Reference Technology Sets, Free Disposal Hulls and Productivity Decompositions

September 16, 2013

Erwin Diewert,[1]
Discussion Paper 13-10,
School of Economics,
University of British Columbia,
Vancouver, B.C.,
Canada, V6N 1Z1.
Email: erwin.diewert@ubc.ca

Kevin Fox,
School of Economics,
University of New South Wales,
Sydney 2052,
Australia.
Email : K.Fox@unsw.edu.au

Abstract

Diewert and Fox (2013) proposed decompositions of a Malmquist-type productivity index into explanatory factors, with a focus on extracting technical progress, technical efficiency change and returns to scale components. A major problem with their decompositions is that it may be difficult to determine the appropriate reference technologies. Using relatively unrestrictive regularity conditions, the paper develops a data envelopment type approach for decomposing productivity growth for a panel of production units into explanatory factors based on the Free Disposal Hull methods pioneered by Tulkens and his co-authors.

JEL Classification: C43, D24, E23

Key Words: Productivity indexes, Free Disposal Hulls, technical efficiency, technical progress, returns to scale, nonparametric approaches to production theory, distance functions.

[1] The authors gratefully acknowledge financial support from the Social Sciences and Humanities Research Council of Canada and the Australian Research Council (LP0884095).. The authors also thank Rolf Färe, Shawna Grosskopf and Knox Lovell for helpful comments.

1. Introduction

Following the definition of Malmquist input, output and productivity indexes by Caves, Christensen and Diewert (1982), there has been much interest in the specification and decomposition of productivity growth based on these theoretical indexes; see e.g. Färe et al. (1994) and Grosskopf (2003). In proposing their decompositions of a Malmquist-type productivity index into explanatory factors of technical progress, technical efficiency change and returns to scale components, Diewert and Fox (2013) followed Bjurek (1996) in defining Malmquist indexes for a production unit when knowledge of the reference best practice technology is available for the two periods under consideration. However, they did not address the problems associated with the determination of the appropriate reference technologies.

In this paper, we assume a panel data set is available for two periods on K production units that are assumed to be producing the same set of M outputs and using the same set of N inputs and we assume that the *Free Disposal Hulls* (FDHs) generated by these data form the reference technology sets for the two periods under consideration; see e.g. Deprins, Simar and Tulkens (1984), Tulkens (1993) and Tulkens and Eeckaut (1995a) (1995b) on the FDH approach, and Lovell (2006) for a review. Thus our methodology follows the same path as the better-known Data Envelopment Analysis (DEA) methodologies (see e.g. Charnes and Cooper, 1985) except that the reference technology sets use Free Disposal Hulls instead of Convex Free Disposal Hulls. Russell and Schworm (2009) informatively refer to these types of reference technology sets as *data generated technologies*.

We follow in the footsteps of Tulkins and his coauthors but we also indicate how the various output and input distance functions that form the theoretical Malmquist indexes can be constructed using these reference technology sets so that we can obtain the productivity decompositions suggested by Diewert and Fox (2013).

2. Productivity Growth Decomposition

Diewert and Fox (2013) derived explanatory factor decompositions of the Bjurek (2006) productivity index, which they defined as follows:

$$(1)\ \Pi_B(x^0,x^1,y^0,y^1) \equiv \{[d^0(y^1,x^0)/d^0(y^0,x^0)][D^0(y^0,x^0)/D^0(y^0,x^1)][d^1(y^1,x^1)/d^1(y^0,x^1)][D^1(y^1,x^0)/D^1(y^1,x^1)]\}^{1/2},$$

where, using the period t reference technology set S^t, and given a nonnegative, nonzero output vector $y > 0_M$ and a strictly positive input vector $x >> 0_N$,[2]

$$(2)\ D^t(y,x) \equiv \max_{\delta>0} \{\delta: (y,x/\delta)\in S^t\}$$

is the period t *input distance function* D^t for periods t = 0,1, and

$$(3)\ d^t(y,x) \equiv \min_{\delta>0} \{\delta: (y/\delta,x)\in S^t\},$$

is the period t *output distance function* for periods t = 0,1.

[2] Notation: $y \geq 0_M$ means each component of the vector y is nonnegative, $y >> 0_M$ means that each component is strictly positive, and $y > 0_M$ means $y \geq 0_M$ but $y \neq 0_M$.

Diewert and Fox (2013) showed that the distance functions in (2) and (3), and hence the productivity index in (1), are well defined under the following relatively unrestrictive regularity conditions on the reference technology set S^t.

P1. S is a nonempty closed subset of the nonnegative orthant in Euclidean M+N dimensional space.
P2. For every $y \geq 0_M$, there exists an $x \geq 0_N$ such that $(y,x)\in S$.
P3. $(y,x^1)\in S$, $x^2 \geq x^1$ implies $(y,x^2)\in S$, i.e. free disposability of inputs.
P4. $y > 0_M$ implies that $(y,0_N)\notin S$.
P5. $x \geq 0_N$, $(y,x)\in S$ implies $0_M \leq y \leq b(x)1_M$ where 1_M is a vector of ones of dimension M and $b(x) \geq 0$ is a finite nonnegative bound.
P6. $x >> 0_N$ implies that there exists $y >> 0_M$ such that $(y,x)\in S$.
P7. $(y^1,x)\in S$, $0_M \leq y^0 \leq y^1$ implies $(y^0,x)\in S$, i.e. free disposability of ouputs.

The focus here will be on the Diewert and Fox (2013) exact Fisher type decomposition of the Bjurek productivity index using an output (rather than an input) orientation. The Bjurek (1996; 310-311) productivity index Π_B defined by (1) is the geometric mean of the Laspeyres and Paasche type Bjurek indexes where the latter indexes are basically Malmquist (1953) output indexes divided by Malmquist input indexes. The Bjurek productivity index turns out to have the following decomposition:

(4) $\Pi_B(x^0,x^1,y^0,y^1) = [\varepsilon^1/\varepsilon^0]\tau\rho$

where x^t and y^t are the observed input and output vectors for a production unit for time periods t = 0,1. The period t *technical efficiency* of the unit for period t, ε^t, is defined as:

(5) $\varepsilon^t \equiv d^t(y^t,x^t) = \min_\delta\{\delta: (y^t/\delta, x^t)\in S^t\} \leq 1,$

so that $[\varepsilon^1/\varepsilon^0]$ measures efficiency change over the two periods. The Diewert and Fox *technical change* measure, τ, is defined as:

(6) $\tau \equiv [d^0(y^0,x^0)/d^1(y^0,x^0)][d^0(y^1,x^1)/d^1(y^1,x^1)]\}^{1/2}$

and the Diewert and Fox *returns to scale measure*, ρ, turns out to equal the following expression:[3]

(7) $\rho = [\varepsilon^0/\varepsilon^1]\ \tau^{-1}\ \Pi_B(x^0,x^1,y^0,y^1)$.

A careful examination of the various distance function components of the decomposition shows that it is necessary to be able to calculate the following four *input distances* for a representative production unit that has output and input vectors (y^0,x^0) and (y^1,x^1) for periods 0 and 1: $D^0(y^0,x^0)$, $D^0(y^0,x^1)$, $D^1(y^1,x^1)$ and $D^1(y^1,x^0)$. We also need to be able to compute the following six *output distances*: $d^0(y^0,x^0)$, $d^0(y^1,x^0)$, $d^0(y^1,x^1)$, $d^1(y^1,x^1)$, $d^1(y^0,x^1)$ and $d^1(y^0,x^0)$. We will show how this can be done by using Free Disposal Hulls generated by a panel data

[3] See Diewert and Fox (2013) for their underlying definition of returns to scale. Their measure is basically a geometric average of output growth divided by input growth for the production unit under consideration but the output and input growth measures hold the technology constant using the period 0 and 1 reference technologies.

set of outputs and inputs covering two periods and K production units, (y^{tk},x^{tk}) for t = 0,1 and k = 1,...,K.

3. Approximations to Reference Technology Sets

To avoid regularity problems, we assume that the input vectors are strictly positive and the output vectors are nonnegative with the first component always positive; i.e., we assume that:

(8) $x^{tk} \gg 0_N$; $y^{tk} = [y_1^{tk},...,y_M^{tk}] \geq 0_M$ and $y_1^{tk} > 0$ for t = 0,1 and k = 1,...,K.

We assume that there are industry best practice technology sets S^0 and S^1 satisfying P1-P7 with

(9) $(y^{0k},x^{0k})\in S^0$ and $(y^{1k},x^{1k})\in S^1$ for k = 1,...,K.

We also assume that there is *no technological regress* in the industry going from period 0 to 1 so that S^0 is a subset of S^1:

(10) $S^0 \subset S^1$.

For more on this assumption, see e.g. Diewert (1980; 264) (1981; 27-28) in the context of a DEA approach to measuring technical progress, and Tulkens and Eeckaut (1995a) (1995b).

The set of input vectors x that can produce at least the output vector $y \geq 0_M$ using the period 0 technology, $S^0(y)$, is defined as follows:

(11) $S^0(y) \equiv \{x: (y,x)\in S^0\}$.

Assumptions (9) imply that $x^{0k}\in S^0(y^{0k})$ for k = 1,...,K. Thus by the input free disposability assumption on S^0, the orthant $\{x: x \geq x^{0k}\}$ is a subset of $S^0(y^{0k})$ for k = 1,...,K.

Now let $y \geq 0_M$ and define $\alpha^0(y;y^{01},y^{02},...,y^{0K})$ as the set of indices i such that y^{0i} is equal to or greater than y; i.e.,

(12) $\alpha^0(y;y^{01},y^{02},...,y^{0K}) \equiv \{i: y^{0i} \geq y ; i = 1,2,...,K\}$.

We abbreviate the notation and write $\alpha^0(y;y^{01},y^{02},...,y^{0K})$ as $\alpha^0(y)$. For a general output vector y, $\alpha^0(y)$ may be empty; i.e., none of the production units k in the panel produced an output vector in period 0 such that $y^{0k} \geq y$. However, since $y^{0k} \geq y^{0k}$, it can be seen that

(13) $k\in\alpha^0(y^{0k})$ for k = 1,...,K.

Fix k and suppose that $i\in\alpha^0(y^{0k})$. Thus $y^{0i} \geq y^{0k}$. Using (9), we have $(y^{0i},x^{0i})\in S^0$. Since $y^{0i} \geq y^{0k}$, using the free disposability assumption on outputs, x^{0i} can also produce y^{0k} in period 0. Thus $(y^{0k},x^{0i})\in S^0$ and using definition (11), $x^{0i}\in S^0(y^{0k})$. Using the free disposability assumption on inputs, the set $\{x: x \geq x^{0i}\}$ will be a subset of $S^0(y^{0k})$. This conclusion holds for all indices i such that $i\in\alpha^0(y^{0k})$. Define the set F^{0k} as the union of these input sets:

(14) $F^{0k} \equiv \bigcup_{i\in\alpha^0(y^{0k})} \{x: x \geq x^{0i}\}$.

It can be seen that F^{0k} is a subset of $S^0(y^{0k})$. Assume that $F^{0k} = S^0(y^{0k})$; i.e., assume that the Free Disposal Hull of the period 0 data points is equal to the true technology set $S^0(y^{0k})$. Under this assumption, we can calculate the *period 0 input distance functions*, $D^0(y^{0k},x)$ for k = 1,...,K and for all $x >> 0_N$:

$$\begin{aligned}(15)\ D^0(y^{0k},x) &= \max_{\delta>0} \{\delta: (x/\delta)\in F^{0k}\} \\ &= \max_i \{\delta_i : x/\delta_i \geq x^{0i}\ ; i\in\alpha^0(y^{0k})\} \\ &= \max_i \{\min_n \{x_n/x_n^{0i} : n = 1,...,N\}: i\in\alpha^0(y^{0k})\}.\end{aligned}$$

Since $k\in\alpha^0(y^{0k})$, it can be seen that the following inequality holds:

$$\begin{aligned}(16)\ D^0(y^{0k},x^{0k}) &= \max_{\delta>0} \{\delta: (x^{0k}/\delta)\in F^{0k}\} \\ &= \max_i \{\min_n \{x_n^{0k}/x_n^{0i} : n = 1,...,N\}: i\in\alpha^0(y^{0k})\} \\ &\geq 1.\end{aligned}$$

If $D^0(y^{0k},x^{0k}) > 1$, then production unit k is not technically efficient relative to the FDH technology spanned by the period 0 observations. Any observation $i\in\alpha^0(y^{0k})$ which attains the maximum in (16) serves as an explicit example of an industry production unit that is more efficient than the inefficient observation k.[4]

Since $k\in\alpha^0(y^{0k})$ and thus the index set $\alpha^0(y^{0k})$ is not empty, by using (15) it can be seen that the period 0 input distance functions $D^0(y^{0k},x^{0k})$ and $D^0(y^{0k},x^{1k})$ are well defined for k = 1,...,K. We will discuss how to define the period 1 input distance functions $D^1(y^{1k},x^{1k})$ and $D^1(y^{0k},x^{0k})$ using data generated Free Disposal Hulls after we discuss how the output distance functions are defined.

We turn our attention to the calculation of the period 0 output distance functions, $d^0(y,x)$. Let $x \geq 0_N$. Define the set of outputs that can be produced using the input vector x and the period 0 technology as $s^0(x)$:

(17) $s^0(x) \equiv \{y: (y,x)\in S^0\}$.

Fix the input vector $x \geq 0_N$ and define $\beta^0(x;x^{01},x^{02},...,x^{0K})$ as the set of indices i such that x^{0i} is equal to or less than x:

(18) $\beta^0(x;x^{01},x^{02},...,x^{0K}) \equiv \{i: x^{0i} \leq x\ ; i = 1,2,...,K\}$.

Abbreviate the notation by writing $\beta^0(x;x^{01},x^{02},...,x^{0K})$ as $\beta^0(x)$. If the x vector is too small, it may be the case that $\beta^0(x)$ is an empty set. However, when $x = x^{0k}$, note that $k\in\beta^0(x^{0k})$ for k = 1,...,K.

[4] Tulkens (1993; 191) stressed the importance of this fact when using a FDH reference technology (as opposed to a DEA reference technology): "From a managerial point of view, I should like to point out that the identification of a set of dominating observations, by showing *actually implemented* production plans that are clearly more efficient, gives to the *inefficiency* scores a credibility that the usually lack when reference is only made to an abstract frontier. This argument is illustrated with the observation labelled D on Figure 1: its inefficiency alleged by all forms of DEA measurement is only supported by reference to some fictitious combination of other observations, and not by any real-life production plan."

Suppose $i \in \beta^0(x^{0k})$. Thus $x^{0i} \leq x^{0k}$. From (9), $(y^{0i},x^{0i}) \in S^0$. Using $x^{0k} \geq x^{0i}$ and the free disposability property on inputs, $(y^{0i},x^{0k}) \in S^0$ and hence $y^{0i} \in s^0(x^{0k})$. Using the free disposability property on outputs, $y^{0i} \in s^0(x^{0k})$ implies that the set $\{y: 0_M \leq y \leq y^{0i}\}$ is a subset of $s^0(x^{0k})$. This conclusion holds for all $i \in \beta^0(x^{0k})$. Define the set f^{0k} as the union of these output sets:

(19) $f^{0k} \equiv \bigcup_{i \in \beta^0(x^{0k})} \{y: 0_M \leq y \leq y^{0i}\}$.

It can be seen that f^{0k} is a subset of $s^0(x^{0k})$. Assume that $f^{0k} = s^0(x^{0k})$; i.e., assume that the Free Disposal Hull of the period 0 data points is equal to the true technology set $s^0(x^{0k})$. Under this assumption, we can calculate the *period 0 output distance functions*, $d^0(y,x^{0k})$ for k = 1,...,K and for all $y \equiv [y_1,...,y_M] \geq 0_M$ with $y_1 > 0$:

$$\begin{aligned}(20)\ d^0(y,x^{0k}) &= \min_{\delta>0} \{\delta: (y/\delta) \in f^{0k}\} \\ &= \min_i \{\delta_i : y/\delta_i \leq y^{0i}\ ;\ i \in \beta^0(x^{0k})\} \\ &= \min_i \{\max_m \{y_m/y_m^{0i} : m = 1,...,M \text{ and } y_m y_m^{0i} > 0\}: i \in \beta^0(x^{0k})\}.\end{aligned}$$

Thus the maximum in (20) is taken over all outputs m such that component m of the vectors y and y^{0i} are both positive. By assumptions (8), m = 1 will satisfy these restrictions and so the max is over a finite nonempty set of output indexes m where both components m of y and y^{0i} are positive.

Since $k \in \beta^0(x^{0k})$, it can be seen that the following inequality holds:

$$\begin{aligned}(21)\ d^0(y^{0k},x^{0k}) &= \min_i \{\max_m \{y_m^{0k}/y_m^{0i} : m = 1,...,M \text{ and } y_m^{0k} y_m^{0i} > 0\}: i \in \beta^0(x^{0k})\} \\ &\leq 1.\end{aligned}$$

If $d^0(y^{0k},x^{0k}) < 1$, then production unit k is not technically efficient relative to the FDH technology spanned by the period 0 observations.

Since $k \in \beta^0(x^{0k})$ and thus the index set $\beta^0(y^{0k})$ is not empty, using (20) it can be seen that the period 0 output distance functions $d^0(y^{0k},x^{0k})$ and $d^0(y^{1k},x^{0k})$ are well defined for k = 1,...,K.

We also need to calculate the output distance functions $d^0(y^{1k},x^{1k})$ for k = 1,...,K. In order to use Free Disposal Hulls to form approximations to these distances, we need to define some additional input index sets. Thus for each x^{1k}, define the set of indices $\beta^0(x^{1k};x^{01},x^{02},...,x^{0K}) = \beta^0(x^{1k})$ as the set of period 0 observations i such that the period 0 input vector for production unit i, x^{0i}, is less than or equal to the period 1 input vector for production unit k, x^{1k}; i.e., define $\beta^0(x^{1k})$ as follows:

(22) $\beta^0(x^{1k}) \equiv \{i: x^{0i} \leq x^{1k}\ ;\ i = 1,...,K\}$.

We now encounter a problem: if x^{1k} is smaller than all of the period 0 input vectors x^{0i} in the panel, then the set of indices defined by (22) may be empty. In this case, it will not be possible to use Free Disposal Hulls to calculate an approximation to $d^0(y^{1k},x^{1k})$. In what follows, we rule out this case.

Assuming that $\beta^0(x^{1k})$ is nonempty, suppose that $i \in \beta^0(x^{1k})$. Thus $x^{0i} \leq x^{1k}$. From (9), $(y^{0i},x^{1i}) \in S^0$. Using $x^{1k} \geq x^{0i}$ and the free disposability property on inputs, $(y^{0i},x^{1k}) \in S^0$ and hence $y^{0i} \in \beta^0(x^{1k})$. Using the free disposability property on outputs, $y^{0i} \in \beta^0(x^{1k})$ implies that the set $\{y: 0_M \leq y \leq y^{0i}\}$ is a subset of $s^0(x^{1k})$. This conclusion holds for all $i \in \beta^0(x^{1k})$. Define the set g^{0k} as the union of these output sets:

(23) $g^{0k} \equiv \bigcup_{i \in \beta^0(x^{1k})} \{y: 0_M \leq y \leq y^{0i}\}$.

It can be seen that g^{0k} is a subset of $s^0(x^{1k})$. Assume that $g^{0k} = s^0(x^{1k})$. Under this assumption, we can calculate the *period 0 output distance functions*, $d^{0k}(y,x^{1k})$ for $k = 1,...,K$ and for all $y \geq 0_M$ with $y_1 > 0$:

(24) $d^{0k}(y,x^{1k}) = \min_{\delta > 0} \{\delta: (y/\delta) \in g^{0k}\}$
$= \min_i \{\delta_i : y/\delta_i \leq y^{0i} ; i \in \beta^0(x^{1k})\}$
$= \min_i \{\max_m \{y_m/y_m^{0i} : m = 1,...,M \text{ and } y_m y_m^{0i} > 0 \}: i \in \beta^0(x^{1k})\}$.

Under the assumption that the index set $\beta^0(x^{1k})$ is nonempty and that $g^{0k} = s^0(x^{1k})$, (24) can be used to calculate the output distances $d^0(y^{1k},x^{1k})$ for $k = 1,...,K$.

We now have to show how approximations to the *period 1* input and output distance functions can be calculated using Free Disposal Hulls. We turn our attention to the period 1 reference technology set S^1. All of our definitions that applied to S^0 can be adapted to S^1 up to a certain point. Thus define the set of input vectors x that can produce at least the output vector $y \geq 0_M$ using the period a technology by $S^1(y) \equiv \{x: (y,x) \in S^1\}$. Let $y \geq 0_M$ and define $\alpha^1(y;y^{11},y^{12},...,y^{1K})$ as the set of indices i such that y^{1i} is equal to or greater than y; i.e.,

(25) $\alpha^1(y;y^{11},y^{12},...,y^{1K}) \equiv \{i: y^{1i} \geq y ; i = 1,2,...,K\}$.

We abbreviate the notation and write $\alpha^1(y;y^{11},...,y^{01K})$ as $\alpha^1(y)$. Since $y^{1k} \geq y^{1k}$, it can be seen that $k \in \alpha^1(y^{1k})$ for $k = 1,...,K$. Fix k and suppose that $i \in \alpha^1(y^{1k})$. Thus $y^{1i} \geq y^{1k}$. Using (9), we have $(y^{1i},x^{1i}) \in S^1$. Since $y^{1i} \geq y^{1k}$, using the free disposability assumption on outputs, x^{1i} can also produce y^{1k} in period 1. Thus $(y^{1k},x^{1i}) \in S^1$ and using definition (11), $x^{1i} \in S^1(y^{1k})$. Using the free disposability assumption on inputs, the set $\{x: x \geq x^{1i}\}$ will be a subset of $S^1(y^{1k})$. This conclusion holds for all indices i such that $i \in \alpha^1(y^{1k})$. Define the set F^{1k} as the union of these input sets:

(26) $F^{1k} \equiv \bigcup_{i \in \alpha^1(y^{1k})} \{x: x \geq x^{1i}\}$.

It can be seen that F^{1k} is a subset of $S^1(y^{1k})$. But some of the period 0 observed input vectors could also produce the period 1 output vector y^{1k}. Thus we can again define $\alpha^0(y)$ by (12) and thus $\alpha^0(y^{1k})$ is the set of period 0 production units i that produced output vectors y^{0i} which satisfy the inequalities $y^{0i} \geq y^{1k}$. Following our usual logic, if $i \in \alpha^0(y^{1k})$, then the set $\{x: x \geq x^{1i}\}$ will be a subset of $S^1(y^{1k})$. This conclusion holds for all indices i such that $i \in \alpha^0(y^{1k})$. Define the set F^{01k} as the union of these input sets:

(27) $F^{01k} \equiv \bigcup_{i \in \alpha^0(y^{1k})} \{x: x \geq x^{0i}\}$.

It can be seen that F^{01k} is also a subset of $S^1(y^{1k})$. Assume that the union of F^{1k} and F^{01k} is equal to the input set $S^1(y^{1k})$. Under this assumption, we can calculate the *period 1 input distance functions*, $D^1(y^{1k},x)$ for k = 1,...,K and for all $x >> 0_N$ as follows:

$$(28)\ D^1(y^{1k},x) = \max_{\delta>0} \{\delta: (x/\delta)\in F^{1k}\cup F^{01k}\}$$
$$= \max [\max_i \{\delta_i : x/\delta_i \geq x^{0i} ; i\in\alpha^0(y^{1k})\}, \max \{\delta_i : x/\delta_i \geq x^{1i} ; i\in\alpha^1(y^{1k})\}]$$
$$= \max [\max_i \{\min_n \{x_n/x_n^{0i} : n = 1,...,N\}: i\in\alpha^0(y^{1k})\},$$
$$\max_i \{\min_n \{x_n/x_n^{1i} : n = 1,...,N\}: i\in\alpha^1(y^{1k})\}].$$

If the index set $\alpha^0(y^{1k})$ is empty, then $D^1(y^{1k},x) = \max_i \{\min_n \{x_n/x_n^{1i} : n = 1,...,N\}: i\in\alpha^1(y^{1k})\}$.

Since $k\in\alpha^1(y^{1k})$, it can be verified that

$$(29)\ D^1(y^{1k},x^{1k}) \geq 1.$$

If $D^1(y^{1k},x^{1k}) > 1$, then production unit k is not technically efficient in period 1 relative to the FDH technology spanned by the period 0 and 1 observations.

Since $k\in\alpha^1(y^{1k})$, the index set $\alpha^1(y^{1k})$ is not empty. Using (28), it can be seen that the period 1 input distance functions $D^1(y^{1k},x^{1k})$ and $D^1(y^{1k},x^{0k})$ are well defined for k = 1,...,K.

Finally, we turn our attention to the period 1 output distance functions. Let $x \geq 0_N$. Define the set of outputs that can be produced using the input vector x and the period 1 technology as $s^1(x) \equiv \{y: (y,x)\in S^1\}$. Fix the input vector $x \geq 0_N$ and define the set of indices $\beta^1(x;x^{11},x^{12},...,x^{1K})$ as follows:

$$(30)\ \beta^1(x;x^{11},x^{12},...,x^{1K}) \equiv \{i: x^{1i} \leq x ; i = 1,2,...,K\} = \beta^1(x).$$

When $x = x^{1k}$, note that $k\in\beta^1(x^{1k})$ for k = 1,...,K.

Suppose $i\in\beta^1(x^{1k})$. Thus $x^{1i} \leq x^{1k}$. From (9), $(y^{1i},x^{1i})\in S^1$. Using $x^{1k} \geq x^{1i}$ and the free disposability property on inputs, $(y^{1i},x^{1k})\in S^1$ and hence $y^{1i}\in s^1(x^{1k})$. Using the free disposability property on outputs, $y^{1i}\in s^1(x^{1k})$ implies that the set $\{y: 0_M \leq y \leq y^{1i}\}$ is a subset of $s^1(x^{1k})$. This conclusion holds for all $i\in\beta^1(x^{1k})$. Define the set f^{1k} as the union of these output sets:

$$(31)\ f^{1k} \equiv \bigcup_{i\in\beta^1(x^{1k})} \{y: 0_M \leq y \leq y^{1i}\}.$$

It can be seen that f^{1k} is a subset of $s^1(x^{1k})$. But some of the period 0 observed output vectors could be produced by the period 1 input vector x^{1k}. Define $\beta^0(x)$ by (18) and thus $\beta^0(x^{1k})$ is the set of period 0 production units i using input vectors that satisfied the inequalities $x^{0i} \leq x^{1k}$. This set may be empty. However, this set is unlikely to be empty if we have a long time series of firm observations for the industry under consideration and we add these observations as feasible observations for periods 0 and 1; see the final paragraph of this paper. Following our usual logic, if $i\in\beta^0(x^{1k})$, then the set $\{y: 0_M \leq y \leq y^{0i}\}$ will be a subset of $s^1(x^{1k})$. This conclusion holds for all indices i such that $i\in\beta^0(x^{1k})$. Define the set f^{01k} as the union of these output sets:

(32) $f^{01k} \equiv \bigcup_{i \in \beta^0(x^{1k})} \{0_M \le y \le y^{0i}\}$.

It can be seen that f^{01k} is also a subset of $s^1(x^{1k})$. Assume that the union of f^{1k} and f^{01k} is equal to the output set $s^1(x^{1k})$. Under this assumption, we can calculate the *period 1 output distance functions*, $d^1(y,x^{1k})$ for k = 1,...,K and for all $y \ge 0_M$ with $y_1 > 0$ as follows:

$$\begin{aligned}(33)\ d^1(y,x^{1k}) &= \min_{\delta>0} \{\delta: (y/\delta) \in f^{01k} \cup f^{1k}\} \\ &= \min [\min_i \{\delta_i : y/\delta_i \le y^{0i} ; i \in \beta^0(x^{1k})\}, \min \{\delta_i : y/\delta_i \le y^{1i} ; i \in \beta^1(x^{1k})\}] \\ &= \min [\min_i \{\max_m \{y_m/y_m^{0i} : m = 1,...,M \text{ and } y_m y_m^{0i} > 0\} : i \in \beta^0(x^{1k})\}, \\ &\qquad \min_i \{\max_m \{y_m/y_m^{1i} : m = 1,...,M \text{ and } y_m y_m^{1i} > 0\} : i \in \beta^1(x^{1k})\}].\end{aligned}$$

If $\beta^0(x^{1k})$ is empty, then $d^1(y,x^{1k}) = \min_i \{\max_m \{y_m/y_m^{1i} : m = 1,...,M \text{ and } y_m y_m^{1i} > 0\}: i \in \beta^1(x^{1k})\}$.

Note that (33) can be used to evaluate the output distances $d^1(y^{1k},x^{1k})$ and $d^1(y^{0k},x^{1k})$ for k = 1,...,K.

There is only one additional set of output distance functions that need to be approximated by Free Disposal Hulls and that is the set of output distances, $d^1(y^{0k},x^{0k})$ for k = 1,...,K.

Recall definitions (17)-(19) above. Since S^0 is a subset of S^1, it can be seen that the set f^{0k} is not only a subset of $s^0(x^{0k})$, it is also a subset of $s^1(x^{0k})$. Thus the set f^{0k} defined by (19) is a subset of $s^1(x^{0k})$. However, some of the period 1 output vectors y^{1k} may also belong to $s^1(x^{0k})$. Recall definition (30) which defined the period 1 index set $\beta^1(x)$. If $\beta^1(x^{0k})$ is not empty, then following our usual logic, if $i \in \beta^1(x^{0k})$, then the set $\{y: 0_M \le y \le y^{1i}\}$ will be a subset of $s^1(x^{0k})$. This conclusion holds for all indices i such that $i \in \beta^1(x^{0k})$. Define the set f^{10k} as the union of these output sets:

(34) $f^{10k} \equiv \bigcup_{i \in \beta^1(x^{0k})} \{0_M \le y \le y^{1i}\}$.

It can be seen that f^{01k} is also a subset of $s^1(x^{0k})$. Assume that the union of f^{0k} and f^{01k} is equal to the output set $s^1(x^{0k})$. Under this assumption, we can calculate the *period 1 output distance functions*, $d^1(y,x^{0k})$ for k = 1,...,K and for all $y >> 0_N$ as follows:

$$\begin{aligned}(35)\ d^1(y,x^{0k}) &= \min_{\delta>0} \{\delta: (y/\delta) \in f^{0k} \cup f^{01k}\} \\ &= \min [\min_i \{\delta_i : y/\delta_i \le y^{0i} ; i \in \beta^0(x^{0k})\}, \min \{\delta_i : y/\delta_i \le y^{1i} ; i \in \beta^1(x^{0k})\}] \\ &= \min [\min_i \{\max_m \{y_m/y_m^{0i} : m = 1,...,M \text{ and } y_m y_m^{0i} > 0\} : i \in \beta^0(x^{0k})\}, \\ &\qquad \min_i \{\max_m \{y_m/y_m^{1i} : m = 1,...,M \text{ and } y_m y_m^{0i} > 0\} : i \in \beta^1(x^{0k})\}].\end{aligned}$$

If $\beta^1(x^{0k})$ is empty, then $d^1(y,x^{0k}) = \min_i \{\max_m \{y_m/y_m^{0i} : m = 1,...,M \text{ and } y_m y_m^{0i} > 0\}: i \in \beta^0(x^{0k})\}$.

Equation (35) enables us to calculate the output distances $d^1(y^{0k},x^{0k})$ for k = 1,...,K.

4. Conclusion

In summary: (15) enables us to calculate $D^0(y^{0k},x^{0k})$ and $D^0(y^{0k},x^{1k})$; (28) to calculate $D^1(y^{1k},x^{1k})$ and $D^1(y^{1k},x^{0k})$; (20) to calculate $d^0(y^{0k},x^{0k})$ and $d^0(y^{1k},x^{0k})$; (33) to calculate $d^1(y^{1k},x^{1k})$ and $d^1(y^{1k},x^{0k})$; (35) to calculate $d^1(y^{0k},x^{0k})$ and (24) to calculate $d^0(y^{1k},x^{1k})$, which are the ten distance functions needed for a decomposition of the Bjurek productivity index given by equations (4)–(7). These distance functions can also be used to decompose the output-oriented Bjurek-Laspeyres and Bjurek-Paasche productivity indexes of Diewert and Fox (2013).

We conclude with an observation that will greatly extend the scope of the above analysis. It is not necessary to assume that the industry data come in the form of an exactly balanced panel with the same K production units present in both periods. The period 0 data set can be expanded to the set of all industry observations that are available to the researcher extending back many periods. With the assumption of no technological regress, all of these observations can be used to define the data driven FDH technology set for period 0. The period 1 data can consist of observations on continuing firms and new entrants. The older data and the data for period 1 can be used to define the data driven FDH for period 1. The productivity decomposition defined by (4)–(7) can still be calculated for any production unit k that is present in periods 0 and 1 where we construct the FDH technology sets using the same methodology as was explained above. Tulkens and Eeckaut (1995a) (1995b) discussed this type of setup (and others) for choosing benchmark observations in a time series context.

References

Bjurek, H. (1996), "The Malmquist Total Factor Productivity Index", *Scandinavian Journal of Economics* 98, 303-313.

Caves, D.W., L.R. Christensen and W.E. Diewert (1982), "The Economic Theory of Index Numbers and the Measurement of Input, Output and Productivity", *Econometrica* 50, 1393-1414.

Charnes, A. and W.W. Cooper (1985), "Preface to Topics in Data Envelopment Analysis", *Annals of Operations Research* 2, 59-94.

Deprins, D., L. Simar, and H. Tulkens (1984), "Measuring Labor Efficiency in Post Offices", pp. 243-267 in M. Marchand, P. Pestieau, and H. Tulkens (eds.), *The Performance of Public Enterprises: Concepts and Measurements*, Amsterdam: North-Holland.

Diewert, W.E. (1980), "Capital and the Theory of Productivity Measurement", *American Economic Review* 70, 260-267.

Diewert, W.E. (1981), "The Theory of Total Factor Productivity Measurement in Regulated Industries", pp. 17-44 in *Productivity Measurement in Regulated Industries*, T. Cowing and R. Stephenson (eds.), New York: Academic Press.

Diewert, W.E. and K.J. Fox (2013), "Decomposing Malmquist Productivity Indexes into Explanatory Factors," Discussion Paper 13-09, School of Economics, University of British Columbia, Vancouver, Canada, V6T 1Z1.

Färe, R., S. Grosskopf, M. Norris and Z. Zhang (1994), "Productivity Growth, Technical Progress and Efficiency Change in Industrialized Countries", *American Economic Review* 84, 66-83.

Grosskopf, S. (2003), "Some Remarks on Productivity and its Decompositions", *Journal of Productivity Analysis* 20, 459-474.

Lovell, C.A.K. (2006), "Introduction", pp. 279-284 in *Public Goods, Environmental Externalities and Fiscal Competition*, P. Chander, J. Drèze, C.A.K. Lovell and J. Mintz, New York: Springer.

Malmquist, S. (1953), "Index Numbers and Indifference Surfaces", *Trabajos de Estastistica* 4, 209-242.

Russell, R.R. and W. Schworm (2009), "Axiomatic Foundations of Efficiency Measurement on Data Generated Technologies", *Journal of Productivity Analysis* 31, 77-86.

Russell, R.R. and W. Schworm (2011), "Properties of Inefficiency Indexes on Input, Output Space", *Journal of Productivity Analysis* 36, 143-156.

Tulkens, H. (1993), "On FDH Efficiency Analysis: Some Methodological Issues and Application to Retail Banking, Courts, and Urban Transit", *Journal of Productivity Analysis* 4, 183–210.

Tulkens, H. and P.V. Eeckaut (1995a), "Non-Frontier Measures of Efficiency, Progress and Regress for Time Series Data", *International Journal of Production Economics* 39, 83-97.

Tulkens, H. and P.V. Eeckaut (1995b), "Nonparametric Efficiency, Progress and Regress Measures for Panel Data: Methodological Aspects", *European Journal of Operational Research* 80, 474-499.