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Abstract

The economic approach to index number theory and productivity measurement is based on a ratio concept. This causes no particular difficulties for economists because they are used to this approach. However, the ratio approach is not one that the business and accounting community finds natural; a manager or owner of a firm is typically interested in analyzing profit *differences* rather than *ratios*. Thus interest centers on decomposing cost, revenue or profit changes into *price* and *quantity* (or volume) *effects.* Diewert (1998) looked at the differences approach to index number theory in some detail, both from the axiomatic and economic perspectives. In this note, we extend the economic approach to index numbers to the profit decomposition problem.

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The economic approach to index number theory and productivity measurement is based on a ratio concept. This causes no particular difficulties for economists because they are used to this approach. However, the ratio approach is not one that the business and accounting community finds natural; a manager or owner of a firm is typically interested in analyzing profit *differences* rather than *ratios*. Thus interest centers on decomposing cost, revenue or profit changes into *price* and *quantity* (or volume) *effects*.[2] For example, the owner of an oil exploration company will generally be interested in knowing how much of the difference between current period profits over the previous period profits is due to the change in the price of crude oil and how much of the profit change is due to improvements in the operating efficiency of the company.

Diewert (1998) looked at the differences approach to index number theory in some detail, both from the axiomatic and economic perspectives. However, when he discussed the economic approach to indexes or indicators of price and quantity change, he used the example of a utility maximizing consumer to illustrate the economic approach. He did not develop the difference approach explicitly in the context of decomposing changes in profits into price and quantity change components. In this note, we will extend the economic approach to the profit decomposition problem.

Suppose that in accounting period t, we can observe a vector of positive prices $p^t \equiv (p_1^t,\ldots,p_N^t)$ for the inputs used and outputs produced for a firm along with the corresponding vector of net outputs $q^t \equiv (q_1^t,\ldots,q_N^t)$ [3] for periods t = 0,1. Our goal in this section is to provide a decomposition of the change in profits going from period 0 to 1, $p^1 \bullet q^1 - p^0 \bullet q^0$, into a sum of N price change effects and N quantity change effects.[4] We will use the decomposition (68) in Diewert (1998). In order to apply this decomposition, we first need to decompose the profits ratio, $p^1 \bullet q^1 / p^0 \bullet q^0$, into a product of two ratios, $[r(p^1)/r(p^0)][a^1/a^0]$.[5] We will explain the meaning of these two ratios below.

We will represent the technology of the production unit by means of an input requirements function f; i.e., $f(q) = f(q_1,\ldots,q_N)$ is the amount of an input (that was not included earlier) that is required to produce the vector of net outputs $q = (q_1,\ldots,q_N)$.[6] Before we introduce technological change, we choose the following flexible functional form for f:[7]

$$(1)\quad f(q) \equiv (q \bullet Aq)^{1/2} = [\textstyle\sum_{j=1}^N \sum_{k=1}^N a_{jk}\, q_j\, q_k\,]^{1/2} \quad , a_{jk} = a_{kj} \text{ for all } jk;$$

where $q \equiv (q_1,\ldots,q_N)$. Define the dual unit input net revenue function r(p) as follows[8]:

[2] In the accounting literature, this is known as variance analysis.
[3] If commodity n is an output produced by the production unit, then $q_n > 0$ while if commodity n is an input used by the production unit, then $q_n < 0$.
[4] Note that $p \bullet q \equiv \sum_{n=1}^N p_n q_n$ is the inner product of the vectors p and q.
[5] This product of two ratios is the counterpart to the product $[c(p^1)/c(p^0)][f(q^1)/f(q^0)]$ on the right hand side of (63) in Diewert (1998).
[6] For material on input requirements functions, see Diewert (1974).
[7] The Fisher (1922) ideal price and quantity indexes are exact for this functional form; see the references in Diewert (1976; 116).
[8] See Diewert (1974) for additional material on this duality.

(2) $r(p) \equiv \max_q \{p \bullet q : f(q) = 1\}$.

We now allow for technological change or improvements in managerial efficiency going from the period 0 technology to the period 1 technology: we now assume that the period t input requirements function is f^t defined in terms of the f defined by (1) as follows:

(3) $f^t(q) \equiv f(q)/a^t$; $t = 0,1$

where the a^t are positive technology parameters.[9] If $a^1 > a^0$, then there has been technical progress or managerial improvements.

We think of the N+1st commodity as a fixed factor; it is a composite of all of the inputs that are held fixed during the two periods under consideration.

As is typical in the exact index number literature, we assume that the observed period t net output vector q^t is a solution to the following period t profit maximization problem: for t = 0,1:

$$
\begin{aligned}
(4)\ p^t \bullet q^t &= \max_q \{p^t \bullet q : f^t(q) = 1\} \\
&= \max_q \{p^t \bullet q : f(q)/a^t = 1\} && \text{using (3)} \\
&= \max_q \{p^t \bullet q : f(q/a^t) = 1\} && \text{using the linear homogeneity of f} \\
&= a^t \max_q \{p^t \bullet (q/a^t) : f(q/a^t) = 1\} \\
&= a^t\, r(p^t) && \text{using definition (2).}
\end{aligned}
$$

Now taking ratios of the two equations in (4) yields:

(5) $p^1 \bullet q^1 / p^0 \bullet q^0 = [r(p^1)/r(p^0)][a^1/a^0]$.

The Fisher ideal price index P_F is defined as follows:

(6) $P_F(p^0,p^1,q^0,q^1) \equiv [p^1 \bullet q^1\ p^1 \bullet q^0 / p^0 \bullet q^0\ \ p^0 \bullet q^1]^{1/2} \equiv P_F$.

Using the fact that q^t/a^t solves the period t maximization problem in (4) for t = 0,1along with the fact that f(q) is defined by (1) leads to the following standard Fisher exactness result:[10]

(7) $r(p^1)/r(p^0) = P_F(p^0,p^1,q^0/a^0,q^1/a^1)$

(8) $\qquad = P_F(p^0,p^1,q^0,q^1)$ since P_F is homogeneous of degree 0 in q^0 and q^1.

Using (5) and (8), we can now determine a^1/a^0 empirically:

(9) $a^1/a^0 = [p^1 \bullet q^1 / p^0 \bullet q^0] / [r(p^1)/r(p^0)] = [p^1 \bullet q^1 / p^0 \bullet q^0] / P_F(p^0,p^1,q^0,q^1) \equiv Q_F(p^0,p^1,q^0,q^1)$.

Now rewrite (9) as:

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[9] Typically, we assume that $a^0 = 1$. Note that we are assuming a type of neutral technological change.

[10] See Diewert (1976; 116).

(10) $[p^1 \bullet q^1 / p^0 \bullet q^0] = P_F(p^0,p^1,q^0,q^1)\ Q_F(p^0,p^1,q^0,q^1)$.

Note that both P_F and Q_F on the right hand side of (10) can be determined numerically from observable data. Now apply Bennet's (1920) Identity to the measure of price and quantity change, $P_FQ_F - (1)(1)$:

(11) $P_FQ_F - 1 = (1/2)[1 + Q_F][P_F - 1] + (1/2)[1 + P_F][Q_F - 1]$.

Now we have all of the ingredients that are required to our economic decomposition of the profit change for the productions unit. We have:

$p^1 \bullet q^1 - p^0 \bullet q^0 = p^0 \bullet q^0\ [\{p^1 \bullet q^1 / p^0 \bullet q^0\} - 1]$

$= p^0 \bullet q^0\ [\{P_FQ_F\} - 1]$ using (10)

$= p^0 \bullet q^0\ [(1/2)[1 + Q_F][P_F - 1] + (1/2)[1 + P_F][Q_F - 1]$ using (11)

$\equiv I_E(p^0,p^1,q^0,q^1) + (1/2)[1 + P_F][Q_F - 1]$

using the Fisher ideal indexes and the definition of the economic price indicator, (72) in Diewert (1998)

(12) $\equiv I_E(p^0,p^1,q^0,q^1) + V_E(p^0,p^1,q^0,q^1)$

using the Fisher ideal indexes and the definition of the economic volume indicator, (75) in Diewert (1998)

(13) $\cong I^B(p^0,p^1,q^0,q^1) + V^B(p^0,p^1,q^0,q^1)$

where the last approximate equality (13) follows from Proposition 9 in Diewert (1998); i.e., the observable expression immediately above (13) approximates the right hand side of (13), the sum of the Bennet indicators of price and quantity change, to the second order at any point where the two price vectors are equal (i.e., $p^0 = p^1$) and where the two quantity vectors are equal (i.e., $q^0 = q^1$). The Bennet indicator of price change is defined as

(14) $I^B(p^0,p^1,q^0,q^1) \equiv (1/2)(q^0 + q^1) \bullet (p^1 - p^0) = \sum_{n=1}^N (1/2)[q_n^0 + q_n^1][p_n^1 - p_n^0]$

and the Bennet indicator of quantity (or volume) change is defined as

(15) $V^B(p^0,p^1,q^0,q^1) \equiv (1/2)(p^0 + p^1) \bullet (q^1 - q^0) = \sum_{n=1}^N (1/2)[p_n^0 + p_n^1][q_n^1 - q_n^0]$.

Since the Bennet indicator of volume change $V^B(p^0,p^1,q^0,q^1)$ approximates $(1/2)[1 + P_F][Q_F - 1]]$ to the second order around an equal price and quantity point by Proposition 9 in Diewert (1998) and since by (9) above, the Fisher quantity index Q_F is equal to the indicator of technological change a^1/a^0, it can be seen that $V^B(p^0,p^1,q^0,q^1) = \sum_{n=1}^N (1/2)[p_n^0 + p_n^1][q_n^1 - q_n^0]$ can be interpreted as an additive measure of overall efficiency change going from period 0 to 1. Note that this Bennet overall measure of efficiency change can be decomposed into a sum of N separate individual efficiency effects, the nth one being $(1/2)[p_n^0 + p_n^1][q_n^1 - q_n^0]$. If commodity n is an output and the production of this commodity has grown going from period 0 to period 1 (so that $q_n^1 > q_n^0$), then the commodity n efficiency effect $(1/2)[p_n^0 + p_n^1][q_n^1 - q_n^0]$ will be positive and will contribute to the overall productivity improvement.]. On the other hand, if commodity n is an input (so the q_n^t are negative numbers) and the utilization of this commodity has fallen going from period 0 to period 1 (so that $-q_n^1 < -q_n^0$), then the commodity n efficiency effect $(1/2)[p_n^0 + p_n^1][q_n^1 - q_n^0]$ will also be positive and will contribute to the overall productivity improvement.

To summarize: equation (12) decomposes the profit change, $p^1\bullet q^1 - p^0\bullet q^0$, for the production unit into the sum of two terms. The first term, I_E, the (Fisher) economic indicator of price change, gives the total contribution of price change to the profit change while the second term, $V_{E,}$ the (Fisher) economic indicator of volume change, is a measure of overall efficiency change. These two terms are approximately equal to the Bennet indicators of price and volume change, I^B and V^B respectively. Thus the (approximate) equation (13) decomposes the profit change, $p^1\bullet q^1 - p^0\bullet q^0$, for the production unit into a sum of N separate commodity specific price change effects, $\sum_{n=1}^N (1/2)[q_n^0 + q_n^1][p_n^1 - p_n^0]$, plus a sum of N commodity specific quantity change effects, $\sum_{n=1}^N (1/2)[q_n^0 + q_n^1][p_n^1 - p_n^0]$. This latter sum can be interpreted as a measure of overall efficiency change. Note that equation (13) can be rewritten as follows:

$$(16)\ V^B(p^0,p^1,q^0,q^1) = p^1\bullet q^1 - p^0\bullet q^0 - \sum_{n=1}^N (1/2)[q_n^0 + q_n^1][p_n^1 - p_n^0].$$

The left hand side of (16) is the overall measure of productivity change. The right hand side of (16) shows how this overall measure can be calculated in *dual form* in terms of input and output price changes. The right hand side can also be interpreted as showing how the benefits of the productivity change are distributed across the various inputs and outputs.[11]

11 The first term on the right hand side of (16), $p^1\bullet q^1 - p^0\bullet q^0$, can be interpreted as the change in the return to the fixed factor.

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