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Chapter 4: Elementary Price Indexes

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1. Introduction

In all countries, the calculation of a Consumer Price Index proceeds in two (or more) stages. In the first stage of calculation, *elementary price indexes* are estimated for the *elementary expenditure aggregates* of a CPI. In the second and higher stages of aggregation, these elementary price indexes are combined to obtain higher level indexes using information on the expenditures on each elementary aggregate as weights. An elementary aggregate consists of the expenditures on a small and relatively homogeneous set of products defined within the consumption classification used in the CPI. Samples of prices are collected within each elementary aggregate, so that elementary aggregates serve as strata for sampling purposes.

Data on the expenditures, or quantities, of the different goods and services are typically not available within an elementary aggregate. As there are no quantity or expenditure weights, most of the index number theory outlined in the previous sections is not directly applicable. An elementary price index is a more primitive concept that relies on price data only.

The question of what is the most appropriate formula to use to estimate an elementary price index is considered in this Chapter.[2] The quality of a CPI or PPI depends heavily on the quality of the elementary indexes, which are the basic building blocks from which CPIs and PPIs are constructed.

CPI and PPI compilers have to select *representative products* within an elementary aggregate and then collect a sample of prices for each of the representative products, usually from a sample of different outlets. The individual products whose prices are actually collected are described as the *sampled products*. Their prices are collected over a succession of time periods. An elementary price index is therefore typically calculated from two sets of matched price observations. It is assumed initially that there are no missing observations and no changes in the quality of the products sampled so that the two sets of prices are perfectly matched.

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[2] The material in this Chapter draws heavily on the contributions of Dalén (1992), Balk (1994) (2002) (2008) and Diewert (1995) (2002) which are reflected in the ILO (2004; 355-371).

2. Elementary Indexes used in Practice

Suppose that there are M lowest level items or specific commodities in a chosen elementary category. Denote the period t price of item m by p_m^t for t = 0,1 and for items m = 1,2,...,M. Define the period t price vector as $p^t \equiv [p_1^t,p_2^t,...,p_M^t]$ for t = 0,1.

The first widely used elementary index number formula is due to the French economist Dutot (1738):

$$(1)\quad P_D(p^0,p^1) \equiv [\sum_{m=1}^M (1/M)\, p_m^1]/[\sum_{m=1}^M (1/M)\, p_m^0] = [\sum_{m=1}^M p_m^1]/[\sum_{m=1}^M p_m^0].$$

Thus the Dutot elementary price index is equal to the arithmetic average of the M period 1 prices divided by the arithmetic average of the M period 0 prices.

The second widely used elementary index number formula is due to the Italian economist Carli (1764):

$$(2)\quad P_C(p^0,p^1) \equiv \sum_{m=1}^M (1/M)(p_m^1/p_m^0).$$

Thus the Carli elementary price index is equal to the *arithmetic* average of the M item price ratios or price relatives, p_m^1/p_m^0. This formula was already encountered in our study of the unweighted stochastic approach to index numbers; recall (15) above.

The third widely used elementary index number formula is due to the English economist Jevons (1865):

$$(3)\quad P_J(p^0,p^1) \equiv \prod_{m=1}^M (p_m^1/p_m^0)^{1/M}.$$

Thus the Jevons elementary price index is equal to the *geometric* average of the M item price ratios or price relatives, p_m^1/p_m^0. Again, this formula was introduced as formula (17) in our discussion of the unweighted stochastic approach to index number theory.

The fourth elementary index number formula P_H is the *harmonic* average of the M item price relatives and it was first suggested in passing as an index number formula by Jevons (1865; 121) and Coggeshall (1887):

$$(4)\quad P_H(p^0,p^1) \equiv [\sum_{m=1}^M (1/M)(p_m^1/p_m^0)^{-1}]^{-1}.$$

Finally, the fifth elementary index number formula is the geometric average of the Carli and harmonic formulae; i.e., it is *the geometric mean of the arithmetic and harmonic means of the M price relatives*:

$$(5)\quad P_{CSWD}(p^0,p^1) \equiv [P_C(p^0,p^1)\, P_H(p^0,p^1)]^{1/2}.$$

This index number formula was first suggested by Fisher (1922; 472) as his formula 101. Fisher also observed that, empirically for his data set, P_{CSWD} was very close to the Jevons index, P_J, and these two indexes were his "best' unweighted index number formulae. In more recent times, Carruthers, Sellwood and Ward (1980; 25) and Dalén (1992; 140) also proposed P_{CSWD} as an elementary index number formula.

Having defined the most commonly used elementary formulae, the question now arises: which formula is "best"? Obviously, this question cannot be answered until desirable properties for elementary indexes are developed. This will be done in a systematic manner in section 4 below (using the test approach) but in the present section, one desirable property for an elementary index will be noted. This is the *time reversal test*, which was noted earlier in Chapter 1. In the present context, this test for the elementary index $P(p^0,p^1)$ becomes:

(6) $P(p^0,p^1)P(p^1,p^0) = 1$.

This test says that if the prices in period 2 revert to the initial prices of period 0, then the product of the price change going from period 0 to 1, $P(p^0,p^1)$, times the price change going from period 1 to 2, $P(p^1,p^0)$, should equal unity; i.e., under the stated conditions, we should end up where we started.[3] It can be verified that the Dutot, Jevons and Carruthers, Sellwood, Ward and Dalén indexes, P_D, P_J and P_{CSWD}, all satisfy the time reversal test but that the Carli and Harmonic indexes, P_C and P_H, fail this test. In fact, these last two indexes fail the test in the following *biased* manner:

(7) $P_C(p^0,p^1)\ P_C(p^1,p^0) \geq 1$;
(8) $P_H(p^0,p^1)\ P_H(p^1,p^0) \leq 1$

with strict inequalities holding in (7) and (8) provided that the period 1 price vector p^1 is not proportional to the period 0 price vector p^0.[4] Thus the Carli index will generally have an *upward bias* while the Harmonic index will generally have a *downward bias*. Fisher (1922; 66 and 383) was quite definite in his condemnation of the Carli index due to its upward bias[5] and perhaps as a result, the use of the Carli index was not permitted in compiling elementary price indexes for the Harmonized Index of Consumer Prices (HICP) that is the official Eurostat index used to compare consumer prices across European Union countries.

In the following section, some numerical relationships between the five elementary indexes defined in this section will be established. Then in the subsequent section, a more

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[3] This test can also be viewed as a special case of Walsh's (1901) Multiperiod Identity Test, T23 in Chapter 1.

[4] These inequalities follow from the fact that a harmonic mean of M positive numbers is always equal to or less than the corresponding arithmetic mean; see Walsh (1901;517) or Fisher (1922; 383-384). This inequality is a special case of Schlömilch's Inequality; see Hardy, Littlewood and Polya (1934; 26).

[5] See also Szulc (1987; 12) and Dalén (1992; 139). Dalén (1994; 150-151) provides some nice intuitive explanations for the upward bias of the Carli index.

comprehensive list of desirable properties for elementary indexes will be developed and the five elementary formulae will be evaluated in the light of these properties or tests.

3. Numerical Relationships between the Frequently Used Elementary Indexes

It can be shown[6] that the Carli, Jevons and Harmonic elementary price indexes satisfy the following inequalities:

(9) $P_H(p^0,p^1) \leq P_J(p^0,p^1) \leq P_C(p^0,p^1)$;

i.e., the Harmonic index is always equal to or less than the Jevons index which in turn is always equal to or less than the Carli index. In fact, the strict inequalities in (46) will hold provided that the period 0 vector of prices, p^0, is not proportional to the period 1 vector of prices, p^1.

The inequalities (9) do not tell us by how much the Carli index will exceed the Jevons index and by how much the Jevons index will exceed the Harmonic index. Hence, in the remainder of this section, some approximate relationships between the five indexes defined in the previous section will be developed that will provide some practical guidance on the relative magnitudes of each of the indexes.

The first approximate relationship that will be derived is between the Jevons index P_J and the Dutot index P_D. For each period t, define the *arithmetic mean of the M prices* pertaining to that period as follows:

(10) $p^{t*} \equiv \sum_{m=1}^{M} (1/M)\, p_m^{\ t}$; t = 0,1.

Now define the *multiplicative deviation of the mth price in period t relative to the mean price in that period*, $e_m^{\ t}$, as follows:

(11) $p_m^{\ t} = p^{t*}(1+e_m^{\ t})$; m = 1,...,M ; t = 0,1.

Note that (10) and (11) imply that the deviations $e_m^{\ t}$ sum to zero in each period; i.e., we have:

(12) $\sum_{m=1}^{M} e_m^{\ t} = 0$; t = 0,1.

Note that the Dutot index can be written as the ratio of the mean prices, p^{1*}/p^{0*}; i.e., we have:

(13) $P_D(p^0,p^1) = p^{1*}/p^{0*}$.

Now substitute equations (11) into the definition of the Jevons index, (3):

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[6] Each of the three indexes P_H, P_J and P_C is a mean of order r where r equals −1, 0 and 1 respectively and so the inequalities follow from Schlömilch's inequality; see Hardy, Littlewood and Polya (1934; 26).

$$(14)\ P_J(p^0,p^1) = \prod_{m=1}^{M} [p^{1*}(1+e_m^{\ 1})/p^{0*}(1+e_m^{\ 0})]^{1/M}$$
$$= [p^{1*}/p^{0*}] \prod_{m=1}^{M} [(1+e_m^{\ 1})/(1+e_m^{\ 0})]^{1/M}$$
$$= P_D(p^0,p^1)\ f(e^0,e^1) \qquad \text{using definition (1)}$$

where $e^t \equiv [e_1^{\ t},...,e_M^{\ t}]$ for t = 0 and 1, and the function f is defined as follows:

$$(15)\ f(e^0,e^1) \equiv \prod_{m=1}^{M} [(1+e_m^{\ 1})/(1+e_m^{\ 0})]^{1/M}.$$

Expand $f(e^0,e^1)$ by a second order Taylor series approximation around $e^0 = 0_M$ and $e^1 = 0_M$. Using (12), it can be verified[7] that we obtain the following second order approximate relationship between P_J and P_D:

$$(16)\ P_J(p^0,p^1) \approx P_D(p^0,p^1)[1 + (1/2M)e^0\cdot e^0 - (1/2M)e^1\cdot e^1]$$
$$= P_D(p^0,p^1)[1 + (1/2)\mathrm{var}(e^0) - (1/2)\mathrm{var}(e^1)]$$

where $\mathrm{var}(e^t)$ is the variance of the period t multiplicative deviations; i.e., for t = 0,1:

$$(17)\ \mathrm{var}(e^t) \equiv (1/M)\sum_{m=1}^{M}(e_m^{\ t} - e^{t*})^2$$
$$= (1/M)\sum_{m=1}^{M}(e_m^{\ t})^2 \qquad \text{since } e^{t*} = 0 \text{ using (12)}$$
$$= (1/M)\ e^t\cdot e^t\ .$$

Under normal conditions[8], the variance of the deviations of the prices from their means in each period is likely to be approximately constant and so under these conditions, the Jevons price index will approximate the Dutot price index to the second order.

Note that with the exception of the Dutot formula, the remaining four elementary indexes defined in section 2 are functions of the relative prices of the M items being aggregated. This fact is used in order to derive some approximate relationships between these four elementary indexes. Thus define the *mth price relative* as

$$(18)\ r_m \equiv p_m^{\ 1}/p_m^{\ 0}\ ; \qquad m = 1,...,M.$$

Define the *arithmetic mean of the m price relatives* as

$$(19)\ r^* \equiv (1/M) \sum_{m=1}^{M} r_m = P_C(p^0,p^1)$$

where the last equality follows from the definition (2) of the Carli index. Finally, define the *deviation* e_m *of the mth price relative* r_m *from the arithmetic average of the M price relatives* r^* as follows:

$$(20)\ r_m = r^*(1+e_m)\ ; \qquad m = 1,...,M.$$

[7] This approximate relationship was first obtained by Carruthers, Sellwood and Ward (1980; 25).

[8] If there are significant changes in the overall inflation rate, some studies indicate that the variance of deviations of prices from their means can also change. Also if M is small, then there will be sampling fluctuations in the variances of the prices from period to period, leading to random differences between the Dutot and Jevons indexes.

Note that (19) and (20) imply that the deviations e_m sum to zero; i.e., we have:

(21) $\sum_{m=1}^{M} e_m = 0$.

Now substitute equations (20) into the definitions of P_C, P_J, P_H and P_{CSWD}, (2)-(5) above, in order to obtain the following representations for these indexes in terms of the vector of deviations, $e \equiv [e_1,...,e_M]$:

(22) $P_C(p^0,p^1) = \sum_{m=1}^{M} (1/M)r_m = r^*1 \equiv r^*f_C(e)$;
(23) $P_J(p^0,p^1) = \prod_{m=1}^{M} r_m^{1/M} = r^*\prod_{m=1}^{M} (1+e_m)^{1/M} \equiv r^*f_J(e)$;
(24) $P_H(p^0,p^1) = [\sum_{m=1}^{M} (1/M)(r_m)^{-1}]^{-1} = r^*[\sum_{m=1}^{M} (1/M)(1+e_m)^{-1}]^{-1} \equiv r^*f_H(e)$;
(25) $P_{CSWD}(p^0,p^1) = [P_C(p^0,p^1)P_H(p^0,p^1)]^{1/2} = r^*[f_C(e)f_H(e)]^{1/2} \equiv r^*f_{CSWD}(e)$

where the last equation in (22)-(25) serves to define the deviation functions, $f_C(e)$, $f_J(e)$, $f_H(e)$ and $f_{CSWD}(e)$. The second order Taylor series approximations to each of these functions[9] around the point $e = 0_M$ are:

(26) $f_C(e) \approx 1$;
(27) $f_J(e) \approx 1 - (1/2M)e\cdot e = 1 - (1/2)\mathrm{var}(e)$;
(28) $f_H(e) \approx 1 - (1/M)e\cdot e = 1 - \mathrm{var}(e)$;
(29) $f_{CSWD}(e) \approx 1 - (1/2M)e\cdot e = 1 - (1/2)\mathrm{var}(e)$

where we have made repeated use of (21) in deriving the above approximations.[10] Thus to the second order, the Carli index P_C will *exceed* the Jevons and Carruthers Sellwood Ward Dalén indexes, P_J and P_{CSWD}, by $(1/2)r^*\mathrm{var}(e)$, which is r^* times one half the variance of the M price relatives p_m^1/p_m^0. Similarly, to the second order, the Harmonic index P_H will *lie below* the Jevons and Carruthers Sellwood Ward Dalén indexes, P_J and P_{CSWD}, by r^* times one half the variance of the M price relatives p_m^1/p_m^0.

Thus empirically, it is expected that the Jevons and Carruthers Sellwood Ward and Dalén indexes will be very close to each other. Using the previous approximation result (16), it is expected that the Dutot index P_D will also be fairly close to P_J and P_{CSWD}, with some fluctuations over time due to changing variances of the period 0 and 1 deviation vectors, e^0 and e^1. Thus it is expected that these three elementary indexes will give much the same numerical answers in empirical applications. On the other hand, the Carli index can be expected to be substantially *above* these three indexes, with the degree of divergence growing as the variance of the M price relatives grows. Similarly, the Harmonic index can be expected to be substantially *below* the three middle indexes, with the degree of divergence growing as the variance of the M price relatives grows.

[9] From (22), it can be seen that $f_C(e)$ is identically equal to 1 so that (26) will be an exact equality rather than an approximation.
[10] These second order approximations are due to Dalén (1992; 143) for the case $r^* = 1$ and to Diewert (1995; 29) for the case of a general r^*.

4. The Test Approach to Elementary Indexes

Recall that in Chapter 1, the axiomatic approach to bilateral price indexes $P(p^0,p^1,q^0,q^1)$ was developed. In the present section, the elementary price index $P(p^0,p^1)$ depends only on the period 0 and 1 price vectors, p^0 and p^1 respectively so that the elementary price index does not depend on the period 0 and 1 quantity vectors, q^0 and q^1. One approach to obtaining new tests or axioms for an elementary index is to look at the twenty or so axioms that were listed in Chapter 1 for bilateral price indexes $P(p^0,p^1,q^0,q^1)$ and adapt those axioms to the present context; i.e., use the old bilateral tests for $P(p^0,p^1,q^0,q^1)$ that do not depend on the quantity vectors q^0 and q^1 as tests for an elementary index $P(p^0,p^1)$.[11] This approach will be utilized in the present subsection.

The first eight tests or axioms are reasonably straightforward and uncontroversial:

T1: *Continuity*: $P(p^0,p^1)$ is a continuous function of the M positive period 0 prices $p^0 \equiv [p_1{}^0,...,p_M{}^0]$ and the M positive period 1 prices $p^1 \equiv [p_1{}^1,...,p_M{}^1]$.

T2: *Identity*: $P(p,p) = 1$; i.e., the period 0 price vector equals the period 1 price vector, then the index is equal to unity.

T3: *Monotonicity in Current Period Prices*: $P(p^0,p^1) < P(p^0,p)$ if $p^1 < p$; i.e., if any period 1 price increases, then the price index increases.

T4: *Monotonicity in Base Period Prices*: $P(p^0,p^1) > P(p,p^1)$ if $p^0 < p$; i.e., if any period 0 price increases, then the price index decreases.

T5: *Proportionality in Current Period Prices*: $P(p^0,\lambda p^1) = \lambda P(p^0,p^1)$ if $\lambda > 0$; i.e., if all period 1 prices are multiplied by the positive number λ, then the initial price index is also multiplied by λ.

T6: *Inverse Proportionality in Base Period Prices*: $P(\lambda p^0,p^1) = \lambda^{-1} P(p^0,p^1)$ if $\lambda > 0$; i.e., if all period 0 prices are multiplied by the positive number λ, then the initial price index is multiplied by $1/\lambda$.

T7: *Mean Value Test*: $\min_m\{p_m{}^1/p_m{}^0 : m = 1,...,M\} \leq P(p^0,p^1) \leq \max_m\{p_m{}^1/p_m{}^0 : m = 1,...,M\}$; i.e., the price index lies between the smallest and largest price relatives.

T8: *Symmetric Treatment of Outlets*: $P(p^0,p^1) = P(p^{0*},p^{1*})$ where p^{0*} and p^{1*} denote the *same* permutation of the components of p^0 and p^1; i.e., if we change the ordering of the outlets (or households) from which we obtain the price quotations for the two periods, then the elementary index remains unchanged.

[11] This was the approach used by Diewert (1995; 5-17), who drew on the earlier work of Eichhorn (1978; 152-160) and Dalén (1992).

Eichhorn (1978; 155) showed that Tests 1, 2, 3 and 5 imply Test 7, so that not all of the above tests are logically independent.

The following tests are more controversial and are not necessarily accepted by all price statisticians.

T9: *The Price Bouncing Test*: $P(p^0,p^1) = P(p^{0*},p^{1**})$ where p^{0*} and p^{1**} denote possibly *different* permutations of the components of p^0 and p^1; i.e., if the ordering of the price quotes for both periods is changed in possibly different ways, then the elementary index remains unchanged.

Obviously, T8 is a special case of T9 where the two permutations of the initial ordering of the prices are restricted to be the same. Thus T9 implies T8. Test T9 is due to Dalén (1992; 138). He justified this test by suggesting that the price index should remain unchanged if outlet prices "bounce" in such a manner that the outlets are just exchanging prices with each other over the two periods. While this test has some intuitive appeal, it is not consistent with the idea that outlet prices should be matched to each other in a one to one manner across the two periods.[12]

The following test was also proposed by Dalén (1992) in the elementary index context:

T10: *Time Reversal*: $P(p^1,p^0) = 1/P(p^0,p^1)$; i.e., if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index.

It is difficult to accept an index that gives a different answer if the ordering of time is reversed.

T11: *Circularity*: $P(p^0,p^1)P(p^1,p^2) = P(p^0,p^2)$; i.e., the price index going from period 0 to 1 times the price index going from period 1 to 2 equals the price index going from period 0 to 2 directly.

The circularity and identity tests imply the time reversal test; (just set $p^2 = p^0$). The circularity property would seem to be a very desirable property: it is a generalization of a property that holds for a single price relative.

T12: *Commensurability*: $P(\lambda_1 p_1^0,..., \lambda_M p_M^0; \lambda_1 p_1^1,..., \lambda_M p_M^1) = P(p_1^0,...,p_M^0; p_1^1,...,p_M^1) = P(p^0,p^1)$ for all $\lambda_1 > 0, ... , \lambda_M > 0$; i.e., if we change the units of measurement for each commodity in each outlet, then the elementary index remains unchanged.

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[12] Since a typical official Consumer Price Index consists of approximately 600 to 1000 separate strata where an elementary index needs to be constructed for each stratum, it can be seen that many strata will consist of quite heterogeneous items. Thus for a fruit category, some of the M items whose prices are used in the elementary index will correspond to quite different types of fruit with quite different prices. Randomly permuting these prices in periods 0 and 1 will lead to very odd price relatives in many cases, which may cause the overall index to behave badly unless the Jevons or Dutot formula is used.

In the bilateral index context, virtually every price statistician accepts the validity of this test. However, in the elementary context, this test is more controversial. If the M items in the elementary aggregate are all very homogeneous, then it makes sense to measure all of the items in the same units. Hence, if we change the unit of measurement in this homogeneous case, then test T12 should restrict all of the λ_m to be the same number (say λ) and test T12 becomes the following test:

(30) $P(\lambda p^0,\lambda p^1) = P(p^0,p^1)$; $\lambda > 0$.

Note that (30) will be satisfied if tests T5 and T6 are satisfied.

However, in actual practice, elementary strata will not be very homogeneous: there will usually be thousands of individual items in each elementary aggregate and the hypothesis of item homogeneity is not warranted. Under these circumstances, it is important that the elementary index satisfy the commensurability test, since the units of measurement of the heterogeneous items in the elementary aggregate are arbitrary and hence *the price statistician can change the index simply by changing the units of measurement for some of the items*.

This completes the listing of the tests for an elementary index. There remains the task of evaluating how many tests are passed by each of the five elementary indexes defined in section 2 above.

It is straightforward to show the following results hold:

- The Jevons elementary index P_J satisfies *all* of the above tests.
- The Dutot index P_D satisfies all of the tests with the important exception of the Commensurability Test T12, which it fails.
- The Carli and Harmonic elementary indexes, P_C and P_H, fail the price bouncing test T9, the time reversal test T10 and the circularity test T11 but pass the other tests.
- The geometric mean of the Carli and Harmonic elementary indexes, P_{CSWD}, fails only the (suspect) price bouncing test T9 and the circularity test T11.

Thus the Jevons elementary index P_J satisfies *all* of the tests and hence emerges as being "best" from the viewpoint of the axiomatic approach to elementary indexes.

The Dutot index P_D satisfies all of the tests with the important exception of the Commensurability Test T12, which it fails. If there are heterogeneous items in the elementary aggregate, this is a rather serious failure and hence price statisticians should be careful in using this index under these conditions.

The geometric mean of the Carli and Harmonic elementary indexes fail only the (suspect) price bouncing test T9 and the circularity test T11. The failure of these two tests is probably not a fatal failure and so this index could be used by price statisticians (who used the test approach for guidance in choosing an index formula), if for some reason, it was decided not to use the Jevons formula. As was noted in section 3 above, numerically, P_{CSWD} should be very close to P_J.

The Carli and Harmonic elementary indexes, P_C and P_H, fail the (suspect) price bouncing test T9, the time reversal test T10 and the circularity test T11 and pass the other tests. The failure of T9 and T11 is again not a fatal failure but the failure of the time reversal test T10 (with an upward bias for the Carli and a downward bias for the Harmonic) is a rather serious failure and so price statisticians should avoid using these indexes.

In the following section, we present an argument due originally to Irving Fisher on why it is desirable for an index number formula to satisfy the time reversal test.

5. An Index Number Formula Should be Invariant to the Choice of the Base Period

There is a problem with the Carli and Harmonic indexes which was first pointed out by Irving Fisher:[13] the rate of price change measured by the index number formula between two periods is dependent on which period is regarded as the base period. Thus the Carli index, $P_C(p^0,p^1)$ as defined by (2), takes period 0 as the base period and calculates (one plus) the rate of price change between periods 0 and 1.[14] Instead of choosing period 0 to be the base period, we could equally choose period 1 to be the base period and measure a reciprocal inflation rate going *backwards* from period 1 to period 0 and this *backwards measured inflation rate* would be $\sum_{m=1}^{M} (1/M)(p_m^0/p_m^1)$. In order to make this backwards inflation rate comparable to the forward inflation rate, we then take the reciprocal of $\sum_{m=1}^{M} (1/M)(p_m^0/p_m^1)$ and thus the overall inflation rate going from period 0 to 1 using period 1 as the base period is the following *Backwards Carli index* P_{BC}:[15]

(31) $P_{BC}(p^0,p^1) \equiv [\sum_{n=1}^{N} (1/M)(p_n^1/p_n^0)^{-1}]^{-1} = P_H(p^0,p^1)$;

i.e., the Backwards Carli index turns out to equal the Harmonic index $P_H(p^0,p^1)$ defined earlier by (4).

If the forward and backwards methods of computing price change between periods 0 and 1 using the Carli formula were equal, then we would have the following equality:

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[13] "Just as the very idea of an index number implies a set of commodities, so it implies two (and only two) times (or places). Either one of the two times may be taken as the 'base'. Will it make a difference which is chosen? Certainly it *ought* not and our Test 1 demands that it shall not. More fully expressed, the test is that the formula for calculating an index number should be such that it will give the same ratio between one point of comparison and the other point, *no matter which of the two is taken as the base*." Irving Fisher (1922; 64).

[14] Instead of calculating price inflation between periods 0 and 1, period 1 can be replaced by any period t that follows period 1; i.e., p^1 in the Carli formula $P_C(p^0,p^1)$ can be replaced by p^t and then the index $P_C(p^0,p^t)$ measures price change between periods 0 and t. The arguments concerning $P_C(p^0,p^1)$ which follow apply equally well to $P_C(p^0,p^t)$.

[15] Fisher (1922; 118) termed the backward looking counterpart to the usual forward looking index the *time antithesis* of the original index number formula. Thus P_H is the time antithesis to P_C. The Harmonic index defined by (4) is also known as the Coggeshall (1887) index.

(32) $P_C(p^0,p^1) = P_H(p^0,p^1)$.

Fisher argued that a good index number formula should satisfy (32) since the end result of using the formula should not depend on which period was chosen as the base period.[16] This seems to be a persuasive argument: if for whatever reason, a particular formula is favoured, where the base period 0 is chosen to the period which appears before the comparison period 1, then the same arguments which justify the forward looking version of the index number formula can be used to justify the backward looking version. If the forward and backward versions of the index agree with one another, then it does not matter which version is used and this equality provides a powerful argument in favour of using the formula. If the two versions do not agree, then rather than picking the forward version over the backward version, a more symmetric procedure would be to take an average of the forward and backward looking versions of the index formula.

Fisher provided an alternative way for justifying the equality of the two indexes in equation (32). He argued that the forward looking inflation rate using the Carli formula is $P_C(p^0,p^1) = \sum_{m=1}^{M} (1/M)(p_m^1/p_m^0)$. As noted above, the backwards looking inflation rate using the Carli formula is $\sum_{m=1}^{M} (1/M)(p_m^0/p_m^1) = P_C(p^1,p^0)$. Fisher[17] argued that the product of the forward looking and backward looking indexes should equal unity; i.e., a good formula should satisfy the following equality (which is equivalent to (32)):

(33) $P_C(p^0,p^1)P_C(p^1,p^0) = 1$.

But (33) is the usual *time reversal test* that was listed in the previous section. Thus Fisher provided a reasonably compelling case for the satisfaction of this test.

As we have seen in section 3 above,[18] the problem with the Carli formula is that it not only does not satisfy the equalities (32) or (33) but it *fails* (33) with the following definite inequality:

(34) $P_C(p^0,p^1)P_C(p^1,p^0) > 1$

unless the price vector p^1 is proportional to p^0 (so that $p^1 = \lambda p^0$ for some scalar $\lambda > 0$), in which case, (33) will hold. The main implication of the inequality (34) is that the use of *the Carli index will tend to give higher measured rates of inflation* than a formula which satisfies the time reversal test (using the same data set and the same weighting). We will provide a numerical example in section 8 below which confirms that this result holds.

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[16] "The justification for making this rule is twofold: (1) no reason can be assigned for choosing to reckon in one direction which does not also apply to the opposite, and (2) such reversibility does apply to any *individual* commodity. If sugar costs twice as much in 1918 as in 1913, then necessarily it costs half as much in 1913 as in 1918." Irving Fisher (1922; 64).

[17] "Putting it in still another way, more useful for practical purposes, the forward and backward index number multiplied together should give unity." Irving Fisher (1922; 64).

[18] Recall the inequalities in (46).

Fisher showed how the downward bias in the backwards looking Carli index P_H and the upward bias in the forward looking Carli index P_C could be cured. The Fisher time rectification procedure[19] as a general procedure for obtaining a bilateral index number formula which satisfies the time reversal test works as follows. Given a bilateral price index P, Fisher (1922; 119) defined the *time antithesis* P° for P as follows:

(35) $P^{\circ}(p^0,p^1,q^0,q^1) \equiv 1/P(p^1,p^0,q^1,q^0)$.

Thus P° is equal to the reciprocal of the price index which has reversed the role of time, $P(p^1,p^0,q^1,q^0)$. Fisher (1922; 140) then showed that the geometric mean of P and P°, say $P^* \equiv [P\times P^\circ]^{1/2}$, satisfies the time reversal test, $P^*(p^0,p^1,q^0,q^1)P^*(p^1,p^0,q^1,q^0) = 1$.

In the present context, P_C is only a function of p^0 and p^1, but the same rectification procedure works and the time antithesis of P_C is the harmonic index P_H. Applying the Fisher rectification procedure to the Carli index, the resulting rectified Carli formula, P_{RC}, turns out to equal the Carruthers, Sellwood and Ward (1980) and Dalén elementary index P_{CSWD} defined earlier by (5):

(36) $P_{RC}(p^0,p^1) \equiv [P_C(p^0,p^1)P_{BC}(p^0,p^1)]^{1/2} = [P_C(p^0,p^1)P_H(p^0,p^1)]^{1/2} = P_{CSWD}(p^0,p^1)$.

Thus P_{CSWD} is the geometric mean of the forward looking Carli index P_C and its backward looking counterpart $P_{BC} = P_H$, and of course, P_{CSWD} will satisfy the time reversal test.

6. A Simple Stochastic Approach to Elementary Indexes

In this section, the Jevons elementary index will be derived using an adaptation of the Country Product Dummy model[20] from the context of comparing prices across two countries to the time series context where the comparison of prices is made between two periods. Suppose that the prices of the M items being priced for an elementary aggregate for periods 0 and 1 are approximately equal to the right hand sides of (37) and (38) below:

(37) $p_m^0 \approx \beta_m$; $m = 1,...,M$;
(38) $p_m^1 \approx \alpha\beta_m$; $m = 1,...,M$

where α is a parameter that can be interpreted as the overall level of prices in period 1 relative to a price level of 1 in period 0 and the β_m are positive parameters that can be interpreted as item specific quality adjustment factors. Note that there are 2M prices on the left hand sides of equations (37) and (38) but only M + 1 parameters on the right hand sides of these equations. The basic hypothesis in (37) and (38) is that the two price

[19] Actually, Walsh (1921; 542) showed Fisher how to rectify a formula so it would satisfy the factor reversal test and Fisher simply adapted the methodology of Walsh to the problem of rectifying a formula so that it would satisfy the time reversal test.
[20] See Summers (1973) who introduced the CPD model. Balk (1980) was the first to adapt the CPD method to the time series context.

vectors p^0 and p^1 are proportional (with $p^1 = \alpha p^0$ so that α is the factor of proportionality) except for random multiplicative errors and hence α represents the underlying elementary price aggregate. If we take logarithms of both sides of (37) and (38) and add some random errors e_m^0 and e_m^1 to the right hand sides of the resulting equations, we obtain the following *linear regression model*:

(39) $\ln p_m^0 = \delta_m + e_m^0$; $\quad$ m = 1,...,M;
(40) $\ln p_m^1 = \gamma + \delta_m + e_m^1$; $\quad$ m = 1,...,M

where $\gamma \equiv \ln\alpha$ and $\delta_m \equiv \ln\beta_m$ for m = 1,...,M.

Note that (39) and (40) can be interpreted as a highly simplified *hedonic regression model* where the δ_m can be interpreted as quality adjustment factors for each item m.[21] The only characteristic of each commodity is the commodity itself. This model is also a special case of the *Country Product Dummy method* for making international comparisons between the prices of different countries. A major advantage of this regression method for constructing an elementary price index is that *standard errors* for the index number α can be obtained. This advantage of the stochastic approach to index number theory was stressed by Selvanathan and Rao (1994).

The least squares estimators for the parameters which appear in (39) and (40) are obtained by solving the following unweighted *least squares minimization problem*:

(41) $\min_{\gamma, \delta\text{'s}} \sum_{m=1}^M [\ln p_m^0 - \delta_m]^2 + \sum_{m=1}^M [\ln p_m^1 - \gamma - \delta_m]^2$.

It can be verified that the least squares estimator for γ is

(42) $\gamma^* \equiv \sum_{m=1}^M (1/M)\ln(p_m^1/p_m^0)$.

If γ^* is exponentiated, then the following estimator for the elementary index α is obtained:

(43) $\alpha^* \equiv \prod_{m=1}^M [p_m^1/p_m^0]^{1/M} \equiv P_J(p^0,p^1)$

where $P_J(p^0,p^1)$ is the *Jevons elementary price index* defined in section 2 above. Thus we have obtained a regression model based justification for the use of the Jevons elementary index.

Although the stochastic model defined by (39) and (40) has not led to a new elementary index number formula (since we just ended up deriving the Jevons index which was already introduced in section 2), a generalization of the CPD method adapted to the time

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[21] For an introduction to hedonic regression models, see Griliches (1971) and Diewert, Heravi and Silver (2009). For an extension of the unweighted CPD model to a situation where information on weights is available, see Balk (1980), Rao (1990) (1995) (2001) (2002) (2004), de Haan (2004), Diewert (2004) (2005) (2006) and de Haan and Krsinich (2012).

series context will be introduced in Chapter 7 of these Notes and the present section will serve to introduce the reader to the methods used there.

The following section briefly considers the economic approach to elementary indexes.

7. The Economic Approach to Elementary Indexes

The *Consumer Price Index Manual* has a section in it which describes an economic approach to elementary indexes; see the ILO (2004; 364-369). This section has sometimes been used to justify the use of the Jevons index over the use of the Carli index or vice versa depending on how much substitutability exists between items within an elementary stratum. If it is thought that there is a great deal of substitutability between items, then it is suggested that the Jevons index is the appropriate index to use. If it is thought that there is very little substitutability between items, then it is suggested that the Carli or the Dutot index is the appropriate index to use. This is a misinterpretation of the analysis that is presented in this section of the *Manual*. What the analysis there shows that *if* appropriate sampling of prices can be accomplished over one of the two periods in the comparison, *then* an appropriately probability weighted Carli or Dutot index can approximate a Laspeyres index (which is consistent with preferences that exhibit no substitutability) and an appropriately probability weighted Jevons index can approximate a Cobb-Douglas price index (which is consistent with a Cobb-Douglas subutility function defined over the items in the elementary stratum which has unitary elasticities of substitution). But the appropriate probability weights can only be known *if* knowledge about item quantities purchased is available or *if* information on item expenditures in one or both of the two periods being compared is available. *Such information is typically not available*, which is exactly the reason elementary indexes are used rather than the far superior indexes P_F, P_W or P_T, which require price and quantity information on purchases within the elementary stratum for both periods. *Thus the economic approach cannot be applied at the elementary level unless price and quantity information are both available*. And of course, if such information is available, then the elementary approach becomes irrelevant and can be replaced by one of the four (preferred) approaches that apply to situations where price and quantity (or expenditure) information is available.

In the following two sections, two numerical examples are provided which illustrate how the various elementary indexes might perform in practice.

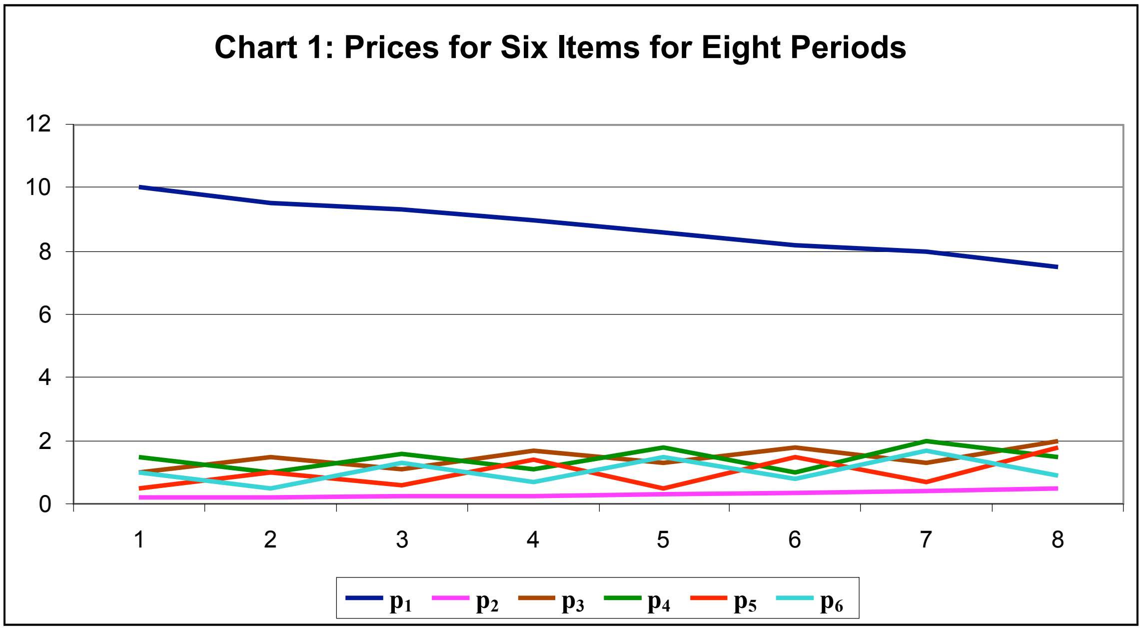
8. Elementary Indexes: A Numerical Example

In this section, we analyze a small data set consisting of 8 periods of price data for six items. The price of item m in period t is p_m^t for t = 1,...,8 and m = 1,...,6. The data are displayed in Table 1 and Chart 1 below.

Table 1: Price Data for Six Items for Eight Periods

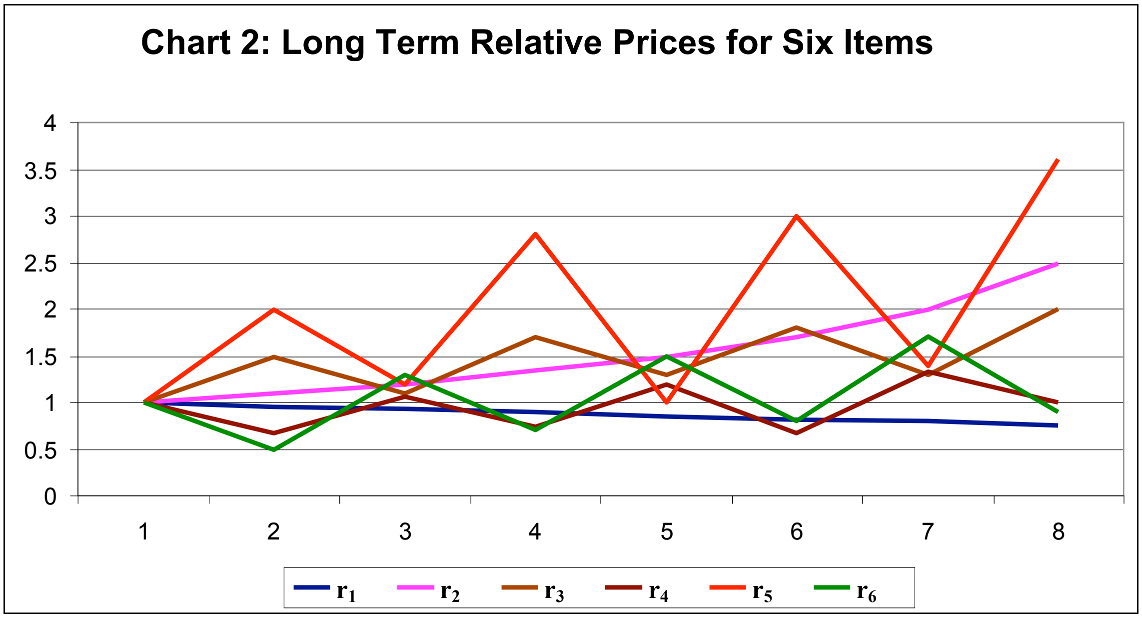
Period t	p_1^t	p_2^t	p_3^t	p_4^t	p_5^t	p_6^t
1	10.0	0.20	1.0	1.5	0.5	1.0

2	9.5	0.22	1.5	1.0	1.0	0.5
3	9.3	0.24	1.1	1.6	0.6	1.3
4	9.0	0.27	1.7	1.1	1.4	0.7
5	8.6	0.30	1.3	1.8	0.5	1.5
6	8.2	0.34	1.8	1.0	1.5	0.8
7	8.0	0.40	1.3	2.0	0.7	1.7
8	7.5	0.50	2.0	1.5	1.8	0.9



Frequently, an elementary category of goods and services can contain a large number of diverse items. For example, the category "musical instruments" could contain grand pianos, electric guitars and flutes so that some items could have very large prices and some items could have rather small prices. The data in Table 1 reflects such a diverse category: item 1 has very large prices (which trend down smoothly), item 2 has very small prices (which trend up smoothly) while items 3-6 have "average" prices (which bounce around from period to period, reflecting periodic sales of the items). All of the prices have an upward trend, with the exception of item 1.

Many statistical agencies compute their elementary indexes using item prices over a year relative to the December or January price of the item. These relative prices are called long term price relatives (as opposed to short term month over month price relatives). For our sample data set, the long term relative prices, $r_n^t \equiv p_n^t/p_n^1$ for n = 1,...,6 and t = 1,...,8, are plotted in Chart 2.



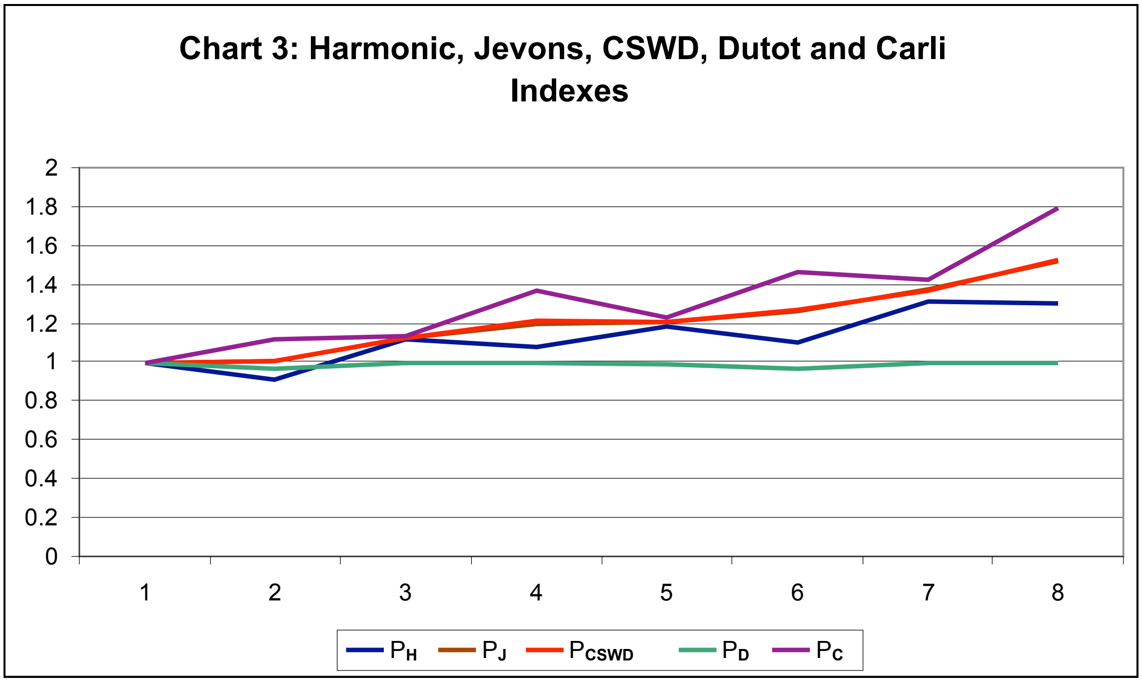
The downward trend in the relative prices for item 1 and the upward trend in the relative prices for item 2 are readily visible in Chart 2 and the price bouncing nature of the long term price relatives for items 3-6 is also visible.

The data in Table 1 are used to calculate the Harmonic, Jevons, Carruthers, Sellwood, Ward and Dalen (CSWD), Dutot and Carli indexes using long term price relatives and the results are listed in Table 2 and plotted in Chart 3 below.[22]

Table 2: Harmonic, Jevons, CSWD, Dutot and Carli Price Indexes Using Long Term Price Relatives for the Artificial Data Set

Period	P_H	P_J	P_{CSWD}	P_D	P_C
1	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.90520	1.00736	1.00664	0.96620	1.11944
3	1.11987	1.12642	1.12631	0.99577	1.13278
4	1.07345	1.19885	1.20999	0.99789	1.36389
5	1.17677	1.20217	1.20146	0.98592	1.22667
6	1.10160	1.26069	1.27013	0.96056	1.46444
7	1.31241	1.36961	1.36621	0.99296	1.42222
8	1.29808	1.51622	1.52503	1.00000	1.79167
Average 2-8	1.14110	1.24020	1.24370	0.98561	1.36020

[22] The index formulae are given by (4), (3), (5), (1) and (2) respectively, except instead of computing $P(p^0,p^1)$, we computed $P(p^1,p^t)$ for the five formulae for t = 1,2,...,8.



As expected, P_J and P_{CWSD} are very close to each other; the two series cannot be distinguished on Chart 3. Also as expected, the Carli index P_C is considerably above the corresponding Jevons and CWSD indexes (for periods 2-8, P_C averages about 12.0 percentage points above the corresponding Jevons index P_J) and the Harmonic index is considerably below the Jevons index (for periods 2-8, P_H averages about 9.9 percentage points below the corresponding Jevons index P_J). Somewhat surprisingly, the Dutot index is far below the other indexes for the later periods (for periods 2-8, P_D averages about 25.5 percentage points below the corresponding Jevons index P_J). This result is due to the very large price for item 1 (and the downward trend in the price of this item): the large price for this item gives the item too much influence in the Dutot index and leads to the anomalous results for this index. However, this example illustrates the problem with the use of the Dutot index in an elementary category: *it will generally not give satisfactory results if the items in the category are quite heterogeneous*.

Recall the results on approximated numerical relationships between the various elementary indexes which were listed in section 3 above. It is of some interest to see how well these approximation results are able to predict the differences between the frequently used elementary indexes when applied to the above artificial data set.

We first consider the difference between the Dutot and Jevons indexes. The period t deviations from the mean (the e_m^t for m = 1,...,6 and t = 1,...,T) can be defined by equations (10) and (11). Denote the period t six dimensional vector of deviations as e^t for t = 1,...,8. The adaptation of (16) to the current context leads to the following approximate relationship between P_J and P_D:[23]

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[23] Note that $e^t \cdot e^t/6$ is $Var(e^t)$, the variance of the deviations from the mean in period t for t = 1,...,8. The vector of variances of the deviations is [2.533, 2.426, 2.126, 1.942, 1.793, 1.682, 1.452, 1.185].

(44) $[P_J(p^1,p^t)/P_D(p^1,p^t)] - 1 \approx (1/2)[e^1 \cdot e^1 - e^t \cdot e^t]/6 = (1/2)[\mathrm{Var}(e^1) - \mathrm{Var}(e^t)]$; t = 2,3,...,8.

The arithmetic mean of the seven ratios less unity on the left hand side of (44) is 0.25751 while the average of the seven differences on the right hand side of (44) is 0.36590. Thus the approximation result (16) in section 3 is not very accurate for describing the expected difference between the Jevons and Dutot indexes. This lack of accuracy for our numerical example is not too surprising since the approximation results depend on the deviations e_m^t being reasonably small. They are not small in the present case due to the very large prices for item 1 and the very small prices for item 2 relative to the other prices.

However, the numerical approximation results given by (26)-(29) for the remaining four elementary indexes are much more accurate as will be seen, because these approximations use relative prices (rather than price levels, which were used in the Dutot approximations). Recall that the period t price relative to the corresponding period 1 price was defined as $r_m^t \equiv p_m^t/p_m^1$ for t =2,3,...,8 and m = 1,...,6. Adapting definitions (19) and (20) to the current situation, we have the following definitions:

(45) $r^{t*} \equiv (1/6) \sum_{m=1}^{6} r_m^t = P_C(p^1,p^t)$; t = 2,3,...,8 ;
(46) $e_m^t \equiv (r_m^t/r^{t*}) - 1$; m = 1,...,6 ; t = 2,3,...,8.

Let $e^t \equiv [e_1^t,...,e_6^t]$ and let $\mathrm{Var}(e^t)$ denote the variance of the entries in the vector e^t for t = 2,...,8.[24] Then adapting (26)-(29) to the current numerical example, we have the following approximate equalities:

(47) $P_J(p^1,p^t) \approx P_C(p^1,p^t) - (1/2)\mathrm{Var}(e^t)$; t = 2,3,...,8;
(48) $P_{CWSW}(p^1,p^t) \approx P_C(p^1,p^t) - (1/2)\mathrm{Var}(e^t)$; t = 2,3,...,8;
(49) $P_H(p^1,p^t) \approx P_C(p^1,p^t) - \mathrm{Var}(e^t)$; t = 2,3,...,8.

The average variance of the e^t over periods 2-8 is 0.2144 and so one half of this variance is 0.1072. Thus on average (over periods 2-8), we expect the Carli indexes to be 10.7 percentage points above the corresponding Jevons indexes and this compares to the average difference between the two indexes of 12.0 percentage points from Table 2. Similarly we expect the Harmonic indexes to be 10.7 percentage points below the corresponding Jevons indexes and this compares to the average difference between the two indexes of 9.9 percentage points from Table 2. Thus the approximation results listed in section 3 above are reasonably accurate for our four indexes that are based on relative prices.

It should be noted that in the above example, there are substantial differences between the Dutot and Jevons indexes, which can be traced to the aggregation of very heterogeneous items (which have different rates of price inflation). There is some evidence that at the aggregate level, there may not be much difference between indexes computed by using

[24] These variances turned out to equal [0.245, 0.013, 0.347, 0.454, 0.375, 0.081, 0.394]. Note that the price bouncing behavior of items 3-6 leads to fairly low variances in some periods.

Jevons elementary indexes versus indexes computed using Dutot elementary indexes. According to Evans (2012), Slovenia uses the Jevons formula exclusively at the elementary levels of their HICP. In their national CPI, however, they exclusively use a Dutot formula. This is because historically the Dutot has always been used in their CPI, and the Statistical Office of the Republic of Slovenia has no plans to change this in the short term. Evans (2012) showed that on average since 1998, the total gap between the Slovenian CPI and HICP has only been 0.1 percentage points.

We have reviewed existing theory dealing with the choice of index number method both when information on prices and quantities is available for two periods (Chapters 1 and 2), for many periods (Chapter 3) and when only price information is available for two periods (the present Chapter). In the following Chapter, we study some of the practical problems that are associated with the construction of a Consumer Price Index.

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