Ambiguity in Corporate Finance:
Real Investment Dynamics*

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Abstract

We examine ambiguity in a model of corporate investment with follow up expansion and contraction options. We compare two specific multi-prior representations of ambiguity: Gilboa and Schmeidler’s (1989) Minimum Expected Utility (MEU) that relies on complete preferences and Bewley’s (1986) Expected Utility (BEU) that relies on incomplete preferences. We show that these two approaches deliver significantly different predictions for real investment decisions. MEU predicts reluctance to expand and eagerness to abandon. BEU predicts reluctance to both expand and abandon projects, consistent with “escalating commitment.” Moreover, anticipation of future reluctance to abandon may induce BEU entrepreneurs to forego initial investment altogether. We show that financial contracts, such as convertible bonds, can offset the reluctance to abandon and thereby enable the initial investment. Importantly, because BEU captures group decisions making under unanimity, the framework is well suited to guide empirical and theoretical studies of team decisions at the corporate level. More broadly, we argue that ambiguity matters for corporate finance because it emerges from the natural incompleteness of preferences of a heterogeneous corporate board.
1 Introduction

Most of finance theory rests on the Bayesian paradigm of Subjective Expected Utility (SEU) axiomatized by Savage (1954). An agent that satisfies Savage’s axioms behaves as if he maximizes expected utility with respect to a unique probability measure, which is interpreted as his “subjective” probability or “beliefs”.

There is widespread concern, however, that SEU often fails to explain observed behavior, a concern that has led to the development of alternative models of decision making. For the most part the alternatives found in corporate finance have preserved SEU but added a behavioral bias to the application of Bayes rule.

A fundamentally different challenge to SEU comes from its inability to capture the degree of confidence that an agent might have in his own subjective probability assessment. Ellsberg (1961), in his famous thought experiments, shows how this limitation may lead individuals to violate Savage’s axioms and express preferences that cannot be represented via a single probability measure. Ellsberg refers to situations in which probabilities are not known as ambiguous, as opposed to risky, which describes situations with known probabilities. Although Ellsberg’s experiments emphasize the subjects’ aversion to ambiguity, there are also studies that have shown situations in which subjects are ambiguity seeking. Regardless of the direction of the impact, it is clear that the existing evidence in several fields suggests that ambiguity and attitudes toward ambiguity are important for choice. Despite this evidence, relatively little is known about the impact of ambiguity on corporate decisions.

In this paper we examine the implications of ambiguity for corporate investment decisions. Because corporations allocate resources between ambiguous and unambiguous opportunities, understanding how ambiguity affects corporate decisions seems essential to understanding allocative efficiency. This understanding is also important for asset prices since real decisions directly influence returns. Moreover, the relevance of ambiguity for corporate investment ex-

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1 Ramsey (1926) and de Finetti (1931) lay the intellectual foundations for the “subjectivist” view of probability that culminated in Savage’s axiomatization.

2 See Baker and Wurgler (2011) for a summary of the behavioral corporate finance literature.

3 One of Ellsberg’s experiments involves two urns with 100 balls each. In the first urn, the unambiguous urn, the subject is told that there are 50 white and 50 blue balls. In the second urn, the ambiguous urn, no information is given on the proportion of white and blue balls. The subject is offered a gamble where they are paid a prize if a white ball is drawn from the urn of their choice. Most subjects choose to draw from the unambiguous urn. They are then offered a second gamble where they are paid the same prize if a blue ball is drawn from an urn of their choice. Again, most subject typically choose to draw from the unambiguous urn. This behavior cannot be justified by any probability distribution since it implies that the subject believes that the probability of a specific outcome (drawing a blue ball from the ambiguous urn) is both less than and greater than 50%. It also cannot be explained by overconfidence since the overconfidence applies to both gambles.

tends well beyond the obvious case of a single entrepreneur deciding on whether to start a new business with ambiguous future prospects. In fact, when corporate decisions are undertaken by a group of members with heterogenous beliefs, e.g., a board, the investment opportunities are *de facto* ambiguous to the board as a whole: the decision maker, i.e., the board, may not be able to assign a unique probability to the outcome of the investment, whether or not the investment is unambiguous for each board member.

We follow a literature that captures ambiguity through a non-Bayesian, “multi-prior” approach to decision making. In this literature, ambiguity is formally modeled as a set of possible priors (beliefs) that an agent entertains with regard to uncertain situations. Unlike a Bayesian, the agent cannot reduce his set of beliefs to a single prior. An important distinction between alternative multi-prior models of decision making is whether or not they require the decision maker to have complete preferences, i.e., whether he can always rank alternative choices. Complete preferences can conveniently be represented by a “utility index”, making them tractable in optimization problems. The same is not true for incomplete preferences: for an agent with incomplete preferences there might be choices he simply cannot rank.5 In this paper we compare and contrast both types of models.

Gilboa and Schmeidler (1989) are the first to axiomatize decision making in the presence of ambiguity in a multi-prior framework with complete preferences. According to their *Minimum Expected Utility* (MEU) model, an agent ranks two choices by computing a utility index equal to the minimum expected utility of each choice over the set of his priors.

Bewley (1986) is the first to axiomatize a multi-prior expected utility representation in the presence of incomplete preferences. In Bewley’s model, preferences are represented by a unanimity criterion: an agent prefers a choice over an alternative if its expected utility is higher under all his priors. To resolve indeterminate situations in which the unanimity criterion prevents the agent from choosing between two alternatives, Bewley (1986) adds an assumption of inertia: a decision maker remains with his status quo unless an alternative is deemed better under all his priors. We refer to this model as the *Bewley Expected Utility* (BEU) model. It is important to note that in BEU the status quo is a device to complete the model of choice and is

5Incomplete preferences were studied originally by Aumann (1962). Incompleteness can be either about beliefs or tastes, or both. Incompleteness about beliefs leads to a multi-prior representation of preferences, while incompleteness about tastes leads to a “multi-utility” representation. In the seminal work of Bewley (1986), incompleteness is only about beliefs, not tastes. In more recent work, Ok, Ortoleva, and Riella (2012) and Galaabaatar and Karni (2013) explore the case of incompleteness in both beliefs and tastes. In our paper the focus is on incompleteness about beliefs.
not part of the preference specification. The treatment of the status quo is therefore conceptually different from that of the behavioral economics literature that relies on “reference-dependent” preferences to analyze the implications of biases such as the endowment effect, loss aversion or framing.\footnote{See, for example, Kahneman and Tversky (1979), Thaler (1980), Kahneman, Knetsch, and Thaler (1991), and Tversky and Kahneman (1991). Sagi (2006) studies the implications of imposing a “no-regret” axiom on a family of complete preferences “anchored” at the status quo.}

We believe that distinguishing between multi-prior models of complete vs. incomplete preferences is particularly important for the study of corporate behavior. As noted above, the BEU model has a natural interpretation as a model of \textit{decision making by a group} such as a board of directors, a management team or a financing syndicate. If this group only accepts proposed actions that are unanimously accepted, the group may act \textit{as if} it were a \textit{single} BEU decision maker. This opens up a completely new role for models of decision making under ambiguity, a role that we feel is particularly important when examining corporate decisions.

In our model, an entrepreneur has a potentially economically valuable idea that involves an initial investment and a future continuation, expansion or contraction option. This problem has been extensively studied in the SEU Bayesian paradigm. We revisit this problem when investment opportunities are ambiguous and contrast the implications for investment dynamics of models with complete preferences, such as MEU, to those of models with incomplete preferences, such as BEU. Our analysis delivers several interesting theoretical and empirical implications for corporate investment and provides new perspectives on security design.

First, we show that in the context of a standard real option model, the investment decisions of an MEU agent are observationally equivalent to those of a pessimistic SEU agent, offering little hope that empirical studies will be able to distinguish one from the other. In contrast, BEU predicts that the entrepreneur will appear to be \textit{pessimistic} when deciding to expand an existing project but \textit{optimistic} when deciding to abandon it. As a consequence, the entrepreneur decides to continue operations even in the presence of negative information. Such behavior is consistent with empirical observations that managers are reluctant to abandon losers and indeed seem to have “escalating commitments”.\footnote{The term “escalating commitment” was coined by Staw (1976) and refers to the idea that decision makers continue to support projects even after receiving negative information about the project.}

Second, we show that the importance of the status quo in a BEU model can result in investment decisions that are \textit{time-inconsistent}. We demonstrate this with an investment that...
is rejected due to an anticipated future change in the status quo. When considering the initial investment, the agent recognizes that he will be reluctant to abandon a project once started, because the status quo will change with investment. If the agent cannot precommit to abandon, it might happen that the initial investment is not undertaken altogether.\(^8\) Time inconsistency has been shown to occur in the MEU model when there is updating in the set of priors.\(^9\) We add to this understanding by showing that BEU may also be time inconsistent if the agent uses the inertia assumption as a criterion to guide his choices when alternatives cannot be ranked.

Third, we show that contract design can change the payoffs of an investment in the presence of time inconsistency so as to neutralize the potential escalating commitment effect. Specifically, we show how a convertible bond can be designed to mitigate the effect of time-inconsistency and hence make the initial investment attractive. This raises an entirely new role for financial contracting, one that anticipates the need to modify payoffs so that future decisions will be time consistent. This result is also important because it shows that while transaction costs can rationalize the static implications of BEU (e.g., reluctance to abandon losers), they cannot rationalize the dynamic implications of the BEU (e.g., the issuance of a financial contract to avoid time inconsistency).

Fourth, relying on the interpretation of BEU as a model of group decision making, we conjecture that a group will be more reluctant to abandon investments as the group increases in size and heterogeneity, and hence in the “number” of priors. This is consistent with studies such as Guler (2007) who, in the context of venture capital syndicates, finds that the probability of terminating an investment decreases as the number of members of the syndicate increases. Our analysis further suggests that, in the context of a group, contracts can be shaped so as to induce the future unanimity needed for investment to proceed in the first place.

Finally, our group interpretation of the BEU and MEU models raises the empirical questions of how groups determine the status quo, what factors contribute to the existence of multiple priors in a group, and the extent to which groups effectively base decisions on a consensus or unanimous view, issues that we leave for future research.

\(^8\)Note the difference in this implication relative to a decision made by an overly optimistic SEU manager. As with BEU, an overly optimistic manager may be reluctant to shut down an investment once it is made but, in contrast to BEU, an overly optimistic manager would also invest in this project initially.

\(^9\)See, for example, Al-Najjar and Weinstein (2009) and Siniscalchi (2011).
Several recent studies have applied the multi-prior approach to finance problems. For the most part, these applications have been in the asset pricing and portfolio choice areas. Epstein and Schneider (2010) and Guidolin and Rinaldi (2012) provide excellent surveys of the literature. Fewer studies, however, have considered how ambiguity could affect corporate investment decisions. Examples include Nishimura and Ozaki (2007), Riedel (2009), Miao and Wang (2011), and Riis Flor and Hesel (2011) who examine the exercise decision in a real option setting but do so only in an MEU framework and do not examine the contracting implications of dynamic inconsistency. Applications of BEU in finance are rare and include Rigotti (2004) who studies financing decisions, Rigotti and Shannon (2005) who study risk sharing and allocative efficiency in general equilibrium, and Easley and O’Hara (2010) who use BEU to study liquidity and market “freezes”. We add to this literature by studying dynamic real investment decisions under incomplete preferences (BEU) and by comparing our results to those obtained with complete preferences (MEU).

The rest of the paper proceeds as follows. Section 2 describes the models of decision making in the presence of ambiguity that we use in our analysis. In Section 3 we solve a two-period real option decision problem with ambiguous outcomes and analyze the implications of ambiguity on real investment and security design. Section 4 develops empirical implications and Section 5 concludes. Appendix A contains basic results from decision theory that are utilized in our analysis and Appendix B contains proofs for all propositions.

2 Models of choice under ambiguity

In this section we describe the different models of decision making (SEU, MEU and BEU) that we use to analyze the dynamic real investment problem in Section 3. To clearly illustrate the difference among these models, we consider an atemporal context where a risk neutral agent chooses between projects with outcomes in two possible future states of the world, labeled $u$ and $d$. A project $f$ is a “gamble” with uncertain outcomes $(f_d, f_u)$. To facilitate the presentation, in what follows we consider a risk-neutral agent and adopt the notation $\mathbb{E}_p[f]$ to indicate the expected value of project $f$ with respect to the probability $p$ of the state $u$. 
2.1 No ambiguity: Subjective Expected Utility (SEU)

The preferences of a decision maker who adheres to Savage’s (1954) axioms are represented via the SEU criterion (see Theorem 2 in Appendix A), i.e., given any two projects $f$ and $g$,

$$ f \text{ is preferred to } g \iff E_p[f] > E_p[g]. \quad (1) $$

The above representation makes clear that a SEU decision maker relies on a unique subjective prior $p$ to choose between alternative projects.

Figure 1, Panel A, graphically illustrates the SEU criterion. The downward sloping straight line through $f$ represents the indifference curve for SEU, i.e., the set of projects $g$ such that $E_p[g] = E_p[f]$. The dark grey region represents the set of acts $g$ preferred to $f$ (i.e., $g \succ f$), while the light grey region represents the set of acts $g$ that are dominated by $f$ (i.e., $f \succ g$).

2.2 Ambiguity

We model ambiguity as a set $\Pi$ of possible probabilities (beliefs) $\pi$ characterizing the realization of state $u$:

$$ \Pi = \{ \pi \in [0, 1] : p - \epsilon \leq \pi \leq p + \epsilon, \; \epsilon \geq 0 \}, \quad (2) $$

where $\epsilon$ captures the “degree” of ambiguity. If the agent adheres to the SEU axioms, then the set $\Pi$ is a singleton, i.e., $\epsilon = 0$ and $\Pi = \{p\}$. Alternatively, under both MEU and BEU, $\epsilon > 0$. The larger is $\epsilon$, the more ambiguous the project is.

2.2.1 Complete preferences: Minimum Expected Utility (MEU)

The preferences of a risk neutral agent who adheres to Gilboa and Schmeidler’s (1989) axioms, can be represented via the MEU model (see Theorem 3 in Appendix A). Specifically, for this agent, there exists a set of priors $\Pi$ such that, given any two projects $f$ and $g$,

$$ f \text{ is preferred to } g \iff \min_{\pi \in \Pi} E_\pi[f] > \min_{\pi \in \Pi} E_\pi[g]. \quad (3) $$

In the representation (3), the quantities $\min_{\pi \in \Pi} E_\pi[f]$ and $\min_{\pi \in \Pi} E_\pi[g]$ are the utility indices for the projects $f$ and $g$ respectively.
Figure 1, Panel B, provides a diagrammatic representation of MEU. Using the set of priors (2) and the representation (3), we have that the utility index of the project $f$ is

$$
\min_{\pi \in [p-\epsilon,p+\epsilon]} \mathbb{E}_\pi[f] = \begin{cases} 
(p - \epsilon)f_u + (1 - p + \epsilon)f_d, & \text{if } f_u \geq f_d \\
(p + \epsilon)f_u + (1 - p - \epsilon)f_d, & \text{if } f_u < f_d \end{cases}.
$$

(4)

Hence, the MEU indifference curve through $f$ is given by the kinked line in Panel B of Figure 1. Note that the kink in the indifference curve occurs at a point along the 45-degree line. When $f_u > f_d$ the minimum expected utility is determined by the pessimistic prior $\pi - \epsilon$ while, when $f_u < f_d$ the lowest expected utility is determined by the optimistic prior $p + \epsilon$.

The dark grey region represents the set of acts $g$ preferred to $f$ (i.e., $g \succ f$), while the light grey region represents the set of acts $g$ that are dominated by $f$ (i.e., $f \succ g$). The figure shows that MEU preferences are complete: any two projects $f$ and $g$ can be ranked. For comparison, we also report the indifference curve through $f$ of a SEU agent (dashed line).

2.2.2 Incomplete preferences: Bewley Expected Utility (BEU)

In an attempt to provide an alternative characterization of Knight’s (1921) distinction between risk and uncertainty, Bewley (1986) develops a theory of choice under uncertainty (ambiguity) that starts from Savage’s (1954) SEU axioms and drops the assumption of completeness.\[^{10}\] Bewley (1986) shows that in this case the preferences of a risk neutral agent can be represented via a \textit{unanimity} criterion (see Theorem 4 in Appendix A). Specifically, there is a set of priors $\Pi$ such that a project $f$ is preferred to a project $g$ if its expected value is higher under \textit{all} probability distributions $\pi$ in the set $\Pi$, i.e.,

$$
f \text{ is preferred to } g \iff \mathbb{E}_\pi[f] > \mathbb{E}_\pi[g], \quad \forall \pi \in \Pi.
$$

(5)

Figure 1, Panel C provides a diagrammatic representation of unanimity preferences. The dark grey region represents the set of projects $g$ that are preferred to $f$. According to (5), this region is the intersection of the “better-than-$f$” sets of an SEU agent with prior $p - \epsilon$ and of an SEU agent with prior $p + \epsilon$. A similar observation can be made for projects that are worse than $f$.

\[^{10}\] In Bewley’s characterization, Savage’s completeness axiom is not dropped in its entirety because it is assumed that the agent has complete preferences over objective lotteries. See Theorem 4 in Appendix A for details.
represented by the light grey region in the figure. For comparison purposes we report also the indifference curve of the SEU agent (dashed line).

The comparison of Panels B and C illustrates the incomplete nature of unanimity preferences. The un-shaded regions in Panel C contain projects $g$ that are not comparable to $f$, i.e., there are priors $\pi' \neq \pi''$ for which $E_{\pi'}[g] > E_{\pi'}[f]$ and $E_{\pi''}[g] < E_{\pi''}[f]$. Unlike SEU and MEU, the unanimity criterion does not assign a unique “value” to a project and therefore cannot always specify what the decision maker will do when faced with a choice. To complete the model of choice Bewley introduces the following inertia assumption:

**Inertia assumption.** A decision maker identifies one of the alternatives as the status quo and will only accept an alternative project if the expected value of the alternative is strictly better than that of the status quo under all possible priors.

To illustrate, assume that $f$ is the status quo in Panel C of Figure 1, and that $g$ is a project in the interior of one of the un-shaded regions. Then the acts $f$ and $g$ are not comparable. Invoking the inertia assumption, the BEU agent will then choose to remain with the status quo $f$ when offered the option to move to $g$. In the remainder of this paper we will use the term Bewley Expected Utility (BEU) model to refer to the unanimity criterion (5) augmented by the Inertia assumption.

Notice that BEU cannot always predict the behavior when facing non binary atemporal decisions. Assume, for example, that $f$ is the status quo in Panel C of Figure 1, and that the agent is contemplating the choice of the projects $g$ and $h$ in the shaded upper region of “better-than-$f$” projects. Then both choices of $g$ and $h$ are preferred to the status quo $f$ and as a result, both choices are possible under BEU. Bewley (1986) refers to this situation as “indeterminateness”. In some cases this indeterminateness can be resolved if $g$ is in the “better-than-$h$” region or $h$ is in the “better-than-$g$” region. When the act $g$ and $h$ are not comparable under BEU, the indeterminateness cannot be resolved and the BEU model fails to prescribe a specific choice.\(^{11}\)

\(^{11}\)Indeterminate choices are not surprising under an incomplete preference models. To resolve the indeterminateness, one would need to introduce a specific selection process. For example, Masatlioglu and Ok (2005) impose a set of behavioral axioms to determine a selection criterion that describes how agents respond to indeterminate- ness. Following Rigotti and Shannon (2005) and Easley and O’Hara (2010), we do not introduce any selection rule in addition to the standard Bewley inertia assumption.
Figure 1: SEU, MEU and BEU

The figure reports the indifference curve through the project $f$ for SEU (Panel A) and MEU (Panel B). Panel C reports the preference ordering according to BEU. In all panels, the dark grey region indicates “better-than-$f$” projects and the light grey region indicates “worse-than-$f$” projects.

Panel A: SEU

Panel B: MEU

Panel C: BEU
2.3 Ambiguity and group decision making

The multi-prior decision models described above refer to a single decision maker but could emerge in the context of group decisions. Consider, for example, a board of directors evaluating two alternative projects and suppose that each board member adheres to SEU and has a distinct prior.\textsuperscript{12} If the group selects one of the alternatives only when there is unanimous agreement it will behave \textit{as if} it was a single agent making decisions on the basis of multiple priors where the set of priors is the collection of all individual priors. Consequently, even if a choice is not considered ambiguous by \textit{any} member of the board, it is \textit{de facto} ambiguous for a group that decides according to the unanimity rule. The “preferences” of the board are, therefore, effectively incomplete.

Despite this incompleteness, decisions eventually have to be made in the real world, and the incomplete preferences of the board have to be operationalized through a decision rule. Both the BEU and MEU models correspond to distinct decision rules of the board.

In the BEU model, the decision rule is the Inertia assumption. In terms of a corporate board, this rule implies that the board identifies one of the alternatives as the status quo. The board then votes on an alternative and accepts it only if there is unanimous approval. In case of disagreement the board will stay with the status quo.

In the MEU model, the decision rule of the board is more complex. One such rule can be inferred from the work of Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) who, at the individual level, derive a set of axioms that complete the unanimity preferences and deliver MEU as a model of choice. To heuristically illustrate the decision rule implied by these axioms for a single multi-prior agent, consider two ambiguous gambles, \( f \) and \( g \) that cannot be ranked based on unanimity. Assume, however, that the agent’s preferences over unambiguous gambles are complete. For gamble \( f \) assume the existence of a set of unambiguous gambles that are not preferred to \( f \). Define the unambiguous gamble \( f_c \) as one of the most preferred elements of this set. In a similar way, determine \( g_c \). The preference of \( f \) relative to \( g \) is then determined by the complete ordering of the unambiguous gambles \( f_c \) and \( g_c \). Gilboa, Maccheroni, Marinacci, and

\textsuperscript{12} The individual beliefs can be dogmatically held by an individual or can be the outcome of learning both among group members and through the observation of public signals, such as market prices. The difference in beliefs in the group can then be interpreted as the residual disagreement that cannot be eliminated through deliberation or price mediation.
Schmeidler (2010) show that the ordering over ambiguous gambles thus obtained is equivalent to MEU.

When cast in terms of group decisions, this decision rule would require the group to agree on a set of alternatives $\mathcal{M}$ that the group can unanimously order, such as, for example, certain monetary amounts. These become the “measuring devices” for the group. For a gamble $f$, each member of the group will report a maximal element of the subset of measuring devices $\mathcal{M}$ that are weakly dominated by $f$. After this phase, the group will obtain the collection of measuring devices reported by each group member. Given the unanimous ordering over the set $\mathcal{M}$ of measuring devices, the group can then define $f_c$ as the least preferred among the reported measuring devices. In a similar manner, the group defines $g_c$ for the gamble $g$. The group ordering between $f$ and $g$ is then determined by the complete ordering of $f_c$ and $g_c$. Under these conditions the group choice will look like as if it was implemented by a single MEU decision maker whose set of priors is the collection of all group members’ priors.

In what follows, we will analyze a real investment decision in the presence of ambiguity. We conduct our analysis from the point of view of a single agent, the entrepreneur. However, given the group interpretation provided in this section, it is important to keep in mind that, all the results will also apply to a group that decides by unanimity and resolves the indeterminateness by either resorting to the status quo, as in the BEU model, or via a suitable decision rule that will result in choice consistent with the MEU model. As we discuss in Section 4, the group interpretation of MEU and BEU is important for suggesting empirical predictions of the two models.

### 3 A dynamic model of ambiguous real investment

We consider a risk-neutral self-financed entrepreneur who has monopoly access to a project. The project requires an investment at time $t = 0$, an expansion or contraction decision at time $t = 1$, and delivers cash flows at $t = 2$. We study the effect of ambiguity on these real investment decisions by considering an entrepreneur who believes that investment outcomes are governed by a set of prior distributions. If the set of priors is a singleton the outcome is risky, not ambiguous, and the entrepreneur follows Savage’s (1954) SEU criterion to determine his choice.
An investment outcome is ambiguous if the set of priors contains more than one element. We now describe the information structure, the technology and the entrepreneur’s choices.

3.1 Information structure

We carry out the analysis in the context of a simple two-period information structure. The state of the world evolves over the two periods according to the binomial tree described in Figure 2.

Figure 2: Information structure

To simplify exposition, we assume that the same set $\Pi$ defined in (2) describes conditional one-step-ahead priors at time $t = 0$, $\pi_0$, and at time $t = 1$ in both $u$ and $d$ states, $\pi_u$ and $\pi_d$, respectively. This assumption essentially imposes independence between successive realizations of the state. Upon observing the realization of the state at time 1, the agent can only refine the set of future paths (binomial subtrees) but cannot learn more about the set of priors governing the ambiguous subsequent states.
3.2 Technology and investment decisions

The underlying technology delivers, for each unit of capital, a cash flow at time $t = 2$ that is given by the ambiguous random variable $\tilde{s}_2$ with outcomes $\{s_{uu}, s_{ud}, s_{du}, s_{dd}\}$ satisfying

$$s_{uu} > s_{ud} > s_{du} > s_{dd}.$$  \hfill (6)

We define $\tilde{s}_2^j$ for $j = u, d$, as the random variable $\tilde{s}_2$ conditional to state $j$ at time $t = 1$ i.e., $\tilde{s}_2^j$ is the ambiguous random variable with outcomes $s_{ju}$ and $s_{jd}, j = u, d$.

We study decisions made by the entrepreneur at two different points in time. The first decision is whether or not to invest $I_0$ to acquire one unit of capital at $t = 0$. If the initial investment is made, the entrepreneur faces a second decision at $t = 1$, when the scale of the initial investment can be maintained, expanded or contracted. Let $A = \{E, C, R\}$ be the set of possible actions at time time $t = 1$, where $E$ indicates Expansion, $C$ indicates Continuation, and $R$ indicates Contraction. If the initial investment is made at $t = 0$ the project delivers net cash flows $\tilde{C}_2$ at time $t = 2$ that depend on the cash flow of the underlying technology, $\tilde{s}_2$, and the scale of the firm, $\lambda$. Specifically,

1. $E$ requires payment of the amount $I_1$ to increase the scale of the firm at $t = 1$ to $\lambda > 1$ and generates the net cash flow $\tilde{C}_2 = \lambda \tilde{s}_2 - I_1$ at time $t = 2$.

2. $C$ requires a maintenance payment $m$ to keep capacity in place and to generate the net cash flow $\tilde{C}_2 = \tilde{s}_2 - m$ at time $t = 2$.\footnote{The payment $m$ can be thought of as a maintenance cost or as the second round of investment required to keep operating the firm. While this term has no affect on our results we include it to capture the notion that in most real investment decisions 'doing nothing' is not an option as each alternative involves some cost. Moreover, this specification is descriptive of the notion of escalating committment as the initial investment $I_0$ requires a follow on investment $m$.}

3. $R$ requires scrapping the firm by setting $\lambda = 0$ and generates a constant recovery amount $\tilde{C}_2 = R$ at time $t = 2$.

Figure 3 provides a diagrammatic representation of the choice available at time $t = 1$ and in node $j = u, d$. The point $f$ denotes the pair of cash flows $(s_{jd}, s_{ju})$, obtained at $t = 2$ if the entrepreneur could continue operations at the scale $\lambda = 1$ without incurring in maintenance costs. The payment of the maintenance/completion cost $m$ yields a continuation cash flow equal to $f - m$. Because $s_{ju} > s_{jd}$, $f$ lies above the 45-degree line.
Figure 3: Expansion and contraction decisions

The figure illustrates the expansion and contraction decisions at time $t = 1$. In the figure, $f = s_j$, $j = u, d$, $\lambda > 1$ is the scale of the expanded firm, $I_1$ the expansion cost, and $R$ the recovery value if the firm is shut down.

A convenient way to understand a change in scale (expansion or contraction) is to consider the line going through $f$. The decision to expand at time $t = 1$ corresponds to moving upward along the ray from $f$ to $\lambda f$, $\lambda > 1$, while the decision to contract corresponds to moving to point $R$ on the 45-degree line where the payoffs are state independent. If the entrepreneur decides to expand, he will have to pay an unambiguous cost $I_1$ to obtain a “scaled” up version of the current firm. The expansion decision therefore entails comparing the cash flow from current operations $f$, net of maintenance costs $m$, with the alternative $\lambda f - I_1$. Given that $s_{ju} > s_{jd}$ and that the expansion cost $I_1$ is unambiguous, the “gamble” $\lambda f - I_1$ lies always to the left of the ray going through $f$. If the entrepreneur decides to contract, he selects a scale $\lambda = 0$ and receives an unambiguous recovery value of $R$. Hence, the contraction decision entails comparing the cash flow from current operations $f$ net of maintenance costs $m$ with the alternative $R$ on the 45-degree line.\(^\text{14}\)

\(^\text{14}\)In general we could also consider a partial contraction, in which the entrepreneur selects a scale $\lambda \in [0, 1)$ and receives a scrap value $R$ in exchange of selling a fraction $1 - \lambda$ of the firm. Graphically this would entail comparing the cash flow from current operations $f - m$ with the alternative $\lambda f + R$, $\lambda \in [0, 1)$. Because the contraction payoff $R$ is unambiguous, the gamble $\lambda f + R$ lies always to the right of the line going through $f$. 

3.3 Solutions of the dynamic real investment problem

We solve the investment decision problem recursively determining first the expansion, continuation or contraction decisions at time \( t = 1 \) and then the initial investment decision at time \( t = 0 \).

3.3.1 Time 1 decisions

Figure 4 illustrates the difference between investment decisions under MEU (Panel A) and BEU (Panel B) at time \( t = 1 \). The figure combines the preference description in Figure 1, with the technology description in Figure 3. In both panels, the point \( f - m \) represents the action \( C \), Continue, which provides the net cash flow of the underlying project at time \( t = 2 \) in each state \( u \) or \( d \).

Panel A compares MEU vs. SEU. The kinked line is the MEU indifference curve through \( f - m \) and the dotted line is the indifference curve for a SEU agent. The choices of an SEU entrepreneur are

\[
\mathcal{E} \text{ is preferred to } C \text{ if } \lambda \mathbb{E}_p[f] - I_1 > \mathbb{E}_p[f] - m, \text{ and} \tag{7}
\]

\[
\mathcal{R} \text{ is preferred to } C \text{ if } R > \mathbb{E}_p[f] - m, \tag{8}
\]

where the entrepreneur uses the unique probability \( p \) in assessing alternatives.

Panel A shows that if the entrepreneur has MEU preferences he will use the prior \( \pi = p - \epsilon \) to decide whether to expand or contract. For a sufficiently high level of investment cost \( I_1 \) and a sufficiently low level of maintenance cost \( m \), an MEU entrepreneur will not expand while an SEU entrepreneur will. Similarly, for sufficiently high recovery values \( R \) an MEU entrepreneur will contract while an SEU entrepreneur will not. The choices of an MEU entrepreneur are

\[
\mathcal{E} \text{ is preferred to } C \text{ if } \lambda \mathbb{E}_{p-\epsilon}[f] - I_1 > \mathbb{E}_{p-\epsilon}[f] - m, \text{ and} \tag{9}
\]

\[
\mathcal{R} \text{ is preferred to } C \text{ if } R > \mathbb{E}_{p-\epsilon}[f] - m. \tag{10}
\]

In summary, the analysis in Figure 4, Panel A, shows that with regards to expansion, continuation and contraction decisions, an MEU entrepreneur behaves as a pessimistic SEU entrepreneur with subjective probability \( p - \epsilon \).
Figure 4: Time 1 decisions

Panel A (B) reports the expansion/contraction decision under MEU (BEU) preferences. In both panels, the dotted line refers to the indifference curve under SEU. In the figure, $f = \tilde{s}_j$, $j = u, d$, $\lambda > 1$ is the scale of the expanded firm, $I_1$ the expansion cost, and $R$ the recovery value if the firm is shut down.

Panel A: MEU

Panel B: BEU
Panel B of Figure 4 illustrates the key difference between MEU and BEU. We assume that for the BEU entrepreneur the status quo is to continue operation, i.e., the point $f - m$. As in the MEU case, the relevant prior that determines whether an expansion decision is taken or not is $\pi = p - \epsilon$. However, the relevant prior for deciding whether to contract or continue is now $\pi = p + \epsilon$ and not $\pi = p - \epsilon$, as in the case of MEU preferences. When offered the option to sell an ambiguous firm, the BEU agent compares the best possible scenario to the unambiguous amount of cash $R$. The choices of a BEU entrepreneur are then

$$\mathcal{E} \text{ is preferred to } \mathcal{C} \text{ if } \lambda\mathbb{E}_{p-\epsilon}[f] - I_1 > \mathbb{E}_{p-\epsilon}[f] - m, \text{ and}$$

$$\mathcal{R} \text{ is preferred to } \mathcal{C} \text{ if } R > \mathbb{E}_{p+\epsilon}[f] - m.$$

In summary, the above analysis shows that the condition for expansion under MEU and BEU, respectively, equations (9) and (11), are identical and coincide with the expansion condition for a pessimistic SEU agent, i.e., equation (7) with prior $p$ set to $p - \epsilon$. In contrast, the contraction decisions are markedly different. While the MEU contraction decision (10) is indistinguishable from the SEU contraction decision (8) with a unique “pessimistic” prior $\pi = p - \epsilon$, the BEU decision is clearly different from both SEU and MEU. A comparison of (10) with (12) shows that when facing a contraction, an MEU entrepreneur behaves pessimistically, while a BEU entrepreneur behaves optimistically. Hence, even when the BEU entrepreneur can sell a firm for $R$ and may even consider $R$ to be larger than his expected payoff from continuing under some priors, he will not shut down if he entertains just one prior under which the expected continuation value of the firm is higher than $R$.

It has been observed that managers are reluctant to divest or shut down projects that have not done well. These observations have motivated explanations based on agency problems, reputation concerns and asymmetric information (see, for example, Boot (1992) and Weisbach (1995)). BEU can provide an alternative explanation in terms of ambiguity even in the absence of asymmetric information. As we discussed in Section 2.3, BEU behaviour is consistent with corporate board decisions based on unanimity even when board members have SEU preferences. In this case, reluctance to terminate a project is due to disagreement in project valuation within a heterogeneous board.
3.3.2 Time 0 decision

At time $t = 0$ the entrepreneur is endowed with an unambiguous quantity $I_0$ which serves as the status quo and faces the choice of whether to invest or not. Without loss of generality, we assume henceforth that $m = 0$. Let us define an investment plan as a pair of actions $\varphi = (a_u, a_d) \in \mathcal{P}$ where $\mathcal{P} = \mathcal{A} \times \mathcal{A}$ is the set of all possible investment plans and where for each $j = u, d$, $a_j$ is the action in state $j$ at time $t = 1$. For instance, we indicate by $(E, E)$ the investment plan: 'Expand in both states $u$ and $d$, and similarly for all the other eight possible investment plans at $t = 1$. We denote by $\varphi_0$ the plan in which the entrepreneur does not invest at time $t = 0$.

The outcome of every investment plan $\varphi = (a_u, a_d)$ is a random payoff at time $t = 2$ defined by

$$\tilde{C}_\varphi = \begin{cases} C^{a_u}(\tilde{s}_2^u) & \text{on nodes following } u \\ C^{a_d}(\tilde{s}_2^d) & \text{on nodes following } d \end{cases},$$

where $C^a(\cdot)$, is a deterministic function defined by

$$C^a(x) = \begin{cases} \lambda x - I_1 & \text{if } a = E \\ x & \text{if } a = C \\ R & \text{if } a = R \end{cases}.$$  

Notice that $\tilde{C}_\varphi$ is the unconditional payoff of an investment plan, i.e., a random variable with four possible outcomes, while $C^{a_u}(\tilde{s}_2^u)$ and $C^{a_d}(\tilde{s}_2^d)$ are payoffs conditional on the state realized at time $t = 1$ and on the action taken, i.e., they are random variables with two possible outcomes.

The choices of the entrepreneur at time $t = 1$, characterized in Section 3.3.1, restrict the set of possible investment plans that the entrepreneur will consider at time $t = 0$. We refer to an implementable investment plan as one that is incentive compatible for the entrepreneur. That is, he will find it in his best interest to carry out the investment plan at time $t = 1$. We now formally define this set for each of the preferences considered above.

**Definition 1.** An investment plan $\varphi = (a_u, a_d) \in \mathcal{P}$ is

1. SEU-implementable if

$$E_p[C^{a_j}(\tilde{s}_2^j)] \geq E_p[C^{a_j'}(\tilde{s}_2^j)], \quad \forall a_j' \in \mathcal{A}, \ j = u, d.$$  

---

15It can be verified that the investment decisions in a model where $m \neq 0$ are identical to the investment decisions of a model where $m = 0$ and where $I_1$ is replaced by $I_1 - m$ and $R$ is replaced by $R + m$. 

We denote by $P_{\text{SEU}}^*$ the set of all SEU-implementable plans.

2. MEU-implementable if

$$\min_{\pi \in \Pi} \mathbb{E}_\pi[C^{a_j}(\tilde{s}_j^2)] \geq \min_{\pi \in \Pi} \mathbb{E}_\pi[C^{a_j'}(\tilde{s}_j^2)], \quad \forall a_j' \in \mathcal{A}, \ j = u, d.$$  

We denote by $P_{\text{MEU}}^*$ the set of all MEU-implementable plans.

3. BEU-implementable with status quo $(C, C)$ if, for $j = u, d$,

(a) when $a_j \neq C$,

$$\mathbb{E}_\pi[C^{a_j}(\tilde{s}_j^2)] > \mathbb{E}_\pi[C^{C}(\tilde{s}_j^2)] = \mathbb{E}_\pi[\tilde{s}_j^2], \quad \forall \pi \in \Pi$$  

(17)

(b) when $a_j = C$, for all $a_j' \in \mathcal{A}$ there exists a prior $\pi \in \Pi$ such that

$$\mathbb{E}_\pi[C^{C}(\tilde{s}_j^2)] \geq \mathbb{E}_\pi[C^{a_j'}(\tilde{s}_j^2)].$$  

(18)

We denote by $P_{\text{BEU}}^*$ the set of all BEU-implementable plans.

Note that the definition of BEU-implementability depends on the agent’s status quo $(C, C)$ at time $t = 1$. In our real investment problem, the status quo is the action $C$ at time $t = 1$ in each state $j = u, d$ and therefore, as stated in (17), a plan is implementable if its expected conditional payoff is higher than that of the status quo $(C, C)$. Condition (18) formalizes the inertia assumption: when satisfied this condition implies that the status quo $C$ is chosen in state $j$. In contrast, for SEU and MEU, the notion of implementability does not depend on the status quo. Conditions (15) and (16) require global optimization rather than comparability with respect to the status quo.\footnote{In principle, one can generalize the definition of BEU-implementability to account for any status quo $(a_u, a_d) \in \mathcal{P}$. Moreover, the definition can be refined to address the potential indeterminateness that arises when there are multiple BEU-implementable plans according to (17)-(18). In a more restrictive definition a plan is BEU-implementable if it is undominated when compared to all the plans that satisfy (17)-(18).}

Given an implementable plan $\varphi = (a_u, a_d)$, the decision of whether to invest or not at time $t = 0$ is similar to that described at time $t = 1$. Specifically, given any implementable investment plan the SEU agent will choose an investment plan $\varphi$ at time $t = 0$ if

$$\mathbb{E}_\pi[\tilde{C}^{\varphi}] > I_0,$$  

(19)

where $\mathbb{E}_\pi[\tilde{C}^{\varphi}] = p\mathbb{E}_p[C^{a_u}(\tilde{s}_2^u)] + (1 - p)\mathbb{E}_p[C^{a_d}(\tilde{s}_2^d)]$.\footnotemark
An MEU agent with priors \( \pi_0 \in \Pi, \pi_u \in \Pi \) and \( \pi_d \in \Pi \), with \( \Pi \) the set of priors defined in (2), will choose an investment plan \( \varphi \) at time \( t = 0 \) if

\[
\min_{(\pi_0, \pi_u, \pi_d) \in \Pi^3} \mathbb{E}_{\pi_0, \pi_u, \pi_d} [\tilde{C}^\varphi] > I_0,
\]

where

\[
\mathbb{E}_{\pi_0, \pi_u, \pi_d} [\tilde{C}^\varphi] = \pi_0 \mathbb{E}_{\pi_u} [C_{\pi_u}(s^u_2)] + (1 - \pi_0)\mathbb{E}_{\pi_d} [C_{\pi_d}(s^d_2)].
\]

A BEU agent will choose the investment plan \( \varphi \) at time \( t = 0 \) if

\[
\mathbb{E}_{\pi_0, \pi_u, \pi_d} [\tilde{C}^\varphi] > I_0, \quad \text{for all} \quad (\pi_0, \pi_u, \pi_d) \in \Pi^3.
\]

If there is at least an implementable plan \( \varphi \) that is chosen at time \( t = 0 \), then the entrepreneur will invest at time zero. We refer to a plan thus chosen as a recursive solution of the dynamic real investment problem. If there is more than one plan \( \varphi \) that the entrepreneur would choose, then a potential indeterminateness arises. For the case of SEU and MEU, such indeterminateness can be resolved by choosing the plan with the highest value (i.e., the plan(s) that maximizes expected payoff for SEU or the plan(s) that maximize the minimum expected payoff for MEU). For the case of BEU we may be left with an indeterminateness, as we discussed in Section 2.2.2.

We illustrate the construction of the set of implementable investment plans and the solution of the dynamic investment problem with the help of Figure 5. In the figure we assume that the recovery value \( R \) and expansion cost \( I_1 \) satisfy the following conditions:

**Assumption 1.** The recovery value \( R \) for shutting down the firm at \( t = 1 \) is such that

\[
\mathbb{E}_{p-\epsilon}[s^d_2] < \mathbb{E}_p[s^d_2] < R < \mathbb{E}_{p+\epsilon}[s^d_2].
\]

The investment cost \( I_1 \) for expanding the scale to \( \lambda \) at time \( t = 1 \) is such that

\[
\lambda\mathbb{E}_p[s^d_2] - I_1 < \mathbb{E}_p[s^u_2], \quad \text{and} \quad \mathbb{E}_{p-\epsilon}[s^u_2] < \lambda\mathbb{E}_{p-\epsilon}[s^u_2] - I_1.
\]

Condition (23) implies that, at time \( t = 1 \) and in state \( d \), \( R \) is undertaken by SEU and MEU but not by BEU who prefers the status quo \( C \) (see Panel A of Figure 5). The first inequality of (24) implies that, in state \( d \), \( E \) is not undertaken under any model of choice. The second inequality in (24) implies that, in state \( u \), \( E \) is undertaken under all models (see Panel B
Figure 5: Time 1 decisions: MEU vs BEU

The shaded region in each panel represents the set of gambles over which MEU and BEU will disagree. The dotted line represents the SEU indifference curve through $\tilde{s}_2^d$ (Panel A) and $\tilde{s}_2^u$ (Panel B).

Panel A: time $t = 1$, state $d$

Panel B: time $t = 1$, state $u$
of Figure 5). In this example, the set of SEU- and MEU-implementable plans coincide, i.e., $\mathcal{P}^*_{SEU} = \mathcal{P}^*_{MEU} = \{(\mathcal{E}, \mathcal{R})\}$ while the set of BEU implementable plans is $\mathcal{P}^*_{BEU} = \{(\mathcal{E}, \mathcal{C})\}$. In other words, SEU and MEU decision makers expand in state $u$ and shut down in state $d$, while BEU decision makers expand in state $u$ and continue in state $d$.

Applying the choice criterion (19) we have that a SEU agent invests at time $t = 0$ if $\mathbb{E}_p[\tilde{C}(\mathcal{E}, \mathcal{R})] > I_0$, i.e., if

$$p(\lambda \mathbb{E}_p[\tilde{s}_2^u] - I_1) + (1 - p)R > I_0. \quad (25)$$

By the criterion (20) an MEU entrepreneur invests at time $t = 0$ if $\min_{(\pi_0, \pi_u, \pi_d) \in \Pi^3} \mathbb{E}_{\pi_0, \pi_u, \pi_d}[\tilde{C}(\mathcal{E}, \mathcal{R})] > I_0$. Under Assumption 1 and condition (6) this implies that the MEU agent invests if and only if

$$(p - \epsilon)(\lambda \mathbb{E}_{p - \epsilon}[\tilde{s}_2^u] - I_1) + (1 - p + \epsilon)R > I_0. \quad (26)$$

By the criterion (22) a BEU agent invests at time $t = 0$ if $\mathbb{E}_{\pi_0, \pi_u, \pi_d}[\tilde{C}(\mathcal{E}, \mathcal{C})] > I_0$, for all $(\pi_0, \pi_u, \pi_d) \in \Pi^3$. Under Assumption 1 this implies that investment occurs if

$$(p - \epsilon)(\lambda \mathbb{E}_{p - \epsilon}[\tilde{s}_2^d] - I_1) + (1 - p + \epsilon)\mathbb{E}_{p - \epsilon}[\tilde{s}_2^d] > I_0. \quad (27)$$

Comparing the MEU investment condition (26) to the BEU investment condition (27) we see that, because $\mathbb{E}_{p - \epsilon}[\tilde{s}_2^d] < R$ by Assumption 1, the threshold investment cost $I_0$ above which investment is not undertaken is lower for BEU than for both SEU and MEU. In other words, for values of the investment cost $I_0$ in the interval

$$(p - \epsilon)(\lambda \mathbb{E}_{p - \epsilon}[\tilde{s}_2^d] - I_1) + (1 - p + \epsilon)\mathbb{E}_{p - \epsilon}[\tilde{s}_2^d] < I_0 < (p - \epsilon)(\lambda \mathbb{E}_{p - \epsilon}[\tilde{s}_2^u] - I_1) + (1 - p + \epsilon)R, \quad (28)$$

an SEU and MEU agent would invest while a BEU agent would not. This happens because the above recursive solutions restrict the choice of plans to only those that are implementable. For a BEU agent this restriction excludes investment plans that, if implementable, would be attractive. In fact, if we consider the plan $(\mathcal{E}, \mathcal{R})$ and apply the BEU choice criterion (22) we can see that, under condition (28) this plan will be undertaken by the BEU entrepreneur if it were implementable because $I_0 < \mathbb{E}_{\pi_0, \pi_u, \pi_d}[\tilde{C}(\mathcal{E}, \mathcal{R})]$ for all $(\pi_0, \pi_u, \pi_d) \in \Pi^3$. However because $(\mathcal{E}, \mathcal{R}) \notin \mathcal{P}^*_{BEU}$ the agent will not consider such a plan when constructing its investment problem recursively. In other words, if the BEU agent were able to precommit to the plan $(\mathcal{E}, \mathcal{R})$ he would undertake it. For the BEU agent the implementability restriction acts as a constraint...
that prevents him from undertaking actions at time $t = 1$ that he would consider attractive at time $t = 0$. This phenomenon happens because of the inertia assumption in the BEU model and raises the important issue of time consistency to which we now turn.

3.4 Dynamic consistency

The above example highlights an important tension in the BEU model between the entrepreneur at time $t = 0$, or “early self”, and the entrepreneur at time $t = 1$, or “late self”. The early self would like to undertake the plan $(\mathcal{E}, \mathcal{R})$ and understands that investing will result in the status quo $(\mathcal{C}, \mathcal{C})$ at time $t = 1$. Hence, to implement $(\mathcal{E}, \mathcal{R})$ he will have to leave the status quo $(\mathcal{C}, \mathcal{C})$ in both future states. Under Assumption 1, however, the early self knows that the late self will not consider the choice $\mathcal{R}$ acceptable in state $d$ relative to the status quo, thus making the plan $(\mathcal{E}, \mathcal{R})$ not implementable. The example illustrates that, the passage of time and the resulting change in status quo, induces a choice reversal for the BEU entrepreneur. This choice reversal does not, however, occur with SEU or MEU preferences under our information structure.

The next definition formally defines the dynamic consistency condition used in our problem.

**Definition 2.** Let $q$ be any investment plan from the set $\mathcal{P}$, or the ‘Do not invest’ plan, $\varnothing_0$. A model of choice is dynamically consistent if the following condition is satisfied:

$$
\text{If } q \text{ is chosen over any implementable plan, then } q \text{ is chosen over any plan, including } \varnothing_0.
$$

(29)

The following proposition shows that SEU and MEU preferences are dynamically consistent.

**Proposition 1.** Both SEU and MEU preferences satisfy the dynamic consistency condition (29).

Unlike SEU and MEU, BEU is not always dynamically consistent. The example of the previous section illustrates that condition (29) is violated for BEU because the plan $(\mathcal{E}, \mathcal{R})$ is non- implementable but preferred to not investing (i.e., the plan $q = \varnothing_0$) at time 0 while the only implementable plan, $(\mathcal{E}, \mathcal{C})$ is not preferred to $\varnothing_0$.

---

17It is well known (see, for example, Al-Najjar and Weinstein (2009) and Siniscalchi (2011)) that, under a general information structure, MEU preferences can lead to time inconsistent choices. Our information structure is built by taking the one-step-ahead priors as primitive in our description of ambiguity. As shown by Epstein and Schneider (2003), the resulting unconditional set of priors over the state space will satisfy the “rectangularity condition” that guarantees time consistency of MEU preferences.
To understand the source of dynamic inconsistency for the BEU model consider a simplified version of the investment problem discussed in the previous section in which we remove the $u$ node in the decision tree (i.e., set $\pi_0 = 0$). At time $t = 0$ the agent is endowed with the gamble $f_0$ which is his status quo, and has to choose between gambles $f_0$ and $f_1$. If he chooses $f_1$ at time $t = 0$, the agent will face a subsequent choice between $f_1$ and $f_2$ at time $t = 1$, with $f_1$ being the new status quo. If he chooses $f_0$ at time $t = 0$ there are no subsequent choices to be made. All gambles are defined on a two dimensional state space. Without loss of generality let us assume that $f_0$, the status quo at time $t = 0$, is an unambiguous gamble. Figure 6 reports $f_0$ on the 45-degree line. The solid line with a kink at $f_0$ represents the contour of the set containing gambles that are preferred to $f_0$.

**Figure 6: Dynamic inconsistency of BEU model**

The highlighted region in the figure refers to acts at time 2 that will lead to disagreement between commitment and recursive solution.

The shaded region in the figures denotes the regions of gambles with $t = 2$ payoffs that will give rise to dynamically inconsistent choices. To see this, consider first the recursive solution, that is, the solution obtained by solving the choice problem by backward induction. To obtain this solution let us start from the decision at time $t = 1$ and compare the status quo $f_1$ to $f_2$. Because $f_2$ is not preferred to $f_1$, the agent at time $t = 1$ will stick with the status quo $f_1$, which is the only implementable plan. Given this, the agent at time $t = 0$ will then have to compare
$f_0$ with $f_1$. Because $f_1$ is not preferred to $f_0$ the early self will stick with the status quo $f_0$. Hence $f_0$ is the recursive solution for the dynamic problem portrayed in Figure 6.

Let us now suppose that at time $t = 0$ the agent can precommit to either gamble $f_1$ or $f_2$. Because $f_1$ is not preferred to $f_0$, $f_1$ will not be chosen over $f_0$. However $f_2$ is preferred to $f_0$, and it will be the precommitment solution of the dynamic choice problem. The pre-commitment solution $f_2$ and the recursive solution $f_0$ disagree. In general, therefore, as stated in Definition 2, dynamic inconsistency arises when: (i) $f_1$ is not preferred to $f_0$ and (ii) $f_2$ is not preferred to $f_1$, and (iii) $f_2$ is preferred to $f_0$. Note that these conditions imply a violation of the dynamic consistency condition (29). Intuitively, dynamic inconsistency arises when there is a change in status quo over time and the two status quibus cannot be ranked. From the figure it is also clear to see that such inconsistency will not arise with SEU and MEU because the indifference curves cannot cross. Notice, however, that for MEU, the indifference curves can cross if we allow updating of the priors (see footnote 17 for further details).

Finally, Figure 6 also helps us understand how contracts can help resolve the dynamic inconsistency problem faced by a BEU entrepreneur. By signing contracts at time 0, the early self can affect the payoff of the gamble $f_2$. A contract can resolve inconsistency if it can manipulate either the status quo $f_1$ and/or the payoff of the gamble $f_2$, so that dynamic consistency can be achieved. In the next section we show how derivative contracts can help the early self to discipline the decisions of the late self. Because SEU and MEU are time-consistent under our information structure, in the next section we focus only on the BEU model.

### 3.5 Convertible bonds as a precommitment mechanism

An implication of the time inconsistency discussed in Section 3.4 is that under some conditions there is an investment plan $(E, R)$ that the entrepreneur would like to undertake at time $t = 0$ but is not implementable. In this subsection we discuss the way in which securities can be issued to make the plan implementable. The problem that we consider can be thought of as a principal-agent problem where the late self is the agent of the early self. The early self issues a convertible bond. The late self responds to the incentives created by the convertible bond. The principal has to create the incentives to shut down the firm in the down state without twisting the incentives to expand the firm in the up state.
To illustrate the role of securities in this specific case, recall that, in the absence of outside securities, in state $d$ the late self compares the payoffs from the status quo $C$ to that of the alternative $R$, i.e.,

$$s^d_2 = \begin{cases} s_{du} & \text{in state } du \\ s_{dd} & \text{in state } dd \end{cases}$$

versus $R$.}

(30)

Because, by Assumption 1, $R < \mathbb{E}_{p+\epsilon}[s^d_2]$, the status quo $C$ is chosen over $R$.

Now consider a convertible bond issued by the early self to a BEU financier with identical set of priors $\Pi$. The bond promises a repayment of $X$ at time $t = 2$ and, at the option of the bondholder, is convertible into a fraction $\alpha$ of the firm’s equity at time $t = 2$. The existence of the bond generates time $t = 2$ payoffs in states $du$ and $dd$ summarized in Table 1. Each entry represents the pair of payoffs to bondholder and entrepreneur at time $t = 2$, depending on the two agents’ decisions.

### Table 1: Entrepreneur’s and bondholder’s payoffs

The table reports the time $t = 2$ payoff to the entrepreneur and the bondholder, following the entrepreneur’s decisions $C$ or $R$ at time $t = 1$ and the bondholder’s conversion strategy at time $t = 2$. The boxed entry represents the payoffs implied by the bondholder’s conversion policy when condition (32) of Proposition 2 holds.

<table>
<thead>
<tr>
<th>Bondholder:</th>
<th>Entrepreneur’s decision at time $t = 1$, state $d$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Continue ($C$)</td>
</tr>
<tr>
<td></td>
<td>Shut down ($R$)</td>
</tr>
<tr>
<td>(Bondholder’s payoff, Entrepreneur’s payoff) at time $t = 2$</td>
<td></td>
</tr>
<tr>
<td>$du$</td>
<td>$dd$</td>
</tr>
<tr>
<td>Convert</td>
<td>$(\alpha s_{du}, (1 - \alpha)s_{du})$ $(\alpha s_{dd}, (1 - \alpha)s_{dd})$</td>
</tr>
<tr>
<td>Does not</td>
<td>$(X, s_{du} - X)$</td>
</tr>
<tr>
<td>Convert</td>
<td>$(\alpha R, (1 - \alpha)R)$</td>
</tr>
<tr>
<td>Does not</td>
<td>$(X, R - X)$</td>
</tr>
</tbody>
</table>

\[18\] Notice that the preferences of the financier do not affect the entrepreneur’s decisions at time $t = 1$ since the proceeds from selling the convertibles are received at $t = 0$. Alternative assumptions about preferences of the financier would not change the essence of our results.
Given the payoffs in Table 1, the following proposition characterizes the set of convertible bonds that the early self can issue to insure that, (i) the late self shuts down the firm at time \( t = 1 \) in state \( d \) and (ii) the early self invests at \( t = 0 \).

**Proposition 2.** Suppose that (i) the underlying technology cash flow \( \tilde{s}_2 \) satisfies condition (6); (ii) the recovery value \( R \) and the time \( t = 1 \) investment cost \( I_1 \) satisfy Assumption 1; and (iii) the time \( t = 0 \) investment cost \( I_0 \) satisfies condition (28). Suppose further that

\[
\lambda s_{du} - I_1 < s_{du} < s_{ud} < \lambda s_{ud} - I_1. \tag{31}
\]

Then, by issuing a convertible bond to a BEU financier with conversion ratio \( \alpha \in (0, 1) \) and face value \( X \) such that

\[
\alpha s_{dd} < X < \min\{s_{dd}, \alpha R\}, \tag{32}
\]

\[
(p + \epsilon) (1 - \alpha) s_{du} + (1 - p - \epsilon) (s_{dd} - X) < (1 - \alpha) R, \tag{33}
\]

the early self will invest at time \( t = 0 \), and the late self will expand in state \( u \) and shut down in state \( d \).

The main point of the proposition is that if the convertible bond’s face value \( X \) and conversion ratio \( \alpha \) satisfy condition (32), the entrepreneur’s best response at time \( t = 1 \) to the bondholder’s conversion strategy at time \( t = 2 \) is to shut down in state \( d \) and expand in state \( u \). The boxed entries in Table 1 represent the payoff to bondholder and entrepreneur under the conversion policy implied by the assumption of the proposition. By Assumption 1, \( s_{du} > R \) and by the RHS of (32), \( \alpha R > X \), hence \( \alpha s_{du} > X \); by the LHS of (32), \( X > \alpha s_{dd} \). Therefore, (i) if the entrepreneur continues at time \( t = 1 \), the bondholder converts at time \( t = 2 \) in state \( du \) but not in state \( dd \); (ii) if the entrepreneur shuts down at time \( t = 1 \), the bondholder converts at time \( t = 2 \) in both states. Anticipating these reactions from the bondholder, in state \( d \) the late self compares the payoff from the status quo \( \mathcal{C} \) to that of the alternative \( \mathcal{R} \), i.e.,

\[
\tilde{\xi}_{2}^{C, d} = \begin{cases} 
(1 - \alpha) s_{du} & \text{in state } du, \\
 s_{dd} - X & \text{in state } dd,
\end{cases}
\text{ versus } (1 - \alpha) R. \tag{34}
\]

Because \((1 - \alpha) s_{du} > (1 - \alpha) s_{dd} - (X - \alpha s_{dd}) = s_{dd} - X\), when deciding whether to shut down, the entrepreneur assesses the ambiguous payoff \( \tilde{\xi}_{2}^{C, d} \) of the status quo \( \mathcal{C} \) by using his optimistic
prior $\pi_d = p + \epsilon$. If condition (33) is satisfied, the entrepreneur chooses to leave the status quo and shut down.

In the above we did not discuss the possibility that the convertible bond may induce the entrepreneur to expand in state $d$. As we show in Lemma B.2, the left inequality of constraint (31) rules out this possibility. Lemma B.1 shows that the right inequality of (31) guarantees that entrepreneur will always expand in state $u$.\footnote{Constraint (31) is imposed to simplify the analysis and can easily be relaxed without affecting qualitatively the result of the proposition.}

4 Discussion and empirical implications

Although the results presented above are based on the analysis of a single decision maker, as we discussed in Section 2.3, an equivalent interpretation is that of a corporate board making decisions. We believe that the link between models of decision making under ambiguity and models of group behavior represent a fruitful direction for further theoretical and empirical research. On the theoretical front, we are not aware of any work that tries to formalize the link between observed board behavior and models of choice under ambiguity.\footnote{Crès, Gilboa, and Vieille (2011) consider the question of how a single decision maker would include the opinions of experts and argue that MEU preferences can be thought of as the result of weighting of these opinions. In contrast, we take the weighting as exogenous, i.e., each member has one vote and unanimity is required to make decisions.} Our results, however, rest on heterogenous beliefs, common tastes, unanimity, and an exogenously specified status quo. Relaxing these assumptions in order to more accurately match observed practice of boards should generate interesting insights and empirical implications.

On the empirical front, our paper suggests that a group interpretation of ambiguity delivers interesting implications that are potentially testable in a corporate laboratory. First, a group interpretation of ambiguity can be useful in guiding the empirical assessment of the set of priors. An obvious obstacle encountered in bringing models of individual decision making to the data is the determination of the set of priors of an agent. Bertrand and Schoar (2003), and Malmendier and Tate (2005) look at the characteristic of individual managers such as age, education, and whether or not the decision maker was a “depression baby”, as proxies for risk attitude and “style”. In the context of a group, an alternative interpretation is that the distribution of these characteristics within the group as well as the number of members are natural empirical proxies for the set of priors of the group.
Second, our paper suggests that the decision rule a group adopts to aggregate individual preferences is crucial to establishing the link between individual models of decision making under ambiguity (MEU and BEU) and group decisions. The internal charter of a group can be examined in order to extract properties of the decision rule. Decision rules that lead to group behaving as in the BEU model would result in a reluctance to undertake investment, escalating commitment and “hanging on to losers” behavior. In addition, group heterogeneity would create an additional motive for corporation to issue convertible securities: the desire of the current board to commit future boards. Decision rules that lead to group behaving as in the MEU model would result also in a reluctance to undertake investment but, unlike BEU, exhibit eagerness to dispose of losers and shut down. Unlike BEU, the MEU model without learning does not suffer from time inconsistency and hence does not predict the contract structure suggested by the BEU model.

Third, because the phenomena we studied in this paper are more pronounced the wider is the set of priors, a group interpretation suggests that these phenomena will be more pronounced when the group is larger and/or more heterogeneous. Some preliminary evidence on group behavior consistent with the BEU model can be found in the venture capital literature. For example, in the context of venture capital syndicates, Guler (2007) finds that (i) the probability of success and the eventual returns to investments decline with the number of financing rounds and that (ii) the probability of terminating an investment at any stage decreases as the number of members of the syndicate increases. Guler (2007) explains his finding as an example of political and institutional influences on investment decisions. The alternative that BEU provides is that as the number of syndicate members increases, the groups’ set of priors increases and as a result, the effective amount of ambiguity also increases. With more priors to satisfy, leaving the status quo becomes less likely.

Fourth, our group interpretation of ambiguity can help in providing empirical foundations to the concept of status quo on which the BEU model rests. For instance, when faced with the choice of paying a fixed cost to continue producing relative to shutting down a factory, the group is not able to define the status quo as “doing nothing”. How does a group define the status quo? Is the status quo what the board chair decides? Answers to these questions could provide a foundation for further theoretical investigation on the determination of status quo.
Finally, because unanimity is at the root of incompleteness in preferences at the group level, an empirical question that is important to address is whether groups do indeed follow a unanimity rule. Although boards may formally be governed by a majority rule, in reality, do they, as an empirical matter, only make decisions when there is unanimous agreement?

5 Conclusion

In this paper we examine a simple model of corporate investment with follow up continuation, expansion and contraction options in the presence of ambiguity. We consider two approaches to decision making under ambiguity which derive from two different deviations from the Bayesian paradigm: MEU, based on the relaxation of the independence axiom, and BEU, based on the relaxation of the completeness axiom. Our findings indicate that, depending on the model of choice used to accommodate such deviations, ambiguity can have significantly different implications on real investment and security design.

In a static setting, MEU predicts reluctance to expand and eagerness to shut down while BEU predicts reluctance to both expand and shut down. In a dynamic setting, anticipation of future reluctance to shut down induces a BEU entrepreneur to underinvest, relative to (i) an MEU entrepreneur and, (ii) a BEU entrepreneur who can precommit to shut down the firm in future bad states of the world. We show that financial contracts, such as convertible bonds, can be designed to offset the reluctance to abandon and mitigate the potential time inconsistency of the BEU entrepreneur. This raises an entirely new role for financial contracting, one that anticipates the need to modify payoffs so that future decisions will be time consistent. Because of the different predictions of the two models of choice analyzed, these results suggest care when relying on ambiguity to “explain” empirically observed phenomena.

To enrich the predictive power of models of choice under ambiguity, we propose a conceptual link between models of decision making under ambiguity and models of group behavior. For instance, a group of SEU agents with heterogeneous beliefs deciding on the basis of unanimity behaves as a single individual agent with multiple priors and incomplete preferences. Both the MEU and BEU models can be thought of as distinct models that can be used to study how a group makes decisions when unanimity is required. A group interpretation of ambiguity can help in the empirical determination of the set of priors and in the identification of the status
quo. Furthermore, when cast in terms of group decisions, our results suggest that financial contracts can be shaped so that to induce unanimity within a board. We believe that the group interpretation of ambiguity dictates the need to relax the Bayesian paradigm for the study of corporate decisions and represents a fruitful direction for both theoretical and empirical research.
A Appendix. Decision theory toolkit

In this appendix we review the theoretical foundations for the three decision models used in the paper, Subjective Expected Utility (SEU), Minimum Expected Utility (MEU) and Bewley Expected Utility (BEU). Our analysis is cast in the framework developed by Anscombe and Aumann (1963) and draws heavily on the review article by Gilboa and Marinacci (2011).

A.1 Preliminaries

Let $S$ denote a set describing the possible states of the world, endowed with an event algebra $\Sigma$, $Y$ be a set describing possible outcomes, and $\Delta(Y)$ the space of lotteries over outcomes $Y$. A lottery is a random variable with outcomes in $Y$ whose probabilities are objectively known by the agent. For simplicity of exposition, we take $S$ and $Y$ as discrete sets.

The agent makes choices over acts, which are functions that map states into lotteries. Formally, an act $f$ is a $\Sigma$-measurable function $f : S \rightarrow \Delta(Y)$. Because the space $\Delta(Y)$ is convex one can construct convex combination of acts, i.e., given any two acts $f$ and $g$ and $\alpha \in [0, 1]$ the mixed act $\alpha f + (1 - \alpha)g$ is a function from $S$ to $\Delta(Y)$ defined as

$$ (\alpha f + (1 - \alpha)g)(s) = \alpha f(s) + (1 - \alpha)g(s), \forall s \in S. $$

Let $F$ be the space of acts. The agent has preferences $\succeq$ on $F$, i.e., $f \succeq g$ means that act $f$ is weakly preferred to $g$. Note that a preference $\succeq$ over acts induces a preference $\succeq_\Delta$ over lotteries if, for all $p$ and $q$ in $\Delta(Y)$ one defines

$$ p \succeq_\Delta q \iff f \succeq g, $$

where $f$ and $g$ are constant acts, i.e., $f(s) = p$ and $g(s) = q$ for all $s \in S$. Because constant act are not subject to state uncertainty, the preference $\succeq_\Delta$ captures preference with respect to risk, as opposed to uncertainty.

Example 1. Let us consider a risk neutral manager who has $I$ dollars and considers whether or not to invest in a project that yields random outcomes. Suppose there are two possible state of nature $u$ and $d$. In state $u$ the project outcome is $y_u$ and in state $d$ the outcome is $y_d$. If the manager does not invest, the outcome is simply $I$. 
In terms of the above notation, the state space is $S = \{d, u\}$ and the outcome space is $Y = \{y_u, y_d, I\}$. “Investing” is an act $f$ such that:

\[
\begin{align*}
    f(u) &= \begin{cases} 
        1, & \text{for outcome } y_u \\
        0, & \text{for outcome } y_d \\
        0, & \text{for outcome } I 
    \end{cases}, \\
    f(d) &= \begin{cases} 
        0, & \text{for outcome } y_u \\
        1, & \text{for outcome } y_d \\
        0, & \text{for outcome } I 
    \end{cases}.
\end{align*}
\]  

(A3)

while “Not investing” is an act $g$ such that

\[
\begin{align*}
    g(u) &= \begin{cases} 
        0, & \text{for outcome } y_u \\
        0, & \text{for outcome } y_d \\
        1, & \text{for outcome } I 
    \end{cases}, \\
    g(d) &= \begin{cases} 
        0, & \text{for outcome } y_u \\
        1, & \text{for outcome } y_d \\
        1, & \text{for outcome } I 
    \end{cases}.
\end{align*}
\]  

(A4)

In other words $f$ and $g$ are degenerate lotteries, which is a formal way to express the fact that upon investing we know for sure the outcome we obtain in each state, although we do not know a priori which one of the two states, $u$ or $d$ will materialize.

In the sequel we summarize the axioms on the primitive preferences $\succeq$ on which the three model of choice we consider in the paper rest, and provide the representation of the preference relation $\succeq$.

A.2 Subjective Expected Utility (SEU)

Savage’s SEU model of decision making rests on the following axioms:

S1. **Weak order.** $\succeq$ on $\mathcal{F}$ is complete and transitive.

S2. **Monotonicity.** For any $f, g \in \mathcal{F}$, if $f(s) \succeq_D g(s)$ for each $s \in S$, then $f \succeq g$.

S3. **Independence.** For any $f, g, h \in \mathcal{F}$, and any $\alpha \in (0, 1)$ we have

\[
    f \succ g \Rightarrow \alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h. 
\]  

(A5)

S4. **Archimedean.** Let $f, g, h \in \mathcal{F}$ be such that $f \succeq g \succeq h$. Then there are $\alpha, \beta \in (0, 1)$ such that

\[
    \alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \alpha)h. 
\]  

(A6)
S5. **Non degeneracy.** There are \( f, g \in \mathcal{F} \) such that \( f \succ g \).

The following theorem characterizes the SEU representation of preferences.

**Theorem 2.** Let \( \succeq \) be a preference defined on \( \mathcal{F} \). The following conditions are equivalent:

1. The preference \( \succeq \) satisfies the axioms S1–S5.

2. There exist a non-constant function \( U : Y \to \mathbb{R} \) and a unique probability measure \( \pi : \Sigma \to [0, 1] \) such that for all \( f, g \in \mathcal{F}, \)

\[
f \succeq g \iff \sum_{s \in S} \left( \sum_{x \in \text{supp}(f)} U(x)f(s) \right) \pi(s) \geq \sum_{s \in S} \left( \sum_{x \in \text{supp}(f)} U(x)g(s) \right) \pi(s) \tag{A7}\]

where the inner part

\[
\sum_{x \in \text{supp}(f)} U(x)f(s)
\]

is the expected utility under the probability \( f(s) \) that the act \( f \) delivers in state \( s \).

In the case of the risk neutral manager considered in Example 1, the two acts: invest, \( f \) and don’t invest, \( g \), map into degenerate lotteries on the outcome space and hence the above theorem implies that

\[
\text{Invest} \succeq \text{Don’t invest} \iff \mathbb{E}_{\pi}[\tilde{x}] \geq I, \tag{A8}\]

where \( \mathbb{E}_{\pi}[\tilde{x}] = \pi(u)y_u + \pi(d)y_d \).

**A.3 Minimum Expected Utility (MEU)**

Gilboa and Schmeidler (1989) start from Savage axioms and

i. Replace the independence axiom S3 over acts with the following independence axiom over lotteries:

**M3. C-Independence.** For all acts \( f, g \in \mathcal{F} \) and all constant acts (lotteries) \( p \)

\[
f \succ g \Rightarrow \alpha f + (1 - \alpha)p \succ \alpha g + (1 - \alpha)p, \quad \forall \alpha \in [0, 1] \tag{A9}\]

ii. Introduce a new axiom to capture aversion to uncertainty:
M6. **Uncertainty Aversion.** For any \( f, g \in \mathcal{F} \) and any \( \alpha \in (0, 1) \)

\[
f \sim g \Rightarrow \alpha f + (1 - \alpha)g \geq f.
\] (A10)

The following theorem characterize the MEU representation of preferences.

**Theorem 3.** Let \( \succeq \) be a preference defined on \( \mathcal{F} \). The following conditions are equivalent:

1. The preference \( \succeq \) satisfies the axioms S1, S2, M3, S4, S5, M6.

2. There exist a non constant function \( U : Y \to \mathbb{R} \) and a convex and compact set \( \Pi \subseteq \Delta(\Sigma) \) of probability measures such that for all \( f, g \in \mathcal{F} \),

\[
f \succeq g \iff \min_{\pi \in \Pi} \sum_{s \in S} \left( \sum_{x \in \text{supp}(f)} U(x) f(s) \right) \pi(s) \geq \min_{\pi \in \Pi} \sum_{s \in S} \left( \sum_{x \in \text{supp}(f)} U(x) g(s) \right) \pi(s)
\] (A11)

In the case of the risk neutral manager considered in Example 1 the above theorem implies that

\[
\text{Invest} \succeq \text{Don’t invest} \iff \min_{\pi \in \Pi} \mathbb{E}_\pi[\tilde{x}] \geq I,
\] (A12)

where \( \mathbb{E}_\pi[\tilde{x}] = \pi(u)y_u + \pi(d)y_d \).

**A.4 Bewley Expected Utility (BEU)**

Bewley (1986) start from Savage’s axioms and replaces the completeness axiom over acts with a completeness axiom over lotteries. Formally,

C1. **C-Completeness.** For every constant act (lottery) \( p, q \in \Delta(Y) \), \( p \succeq q \) or \( q \succeq p \).

The following theorem characterize the BEU representation of preferences.\(^{21}\)

**Theorem 4.** Let \( \succeq \) be a preference defined on \( \mathcal{F} \). The following conditions are equivalent:

1. The preference \( \succeq \) satisfies the axioms C1 and S2–S5.

\(^{21}\) The theorem is based on the characterization provided by Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) who represent weak preferences. Bewley (1986) provides characterization of strict preferences.
2. There exist a non constant function $U : Y \to \mathbb{R}$ and a convex and compact set $\Pi \subseteq \Delta(\Sigma)$ of probability measures such that for all $f, g \in F$,

$$f \succeq g \iff \sum_{s \in S} \left( \sum_{x \in \text{supp}(f(s))} U(x)f(s) \right) \pi(s) \geq \sum_{s \in S} \left( \sum_{x \in \text{supp}(f(s))} U(x)g(s) \right) \pi(s), \forall \pi \in \Pi.$$  

(A13)

In the case of the risk neutral manager considered in Example 1 the above theorem implies that

$$\text{Invest} \succeq \text{Don’t invest} \iff \mathbb{E}_\pi[\tilde{x}] \geq I, \forall \pi \in \Pi,$$

(A14)

where $\mathbb{E}_\pi[\tilde{x}] = \pi(u)y_u + \pi(d)y_d$.

As discussed in the paper, we refer to the BEU model as the unanimity criterion (A13) augmented by the Inertia assumption.

B Appendix. Proofs

Proof of Proposition 1

The proof is by contradiction. Consider SEU and suppose there exists a plan $\varphi' \in \mathcal{P} \cup \{\varphi_0\}$ such that $\varphi'$ is strictly preferred to $q$ for a given plan $q \in \mathcal{P} \cup \{\varphi_0\}$ and that $q$ is chosen over all $\varphi = (a_u, a_d) \in \mathcal{P}^\ast_{\text{SEU}}$. We will now construct a maximal plan in the set of implementable plan: Because $\mathcal{A}$ is finite, for each node $j$, there must be an action $a''_j$ such that

$$\mathbb{E}_p[\tilde{C}a''_j(s_j^2)] = \sup_{a \in \mathcal{A}} \mathbb{E}_p[\tilde{C}a(s_j^2)].$$

By construction, the plan $\varphi'' = (a''_u, a''_d) \in \mathcal{P}^\ast_{\text{SEU}}$ and satisfies

$$\mathbb{E}_p(\tilde{C} \varphi'') \geq \mathbb{E}_p(\tilde{C} \varphi') > \mathbb{E}_p(\tilde{C} q).$$

As a result, $\varphi'' \in \mathcal{P}^\ast_{\text{SEU}}$ is preferred to $q$. This is a contradiction because by assumption $q$ is preferred to all $\varphi \in \mathcal{P}^\ast_{\text{SEU}}$.

Consider MEU and suppose there exists a plan $\varphi' \in \mathcal{P} \cup \{\varphi_0\}$ such that $\varphi'$ is strictly preferred to $q$ for a given plan $q \in \mathcal{P} \cup \{\varphi_0\}$ and that $q$ is chosen overall $\varphi = (a_u, a_d) \in \mathcal{P}^\ast_{\text{MEU}}$.
The procedure is similar for MEU and we will construct similarly a maximal plan in the set of MEU-implementable plans: For each node $j$, there must be an action $a''_j$ such that

$$\min_{\pi \in \Pi} E_{\pi}[\tilde{C}''(\tilde{s}_2^j)] = \sup_{a \in A} \min_{\pi \in \Pi} E_{\pi}[\tilde{C}(\tilde{s}_2^j)].$$

(B15)

By construction, the plan $\mathcal{\varphi}'' = (a''_u, a''_d)$ is MEU-implementable. Let us now prove that the plan $\mathcal{\varphi}''$ is preferred to $q$ by the MEU agent thus establishing a contradiction:

$$\min_{(\pi_0, \pi_u, \pi_d) \in \Pi^3} \mathbb{E}_{\pi_0, \pi_u, \pi_d}(\mathcal{\varphi}''') = \min_{(\pi_0, \pi_u, \pi_d) \in \Pi^3} \left[ \pi_0 \mathbb{E}_{\pi_u}[\tilde{C}''(\tilde{s}_2^u)] + (1 - \pi_0) \mathbb{E}_{\pi_d}[\tilde{C}''(\tilde{s}_2^d)] \right]$$

(B16)

$$= \min_{\pi_0 \in \Pi} \left[ \pi_0 \min_{\pi_u \in \Pi} \mathbb{E}_{\pi_u}[\tilde{C}''(\tilde{s}_2^u)] + (1 - \pi_0) \min_{\pi_d \in \Pi} \mathbb{E}_{\pi_d}[\tilde{C}''(\tilde{s}_2^d)] \right]$$

(B17)

$$\geq \min_{\pi_0 \in \Pi} \left[ \pi_0 \min_{\pi_u \in \Pi} \mathbb{E}_{\pi_u}[\tilde{C}''(\tilde{s}_2^u)] + (1 - \pi_0) \min_{\pi_d \in \Pi} \mathbb{E}_{\pi_d}[\tilde{C}''(\tilde{s}_2^d)] \right]$$

(B18)

$$= \min_{(\pi_0, \pi_u, \pi_d) \in \Pi^3} \mathbb{E}_{\pi_0, \pi_u, \pi_d}(\mathcal{\varphi}') > \min_{(\pi_0, \pi_u, \pi_d) \in \Pi^3} \mathbb{E}_{\pi_0, \pi_u, \pi_d}(\mathcal{\varphi})$$

(B19)

and as a result, $\mathcal{\varphi}''$ is preferred to $q$. This is a contradiction because $\mathcal{\varphi}'' \in \mathcal{P}^{\ast}_{MEU}$ and so it cannot be chosen over $q$. Notice that equality (B16) is just the definition of the utility derived by the MEU agent. Equality (B17) follows from the fact that $\pi$ and $1 - \pi$ are non negative and so we can optimize sequentially over the triplet $(\pi, \pi_u, \pi_d)$. Inequality (B18) is implied by the fact that $\mathcal{\varphi}''$ is maximal in the set of implementable plan (e.g., equation (B15)). The equality in (B19) is the definition of the derived utility from the outcome of the action $\mathcal{\varphi}'$ and the last inequality follows from the fact that we assumed that $\mathcal{\varphi}'$ is strictly preferred to $q$.

Proof of Proposition 2

Let us start with a the following two lemmata:

**Lemma B.1 (State $u$ with convertible bond).** Suppose that Assumption 1 and conditions (6), (28) and (31) are satisfied. Assume that the entrepreneur issues a convertible bond with conversion ratio $\alpha \in (0, 1)$ and face value $X$ satisfying condition (32) and undertakes the investment at time $t = 0$. The entrepreneur’s real investment decision at time $t = 1$ in state $u$ is to expand the firm. The bondholder’s conversion choice is to convert the bond in all subsequent states at time $t = 2$. 


**Proof:** When the entrepreneur reaches the node $u$ at time $t = 1$, the subsequent firms cash flow $\tilde{C}_2$ and payoff to investors and the entrepreneur can be described for each real investment choice at time $t = 1$:

(i) If $a_u = E$, then the firm cash flows are $\tilde{C}_2 = \lambda \tilde{s}_u - I_1$. The investors will have to decide whether to convert the bond or take its face value. It is easy to check that conditions (31) and (32) imply

$$X < \alpha R < \alpha (\lambda s_{ud} - I_1) < \alpha (\lambda s_{uu} - I_1)$$

and the financier will choose to convert the bond in both states $uu$ and $ud$. Consequently, if the entrepreneur chooses to expand the firm, he will be left with the cash flow $\left(1 - \alpha\right)\left(\lambda s_{u} - I_1\right)$ as time $t = 2$.

(ii) If $a_u = C$, the firm cash flows are given by $\tilde{C}_2 = \tilde{s}_u^2$. Under Assumption 1 and conditions (6), $R < \tilde{s}_u^2$. Moreover condition (32) implies that $X < \alpha R$. Hence the investor will convert the bond and the entrepreneur is left with $\left(1 - \alpha\right)\tilde{s}_u^u$.

(iii) If $a_d = R$, the firm cash flow is $\tilde{C}_2 = R$ and the share $\alpha R$ goes to the financier whereas the share $\left(1 - \alpha\right)R$ goes to the entrepreneur.

To summarize, for all actions, the entrepreneur’s cash flow subsequent to the node $u$ is given by $\left(1 - \alpha\right)\tilde{C}_2$. When the convertible is issued, the entrepreneur’s payoff subsequent to node $u$ is simply a scaled version of the entrepreneur’s payoff when the convertible is not issued. Consequently, the entrepreneur’s incentives are not distorted in node $u$ and his investment plan is identical to that in the absence of a convertible. Conditions (11) and (12), and Assumption 1 imply that, in the absence of convertibles, the BEU entrepreneur expands the firm in node $u$. This concludes the proof of Lemma B.1.

**Lemma B.2 (State $d$ with convertible bond).** Suppose that Assumption 1 and conditions (6), (28) and (31) are satisfied. Suppose the entrepreneur issues a convertible bond with conversion ratio $\alpha \in (0, 1)$ and face value $X$ satisfying condition (32) and undertake the investment at time $t = 0$. The entrepreneur’s real investment decision at time time $t = 1$ in state $d$ is to shut down the firm and the bondholders are left with a proportion $\alpha$ of the scrap value $R$ at time $t = 2$. 

$\blacksquare$
Proof: When the entrepreneur reaches the node $d$ at time $t = 1$, the subsequent firms cash flow $\tilde{C}_2$ and payoff to investors and the entrepreneur can be described for each real investment choice at time $t = 1$:

(i) If $a_d = E$, the firm’s cash flow is $\tilde{C}_2 = \lambda \tilde{s}_2^d - I_1$ at $t = 2$. The financier will then receive the cash flow $\varphi(\lambda \tilde{s}_2^d - I_1)$ where the function $\varphi$ summarizes the convertible exercise policy of the bondholder and is defined by

$$\varphi(y) = \max\{\alpha y, \min\{X, y\}\}. \quad (B20)$$

The entrepreneur ends up then with the cash flow

$$\tilde{\xi}_2^{E,d} = \psi(\lambda \tilde{s}_2^d - I_1) \quad (B21)$$

where the function $\psi$ defined by

$$\psi(y) = y - \varphi(y). \quad (B22)$$

describes the payoff to the entrepreneur when the bond is optimally exercised by the bondholder. Notice that the function $\psi$ is non-decreasing because $\alpha \in (0, 1)$.

(ii) If $a_d = C$, the firms cash flow is $\tilde{C}_2 = \tilde{s}_2^d$ at time $t = 2$. Assumption 1 and condition (32) imply that $\alpha s_{dd} < X < \alpha R < \alpha s_{du}$ and therefore the bondholder converts in node $du$ but not at node $dd$. This yield the bondholder’s payoff

$$\tilde{B}_2^{C,d} = \begin{cases} \alpha s_{du} & \text{if } \tilde{s}_2^d = s_{du} \\ X & \text{if } \tilde{s}_2^d = s_{dd} \end{cases} \quad (B23)$$

and the resulting payoff to the entrepreneur

$$\tilde{\xi}_2^{C,d} = \begin{cases} (1 - \alpha)s_{du} & \text{if } \tilde{s}_2^d = s_{du} \\ s_{dd} - X & \text{if } \tilde{s}_2^d = s_{dd} \end{cases} \quad (B24)$$

(iii) If $a_d = R$, the bond holders get $\alpha R$ and the entrepreneur gets $(1 - \alpha)R$. 
First we show that, when facing the binary decision to expand relative to the status quo $C$ at time $t = 1$ in state $d$, the entrepreneur does not expand and chooses the status quo. The difference is cash flow to the entrepreneur in state $du$ satisfies

$$
\left[ \tilde{\xi}_2^C, du \right] - \left[ \tilde{\xi}_2^E, du \right] = (1 - \alpha)s_{du} - \psi(\lambda s_{du} - I_1)
\geq (1 - \alpha)s_{du} - \psi(s_{du})
= \varphi(s_{du}) - \alpha s_{du} \geq 0,
$$

(B25)

where inequality (B25) follows from condition (31) and the monotonicity of $\psi$. The equality in (B26) follows from the definition of $\psi$ in (B22) and the last inequality in (B26) follows from the definition of $\varphi$ in (B20).

Similarly, in state $dd$, the difference in cash flow to the entrepreneur is

$$
\left[ \tilde{\xi}_2^C, dd \right] - \left[ \tilde{\xi}_2^E, dd \right] = s_{dd} - X - \psi(\lambda s_{dd} - I_1)
\geq s_{dd} - X - \psi(s_{dd})
= \varphi(s_{dd}) - X \geq 0,
$$

(B27)

where the inequality (B27) follows from condition (31) and the monotonicity of $\psi$. The equality in (B28) follows from the definition of $\psi$ in (B22). The last inequality in (B28) follows from the definition of $\varphi$ in (B20) and from the fact that condition (32) implies $X < s_{dd}$.

To summarize, when the firm is started after the convertible is issued, the entrepreneur’s payoff resulting from expanding the firm at node $d$ is dominated by the payoff resulting from continuing the firm. Consequently, at time $t = 1$, the BEU entrepreneur will not expand the firm in the ‘down state’.

Second, we show that, facing the binary decision $R$ vs. $C$ (status quo) at time $t = 1$ in state $d$, the entrepreneur does not choose the status quo but instead shuts down the firm. The entrepreneur must compare the payoff $\tilde{\xi}_2^C, d$ defined in (34) with the payoff $(1 - \alpha)R$. He will shut down at time $t = 1$ in state $d$ if and only if

$$
(1 - \alpha)R > \pi (1 - \alpha)s_{du} + (1 - \pi)(s_{dd} - X), \quad \forall \pi \in \Pi.
$$

(B29)
Because \((1 - \alpha)s_{du} > (1 - \alpha)s_{dd} - (X - \alpha s_{dd}) = s_{dd} - X\), the entrepreneur will assess inequality (B29) by using his optimistic prior \(\pi_d = p + \epsilon\) and will shut down in state \(d\) if and only if

\[
(1 - \alpha)R > (p + \epsilon)(1 - \alpha)s_{du} + (1 - p - \epsilon)(s_{dd} - X)
\]

or

\[
X - \alpha s_{dd} > \frac{(1 - \alpha)(E_{p+\epsilon}[\tilde{s}_2^d] - R)}{1 - p - \epsilon}.
\] (B30)

The last inequality is the left hand side of condition (32). The presence of the convertible bond induced the entrepreneur to shut down the firm in state \(d\). This concludes the proof of Lemma B.2.

Now let us turn to the proof of Proposition 2. At time \(t = 0\), the entrepreneur contemplates starting the firm after issuing the convertible or remain with the status quo \(\varphi_0\) and get \(I_0\). The results in Lemma B.1 and B.2 show that the entrepreneur’s anticipated investment plan is to expand in state \(u\) and shut down in state \(d\). Therefore, the convertible bond is a gamble \(\tilde{B}_2\) with time \(t = 2\) payoffs \(\alpha(\lambda \tilde{s}_{uu}^2 - I_1)\) in states \(uu\) and ‘up-down’ and \(\alpha R\) in the states \(du\) and \(dd\).

It follows that a BEU financier purchasing such a convertible bond will be willing to pay a price \(P\) such that

\[
P < \alpha \left[ (p - \epsilon) (\lambda E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p + \epsilon)R \right].
\] (B31)

Notice that the face value of the bond \(X\) does not directly appear in the bond price formula. This should not be surprising because bond holders always convert the bond given the entrepreneur’s investment exercise policy. The face value only plays the role of a “threat” to induce the entrepreneur to avoid taking actions that will result in having to pay \(X\) to the bondholder. The entrepreneur undertakes the investment at time \(t = 0\) if

\[
I_0 - P < (1 - \alpha) \left[ (p - \epsilon) (\lambda E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p + \epsilon)R \right].
\] (B32)

Using (B31), inequality (B32) is equivalent to

\[
I_0 < (p - \epsilon)(\lambda E_{p-\epsilon}[\tilde{s}_2^u] - I_1) + (1 - p + \epsilon)R,
\] (B33)
When condition (28) is satisfied, conditions (B33) holds and as a result the entrepreneur issues the convertible and starts the project. This concludes the proof of Proposition 2.
References


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