construct their own cost benefit models.[7] Many students will end up being applied economists, working for either an economic consulting

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[7] It is interesting to note that Dupuit, along with Cournot, were two of the first economists to appreciate the usefulness of mathematics in the study of economics. Dupuit concludes his paper with the following observations on the usefulness of mathematical modeling: "In presenting, in this note, some of the principles of our science in this particular form, it was our wish to try and make clear how great would be the advantages of an alliance with mathematics, despite the anathema which economists of all times have pronounced against the latter. So soon as it is realized, with J.B. Say, that political economy is concerned with quantities susceptible of a more or a less, it must also be recognized that it is in the realm of mathematics. If one has gone astray in political economy every time one has relied on mathematical calculations, it is because there are mathematicians who make false calculations, just as there are logicians who produce false arguments: the former no more invalidate mathematics than the latter invalidate logic, which alone is sometimes regarded as a science. Not only do the symbols and drawings of mathematics give body and form to abstract ideas and thereby call the senses to the aid of man's intellectual power, but its formulae take hold of these ideas, modify them, and transform them, and bring to light everything which is true, right, and precise in them, without forcing the mind to follow all the motions of a wheelwork the course of which has been established once for all. They are machines which, at a certain stage, can think for us, and there is as much advantage in using them as there is in using those which, in industry, labour for us." Jules Dupuit (1969; 283).

firm or for governments or regulators. In each of these alternative lines of work, it is likely that the materials to be covered will be found to be at least somewhat useful.

The most important part of being an applied economist is the ability to construct a model that captures *the most important aspects of the problem under consideration*. Unfortunately, the real world is a lot more complicated than any model that can be constructed so model construction is not a completely scientific undertaking; it has large artistic elements in it.

There are many practical issues in cost benefit analysis that we will not be able to cover in this course (which is really a course in advanced microeconomic theory). Thus the student is advised to read a good undergraduate textbook on cost benefit analysis if he or she wants to work in this area as an analyst. Two such texts are Campbell and Brown (2003) and Boardman, Greenberg, Vining and Weimer (2006).

In order to give students a flavour of what will be covered in subsequent chapters (and the techniques that will be used), in the next section, we will provide an introduction to *measures of welfare change*. These measures are an important part of cost benefit analysis.

3. Measures of Welfare Change and Willingness to Pay for a Single Consumer

The starting point for deriving practical measures of welfare change between two situations that are characterized by the price vectors that the consumer faces in periods 0 and 1, p^0 and p^1, and the observed consumption quantity vectors q^0 and q^1 for this consumer is the *consumer's cost or expenditure function* C.[8] Thus suppose that the consumer has preferences that are defined by the *utility function* f(q) over all nonnegative N dimensional quantity vectors $q \equiv [q_1,...,q_N] \geq 0_N$.[9] In addition, suppose that f is a nonnegative, increasing,[10] continuous and quasiconcave function over the nonnegative orthant $\Omega \equiv \{q : q \geq 0_N\}$. Now suppose that the consumer faces the positive vector of commodity prices $p >> 0_N$ and suppose that the consumer wishes to attain the utility level u belonging to the range of f as cheaply as possible. Then the consumer will solve the following *cost minimization problem* and the consumer's *cost or expenditure function*, C(u,p), will be the minimum cost of achieving the target utility level u:

(2) $C(u,p) \equiv \min_q \{p \cdot q : f(q) \geq u ; q \geq 0_N\}$.

It can be shown[11] that C(u,p) will have the following properties: (i) C(u,p) is jointly continuous in u,p for $p >> 0_N$ and $u \in U$ where U is the range of f and is a nonnegative function over this domain of definition set; (ii) C(u,p) is increasing in u for each fixed p

[8] The analysis in this section and the following section follows that of Diewert and Mizobuchi (2009).
[9] Notation: $q \geq 0_N$ means each component of q is nonnegative; $q >> 0_N$ means each component of q is positive and $q > 0_N$ means $q \geq 0_N$ but $q \neq 0_N$ where 0_N denotes an N dimensional vector of zeros. Also $p \cdot q$ denotes the inner product of the vectors p and q; i.e., $p \cdot q = p^T q \equiv \sum_{n=1}^{N} p_n q_n$.
[10] Thus if $q^2 >> q^1 \geq 0_N$, then $f(q^2) > f(q^1)$.
[11] See Diewert (1993; 124) and Chapter 4 below.

and (iii) $C(u,p)$ is nondecreasing, linearly homogeneous and concave function of p for each $u \in U$.[12] Conversely, if a cost function is given and satisfies the above properties, then the utility function f that is dual to C can be recovered as follows.[13] For $u \in U$ and $q \gg 0_N$, define the function $F(u,q)$ as follows:

(3) $F(u,q) \equiv \max_p \{C(u,p) : p \cdot q \le 1 \,;\, p \ge 0_N\}$.

Now solve the equation:

(4) $F(u,q) = 1$

for u^* and this solution u^* will equal $f(q)$.

Suppose the consumer has general preferences defined by the utility function $f(q)$ and the general cost function $C(u,p)$ is dual to f. Let p^t and q^t be the observed price and quantity data pertaining to period t and define the period t level of utility $u^t \equiv f(q^t)$ for $t = 0,1$. We assume that the consumer is minimizing the cost of achieving the utility level u^t in period t so we have:

(5) $p^t \cdot q^t = C(f(q^t),p^t)$; $\qquad t = 0,1.$

Our task in the present section is to decompose the consumer's observed value change over the two periods under consideration, $p^1 \cdot q^1 - p^0 \cdot q^0$, into the sum of two terms, one of which is the part of the value change that is due to price change and the other part due to quantity change. This is the *difference approach* to explaining a change in a value aggregate as opposed to the usual *ratio approach* used in index number theory.[14]

The difference counterpart to the Allen (1949) quantity index[15] is the following *Hicks Samuelson quantity variation* Q_S: for each strictly positive reference price vector $p \gg 0_N$, define $Q_S(q^0,q^1,p)$ as follows:[16]

(6) $Q_S(q^0,q^1,p) \equiv C(f(q^1),p) - C(f(q^0),p)$.

Just as the Allen quantity index $Q_A(q^0,q^1,p)$ is an entire family of indexes (one for each reference price vector p), so too is the family of quantity variations, Q_S. Two special cases of (6) are of particular importance, the *equivalent and compensating variations*, Q_E and Q_C, defined as follows:[17]

[12] Call these conditions on the cost function Conditions I.
[13] See Diewert (1974; 119) (1993; 129) for the details and for references to various duality theorems.
[14] Hicks (1942) seems to have been the first to explore the similarities between the two approaches.
[15] For each reference price vector $p \gg 0_N$, the Allen quantity index is defined as $C(f(q^1),p)/C(f(q^0),p)$.
[16] McKenzie (1957) and Samuelson (1974) recognized that $C(f(q),p)$ was a valid cardinalization of utility for any reference price vector p and thus (6) is a valid cardinal measure of the utility difference between periods 0 and 1. Hicks on the other hand only considered the special cases (7) and (8) defined below.
[17] Henderson (1941; 120) introduced these variations in the N = 2 case and Hicks (1942) introduced them in the general case, although his exposition is difficult to follow. The term compensating variation is due to Henderson (1941; 118) and the term equivalent variation is due to Hicks (1942; 128). Hicks (1939; 40-41)

(7) $Q_E(q^0,q^1,p^0) \equiv Q_S(q^0,q^1,p^0) = C(f(q^1),p^0) - C(f(q^0),p^0)$;
(8) $Q_C(q^0,q^1,p^1) \equiv Q_S(q^0,q^1,p^1) = C(f(q^1),p^1) - C(f(q^0),p^1)$.

Thus the equivalent variation uses the period 0 price vector p^0 as the reference price vector while the compensating variation uses the period 1 price vector p^1 as the reference price vector. It is clear that the measures defined by (7) and (8) are cardinal measures of welfare change.

We will now define a family of measures of consumer *willingness to pay* for changes in prices going from the period 0 situation where prices are p^0 and the period 1 situation where prices are p^1. This family of willingness to pay measures is also known as a family of *price variation functions*; see Diewert and Mizobuchi (2009).[18] Thus generalizing Hicks (1939; 40-41) (1946; 331-332), define a family of *Hicksian price variation functions* $P_H(p^0,p^1,f(q))$ as follows: for each nonnegative, nonzero reference quantity vector q, define $P_H(p^0,p^1,f(q))$ as follows:[19]

(9) $P_H(p^0,p^1,f(q)) \equiv C(f(q),p^1) - C(f(q),p^0)$.

The interpretation of the measure (9) is as follows: if the consumer is initially at a situation where his or her utility is f(q) and he or she faces prices p^0, then $P_H(p^0,p^1,f(q))$ is the additional amount of income the consumer will need to exactly compensate him or her for a change in prices from p^0 to p^1, while holding utility constant at the level f(q). Two special cases of (9) are of particular importance, the *Laspeyres and Paasche price variation functions*, P_{HL} and P_{HP}, defined as follows:[20]

(10) $P_{HL}(p^0,p^1,f(q^0)) \equiv P_H(p^0,p^1,f(q^0)) = C(f(q^0),p^1) - C(f(q^0),p^0)$;[21]
(11) $P_{HP}(p^0,p^1,f(q^1)) \equiv P_H(p^0,p^1,f(q^1)) = C(f(q^1),p^1) - C(f(q^1),p^0)$.[22]

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initially defined the compensating variation as a measure of price change (see (9) below): "As we have seen, the best way of looking at consumer's surplus is to regard it as a means of expressing, in terms of money income, the gain which accrues to the consumer as a result of a fall in price. Or better, it is the *compensating variation* in income, whose loss would just offset the fall in price and leave the consumer no better off than before." However, later, Hicks (1942; 127-128), following Henderson (1941; 120) defined (geometrically) the compensating variation as $C(u^1,p^1) - C(u^0,p^1)$ and the equivalent variation as $C(u^1,p^0) - C(u^0,p^0)$, which are measures of welfare (or quantity) change. The terminology here is a bit confusing!

[18] It should be emphasized that the family of welfare change measures defined by (6) is of fundamental importance whereas the family of willingness to pay measures (9) is of secondary importance.

[19] The family of price variation functions defined by (9) is the difference counterpart to the family of Konüs (1939; 17) *true cost of living indexes*, which uses a ratio concept.

[20] In the index number literature, $C(u^0,p^1)/C(u^0,p^0)$ is known as the Laspeyres Konüs (1939; 17) *true cost of living index* or price index and $C(u^1,p^1)/C(u^1,p^0)$ is known as the Paasche Konüs theoretical price index; see Pollak (1983). It can be seen that (10) and (11) are the difference counterparts to these ratio type indexes.

[21] Hicks (1945-46; 68) called this measure the 'price compensating variation' and distinguished this measure from a quantity compensating variation, which he did not define in a very clear manner. Hicks also considered price and quantity variations in Hicks (1943).

[22] Hicks (1945-46; 69) called this measure the "price equivalent variation".

Thus the Laspeyres price variation uses the period 0 quantity vector q^0 as the reference quantity vector while the Paasche price variation uses the period 1 quantity vector q^1 as the reference quantity vector.

Let $M^0 \equiv p^0 \cdot q^0$ be the consumer's nominal "income" or expenditure on the N commodities in period 0. Then $P_{HL}(p^0,p^1,f(q^0))$ is the amount of nominal income that must be added to the period 0 income M^0 in order to allow the consumer, facing period 1 prices p^1, to achieve the same utility level as was achieved in period 0, which is $u^0 \equiv f(q^0)$. Similarly, let $M^1 \equiv p^1 \cdot q^1$ be the consumer's nominal "income" in period 1. Then $P_{HP}(p^0,p^1,f(q^1))$ is the amount of nominal income that must be subtracted from the period 1 income M^1 in order to allow the consumer, facing period 0 prices p^0, to achieve the same utility level as was achieved in period 1, which is $u^1 \equiv f(q^1)$.[23]

Note that the equivalent quantity variation defined by (7) matches up with the Paasche price variation defined by (11) in order to provide an *exact decomposition* of the value change going from period 0 to 1; i.e., using these definitions and assumptions (5), it can be seen that:

(12) $p^1 \cdot q^1 - p^0 \cdot q^0 = C(f(q^1),p^1) - C(f(q^0),p^0) = Q_E(q^0,q^1,p^0) + P_{HP}(p^0,p^1,f(q^1)).$

Similarly, the compensating quantity variation defined by (8) matches up with the Laspeyres price variation defined by (10) in order to provide another *exact decomposition* of the value change going from period 0 to 1:

(13) $p^1 \cdot q^1 - p^0 \cdot q^0 = C(f(q^1),p^1) - C(f(q^0),p^0) = Q_C(q^0,q^1,p^1) + P_{HL}(p^0,p^1,f(q^0)).$

We will conclude this section by relating the willingness to pay measures (or measures of expenditure change due to price changes holding utility constant) defined by (10) and (11) to areas under demand curves. But first, we require a result from microeconomic theory; namely Shephard's Lemma.[24]

Shephard's (1953; 11) *Lemma*: If the cost function C(u,p) satisfies the properties listed below equation (2) above and in addition is once differentiable with respect to the components of prices at the point (u,p) where u is in the range of the utility function f and $p >> 0_N$, then

(14) $q^* = \nabla_p C(u,p)$

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[23] Note that these arguments provide a generalization (to the case of arbitrary price changes) of the arguments made in section 1 about consumer willingness to pay for changes in a single price.

[24] We will give a proof of this result in Chapter 4 below. Shephard (1953) was the first person to establish the above result starting with just a cost function satisfying the appropriate regularity conditions. However, Hotelling (1932; 594) stated a version of the result in the context of profit functions and Hicks (1946; 331) and Samuelson (1953-54; 15-16) established the result starting with a differentiable utility or production function.

where $\nabla_p C(u,p)$ is the vector of first order partial derivatives of expenditure with respect to prices, $[\partial Cu^*,p)/\partial p_1,..,\partial C(u,p)/\partial p_N]^T$, and q^* is any solution to the cost minimization problem

(15) $\min_q \{p^Tq: f(q) \geq u\} \equiv C(u,p)$.

Under these differentiability hypotheses, it turns out that the q^* solution to (15) is unique.

Thus if C(u,p) satisfies the appropriate regularity conditions and the partial derivatives of C(u,p) with respect to the components of $p \equiv [p_1,p_2,\ldots,p_N]$ exist, then the system of *Hicksian demand functions*, $q_n(u,p)$ for $n = 1,2,\ldots,N$ can be formed by simple partial differentiation; i.e., we have

(16) $q_n(u,p) = \partial C(u,p)/\partial p_n$; $\qquad n = 1,\ldots,N.$

These demand functions tell us how the consumer will change his or her demands for commodities as prices change, holding utility constant.

Using (16), define the *Hicksian demand function for commodity 1* as a function of the price for commodity 1, p_1, holding utility constant at u^0 and the price of commodity 2,…,N constant at their period 0 levels, $p_2^0,\ldots,p_N^0$, as follows:

(17) $q_1(p_1) \equiv q_1(u^0,p_1,p_2^0,\ldots,p_N^0) = \partial C(u^0,p_1,p_2^0,\ldots,p_N^0)/\partial p_1$.

Now consider a change in the price of commodity 1 from p_1^0 to p_1^1. From (17), it can be seen that $q_1(p_1)$ is an exact differential and thus the theory of integration tells us that the following equality is satisfied:

$$(18) \int_{p_1^0}^{p_1^1} q_1(p_1)dp_1 = \int_{p_1^0}^{p_1^1} [\partial C(u^0,p_1,p_2^0,\ldots,p_N^0)/\partial p_1]dp_1$$
$$= C(u^0,p_1^1,p_2^0,\ldots,p_N^0) - C(u^0,p_1^0,p_2^0,\ldots,p_N^0)$$
$$= P_H(p^0;\, p_1^1,p_2^0,\ldots,p_N^0;\, f(q^0)) \qquad \text{using definition (9).}$$

Thus the *area under the Hicksian demand curve* $q_1(p_1)$ defined by (17), going from the price p_1^0 to p_1^1, is equal to a *Hicksian price variation* where the utility level is held constant at u^0, the base period price vector is $p^0 \equiv [p_1^0,p_2^0,\ldots,p_N^0]$ and the comparison price vector is the "mixed" vector $[p_1^1,p_2^0,\ldots,p_N^0]$. Note that the integral on the left hand side of (18) can be approximated by the following expression:

$$(19) \int_{p_1^0}^{p_1^1} q_1(p_1)dp_1 \approx (1/2)[q_1(p_1^0) + q_1(p_1^1)][p_1^1 - p_1^0] = (1/2)[q_1^0 + q_1(p_1^1)][p_1^1 - p_1^0].$$

The last expression on the right hand side of (19) is *almost* an observable approximation to the integral; the problem is that $q_1(p_1^1) = \partial C(u^0,p_1^1,p_2^0,\ldots,p_N^0)/\partial p_1$ cannot be evaluated without a knowledge of the consumer's expenditure function C or the dual utility function f.

All of the price and quantity variations defined above cannot be evaluated in general using observed price and quantity data pertaining to the two periods under consideration. Thus we now turn our attention to the problem of finding *observable approximations* to the above theoretical variation functions. We will assume that we can observe the price vectors that the consumer faces in two situations, say p^0 and p^1, and the corresponding two consumption vectors, q^0 and q^1. We will look for approximations to (7) and (8) and (10) and (11) that depend only on these 4 vectors.

Problems

1. Prove that (12) and (13) hold (under appropriate assumptions).

2. Let N = 2 and suppose that the consumer's cost function $C(u,p_1,p_2)$ satisfies the regularity conditions listed below equation (2) in the text and in addition, is differentiable with respect to the commodity prices p_1 and p_2. Assume also that conditions (5) hold. Define the Hicksian demand function for commodity 2, $q_2(p_2)$ as follows:

(i) $q_2(p_2) \equiv q_2(u^0,p_1^{\,1},p_2) = \partial C(u^0,p_1^{\,1},p_2)/\partial p_2$.

(a) Prove that $\int_{p_1^0}^{p_1^1} q_1(p_1)dp_1 + \int_{p_2^0}^{p_2^1} q_2(p_2)dp_2 = P_{HL}(p^0,p^1,f(q^0))$

where $P_{HL}(p^0,p^1,f(q^0))$ is the Laspeyres price variation for N = 2 defined above by (10).
(b) Prove a result that is similar to (a) for the case N = 2, except that the Paasche price variation defined by (11) is on the right hand side of your counterpart to (a).
(c) Generalize the result (a) to the case where N = 3.

4. Observable First Order Approximations to the Price and Quantity Variations

Looking at definition (7) for the equivalent variation, it can be seen that the term $C(f(q^0),p^0)$ is equal to period 0 expenditure on the N commodities, $p^0 \cdot q^0$, and hence this term is observable under our assumptions (5), which we maintain throughout this section. The remaining term, $C(f(q^1),p^0)$, is not observable but we can use Shephard's Lemma if $C(u,p)$ is differentiable with respect to prices in order to obtain the following first order approximation to this term:

$$
\begin{aligned}
(20)\ C(f(q^1),p^0) &\approx C(f(q^1),p^1) + \nabla_p C(f(q^1),p^1) \cdot [p^0 - p^1] \\
&= C(f(q^1),p^1) + q^1 \cdot [p^0 - p^1] && \text{using Shephard's Lemma} \\
&= p^1 \cdot q^1 + p^0 \cdot q^1 - p^1 \cdot q^1 && \text{using (5) for } t = 1 \\
&= p^0 \cdot q^1 .
\end{aligned}
$$

Using (5) for t = 0, (20) and definition (7), we obtain the following *first order approximation to the equivalent variation*:

$$
\begin{aligned}
(21)\ Q_E(q^0,q^1,p^0) &\approx p^0 \cdot q^1 - p^0 \cdot q^0 \\
&= p^0 \cdot [q^1 - q^0]
\end{aligned}
$$

$$\equiv V_L(p^0,p^1,q^0,q^1)$$

where the observable *Laspeyres indicator of quantity change*, $V_L(p^0,p^1,q^0,q^1)$, is defined as $p^0\cdot[q^1 - q^0]$, the inner product of the base period prices p^0 with the quantity change vector, $q^1 - q^0$. In a similar fashion, it can be shown that a first order approximation to the term $C(f(q^0),p^1)$ is $p^1\cdot q^0$ and so a *first order approximation to the compensating variation* $Q_C(q^0,q^1,p^1)$ defined by (8) is:[25]

$$(22)\ Q_C(q^0,q^1,p^1) \approx p^1\cdot q^1 - p^1\cdot q^0 = p^1\cdot[q^1 - q^0] \equiv V_P(p^0,p^1,q^0,q^1)$$

where the observable *Paasche indicator of quantity change*, $V_P(p^0,p^1,q^0,q^1)$, is defined as $p^1\cdot[q^1 - q^0]$, the inner product of the current period prices p^1 with the quantity change vector, $q^1 - q^0$.

Note that V_L and V_P are the difference counterparts to the ordinary *Laspeyres* and *Paasche quantity indexes*, Q_L, and Q_P, defined as follows:

$$(23)\ Q_L(p^0,p^1,q^0,q^1) \equiv p^0\cdot q^1/p^0\cdot q^0\ ;\ \ Q_P(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/p^1\cdot q^0.$$

We now turn our attention to the problem of finding observable approximations for the Laspeyres and Paasche price variation functions defined by (10) and (11) above. An observable first order approximation to the term $C(f(q^0),p^1)$ in (10) is

$$\begin{aligned}(24)\ C(f(q^0),p^1) &\approx C(f(q^0),p^0) + \nabla_p C(f(q^0),p^0)\cdot[p^1 - p^0] \\ &= C(f(q^0),p^0) + q^0\cdot[p^1 - p^0] && \text{using Shephard's Lemma} \\ &= p^0\cdot q^0 + p^1\cdot q^0 - p^0\cdot q^0 && \text{using (5) for } t = 0 \\ &= p^1\cdot q^0.\end{aligned}$$

Using (5) for $t = 0$, (24) and definition (10), we obtain the following *first order approximation to the Laspeyres price variation*:

$$(25)\ P_{HL}(p^0,p^1,f(q^0)) \approx p^1\cdot q^0 - p^0\cdot q^0 = q^0\cdot[p^1 - p^0] \equiv I_L(p^0,p^1,q^0,q^1)$$

where the observable *Laspeyres indicator of price change*, $I_L(p^0,p^1,q^0,q^1)$, is defined as $q^0\cdot[p^1 - p^0]$, the inner product of the base period quantity vector q^0 with the price change vector, $p^1 - p^0$. In a similar fashion, it can be shown that a first order approximation to the term $C(f(q^1),p^0)$ is $p^0\cdot q^1$ and so a *first order approximation to the Paasche price variation* $P_{HP}(p^0,p^1,f(q^1))$ defined by (11) is:

[25] The first order approximations (20) and (21) were obtained by Hicks (1942; 127-134); see also Diewert (1992a; 568).

$$(26)\ P_{HP}(p^0,p^1,f(q^1)) \approx p^1\cdot q^1 - p^0\cdot q^1 = q^1\cdot[p^1 - p^0] \equiv I_P(p^0,p^1,q^0,q^1)$$

where the observable *Paasche indicator of price change*, $I_P(p^0,p^1,q^0,q^1)$, is defined as $q^1\cdot[p^1 - p^0]$, the inner product of the current period quantity vector q^1 with the price change vector, $p^1 - p^0$.[26]

Note that I_L and I_P[27] are the difference counterparts to the ordinary *Laspeyres* and *Paasche price indexes*, P_L, and P_P, defined as follows:

$$(27)\ P_L(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^0/p^0\cdot q^0\ ;\ \ P_P(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/p^0\cdot q^1.$$

In the usual approach to index number theory, it proves to be useful to take the *geometric* average of the Laspeyres and Paasche price indexes, leading to the Fisher (1922) ideal price index P_F, since the Fisher index has very good properties from the viewpoint of the test or axiomatic approach to index number theory; see Diewert (1992b) and Balk (1995). However, in the axiomatic approach[28] to price and quantity measurement in the difference context, it proves to be better to take the *arithmetic* average of the Paasche and Laspeyres indicators. This leads to the *Bennet* (1920) *indicators of price and quantity change* defined as follows:[29]

$$(28)\ I_B(p^0,p^1,q^0,q^1) \equiv (1/2)I_L(p^0,p^1,q^0,q^1) + (1/2)I_P(p^0,p^1,q^0,q^1) = (1/2)[q^0+q^1]\cdot[p^1-p^0]\ ;$$
$$(29)\ V_B(p^0,p^1,q^0,q^1) \equiv (1/2)V_L(p^0,p^1,q^0,q^1) + (1/2)V_P(p^0,p^1,q^0,q^1) = (1/2)[p^0+p^1]\cdot[q^1-q^0].$$

We note that Hicks (1942; 134) (1945-46; 73) obtained the Bennet quantity indicator V_B as an approximation to the arithmetic average of the equivalent and compensating variations and he also identified V_B as a generalization to many markets of Marshall's consumer surplus concept.

[26] The first order approximations (25) and (26) were obtained by Hicks (1945-46; 72-73) (1946; 331).

[27] Hicks (1942; 128) (1945-46; 71) called I_L and I_P the Laspeyres and Paasche variations but we will reserve the term "variation" for the (unobservable) theoretical measures of price and quantity change defined by (6) for changes in quantities and by (9) for changes in prices. We will follow Diewert (1992a; 556) (2005; 313) and use the term "indicator" to denote a given function of the price and quantity data pertaining to the two periods under consideration so that the term *indicator* becomes the difference theory counterpart to an *index number formula* in the ratio approach to the measurement of price and quantity change. Since P and Q are usually used to denote price and quantity indexes, a different notation is required to denote price and quantity indicators. Using I to denote a price indicator and V to denote a quantity (or volume) indicator follows the conventions used by Diewert (2005). Note that national income accountants use the term "volume index" to denote a quantity index.

[28] See Diewert (2005) and Balk (2007) on the axiomatic approach to measures of price and quantity change using differences.

[29] Note that (28) is a generalization of (1) above.

It can be verified that the Laspeyres, Paasche and Bennet price and quantity indicators can be used in order to obtain the following exact decompositions of the value change in the aggregate over the two periods under consideration:

(30) $p^1\cdot q^1 - p^0\cdot q^0 = I_L(p^0,p^1,q^0,q^1) + V_P(p^0,p^1,q^0,q^1)$;
(31) $p^1\cdot q^1 - p^0\cdot q^0 = I_P(p^0,p^1,q^0,q^1) + V_L(p^0,p^1,q^0,q^1)$;
(32) $p^1\cdot q^1 - p^0\cdot q^0 = I_B(p^0,p^1,q^0,q^1) + V_B(p^0,p^1,q^0,q^1)$.

It would seem that taking an average of two first order approximations as in (29) should give us close to a second order approximation to either the equivalent or the compensating variation. However, this turns out not to be the case. The problem is this: suppose $p^0\cdot[q^1 - q^0]$ is positive while $p^1\cdot[q^1 - q^0]$ is negative. Now suppose there can be inflation or deflation in period 1 so that the price vector for period 1 is $\lambda > 0$ times the initial p^1. Under these circumstances, the Bennet indicator of quantity change, V_B defined by (29) becomes:

(33) $V_B(p^0,\lambda p^1,q^0,q^1) = (1/2)[p^0 + \lambda p^1]\cdot[q^1-q^0]$.

As λ approaches 0, $V_B(p^0,\lambda p^1,q^0,q^1)$ will be positive, indicating that the consumer has experienced a welfare gain going from the period 0 situation to the period 1 situation but as λ approaches infinity, $V_B(p^0,\lambda p^1,q^0,q^1)$ will become negative, indicating a welfare loss. Thus changing the scale of prices in period 1 can change the sign of the welfare change, which is not a satisfactory situation! We will return to this apparent problem with our analysis here in Chapter 6 and provide a satisfactory solution to this paradox. But before proceeding further, we will backtrack a bit an review some microeconomic theory which will prove to be useful later on.

Problem

3. Show that (30)-(32) hold.

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