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Chapter 2: The Economic Approach to Index Number Theory

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1. Introduction

In the economic approach, the period 0 quantity vector q^0 is determined by the consumer's preference function f and the period 0 vector of prices p^0 that the consumer faces and the period 1 quantity vector q^1 is determined by the consumer's preference function f and the period 1 vector of prices p^1.

In this Chapter, we will give an overview of the economic approach for the conceptual foundations of consumer and producer price indexes. Sections 2-11 are devoted to Cost of Living Indexes (or Consumer Price Indexes) while sections 12-14 are devoted to Producer Output Price Indexes. Most of the material (but not all) in this Chapter can be found in the Consumer Price Index Manual and the Producer Price Index Manual; see the ILO/IMF/OECD/UNECE/Eurostat/World Bank (2004) and the IMF/ILO/OECD/UNECE /Eurostat/World Bank (2004).

Section 2 defines the family of Konüs (1924) true Cost of Living Indexes for a single consumer or household for the case where the consumer has general preferences. Section 3 specializes the general preference case to the case where preferences are homothetic.[2]

Section 4 is a technical section which derives Wold's Identity and Shephard's Lemma. These theoretical results are required in order to derive the results in the subsequent sections.

Sections 5,6 and 7 show that certain flexible functional forms[3] that can be used to represent consumer preferences are consistent with certain functional forms for bilateral index number formulae. The resulting index number formulae are called superlative formulae and they are useful target indexes. Section 5 shows that the Fisher price index P_F is superlative, section 6 shows that the Walsh index P_W is superlative and section 7 shows that the Törnqvist Theil index P_T is also superlative.

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[2] Essentially, this case implies that the preferences of the consumer can be represented by a utility function that is linearly homogeneous.
[3] A flexible functional form is one that can approximate an arbitrary linearly homogeneous twice continuously differentiable functional form to the second order at an arbitrary point in the function's domain of definition.

In section 8, we turn our attention to the case where the consumer's preferences are nonhomothetic; i.e., they cannot be represented by a linearly homogeneous function. It turns out that the Törnqvist Theil price index can be given a strong justification in this context.

In section 9, we turn our attention to quantity indexes. There are various ways of defining a theoretical quantity index when the consumer has nonhomothetic preferences. One method is to use the concept of an Allen (1949) quantity index. In section 9, we show that observable Törnqvist Theil implicit quantity index, $Q_{T*}(p^0,p^1,q^0,q^1)$,[4] is exactly equal to the geometric mean of two relevant Allen quantity indexes.

Sections 10 and 11 generalize the results in section 2 for a single consumer or household to the case of many households.

Sections 12-14 adapt the consumer theory results in sections 2 and 8 to the producer context.

2. The Konüs Cost of Living Index for a Single Consumer

In this section, we will outline the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, A. A. Konüs (1924). This theory relies on the assumption of *optimizing behavior* on the part of economic agents (consumers or producers). Thus given a vector of commodity or input prices p^t that the agent faces in a given time period t, it is assumed that the corresponding observed quantity vector q^t is the solution to a cost minimization problem that involves either the consumer's preference or utility function f or the producer's production function f.[5] Thus in contrast to the axiomatic approach to index number theory, the economic approach does *not* assume that the two quantity vectors q^0 and q^1 are independent of the two price vectors p^0 and p^1. In the economic approach, the period 0 quantity vector q^0 is determined by the consumer's preference function f and the period 0 vector of prices p^0 that the consumer faces and the period 1 quantity vector q^1 is determined by the consumer's preference function f and the period 1 vector of prices p^1.

We assume that "the" consumer has well defined *preferences* over different combinations of the N consumer commodities or items.[6] Each combination of items can be represented by a positive vector $q \equiv [q_1,\ldots,q_N]$. The consumer's preferences over alternative possible consumption vectors q are assumed to be representable by a continuous, increasing and

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[4] Q_{T*} is defined as $Q_{T*}(p^0,p^1,q^0,q^1) \equiv [p^1 \cdot q^1/p^0 \cdot q^0]/P_T(p^0,p^1,q^0,q^1)$ where $P_T(p^0,p^1,q^0,q^1)$ is the Törnqvist Theil price index that was defined in Chapter 1.

[5] For a description of the economic theory of the input and output price indexes, see Balk (1998). In the economic theory of the output price index, q^t is assumed to be the solution to a revenue maximization problem involving the output price vector p^t.

[6] In this section, these preferences are assumed to be invariant over time. In chapter 5 when we introduce environmental variables, this assumption will be relaxed (one of the environmental variables could be a time variable that shifts tastes).

concave[7] utility function f. Thus if $f(q^1) > f(q^0)$, then the consumer prefers the consumption vector q^1 to q^0. We further assume that the consumer minimizes the cost of achieving the period t utility level $u^t \equiv f(q^t)$ for periods t = 0,1. Thus we assume that the observed period t consumption vector q^t solves the following *period t cost minimization problem*:

$$(1)\quad C(u^t,p^t) \equiv \min_q \{\textstyle\sum_{i=1}^N p_i^t q_i : f(q) = u^t \equiv f(q^t)\} = \textstyle\sum_{i=1}^N p_i^t q_i^t = p^t \cdot q^t ; \qquad t = 0,1.$$

The period t price vector for the n commodities under consideration that the consumer faces is p^t. Note that the solution to the cost or expenditure minimization problem (1) for a general utility level u and general vector of commodity prices p defines the *consumer's cost function*, C(u,p). We shall use the cost function in order to define the consumer's cost of living price index.

The Konüs (1924) family of *true cost of living indexes* pertaining to two periods (where the consumer faces the strictly positive price vectors $p^0 \equiv (p_1^0,\ldots,p_N^0)$ and $p^1 \equiv (p_1^1,\ldots,p_N^1)$ in periods 0 and 1 respectively) is defined as the ratio of the minimum costs of achieving the same utility level $u \equiv f(q)$ where $q \equiv (q_1,\ldots,q_N)$ is a positive reference quantity vector; i.e., a Konüs true cost of living index is defined as follows:

$$(2)\quad P_K(p^0,p^1,q) \equiv C[f(q),p^1]/C[f(q),p^0].$$

We say that definition (2) defines a *family* of price indexes because there is one such index for each reference quantity vector q chosen.

It is natural to choose two specific reference quantity vectors q in definition (2): the observed base period quantity vector q^0 and the current period quantity vector q^1. The first of these two choices leads to the *following Laspeyres-Konüs true cost of living index*:

$$\begin{aligned}(3)\quad P_K(p^0,p^1,q^0) &\equiv C[f(q^0),p^1]/C[f(q^0),p^0] \\ &= C[f(q^0),p^1]/p^0\cdot q^0 && \text{using (1) for } t = 0 \\ &= \min_q\{p^1\cdot q : f(q) = f(q^0)\}/p^0\cdot q^0 \\ & && \text{using the definition of the cost minimization} \\ & && \text{problem that defines } C[f(q^0),p^1] \\ &\le p^1\cdot q^0/p^0\cdot q^0 && \text{since } q^0 \text{ is feasible for the minimization problem} \\ &\equiv P_L(p^0,p^1,q^0,q^1)\end{aligned}$$

where P_L is the Laspeyres price index defined in Chapter 1. *Thus the (unobservable) Laspeyres-Konüs true cost of living index is bounded from above by the observable Laspeyres price index.*[8]

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[7] f is concave if and only if $f(\lambda q^1 + (1-\lambda)q^2) \ge \lambda f(q^1) + (1-\lambda)f(q^2)$ for all $0 \le \lambda \le 1$ and all $q^1 >> 0_N$ and $q^2 >> 0_N$. Note that $q \ge 0_N$ means that each component of the N dimensional vector q is nonnegative, $q >> 0_N$ means that each component of q is positive and $q > 0_N$ means that $q \ge 0_N$ but $q \ne 0_N$; i.e., q is nonnegative but at least one component is positive.

[8] This inequality was first obtained by Konüs (1924) (1939; 17). See also Pollak (1983).

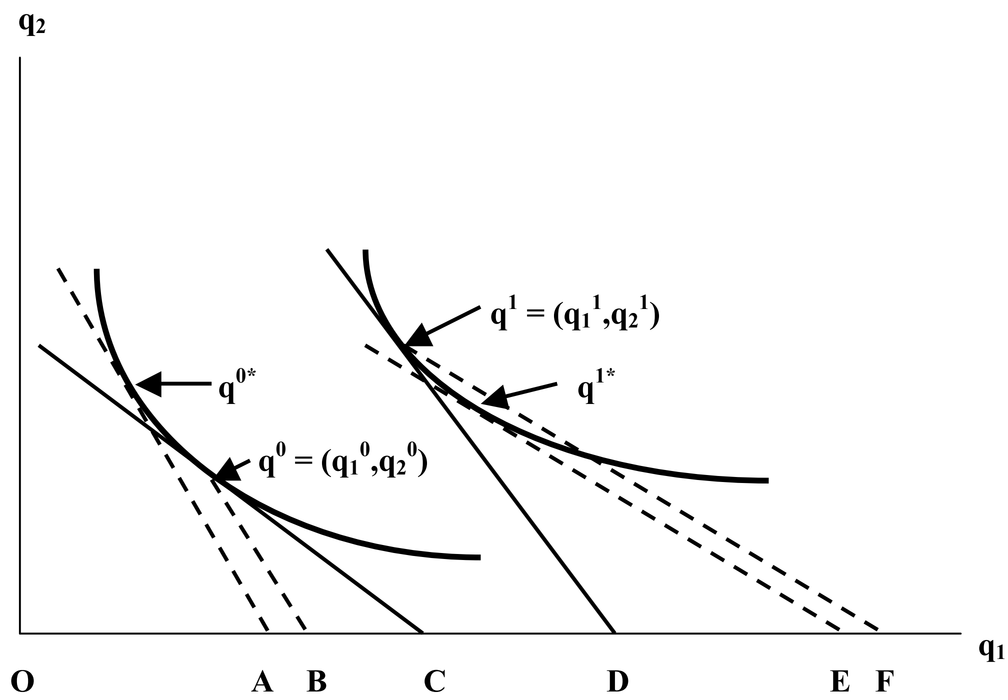
The second of the two natural choices for a reference quantity vector q in definition (2) leads to the *following Paasche-Konüs true cost of living index*:

(4) $P_K(p^0,p^1,q^1) \equiv C[f(q^1),p^1]/C[f(q^1),p^0]$
$= p^1 \cdot q^1/C[f(q^1),p^0]$ using (1) for t = 1
$= p^1 \cdot q^1/\min_q \{p^0 \cdot q : f(q) = f(q^1)\}$
using the definition of the cost minimization problem that defines $C[f(q^1),p^0]$
$\geq p^1 \cdot q^1/p^0 \cdot q^1$ since q^1 is feasible for the minimization problem so $p^0 \cdot q^1 \geq C[f(q^1),p^0]$ and hence $1/C[f(q^1),p^0] \geq 1/ p^0 \cdot q^1$
$\equiv P_P(p^0,p^1,q^0,q^1)$

where P_P is the Paasche price index defined in Chapter 1. *Thus the (unobservable) Paasche-Konüs true cost of living index is bounded from below by the observable Paasche price index.*[9]

It is possible to illustrate the two inequalities (3) and (4) if there are only two commodities; see Figure 1 below.

Figure 1: The Laspeyres and Paasche bounds to the True Cost of Living



The solution to the period 0 cost minimization problem is the vector q^0 and the straight line through C represents the consumer's period 0 budget constraint, the set of quantity

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[9] This inequality is also due to Konüs (1924) (1939; 19). See also Pollak (1983).

points q_1,q_2 such that $p_1^0q_1+p_2^0q_2 = p_1^0q_1^0+p_2^0q_2^0$. The curved line through q^0 is the consumer's period 0 indifference curve, the set of points q_1,q_2 such that $f(q_1,q_2) = f(q_1^0,q_2^0)$; i.e., it is the set of consumption vectors that give the same utility as the observed period 0 consumption vector q^0. The solution to the period 1 cost minimization problem is the vector q^1 and the straight line through D represents the consumer's period 1 budget constraint, the set of quantity points q_1,q_2 such that $p_1^1q_1+p_2^1q_2 = p_1^1q_1^1+p_2^1q_2^1$. The curved line through q^1 is the consumer's period 1 indifference curve, the set of points q_1,q_2 such that $f(q_1,q_2) = f(q_1^1,q_2^1)$; i.e., it is the set of consumption vectors that give the same utility as the observed period 1 consumption vector q^1. The point q^{0*} solves the hypothetical cost minimization problem of minimizing the cost of achieving the base period utility level $u^0 \equiv f(q^0)$ when facing the period 1 price vector $p^1 = (p_1^1,p_2^1)$. Thus we have $C[u^0,p^1] = p_1^1q_1^{0*}+p_2^1q_2^{0*}$ and the dashed line through A is the corresponding isocost line $p_1^1q_1+p_2^1q_2 = C[u^0,p^1]$. Note that the hypothetical cost line through A is parallel to the actual period 1 cost line through D. From (3), the Laspeyres-Konüs true index is $C[u^0,p^1]/[p_1^0q_1^0+p_2^0q_2^0]$ while the ordinary Laspeyres index is $[p_1^1q_1^0+ p_2^1q_2^0]/[p_1^0q_1^0+p_2^0q_2^0]$. Since the denominators for these two indexes are the same, the difference between the indexes is due to the differences in their numerators. In Figure 1, this difference in the numerators is expressed by the fact that the cost line through A lies *below* the parallel cost line through B. Now if the consumer's indifference curve through the observed period 0 consumption vector q^0 were L shaped with vertex at q^0, then the consumer would not change his or her consumption pattern in response to a change in the relative prices of the two commodities while keeping a fixed standard of living. In this case, the hypothetical vector q^{0*} would coincide with q^0, the dashed line through A would coincide with the dashed line through B and the true Laspeyres-Konüs index would *coincide* with the ordinary Laspeyres index. However, L shaped indifference curves are not generally consistent with consumer behavior; i.e., when the price of a commodity decreases, consumers generally demand more of it. Thus in the general case, there will be a gap between the points A and B. The magnitude of this gap represents the amount of *substitution bias* between the true index and the corresponding Laspeyres index; i.e., the Laspeyres index will generally be *greater* than the corresponding true cost of living index, $P_K(p^0,p^1,q^0)$.

Figure 1 can also be used to illustrate the inequality (4). First note that the dashed lines through E and F are parallel to the period 0 isocost line through C. The point q^{1*} solves the hypothetical cost minimization problem of minimizing the cost of achieving the current period utility level $u^1 \equiv f(q^1)$ when facing the period 0 price vector $p^0 = (p_1^0,p_2^0)$. Thus we have $C[u^1,p^0] = p_1^0q_1^{1*}+p_2^0q_2^{1*}$ and the dashed line through E is the corresponding isocost line $p_1^1q_1+p_2^1q_2 = C[u^0,p^1]$. From (4), the Paasche-Konüs true index is $[p_1^1q_1^1+p_2^1q_2^1]/C[u^1,p^0]$ while the ordinary Paasche index is $[p_1^1q_1^1+p_2^1q_2^1]/ [p_1^0q_1^1+p_2^0q_2^1]$. Since the numerators for these two indexes are the same, the difference between the indexes is due to the differences in their denominators. In Figure 1, this difference in the denominators is expressed by the fact that the cost line through E lies *below* the parallel cost line through F. The magnitude of this difference represents the amount of *substitution bias* between the true index and the corresponding Paasche index; i.e., the Paasche index will generally be *less* than the corresponding true cost of living index, $P_K(p^0,p^1,q^1)$. Note that this inequality goes in the opposite direction to the previous inequality between the two Laspeyres indexes. The reason for this change in direction is due to the fact that one set of differences between the two indexes takes place in the numerators of the two indexes (the Laspeyres inequalities) while the other set takes place in the denominators of the two indexes (the Paasche inequalities).

The bound (3) on the Laspeyres-Konüs true cost of living $P_K(p^0,p^1,q^0)$ using the base period level of utility as the living standard is *one sided* as is the bound (4) on the Paasche-Konüs true cost of living $P_K(p^0,p^1,q^1)$ using the *current period* level of utility as the living standard. In a remarkable result, Konüs (1924; 20) showed that there exists an intermediate consumption vector q^* that is on the straight line joining the base period consumption vector q^0 and the current period consumption vector q^1 such that the corresponding (unobservable) true cost of living index $P_K(p^0,p^1,q^*)$ is between the observable Laspeyres and Paasche indexes, P_L and P_P.[10] Thus we have:

Proposition 1: There exists a number λ^* between 0 and 1 such that

(5) $P_L \le P_K(p^0,p^1, \lambda^*q^0 + (1-\lambda^*)q^1) \le P_P$ or $P_P \le P_K(p^0,p^1, \lambda^*q^0 + (1-\lambda^*)q^1) \le P_L$.

Proof: Define $g(\lambda)$ for $0 \le \lambda \le 1$ by $g(\lambda) \equiv P_K(p^0,p^1,(1-\lambda)q^0 + \lambda q^1)$. Note that $g(0) = P_K(p^0,p^1,q^0)$ and $g(1) = P_K(p^0,p^1,q^1)$. There are 24 = (4)(3)(2)(1) possible a priori inequality relations that are possible between the four numbers $g(0)$, $g(1)$, P_L and P_P. However, the inequalities (3) and (4) above imply that $g(0) \le P_L$ and $P_P \le g(1)$. This means that there are only six possible inequalities between the four numbers:

(6) $g(0) \le P_L \le P_P \le g(1)$;
(7) $g(0) \le P_P \le P_L \le g(1)$;
(8) $g(0) \le P_P \le g(1) \le P_L$;
(9) $P_P \le g(0) \le P_L \le g(1)$;
(10) $P_P \le g(1) \le g(0) \le P_L$;
(11) $P_P \le g(0) \le g(1) \le P_L$.

Using the assumptions that: (a) the consumer's utility function f is continuous over its domain of definition; (b) the utility function is increasing in the components of q and hence is subject to local nonsatiation and (c) the price vectors p^t have strictly positive components, it is possible to use Debreu's (1959; 19) Maximum Theorem (see also Diewert (1993; 112-113) for a statement of the Theorem) to show that the consumer's cost function $C(f(q),p^t)$ will be continuous in the components of q. Thus using definition (2), it can be seen that $P_K(p^0,p^1,q)$ will also be continuous in the components of the vector q. Hence $g(\lambda)$ is a continuous function of λ and assumes all intermediate values between g(0) and g(1). By inspecting the inequalities (6)-(11) above, it can be seen that we can choose λ between 0 and 1, λ^* say, such that $P_L \le g(\lambda^*) \le P_P$ for case (6) or such that $P_P \le g(\lambda^*) \le P_L$ for cases (7) to (11). Thus at least one of the two inequalities in (5) holds. Q.E.D.

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[10] For more recent applications of the Konüs method of proof, see Diewert (1983a;191) for an application to the consumer context and Diewert (1983b; 1059-1061) for an application to the producer context.

The above inequalities are of some practical importance. If the observable (in principle) Paasche and Laspeyres indexes are not too far apart, then taking a symmetric average of these indexes should provide a good approximation to a true cost of living index where the reference standard of living is somewhere between the base and current period living standards. To determine the precise symmetric average of the Paasche and Laspeyres indexes, we can appeal to the results in Chapter 1 above and take the geometric mean, which is the Fisher price index. Thus the Fisher ideal price index receives a fairly strong justification as a good approximation to an unobservable theoretical cost of living index.

The bounds (3)-(5) are the best bounds that we can obtain on true cost of living indexes without making further assumptions. In a subsequent section, we will make further assumptions on the class of utility functions that describe the consumer's tastes for the N commodities under consideration. With these extra assumptions, we are able to determine the consumer's true cost of living exactly.

3. The True Cost of Living Index when Preferences are Homothetic

Up to now, the consumer's preference function f did not have to satisfy any particular homogeneity assumption. In this section, we assume that f is (positively) *linearly homogeneous*[11]; i.e., we assume that the consumer's utility function has the following property:

(12) $f(\lambda q) = \lambda f(q)$ for all $\lambda > 0$ and all $q >> 0_N$.

Given the continuity of f, it can be seen that property (12) implies that $f(0_N) = 0$ so that our old u_0 is now equal to 0. Furthermore, f also satisfies $f(q) > 0$ if $q > 0_N$.

In the economics literature, assumption (12) is known as the assumption of *homothetic preferences*.[12] This assumption is not strictly justified from the viewpoint of actual economic behavior, but it leads to economic price indexes that are independent from the consumer's standard of living.[13] Under this assumption, the consumer's expenditure or cost function, C(u,p) defined by (1) above, decomposes as follows. For positive

[11] This assumption is fairly restrictive in the consumer context. It implies that each indifference curve is a radial projection of the unit utility indifference curve. It also implies that all income elasticities of demand are unity, which is contradicted by empirical evidence.

[12] More precisely, Shephard (1953) defined a homothetic function to be a monotonic transformation of a linearly homogeneous function. However, if a consumer's utility function is homothetic, we can always rescale it to be linearly homogeneous without changing consumer behavior. Hence, we simply identify the homothetic preferences assumption with the linear homogeneity assumption.

[13] This particular branch of the economic approach to index number theory is due to Shephard (1953) (1970) and Samuelson and Swamy (1974). Shephard in particular realized the importance of the homotheticity assumption in conjunction with separability assumptions in justifying the existence of subindexes of the overall cost of living index. It should be noted that if the consumer's change in real income or utility between the two periods under consideration is not too large, then assuming that the consumer has homothetic preferences will lead to a true cost of living index which is very close to Laspeyres-Konüs and Paasche-Konüs true cost of living indexes defined above by (3) and (4).

commodity prices $p >> 0_N$ and a positive utility level u, we have by the definition of C as the minimum cost of achieving the given utility level u:

$$
\begin{aligned}
(13)\quad C(u,p) &\equiv \min_q \{\textstyle\sum_{i=1}^N p_iq_i : f(q_1,\ldots,q_N) \geq u\} \\
&= \min_q \{\textstyle\sum_{i=1}^N p_iq_i : (1/u)f(q_1,\ldots,q_N) \geq 1\} && \text{dividing by } u > 0 \\
&= \min_q \{\textstyle\sum_{i=1}^N p_iq_i : f(q_1/u,\ldots,q_N/u) \geq 1\} && \text{using the linear homogeneity of } f \\
&= u \min_q \{\textstyle\sum_{i=1}^N p_iq_i/u : f(q_1/u,\ldots,q_N/u) \geq 1\} \\
&= u \min_z \{\{\textstyle\sum_{i=1}^N p_iz_i : f(z_1,\ldots,z_N) \geq 1\} && \text{letting } z_i = q_i/u \\
&= uC(1,p) && \text{using definition (1) with } u = 1 \\
&= uc(p)
\end{aligned}
$$

where $c(p) \equiv C(1,p)$ is the *unit cost function* that is corresponds to f.[14] It can be shown that the unit cost function c(p) satisfies the same regularity conditions that f satisfied; i.e., c(p) is positive, concave and (positively) linearly homogeneous for positive price vectors.[15] Substituting (13) into (1) and using $u^t = f(q^t)$ leads to the following equations:

$$(14)\quad p^t \cdot q^t = c(p^t)f(q^t) \qquad \text{for } t = 0,1.$$

Thus under the linear homogeneity assumption on the utility function f, observed period t expenditure on the n commodities (the left hand side of (14) above) is equal to the period t unit cost $c(p^t)$ of achieving one unit of utility times the period t utility level, $f(q^t)$, (the right hand side of (14) above). Obviously, we can identify the period t unit cost, $c(p^t)$, as the period t price level P^t and the period t level of utility, $f(q^t)$, as the period t quantity level Q^t.[16]

The linear homogeneity assumption on the consumer's preference function f leads to a simplification for the family of Konüs true cost of living indices, $P_K(p^0,p^1,q)$, defined by (2) above. Using this definition for an arbitrary reference quantity vector q, we have:

$$
\begin{aligned}
(15)\quad P_K(p^0,p^1,q) &\equiv C[f(q),p^1]/C[f(q),p^0] \\
&= c(p^1)f(q)/c(p^0)f(q) && \text{using (14) twice} \\
&= c(p^1)/c(p^0).
\end{aligned}
$$

[14] Economists will recognize the producer theory counterpart to the result $C(u,p) = uc(p)$: if a producer's production function f is subject to constant returns to scale, then the corresponding total cost function C(u,p) is equal to the product of the output level u times the unit cost c(p).

[15] Obviously, the utility function f determines the consumer's cost function C(u,p) as the solution to the cost minimization problem in the first line of (13). Then the unit cost function c(p) is defined as C(1,p). Thus f determines c. But we can also use c to determine f under appropriate regularity conditions. In the economics literature, this is known as *duality theory.* For additional material on duality theory and the properties of f and c, see Samuelson (1953), Shephard (1953) (1970) and Diewert (1974) (1993b; 107-123).

[16] There is also a producer theory interpretation of the above theory; i.e., let f be the producer's (constant returns to scale) production function, let p be a vector of input prices that the producer faces, let q be an input vector and let $u = f(q)$ be the maximum output that can be produced using the input vector q. $C(u,p) \equiv \min_q \{p\cdot q: f(q) \geq u\}$ is the producer's cost function in this case and $c(p^t)$ can be identified as the period t input price level while $f(q^t)$ is the period t aggregate input level.

Thus under the homothetic preferences assumption, the entire family of Konüs true cost of living indexes collapses to a single index, $c(p^1)/c(p^0)$, the ratio of the minimum costs of achieving unit utility level when the consumer faces period 1 and 0 prices respectively. Put another way, *under the homothetic preferences assumption,* $P_K(p^0,p^1,q)$ *is independent of the reference quantity vector q.*[17]

If we use the Konüs true cost of living index defined by the right hand side of (15) as our price index concept, then the corresponding implicit quantity index defined using the product test has the following form:

$$\begin{aligned}(16)\ Q(p^0,p^1,q^0,q^1,q) &\equiv p^1\cdot q^1/\{p^0\cdot q^0\ P_K(p^0,p^1,q)\} && \text{using the product test and } P_K \text{ as the price index}\\ &= c(p^1)f(q^1)/\{c(p^0)f(q^0)\ P_K(p^0,p^1,q)\} && \text{using (14) twice}\\ &= c(p^1)f(q^1)/\{c(p^0)f(q^0)[c(p^1)/c(p^0)]\} && \text{using (15)}\\ &= f(q^1)/f(q^0).\end{aligned}$$

Thus under the homothetic preferences assumption, the *implicit quantity index* that corresponds to the true cost of living price index $c(p^1)/c(p^0)$ is the *utility ratio* $f(q^1)/f(q^0)$. Since the utility function is assumed to be homogeneous of degree one, this is the natural definition for a quantity index.

Problem 1: Assume that the consumer has homothetic preferences. Show that for any reference quantity vector $q >> 0_N$, we have:

$$\text{(i) } P_P(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/p^0\cdot q^1 \le P_K(p^0,p^1,q) \le p^1\cdot q^0/p^0\cdot q^0 \equiv P_L(p^0,p^1,q^0,q^1)$$

where P_K is the true cost of living index defined by (2) above and P_P and P_L are the ordinary Paasche and Laspeyres price indexes. Thus under the assumption of homothetic preferences, all true cost of living indexes lie between P_P and P_L and we can also deduce that $P_P \le P_L$.

4. Wold's Identity and Shephard's Lemma

In subsequent sections, we will need two additional results from economic theory: Wold's Identity and Shephard's Lemma.

Wold's (1944; 69-71) (1953; 145) *Identity* is the following result. Assuming that the consumer satisfies the cost minimization assumptions (1) for periods 0 and 1 and that the utility function f is differentiable at the observed quantity vectors $q^0 >> 0_N$ and $q^1 >> 0_N$ it can be shown[18] that the following equations hold:

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[17] Pollak (1983) derived this result. It seems likely that Frisch (1936) knew of this result but it is not clear.

[18] To prove this, consider the first order necessary conditions for the strictly positive vector q^t to solve the period t cost minimization problem. The conditions of Lagrange with respect to the vector of q variables are: $p^t = \lambda^t\ \nabla f(q^t)$ where λ^t is the optimal Lagrange multiplier and $\nabla f(q^t)$ is the vector of first order partial derivatives of f evaluated at q^t. Note that this system of equations is the price equals a constant times marginal utility equations that are familiar to economists. Now take the inner product of both sides of this

(17) $p_i^t / \sum_{k=1}^N p_k^t q_k^t = [\partial f(q^t)/\partial q_i] / \sum_{k=1}^N q_k^t \, \partial f(q^t)/\partial q_k$; $t = 0,1$; $k = 1,\ldots,N$

where $\partial f(q^t)/\partial q_i$ denotes the partial derivative of the utility function f with respect to the ith quantity q_i evaluated at the period t quantity vector q^t.

If we make the homothetic preferences assumption and assume that the utility function is linearly homogeneous, then Wold's Identity (17) simplifies into the following equations which will prove to be very useful:[19]

(18) $p_i^t / \sum_{k=1}^N p_k^t q_k^t = [\partial f(q^t)/\partial q_i]/f(q^t)$; $t = 0,1$; $k = 1,\ldots,N$.

Using vector notation, (18) can be rewritten as follows:

(19) $p^t/p^t \cdot q^t = \nabla f(q^t)/f(q^t)$; $t = 0,1$.

Shephard's (1953; 11) *Lemma* is the following result. Consider the period t cost minimization problem defined by (1) above. If the cost function $C(u^t,p^t)$ is differentiable with respect to the components of the price vector p, then the period t quantity vector q^t is equal to the vector of first order partial derivatives of the cost function with respect to the components of p; i.e., we have

(20) $q_i^t = \partial C(u^t,p^t)/\partial p_i$; $i = 1,\ldots,N$; $t = 0,1$.

To explain why (20) holds, consider the following argument. Because we are assuming that the observed period t quantity vector q^t solves the cost minimization problem defined by $C(u^t,p^t)$, then q^t must be feasible for this problem so we must have $f(q^t) = u^t$. Thus q^t is a feasible solution for the following cost minimization problem where the general price vector p has replaced the specific period t price vector p^t:

$$(21)\ C(u^t,p) \equiv \min_q \{\textstyle\sum_{i=1}^N p_i q_i : f(q_1,\ldots,q_n) \geq u^t\}$$
$$\leq \textstyle\sum_{i=1}^N p_i q_i^t$$

where the inequality follows from the fact that $q^t \equiv (q_1^t,\ldots,q_N^t)$ is a feasible (but usually not optimal) solution for the cost minimization problem in (21). Now define for each strictly positive price vector p the function g(p) as follows:

$$(22)\ g(p) \equiv \textstyle\sum_{i=1}^N p_i q_i^t - C(u^t,p)$$

where as usual, $p \equiv (p_1,\ldots,p_n)$. Using (1) and (21), it can be seen that g(p) is minimized (over all strictly positive price vectors p) at $p = p^t$. Thus the first order necessary

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equation with respect to the period t quantity vector q^t and solve the resulting equation for λ^t. Substitute this solution back into the vector equation $p^t = \lambda^t \nabla f(q^t)$ and we obtain (17).

[19] Differentiate both sides of the equation $f(\lambda q) = \lambda f(q)$ with respect to λ and then evaluate the resulting equation at $\lambda = 1$. We obtain the equation $\sum_{i=1}^N f_i(q)q_i = f(q)$ where $f_i(q) \equiv \partial f(q)/\partial q_i$.

conditions for minimizing a differentiable function of N variables hold, which simplify to equations (20).

If we make the homothetic preferences assumption and assume that the utility function is linearly homogeneous, then using (13), Shephard's Lemma (20) becomes:

(23) $q_i^t = u^t \, \partial c(p^t)/\partial p_i$; $\quad$ $i = 1,\ldots,n$; $t = 0,1$.

Equations (14) can be rewritten as follows:

(24) $\sum_{i=1}^N p_i^t q_i^t = c(p^t)f(q^t) = c(p^t)u^t$ $\quad$ for $t = 0,1$.

Combining equations (23) and (24), we obtain the following system of equations:

(25) $q_i^t / \sum_{k=1}^N p_k^t q_k^t = [\partial c(p^t)/\partial p_i]/c(p^t)$; $\quad$ $i = 1,\ldots,N$; $t = 0,1$.

Using vector notation, we can rewrite (25) as follows:

(26) $q^t/p^t \cdot q^t = \nabla c(p^t)/c(p^t)$; $\quad$ $t = 0,1$.

Note the symmetry of equations (19) with equations (26). It is these two sets of equations that we shall use in subsequent material.

Problem 2:[20] Suppose that the consumer's cost function C(u,p) is differentiable with respect to the components of the commodity price vector p. Then the consumer's system of *Hicksian* (1946; 331) *demand functions* is defined as:

(i) $d_i(u,p) \equiv \partial C(u,p)/\partial p_i$; $\quad$ $i = 1,\ldots,N$.

These functions trace out the consumer's demand for goods and services as prices p vary but the standard of living u is held fixed. Now suppose that the consumer has homothetic preferences so that:

(ii) $C(u,p) = uC(1,p)$.

Show that under this assumption of homothetic preferences that all N real income elasticities of demand are one; i.e., prove that

(iii) $[\partial d_i(u,p)/\partial u][u/d_i(u,p)] = 1$; $\quad$ $i = 1,\ldots,N$.

5. Superlative Indexes: The Fisher Ideal Index

Suppose the consumer has the following utility function:

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[20] This problem shows why the assumption of homothetic preferences is not likely to be satisfied empirically.

(27) $f(q) \equiv [q^T A q]^{1/2}$

where $A \equiv [a_{ij}]$ is an N by N symmetric matrix that has one positive eigenvalue (that has a strictly positive eigenvector) and the remaining N−1 eigenvalues are zero or negative. Under these conditions, there will be a *region of regularity* where the function f is positive, concave and increasing and hence f can provide a valid representation of preferences over this region.

Differentiating f(q) defined by (27) with respect to the components of q yields the following system of equations for the vector of derivatives of f, $\nabla f(q)$:

(28) $\nabla f(q) = Aq/[q^T A q]^{1/2} = Aq/f(q)$ using (27).

Now evaluate both sides of (28) evaluated at the observed period t quantity vector q^t and divide both sides of the resulting equation by $f(q^t)$. We obtain the following system of equations:

(29) $\nabla f(q^t)/f(q^t) = Aq^t/f(q^t)^2$ $t = 0,1.$

Assume cost minimizing behaviour for the consumer in periods 0 and 1. Since the utility function defined by (27) is linearly homogeneous and differentiable, equations (19) (Wold's Identity) will hold. Now recall the definition of the Fisher ideal quantity index, Q_F defined by the first line of (30) below:

$$
\begin{aligned}
(30)\ Q_F(p^0,p^1,q^0,q^1) &\equiv [p^0\cdot q^1\ p^1\cdot q^1/p^0\cdot q^0\ p^1\cdot q^0]^{1/2} \\
&= [(p^0\cdot q^1/p^0\cdot q^0)/(p^1\cdot q^0/p^1\cdot q^1)]^{1/2} \\
&= [(\nabla f(q^0)\cdot q^1/f(q^0))/(\nabla f(q^1)\cdot q^0/f(q^1))]^{1/2} && \text{using (19) for } t = 0 \text{ and } 1 \\
&= [(q^{0T}Aq^1/f(q^0)^2)/(q^{1T}Aq^0/f(q^1)^2)]^{1/2} && \text{using (29) for } t = 0 \text{ and } 1 \\
&= f(q^1)/f(q^0).
\end{aligned}
$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function defined by (27), the Fisher ideal quantity index Q_F is *exactly* equal to the true quantity index, $f(q^1)/f(q^0)$.[21]

The price index that corresponds to the Fisher quantity index Q_F using the product test is the Fisher price index P_F. Let c(p) be the unit cost function that corresponds to the homogeneous quadratic utility function f defined by (27). Then using (24), $p^t\cdot q^t = c(p^t)f(q^t)$, the product test and (30), it can be seen that

(31) $P_F(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0)$.

[21] This result was first derived by Konüs and Byushgens (1926). For an alternative derivation and the early history of this result, see Diewert (1976a; 116).

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function $f(q) = (q^TAq)^{1/2}$, the Fisher ideal price index P_F is *exactly* equal to the true price index, $c(p^1)/c(p^0)$. The significance of (30) and (31) is that we can calculate the consumer's true rate of utility growth and his or her true rate of price inflation *without having to undertake any econometric estimation*; i.e., the left hand sides of (30) and (31) can be calculated exactly using observable price and quantity data for the consumer for the two periods under consideration. Thus the present economic approach to index number theory using a *ratio approach* leads to practical solutions to the index number problem whereas the earlier *levels approach* explained in section 3 of Chapter 1 did not lead to practical solutions.

A twice continuously differentiable function f(q) of N variables q can provide a *second order approximation* to another such function $f^*(q)$ around the point q^* if the level and all of the first and second order partial derivatives of the two functions coincide at q^*. It can be shown[22] that the homogeneous quadratic function f can provide a second order approximation to an arbitrary f^* around any point q^* in the class of twice continuously differentiable linearly homogeneous functions. Thus the homogeneous quadratic functional form defined by (28) is a *flexible functional form*.[23] Diewert (1976a; 117) termed an index number formula $Q_F(p^0,p^1,q^0,q^1)$ that was *exactly* equal to the true quantity index $f(q^1)/f(q^0)$ (where f is a flexible functional form) *a superlative index number formula*.[24] Equation (30), and the fact that the homogeneous quadratic function f defined by (27) is a flexible functional form, shows that the Fisher ideal quantity index Q_F is a superlative index number formula. Since the Fisher ideal price index P_F also satisfies (31) where c(p) is the dual unit cost function that is generated by the homogeneous quadratic utility function, P_F is also a superlative index number formula.

It is possible to show that the Fisher ideal price index is a superlative index number formula by a different route. Instead of starting with the assumption that the consumer's utility function is the homogeneous quadratic function defined by (27), we can start with the assumption that the consumer's unit cost function is a homogeneous quadratic. Thus we suppose that the consumer has the following unit cost function:

(32) $c(p) \equiv (p^TBp)^{1/2}$

where the N by N matrix $B \equiv [b_{ij}]$ is symmetric so that

(33) $B^T = B$.

Differentiating c(p) defined by (32) with respect to p_i yields the following expression for the vector of first order partial derivatives of c, $\nabla c(p)$:

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22 See Diewert (1976a; 130) and let the parameter r equal 2.

23 Diewert (1974a; 133) introduced this term to the economics literature.

24 As we have seen earlier, Fisher (1922; 247) used the term superlative to describe the Fisher ideal price index. Thus Diewert adopted Fisher's terminology but attempted to give more precision to Fisher's definition of superlativeness.

(34) $\nabla c(p) = Bp/(p^TBp)^{1/2} = Bp/c(p)$ using (32) and (33).

Now evaluate the second equation in (34) at the observed period t price vector $p^t \equiv (p_1^t,\ldots,p_N^t)$ and divide both sides of the resulting equation by $c(p^t)$. We obtain the following equations:

(35) $\nabla c(p^t)/c(p^t) = Bp^t/c(p^t)^2$; $t = 0,1$.

As we are assuming cost minimizing behavior for the consumer in periods 0 and 1 and since the unit cost function c defined by (32) is differentiable, equations (20) (Shephard's Lemma) will hold. Now recall the definition of the Fisher ideal price index, P_F given by the first line of (36) below:

$$
\begin{aligned}
(36)\ P_F(p^0,p^1,q^0,q^1) &\equiv [p^1\cdot q^0\ p^1\cdot q^1/p^0\cdot q^0\ p^0\cdot q^1]^{1/2} \\
&= [(p^1\cdot q^0/p^0\cdot q^0)/(p^0\cdot q^1/p^1\cdot q^1)]^{1/2} \\
&= [(p^1\cdot\nabla c(q^0)/c(p^0))/(p^0\cdot\nabla c(p^1)/c(p^1))]^{1/2} && \text{using (20) for } t = 0,1 \\
&= [(p^1\cdot Bp^0/c(p^0)^2)/(p^0\cdot Bp^1/c(p^1)^2)]^{1/2} && \text{using (35) for } t = 0,1 \\
&= c(p^1)/c(p^0) && \text{using } B = B^T.
\end{aligned}
$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the unit cost function defined by (32), the Fisher ideal price index P_F is *exactly* equal to the true price index, $c(p^1)/c(p^0)$.[25]

Problem 3: Suppose the consumer's utility function is defined as $f(q) \equiv [q^TAq]^{1/2}$ where $A = A^T$ and A^{-1} exists. Let p be a strictly positive vector of commodity prices and use calculus to solve the following constrained minimization problem:

(i) $\min_q \{p^Tq : [q^TAq]^{1/2} = 1\} \equiv c(p)$.

Show that $c(p) = [p^TA^{-1}p]^{1/2}$.

Since the homogeneous quadratic unit cost function c(p) defined by (32) is also a flexible functional form, the fact that the Fisher ideal price index P_F exactly equals the true price index $c(p^1)/c(p^0)$ means that P_F is a *superlative index number formula*.[26]

---

[25] This result was obtained by Diewert (1976a; 133-134).

[26] Note that we have shown that the Fisher index P_F is exact for the preferences defined by (27) as well as the preferences that are dual to the unit cost function defined by (32). These two classes of preferences do not coincide in general. However, if the N by N symmetric matrix A of the a_{ik} has an inverse, then it can readily be shown that the N by N matrix B of the b_{ik} will equal A^{-1}. See Problem 3 above. Note also that if B is equal to the rank one matrix bb^T where b is a column vector of positive constants, then the Fisher price and quantity indexes will be exact for the preferences that are dual to the unit cost function $c(p) \equiv p^Tb$. These dual preferences are Leontief (no substitution) preferences. Thus if preferences happen to be of the no substitutability variety, the Fisher price index will still be equal to the true cost of living index in this case.

It turns out that there are many other superlative index number formulae; i.e., there exist many quantity indexes $Q(p^0,p^1,q^0,q^1)$ that are exactly equal to $f(q^1)/f(q^0)$ and many price indexes $P(p^0,p^1,q^0,q^1)$ that are exactly equal to $c(p^1)/c(p^0)$ where the aggregator function f or the unit cost function c is a flexible functional form. We will define a family of superlative indexes in the following section.

6. Quadratic Mean of Order r Superlative Indexes

Suppose that the consumer has the *following quadratic mean of order r utility function*:[27]

$$(37)\ f^r(q_1,\ldots,q_N) \equiv [\sum_{i=1}^N \sum_{k=1}^N a_{ik}\, q_i^{r/2}\, q_k^{r/2}\,]^{1/r}$$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976a; 130) showed that the utility function f^r defined by (37) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order.[28] Note that when $r = 2$, f^r equals the homogeneous quadratic function defined by (27) above.

Define the *quadratic mean of order r quantity index* Q^r by:

$$(38)\ \ Q^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^N s_i^0\ (q_i^1/q_i^0)^{r/2}\}^{1/r}\ \{\sum_{i=1}^N s_i^1\ (q_i^1/q_i^0)^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^t q_i^t/\sum_{k=1}^N p_k^t q_k^t$ is the period t expenditure share for commodity i. It can be verified that when $r = 2$, Q^r simplifies to Q_F, the Fisher ideal quantity index. It can be shown that Q^r is exact for the aggregator function f^r defined by (37); i.e., we have

$$(39)\ Q^r(p^0,p^1,q^0,q^1) = f^r(q^1)/f^r(q^0).$$

Problem 4: Show that (39) is true. Hint: use the same arguments that were used to establish (30).

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function defined by (32), the quadratic mean of order r quantity index Q_F is *exactly* equal to the true quantity index, $f^r(q^1)/f^r(q^0)$.[29] Since Q^r is exact for f^r and f^r is a flexible functional form, we see that the quadratic mean of order r quantity index Q^r is a *superlative index* for each $r \neq 0$. Thus there are an infinite number of superlative quantity indexes.

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[27] This terminology is due to Diewert (1976a; 129).

[28] This result holds for any predetermined $r \neq 0$; i.e., we require only the $N(N+1)/2$ independent a_{ik} parameters in order to establish the flexibility of f^r in the class of linearly homogeneous aggregator functions.

[29] See Diewert (1976a; 130).

For each quantity index Q^r, we can use the product test that was described in Chapter 1 in order to define the corresponding *implicit quadratic mean of order r price index* P^{r*}:

(40) $P^{r*}(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/[p^0\cdot q^0 Q^r(p^0,p^1,q^0,q^1)] = c^r(p^1)/c^r(p^0)$

where c^r is the unit cost function that is dual to the aggregator function f^r defined by (37) above. For each $r \neq 0$, the implicit quadratic mean of order r price index P^{r*} is also a superlative index.

When $r = 2$, Q^r defined by (39) simplifies to Q_F, the Fisher ideal quantity index and P^{r*} defined by (40) simplifies to P_F, the Fisher ideal price index. When $r = 1$, Q^r defined by (39) simplifies to

(41) $Q^1(p^0,p^1,q^0,q^1) = [p^1\cdot q^1/p^0\cdot q^0]/P_W(p^0,p^1,q^0,q^1)$

where P_W is the *Walsh* (1901; 398) (1921; 97) *price index* defined earlier by (13) in Chapter 1. Thus P^{1*} is equal to the Walsh price index P_W and the Walsh price index is also a superlative price index.

The above results provide reasonably strong justifications for the Fisher and Walsh price indexes from the viewpoint of the economic approach. An even stronger justification[30] can be provided for the Törnqvist Theil index P_T defined by (18) in Chapter 1 as we will show in the following section.

Suppose the consumer has the *following quadratic mean of order r unit cost function*:[31]

(42) $c^r(p_1,\ldots,p_N) \equiv [\sum_{i=1}^N \sum_{k=1}^N b_{ik}\, p_i^{r/2}\, p_k^{r/2}\,]^{1/r}$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976a; 130) showed that the unit cost function c^r defined by (42) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when $r = 2$, c^r equals the homogeneous quadratic unit cost function defined by (32) above.

Define the *quadratic mean of order r price index* P^r by:

(43) $P^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^N s_i^0\, (p_i^1/p_i^0)^{r/2}\}^{1/r}\, \{\sum_{i=1}^N s_i^1\, (p_i^1/p_i^0)^{-r/2}\}^{-1/r}$

where $s_i^t \equiv p_i^t q_i^t/\sum_{k=1}^N p_k^t q_k^t$ is the period t expenditure share for commodity i as usual. It can be verified that when $r = 2$, P^r simplifies into P_F, the Fisher ideal quantity index.

---

[30] The exact index number formula (55) is stronger than the above results because we no longer have to assume homothetic preferences.

[31] This terminology is due to Diewert (1976a; 130). This unit cost function was first defined by Denny (1974).

Using exactly the same techniques as were used in the previous section, it can be shown that P^r is exact for the unit cost function c^r defined by (42); i.e., we have

(44) $P^r(p^0,p^1,q^0,q^1) = c^r(p^1)/c^r(p^0)$.

Problem 5: Show that (44) is true.

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the n commodities that correspond to the unit cost function defined by (42), the quadratic mean of order r price index P_F is *exactly* equal to the true price index, $c^r(p^1)/c^r(p^0)$.[32] Since P^r is exact for c^r and c^r is a flexible functional form, we see that the quadratic mean of order r price index P^r is a *superlative index* for each $r \neq 0$. Thus again, it can be seen that there are an infinite number of superlative price indexes.

For each price index P^r, we can use the product test in order to define the corresponding *implicit quadratic mean of order r quantity index* Q^{r*}:

$$(45)\quad Q^{r*}(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^N p_i^1q_i^1/\{\sum_{i=1}^N p_i^0q_i^0\, P^r(p^0,p^1,q^0,q^1)\} = f^{r*}(q^1)/f^{r*}(q^0)$$

where f^{r*} is the aggregator function that corresponds to the unit cost function c^r defined by (42) above.[33] For each $r \neq 0$, the implicit quadratic mean of order r quantity index Q^{r*} is also a superlative index.

When r = 2, P^r defined by (43) simplifies to P_F, the Fisher ideal price index and Q^{r*} defined by (45) simplifies to Q_F, the Fisher ideal quantity index. When r = 1, P^r defined by (43) simplifies to:

$$\begin{aligned}(46)\quad P^1(p^0,p^1,q^0,q^1) &\equiv \{\sum_{i=1}^N s_i^0\,(p_i^1/p_i^0)^{1/2}\}/\{\sum_{i=1}^N s_i^1\,(p_i^1/p_i^0)^{-1/2}\}\\ &= \{[\sum_{i=1}^N p_i^0q_i^0/\sum_{i=1}^N p_i^0q_i^0](p_i^1/p_i^0)^{1/2}\}/\{[\sum_{i=1}^N p_i^1q_i^1/\sum_{i=1}^N p_i^1q_i^1](p_i^1/p_i^0)^{-1/2}\}\\ &= \{\sum_{i=1}^N q_i^0(p_i^0p_i^1)^{1/2}/\sum_{i=1}^N p_i^0q_i^0\}/\{\sum_{i=1}^N q_i^1(p_i^0p_i^1)^{1/2}/\sum_{i=1}^N p_i^1q_i^1\}\\ &= [\sum_{i=1}^N p_i^1q_i^1/\sum_{i=1}^N p_i^0q_i^0]/Q_W(p^0,p^1,q^0,q^1)\end{aligned}$$

where Q_W is the *Walsh quantity index*. Thus Q^{1*} is equal to Q_W, the *Walsh quantity index*, and hence it is also a superlative quantity index.

7. Superlative Indexes: The Törnqvist Theil Index

Before we derive our main result, we require a preliminary result. Suppose the function of N variables, $f(z_1,\ldots,z_N) \equiv f(z)$, is quadratic; i.e.,

(47) $f(z) \equiv a_0 + a^Tz + (1/2)\, z^TAz$; $A = A^T$

[32] See Diewert (1976a; 133-134).

[33] The function f^{r*} can be defined by using c^r as follows: $f^{r*}(q) \equiv 1/\max_p \{\sum_{i=1}^N p_iq_i : c^r(p) = 1\}$.

where a is a vector of parameters and A is a symmetric matrix of parameters. It is well known that the second order Taylor series approximation to a quadratic function is *exact*; i.e., if f is defined by (47) above, then for any two points, z^0 and z^1, we have

(48) $f(z^1) - f(z^0) = \nabla f(z^0)^T(z^1 - z^0) + (1/2)(z^1 - z^0)^T\nabla^2 f(z^0)(z^1 - z^0)$.

It is less well known that *an average of two first order Taylor series approximations* to a quadratic function is also *exact*; i.e., if f is defined by (47) above, then for any two points, z^0 and z^1, we have

(49) $f(z^1) - f(z^0) = (1/2)[\nabla f(z^0) + \nabla f(z^1)]^T[z^1 - z^0]$.

Diewert (1976a; 118) and Lau (1979) showed that equation (49) characterized a quadratic function and called the equation the *quadratic approximation lemma*. We will refer to (49) as the *quadratic identity*.

Problem 6: Show that (49) holds. Hint: Substitute (47) and its derivatives into (49).

We now suppose that the consumer's cost function, C(u,p), has the following *translog functional form*:[34]

(50) $\ln C(u,p) \equiv a_0 + \sum_{i=1}^N a_i \ln p_i + (1/2) \sum_{i=1}^N \sum_{k=1}^N a_{ik} \ln p_i \ln p_k$
$\quad + b_0 \ln u + \sum_{i=1}^N b_i \ln p_i \ln u + (1/2)\, b_{00}\, [\ln u]^2$

where ln is the natural logarithm function and the parameters a_i, a_{ik}, and b_i satisfy the following restrictions:

(51) $a_{ik} = a_{ki}$; $\quad$ i,k = 1,…,N;
(52) $\sum_{i=1}^N a_i = 1$;
(53) $\sum_{i=1}^N b_i = 0$;
(54) $\sum_{k=1}^N a_{ik} = 0$; $\quad$ i = 1,…,N.

The parameter restrictions (51)-(54) ensure that C(u,p) defined by (50) is linearly homogeneous in p. It can be shown that the translog cost function defined by (50)-(54) can provide a second order Taylor series approximation to an arbitrary cost function.[35]

We assume that the consumer engages in cost minimizing behavior during periods 0 and 1 so that equations (1) hold. Applying Shephard's Lemma to the translog cost function leads to the following equations:

[34] Christensen, Jorgenson and Lau (1975) introduced this function into the economics literature.
[35] It can also be shown that if $b_0 = 1$ and all of the $b_i = 0$ for i = 1,...,N and $b_{00} = 0$, then $C(u,p) = uC(1,p) \equiv uc(p)$; i.e., with these additional restrictions on the parameters of the general translog cost function, we have homothetic preferences. Note that we also assume that utility u is scaled so that u is always positive. Finally, we assume that for each of our translog results, the regularity region contains the observed price and quantity data.

(55) $s_i^t = a_i + \sum_{k=1}^N a_{ik} \ln p_k^t + b_i \ln u^t$; $\qquad i = 1,\ldots,N$; $t = 0,1$

where as usual, s_i^t is the period t expenditure share on commodity i. Define the geometric average of the period 0 and 1 utility levels as u^*; i.e., define

(56) $u^* \equiv [u^0u^1]^{1/2}$.

Now observe that the right hand side of the equation that defines the natural logarithm of the translog cost function, equation (50), is a quadratic function of the variables $z_i \equiv \ln p_i$ if we hold utility constant at the level u^*. Hence we can apply the quadratic identity, (49), and get the following equation:

(57) $\ln C(u^*,p^1) - \ln C(u^*,p^0)$
$= (1/2)\sum_{i=1}^N [\partial \ln C(u^*,p^0)/\partial \ln p_i + \partial \ln C(u^*,p^1)/\partial \ln p_i][\ln p_i^1 - \ln p_i^0]$
$= (1/2)\sum_{i=1}^N [a_i+\sum_{k=1}^N a_{ik} \ln p_k^0+b_i \ln u^* + a_i+\sum_{k=1}^N a_{ik} \ln p_k^1+b_i \ln u^*][\ln p_i^1 - \ln p_i^0]$
differentiating (48) at the points (u^*,p^0) and (u^*,p^1)
$= (1/2)\sum_{i=1}^N [a_i+\sum_{k=1}^N a_{ik} \ln p_k^0+b_i \ln[u^0u^1]^{1/2}+a_i+\sum_{k=1}^N a_{ik} \ln p_k^1+b_i \ln[u^0u^1]^{1/2}][\ln p_i^1 - \ln p_i^0]$
using definition (56) for u^*
$= (1/2)\sum_{i=1}^N [a_i+\sum_{k=1}^N a_{ik} \ln p_k^0+b_i \ln u^0 + a_i+\sum_{k=1}^N a_{ik} \ln p_k^1+b_i \ln u^1][\ln p_i^1 - \ln p_i^0]$
$= (1/2)\sum_{i=1}^N [\partial \ln C(u^0,p^0)/\partial \ln p_i + \partial \ln C(u^1,p^1)/\partial \ln p_i][\ln p_i^1 - \ln p_i^0]$
differentiating (48) at the points (u^0,p^0) and (u^1,p^1)
$= (1/2)\sum_{i=1}^N [s_i^0 + s_i^1][\ln p_i^1 - \ln p_i^0]$ using (55)
$\equiv \ln P_T(p^0,p^1,q^0,q^1)$.

The last equation in (57) defines the logarithm of an observable index number formula, $P_T(p^0,p^1,q^0,q^1)$, which is the Törnqvist (1936), Törnqvist and Törnqvist (1937) Theil (1967) price index that was introduced in Chapter 1.[36] Hence exponentiating both sides of (57) yields the following equality between the true cost of living between periods 0 and 1, evaluated at the intermediate utility level u^* and the observable price index P_T:[37]

(58) $C(u^*,p^1)/C(u^*,p^0) = P_T(p^0,p^1,q^0,q^1)$.

Since the translog cost function is a flexible functional form, the Törnqvist-Theil price index P_T is also a *superlative index*.[38] The importance of (58) as compared to our earlier exact index number results is that we no longer have to assume that preferences are homothetic. However, we do have to choose a particular utility level on the left hand side of (58) in order to obtain our new exact result, the geometric mean of u^0 and u^1.

It is somewhat mysterious how a ratio of *unobservable* cost functions of the form appearing on the left hand side of the above equation can be *exactly* estimated by an

[36] See Balk (2008; 26) on the history of this index.
[37] This result is due to Diewert (1976a; 122).
[38] Diewert (1978; 888) showed that $P_T(p^0,p^1,q^0,q^1)$ approximates the other superlative indexes P^r and P^{r*} to the second order around an equal price and quantity point; see Problems 7 and 8 below.

observable index number formula but the key to this mystery is the assumption of cost minimizing behavior and the quadratic identity (49) along with the fact that derivatives of cost functions are equal to quantities, as follows using Shephard's Lemma. In fact, all of the exact index number results derived in this section and the previous section can be derived using transformations of the quadratic identity along with Shephard's Lemma (or Wold's identity).[39]

For most empirical applications, assuming that the consumer has (transformed) quadratic preferences will be an adequate assumption so the results presented in sections 5 and 6 are quite useful to index number practitioners who are willing to adopt the economic approach to index number theory.[40] Essentially, the economic approach to index number theory provides a strong justification for the use of the Fisher price index P_F, the Törnqvist-Theil price index P_T, the implicit quadratic mean of order r price indexes P^{r*} defined by (40) (when r = 1, this index is the Walsh price index P_W) and the quadratic mean of order r price indexes P^r defined by (43).

Since the Fisher, Walsh, Törnqvist-Theil indexes and the family of indexes P^r and P^{r*} are all exact indexes for flexible functional form preferences, which index should be chosen in practice? Diewert (1978) showed that all of these indexes numerically approximate each other to the second order around a point (p^0,p^1,q^0,q^1) where the two price vectors are equal and the two quantity vectors are equal so that we have $p^0 = p^1 = p$ and $q^0 = q^1 = q$; i.e., it can be shown that for all $r \neq 0$, the following equations hold at an equal price and quantity point:[41]

(59) $P_F(p,p,q,q) = P_T(p,p,q,q) = P_W(p,p,q,q) = P^r(p,p,q,q) = P^{r*}(p,p,q,q) \quad = 1;$
(60) $\nabla P_F(p,p,q,q) = \nabla P_T(p,p,q,q) = \nabla P_W(p,p,q,q) = \nabla P^r(p,p,q,q) = \nabla P^{r*}(p,p,q,q);$
(61) $\nabla^2 P_F(p,p,q,q) = \nabla^2 P_T(p,p,q,q) = \nabla^2 P_W(p,p,q,q) = \nabla^2 P^r(p,p,q,q) = \nabla^2 P^{r*}(p,p,q,q).$

Note that there are 4N first order partial derivatives in each of the five vectors in (60) and there are $(4N)^2$ second order partial derivatives in each of the five matrices in (61) where each derivative is evaluated at an equal price and quantity point where $p^0 = p^1 = p$ and $q^0 = q^1 = q$. Thus if p^0 is reasonably close to p^1 and q^0 is reasonably close to q^1, then we would expect all of the indexes, P_F, P_T, P_W, P^r and P^{r*}, to be quite close. However, Hill (2006) showed that the close approximation result breaks down for the indexes P^r and P^{r*} if r becomes very large in magnitude. But when working with annual time series data, experience has shown that P_F, P_T and P_W numerically approximate each other very closely. Thus from the viewpoint of the economic approach to index number theory, there

[39] See Diewert (2002).
[40] However, if consumer preferences are nonhomothetic and the change in utility is substantial between the two situations being compared, then we may want to compute separately the Laspeyres-Konüs and Paasche-Konüs true cost of living indexes defined above by (3) and (4), $C(u^0,p^1)/C(u^0,p^0)$ and $C(u^1,p^1)/C(u^1,p^0)$ respectively. In order to do this, we would have to use econometrics and estimate empirically the consumer's cost or expenditure function. However, if we are willing to make the assumption that the consumer's cost function can be adequately represented by a general translog cost function, then we can use (58) to calculate the true index $C(u^*,p^1)/C(u^*,p^0)$ where $u^* \equiv [u^0u^1]^{1/2}$.
[41] The equations (59)-(61) also hold if $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ where λ and μ are arbitrary positive scalars.

is no basis for preferring any one of these three indexes but fortunately, usually, each of these three indexes will give much the same value numerically.[42]

Problem 7. Consider the Laspeyres, Paasche, Fisher, Törnqvist and Walsh price indexes, P_L, P_P, P_F, P_T and P_W as functions of the four sets of variables, p^0,p^1,q^0,q^1. Show that all of the 4N first order partial derivatives of each of these 5 indexes are equal when evaluated at a point where the two price vectors are equal (so that $p^0 = p^1 \equiv p$) and where the two quantity vectors are equal (so that $q^0 = q^1 \equiv q$); i.e., show that

(i) $\nabla P_L(p,p,q,q) = \nabla P_P(p,p,q,q) = \nabla P_F(p,p,q,q) = \nabla P_T(p,p,q,q) = \nabla P_W(p,p,q,q)$.

Comment: It is easy to show that

(ii) $P_L(p,p,q,q) = P_P(p,p,q,q) = P_F(p,p,q,q) = P_T(p,p,q,q) = P_W(p,p,q,q) = 1$.

Equations (i) and (ii) show that the Laspeyres, Paasche, Fisher, Törnqvist and Walsh indexes all approximate each other to the first order around an equal price and quantity point.

Problem 8:

(a) Show that $\nabla^2 P_L(p,p,q,q) \neq \nabla^2 P_P(p,p,q,q)$ and hence the Laspeyres and Paasche indexes do *not* approximate each other to the second order around an equal price and quantity point; i.e., their 4N by 4N matrices of second order partial derivatives are not all equal when evaluated at an equal price and quantity point.
(b) Show that $\nabla^2 P_L(p,p,q,q) \neq \nabla^2 P_F(p,p,q,q)$ and hence the Laspeyres and Fisher indexes do *not* approximate each other to the second order around an equal price and quantity point.
(c) Show that $\nabla^2 P_P(p,p,q,q) \neq \nabla^2 P_F(p,p,q,q)$ and hence the Paasche and Fisher indexes do *not* approximate each other to the second order around an equal price and quantity point.
(d) Show that $\nabla^2 P_F(p,p,q,q) = \nabla^2 P_W(p,p,q,q)$ and hence the Fisher and Walsh indexes *do* approximate each other to the second order around an equal price and quantity point.
(e) Show that $\nabla^2 P_F(p,p,q,q) = \nabla^2 P_T(p,p,q,q)$ and hence the Fisher and Törnqvist indexes *do* approximate each other to the second order around an equal price and quantity point.
Hint: All of the above results can be established by straight forward differentiation. However, it will take a considerable amount of time to establish these results.

In the following section, we will establish a generalization of the results in this section that will enable us to deal with the problems associated with computing cost of living indexes when tastes change.

8. Nonhomothetic Preferences, Cost of Living Indexes and Taste Change

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[42] However, note that the Törnqvist Theil index can be justified in the context of nonhomothetic preferences which is a point in favour of this index. On the other hand, if the underlying data satisfy revealed preference theory, it can be shown that the Fisher quantity index will correctly indicate an increase or decrease in utility under these conditions; see Diewert (1976b).

It is possible to generalize the results in the previous section using some results in Caves, Christensen and Diewert (1982; 1409-1411). We will now those results.

Assume that in period t, the consumer has the utility function $f^t(q,z^t)$ for t = 0,1, where z^t is a period t vector of *environmental or demographic variables* that affect the consumer's choices over market goods and services, q. Note that we write the period t utility function as f^t so we are also allowing for taste changes as we move from period 0 to 1. We assume that $f^t(q,z^t)$ is nonnegative, increasing, continuous and quasiconcave in q for $q \geq 0_N$.

For $p >> 0_N$, and u in the range of $f^t(q,z^t)$, we define the consumer's period t *conditional cost function* C^t as follows:

$$(62)\ C^t(u,p^t,z^t) \equiv \min_q \{p^t \cdot q : f^t(q,z^t) = u\}\ ; \qquad t = 0,1.$$

Let q^t be the consumer's observed market consumption vector for period t and define the period t utility level as:

$$(63)\ u^t \equiv f^t(q^t,z^t)\ ; \qquad t = 0,1.$$

Suppose the consumer faces the market price vector p^t in period t for t = 0,1. As usual, we assume that the observed period t consumption vector q^t solves the following *period t cost minimization problem*:

$$(64)\ C^t(u^t,p^t,z^t) \equiv \min_q \{p^t \cdot q : f^t(q,z^t) = u^t\} = p^t \cdot q^t\ ; \qquad t = 0,1.$$

Define a *family of generalized Konüs true cost of living indexes* between periods 0 and 1 as follows:

$$(65)\ P_{CCD}(p^0,p^1,u,z,t) \equiv C^t(u,p^1,z)/C^t(u,p^0,z).$$

Note that all variables are exactly the same in the numerator and denominator on the right hand side of (65), except that the period 1 price vector p^1 appears in the numerator and the period 0 price vector p^0 appears in the denominator. Thus the resulting index is a valid measure of pure price change.

Caves, Christensen and Diewert (1982; 1409-1410) singled out the two natural special cases of (65), where the common variables in the numerator and denominator on the right hand side of (65) are chosen to be the period 0 variables or the period 1 variables:

$$(66)\ P_{CCD}(p^0,p^1,u^0,z^0,0) \equiv C^0(u^0,p^1,z^0)/C^0(u^0,p^0,z^0)\ ;$$
$$(67)\ P_{CCD}(p^0,p^1,u^1,z^1,1) \equiv C^1(u^1,p^1,z^1)/C^1(u^1,p^0,z^1).$$

It turns out that we will not be able to provide empirical approximations to the individual price indexes defined by (66) and (67) but we will be able to provide an exact index number formula for the geometric mean of these two indexes. In order to accomplish this task, we will require the following generalization of the quadratic identity, (49):

Proposition 2: Let x and y be N and M dimensional vectors respectively and let f^1 and f^2 be two general quadratic functions defined as follows:

(68) $f^1(x,y) \equiv a_0^{\,1} + a^{1T}x + b^{1T}y + (1/2)x^TA^1x + (1/2)\, y^TB^1y + x^TC^1y$; $A^{1T} = A^1$; $B^{1T} = B^1$;
(69) $f^2(x,y) \equiv a_0^{\,2} + a^{2T}x + b^{2T}y + (1/2)x^TA^2x + (1/2)\, y^TB^2y + x^TC^2y$; $A^{2T} = A^2$; $B^{2T} = B^2$

where the $a_0^{\,i}$ are scalar parameters, the a^i and b^i are parameter vectors and the A^i, B^i and C^i are parameter matrices for i = 1,2. Note that the A^i and B^i are symmetric matrices. If $A^1 = A^2$, then the following equation holds for all x^1, x^2, y^1 and y^2:[43]

(70) $f^1(x^2,y^1) - f^1(x^1,y^1) + f^2(x^2,y^2) - f^2(x^1,y^2) = [\nabla_xf^1(x^1,y^1) + \nabla_xf^2(x^2,y^2)]^T[x^2 - x^1]$.

Problem 9: Prove (70). *Hint*: differentiation and substitution establishes (70).

We now suppose that the consumer's period t cost function, $C^t(u,p,z)$, has the following functional form:[44]

(71) $\ln C^t(u,p,z) \equiv a_0^{\,t} + \sum_{n=1}^{N} a_n^{\,t} \ln p_n + b_0^{\,t} \ln u + \sum_{m=1}^{M} b_{0m}^{\,t} z_m \ln u + \sum_{n=1}^{N} b_n^{\,t} \ln p_n \ln u$
$\qquad + (1/2)\, b_{00}^{\,t} [\ln u]^2 + (1/2) \sum_{i=1}^{N} \sum_{n=1}^{N} a_{in}^{\,t} \ln p_i \ln p_n$
$\qquad + (1/2) \sum_{i=1}^{M} \sum_{m=1}^{M} b_{im}^{\,t} z_i z_m + \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm}^{\,t} z_m \ln p_n$

where the parameters satisfy the following restrictions, which impose linear homogeneity in prices p on $C^t(u,p,z)$:

(72) $a_{in}^{\,t} = a_{ni}^{\,t}$; i,n = 1,…,N;
(73) $b_{im}^{\,t} = b_{mi}^{\,t}$; i,m = 1,...,M;
(74) $\sum_{n=1}^{N} a_n^{\,t} = 1$;
(75) $\sum_{n=1}^{N} b_n^{\,t} = 0$;
(76) $\sum_{i=1}^{N} a_{in}^{\,t} = 0$; n = 1,…,N;
(77) $\sum_{n=1}^{N} c_{nm}^{\,t} = 0$; m = 1,...,M.

It can be shown that the $C^t(u,p,z)$ defined by (71) can provide a second order approximation in the variables u,p and z to an arbitrary twice continuously differentiable cost function, C(u,p,z), and hence C^t is a flexible functional form.

If the consumer in period t has preferences that are dual to the C^t defined by (71)-(77), then Shephard's Lemma implies that the period t market expenditure shares, $s_n^{\,t}$, will satisfy the following equations:

[43] Balk (1998; 225-226) established this result using Diewert's (1976) original quadratic identity. The Translog Lemma in Caves, Christensen and Diewert (1982; 1412) is simply a logarithmic version of (70).
[44] Caves, Christensen and Diewert (1982; 1397) assumed that C^t was a general translog functional form whereas we are assuming a "mixed" translog functional form, which allows the components of the z vector to be 0 if this is required.

$$(78)\ s_n^{\,t} = \partial \ln C^t(u^t,p^t,z^t)/\partial \ln p_n = a_n^{\,t} + b_n^{\,t} \ln u^t + \sum_{i=1}^{N} a_{ni}^{\,t} \ln p_i + \sum_{m=1}^{M} c_{nm}^{\,t} z_m\ ; \qquad n = 1,\ldots,N\ ;\ t = 0,1.$$

With the above preliminaries, we can now prove the following Proposition:

Proposition 3: Suppose the consumer has preferences in period t that are dual to the cost function C^t defined by (71)-(77) for t = 0,1 and the consumer engages in cost minimizing behavior in each period so that equations (64) and (78) are satisfied. Finally, suppose that the quadratic coefficients on prices are the same for the two periods under consideration so that:

$$(79)\ a_{in}^{\ 0} = a_{in}^{\ 1}\ ; \qquad i,n = 1,\ldots,N.$$

Then the geometric mean of the two CCD true cost of living indexes defined by (66) and (67) is exactly equal to the observable Törnqvist Theil price index $P_T(p^0,p^1,q^0,q^1)$ defined in (57) above; i.e., we have:

$$(80)\ [P_{CCD}(p^0,p^1,u^0,z^0,0)\ P_{CCD}(p^0,p^1,u^1,z^1,1)]^{1/2} = P_T(p^0,p^1,q^0,q^1).$$

Proof: Take twice the logarithm of the left hand side of (80). Using definitions (66) and (67) and using the quadratic nature of $\ln C^t$ in lnp and z (see (71)), we obtain the following equation:

$$\begin{aligned}(81)\ & \ln C^0(u^0,p^1,z^0) - \ln C^0(u^0,p^0,z^0) + \ln C^1(u^1,p^1,z^1) - C^1(u^1,p^0,z^1) \\ & = \sum_{n=1}^{N} [\partial \ln C^0(u^0,p^0,z^0)/\partial \ln p_n + \partial \ln C^1(u^1,p^1,z^1)/\partial \ln p_n][\ln p_n^{\,1} - \ln p_n^{\,0}] && \text{using assumption (79) and Proposition 2} \\ & = \sum_{n=1}^{N} [s_n^{\,0} + s_n^{\,1}][\ln p_n^{\,1} - \ln p_n^{\,0}] && \text{using (78)} \\ & = 2 \ln P_T(p^0,p^1,q^0,q^1) && \text{using the definition of } P_T \text{ in (57).}\end{aligned}$$

Equation (81) is equivalent to (80). Q.E.D.

The above result is essentially equivalent to Theorem 5 in Caves, Christensen and Diewert (1982; 1410).[45] The result in Proposition 3 provides a reasonably powerful justification for the use of the Törnqvist Theil price index as a measure of a consumer's change in his or her cost of living index even if preferences are nonhomothetic.[46]

Up to this point, we have not studied quantity indexes for the case of nonhomothetic preferences. In the case of a linearly homogeneous aggregator function, f(q) say, we have noted that the companion quantity index to the Konüs price index $c(p^1)/c(p^0)$ (the unit

[45] CCD assumed that their translog cost functions were quadratic in the logs of prices and the logs of the demographic variables. Balk (1989) also obtained a special case of Proposition 3 where there were no demographic variables but there was taste change. Balk's case is also a special case of Theorem 5 in CCD.

[46] Note that we have provided two separate interpretations for Törnqvist Theil price index in the context of nonhomothetic preferences.

cost ratio) was the ratio of the quantity aggregates $f(q^1)/f(q^0)$. In the following section, we will show how to find quantity indexes when preferences are nonhomothetic.[47]

9. Allen Quantity Indexes

Suppose that we make the same assumptions on preferences that we made at the beginning of section 2. Let $C[f(q),p]$ be the consumer's cost function that is dual to the aggregator function $f(q)$. We again assume cost minimizing behavior in periods 0 and 1 so that equations (1) are satisfied.

The *Allen* (1949) *family of true quantity indexes*, $Q_A(q^0,q^1,p)$, is defined for an arbitrary positive reference price vector p as follows:

(82) $Q_A(q^0,q^1,p) \equiv C[f(q^1),p]/C[f(q^0),p]$.

The basic idea of the Allen quantity index dates back to Hicks (1941-42) who observed that if the price vector p were held fixed and the quantity vector q is free to vary, then $C[f(q),p]$ is a perfectly valid cardinal measure of utility.[48]

As was the case with the true cost of living, the Allen definition simplifies considerably if the utility function happens to be linearly homogeneous. In this case, (82) simplifies to:[49]

(83) $Q_A(q^0,q^1,p) = f(q^1)C(1,p)/f(q^0)C(1,p) = f(q^1)/f(q^0)$.

However, in the general case where the consumer has nonhomothetic preferences, we do not obtain the nice simplification given by (83).

It is useful to specialize the general definition of the Allen quantity index and let the reference price vector equal either the period 0 price vector p^0 or the period 1 price vector p^1:

(84) $Q_A(q^0,q^1,p^0) \equiv C[f(q^1),p^0]/C[f(q^0),p^0]$;
(85) $Q_A(q^0,q^1,p^1) \equiv C[f(q^1),p^1]/C[f(q^0),p^1]$.

Index number formula that are exact for either of the theoretical indexes defined by (84) and (85) do not seem to exist, at least for the case of nonhomothetic preferences that can be represented by a flexible functional form. However, we can find an index number formula that is exactly equal to the geometric mean of the Allen indexes defined by (84) and (85) where the underlying preferences are represented by a flexible functional form. Thus assume that the consumer's preferences can be represented by the translog cost function, $C(u,p)$ defined by (50), with the restrictions (51)-(54). This functional form is a special case of the functional form which appears in Proposition 3, with the demographic variables omitted and with no taste changes between periods 0 and 1. Hence we can apply

[47] The material in the following section is taken from Diewert (2009).
[48] Samuelson (1974) called this a money metric measure of utility.
[49] See Diewert (1981) for references to the literature.

Proposition 3 in the present context, and conclude that the following simplified version of equation (80) is satisfied for our plain vanilla translog consumer (but with general nonhomothetic preferences):

$$(86)\ [\{C[f(q^0),p^1]/C[f(q^0),p^0]\}\{C[f(q^1),p^1]/C[f(q^1),p^0]\}]^{1/2} = P_T(p^0,p^1,q^0,q^1).$$

The implicit quantity index, $Q_{T*}(p^0,p^1,q^0,q^1)$ that corresponds to the Törnqvist Theil price index $P_T(p^0,p^1,q^0,q^1)$ is defined as the value ratio, $p^1\cdot q^1/p^0\cdot q^0$, divided by P_T. Thus we have:

$$\begin{aligned}
(87)\ Q_{T*}(p^0,p^1,q^0,q^1) &\equiv [p^1\cdot q^1/p^0\cdot q^0]/P_T(p^0,p^1,q^0,q^1) \\
&= [C[f(q^1),p^1]/C[f(q^0),p^0]]/P_T(p^0,p^1,q^0,q^1) && \text{using (1)} \\
&= [C[f(q^1),p^1]/C[f(q^0),p^0]]/[\{C[f(q^0),p^1]/C[f(q^0),p^0]\}\{C[f(q^1),p^1]/C[f(q^1),p^0]\}]^{1/2} \\
& && \text{using (86)} \\
&= [\{C[f(q^1),p^0]/C[f(q^0),p^0]\}\{C[f(q^1),p^1]/C[f(q^0),p^1]\}]^{1/2} \\
&= [Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2}
\end{aligned}$$

where the last equality follows using definitions (84) and (85). Thus the observable implicit Törnqvist Theil quantity index, $Q_{T*}(p^0,p^1,q^0,q^1)$, is exactly equal to the geometric mean of the two Allen quantity indexes defined by (84) and (85). This is a very useful result.

Note that in general, the geometric mean of the two "natural" Allen quantity indexes defined by (84) and (85) matches up with the geometric mean of the two "natural" Konüs price indexes defined by (3) and (4); i.e., using these definitions, we have:

$$\begin{aligned}
(88)\ [P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}\,[Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2} &= C[f(q^1),p^1]/C[f(q^0),p^0] \\
&= p^1\cdot q^1/p^0\cdot q^0.
\end{aligned}$$

Thus in general, these two "natural" geometric mean price and quantity indexes satisfy the product test. Under our translog assumptions, we have a special case of (88) where $Q_{T*}(p^0,p^1,q^0,q^1)$ is equal to $[Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2}$ and $P_T(p^0,p^1,q^0,q^1)$ is equal to $[P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}$.

There is an alternative concept for a theoretical quantity index in the case of nonhomothetic preferences that appears frequently in the literature and that is the *Malmquist* (1953) *quantity index*. Results that are similar to the results that we have already derived can be obtained for this concept but we will leave these results to the interested reader.[50]

The bottom line is that it can be seen that it is not necessary to use econometric methods in order to form estimates for price and quantity aggregates; instead, exact index numbers can be used. In particular, empirical index number formula can be used to closely

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[50] See Pollak (1983), Diewert (1981) and Caves, Christensen and Diewert (1982) for additional material on this index concept. Diewert (1976a; 123-124) derived an nonhomothetic translog result for this index number concept that is an exact analogue to the result in equation (58) for a nonhomothetic cost function.

approximate a consumer's cost of living index or his or her welfare change, even in the case of nonhomothetic preferences.

In the following section, we derive some results for the economic approach to cost of living indexes when there are many consumers or households.

10. Plutocratic Cost of Living Indexes and Observable Bounds

In this section, we will consider an economic approach to the CPI that is based on the *plutocratic cost of living index* that was originally defined by Prais (1959). This concept was further refined by Pollak (1980; 276) (1981; 328) who defined his *Scitovsky-Laspeyres cost of living index* as the ratio of total expenditure required to enable each household in the economy under consideration to attain its base period indifference surface at period 1 prices to that required at period 0 prices. In the following paragraph, we will make various assumptions and explain this concept more fully.

Suppose that there are H households (or regions) in the economy and suppose further that there are N commodities in the economy in periods 0 and 1 that households consume *and* that we wish to include in our definition of the cost of living. Denote an N dimensional vector of commodity consumption in a given period by $q \equiv (q_1,q_2,\ldots, q_N)$ as usual. Denote the vector of period t market prices faced by household h by $p_h^t \equiv (p_{h1}^t,p_{h2}^t,\ldots,p_{hN}^t)$ for t = 0,1. Note that we are *not* assuming that each household faces the same vector of commodity prices. In addition to the market commodities that are in the vector q, we assume that each household is affected by an M dimensional vector of *environmental*[51] *or demographic*[52] *variables or public goods*, $e \equiv (e_1,e_2,\ldots,e_M)$. We suppose that there are H households (or regions) in the economy during periods 0 and 1 and the preferences of household h over different combinations of market commodities q and environmental variables e can be represented by the continuous utility function $f^h(q,e)$ for $h = 1,2,\ldots,H$.[53] For periods t = 0,1 and for households $h = 1,2,\ldots,H$, it is assumed that the observed household h consumption vector $q_h^t \equiv (q_{h1}^t,\ldots,q_{hN}^t)$ is a solution to the following household h expenditure minimization problem:

$$(89)\ \min_q \{p_h^t\cdot q : f^h(q,e_h^t) \geq u_h^t\} \equiv C^h(u_h^t,e_h^t,p_h^t) = p_h^t\cdot q_h^t\,; \qquad t = 0,1;\ h = 1,2,\ldots H$$

where e_h^t is the environmental vector facing household h in period t, $u_h^t \equiv f^h(q_h^t,e_h^t)$ is the utility level achieved by household h during period t and C^h is the cost or expenditure function that is dual to the utility function f^h. Basically, these assumptions mean that each household has *stable preferences* over the same list of commodities during the two periods under consideration, the same households appear in each period and each household chooses its consumption bundle in the most cost efficient way during each

[51] This is the terminology used by Pollak (1989; 181) in his model of the conditional cost of living concept.
[52] Caves, Christensen and Diewert (1982; 1409) used the terms *demographic variables* or *public goods* to describe the vector of conditioning variables e in their generalized model of the Konüs price index or cost of living index.
[53] We assume that each $f^h(q,e)$ is continuous and increasing in the components of q and e and is concave in the components of q.

period, conditional on the environmental vector that it faces during each period. Note again that the household (or regional) prices are in general different across households (or regions).

With the above assumptions in mind, we follow Diewert (2001) and generalize Pollak (1980) (1981) and Diewert (1983a; 190)[54] by defining the class of *conditional plutocratic cost of living indexes*, $P^*(p^0,p^1,u,e_1,e_2,\ldots,e_H)$, pertaining to periods 0 and 1 for the arbitrary utility vector of household utilities $u \equiv (u_1,u_2,\ldots,u_H)$ and for the arbitrary vectors of household environmental variables e_h for $h = 1,2,\ldots,H$ as follows:

$$(90)\quad P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u,e_1,e_2,\ldots,e_H) \equiv \sum_{h=1}^H C^h(u_h,e_h,p_h^1)/\sum_{h=1}^H C^h(u_h,e_h,p_h^0)\ .$$

The numerator on the right hand side of (90) is the sum over households of the minimum cost, $C^h(u_h,e_h,p_h^1)$, for household h to achieve the arbitrary utility level u_h, given that the household h faces the arbitrary vector of household h environmental variables e_h and also faces the period 1 vector of prices p_h^1. The denominator on the right hand side of (90) is the sum over households of the minimum cost, $C^h(u_h,e_h,p_h^0)$, for household h to achieve the *same* arbitrary utility level u_h, given that the household faces the *same* arbitrary vector of household h environmental variables e_h and also faces the period 0 vector of prices p_h^0. Thus in the numerator and denominator of (90), only the price variables are different, which is precisely what we want in a theoretical definition of a consumer price index.

We now specialize the general definition (90) by replacing the general utility vector u by either the period 0 vector of household utilities $u^0 \equiv (u_1^0,u_2^0,\ldots u_H^0)$ or the period 1 vector of household utilities $u^1 \equiv (u_1^1,u_2^1,\ldots u_H^1)$. We also specialize the general definition (90) by replacing the general household environmental vectors $(e_1,e_2,\ldots e_H) \equiv e$ by either the period 0 vector of household environmental variables $e^0 \equiv (e_1^0,e_2^0,\ldots e_H^0)$ or the period 1 vector of household environmental variables $e^1 \equiv (e_1^1,e_2^1,\ldots,e_H^1)$. The choice of the base period vector of utility levels and base period environmental variables leads to the *Laspeyres conditional plutocratic cost of living index*, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e^0)$[55], while the choice of the period 1 vector of utility levels and period 1 environmental variables leads to the *Paasche conditional plutocratic cost of living index*, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e^1)$. It turns out that these last two indexes satisfy some interesting inequalities, which we derive below.

Using definition (90), the *Laspeyres plutocratic conditional cost of living index*, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e^0)$, may be written as follows:

---

[54] These authors provided generalizations of the plutocratic cost of living index due to Prais (1959). Pollak (1980) (1981) and Diewert (1983a) did not include the environmental variables in their definitions of a group cost of living index.

[55] This is the concept of a cost of living index that Triplett (2000; 27) found most useful for measuring inflation: "One might want to produce a COL *conditional* on the base period's weather experience…. In this case, the unusually cold winter does not affect the *conditional* COL subindex that holds the environment constant. … the COL subindex that holds the environment constant is probably the COL concept that is most useful for an anti-inflation policy." T.P. Hill (1999; 4) endorsed this point of view.

$$
\begin{aligned}
(91)\quad & P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e_1^0,e_2^0,\ldots,e_H^0) \\
& \equiv \textstyle\sum_{h=1}^H C^h(u_h^0,e_h^0,p_h^1)/\sum_{h=1}^H C^h(u_h^0,e_h^0,p_h^0) \\
& = \textstyle\sum_{h=1}^H C^h(u_h^0,e_h^0,p_h^1)/\sum_{h=1}^H p_h^0\cdot q_h^0 && \text{using (89) for } t = 0 \\
& \leq \textstyle\sum_{h=1}^H p_h^1\cdot q_h^0/\sum_{h=1}^H p_h^0\cdot q_h^0 \\
& \qquad \text{since } C^h(u_h^0,e_h^0,p_h^1) \equiv \min_q \{p_h^1\cdot q : f^h(q,e_h^0) \geq u_h^0\} \leq p^1\cdot q_h^0 \text{ and } q_h^0 \\
& \qquad \text{is feasible for the cost minimization problem for } h = 1,2,\ldots,H \\
& \equiv P_{PL}
\end{aligned}
$$

where P_{PL} is defined to be the observable (in principle) *plutocratic Laspeyres price index*, $\sum_{h=1}^H p_h^1\cdot q_h^0/\sum_{h=1}^H p_h^0\cdot q_h^0$, which uses the individual vectors of household or regional quantities for period 0, $(q_1^0,\ldots,q_H^0)$, as quantity weights.[56] If prices are equal across households (or regions), so that

$$(92)\quad p_h^t = p^t ; \qquad t = 0,1 \text{ and } h = 1,2,\ldots,H,$$

then the plutocratic (or disaggregated) Laspeyres price index P_{PL} collapses down to the usual *aggregate Laspeyres index*, P_L. Thus if (92) holds, then P_{PL} in (91) becomes

$$
\begin{aligned}
(93)\quad P_{PL} & \equiv \textstyle\sum_{h=1}^H p_h^1\cdot q_h^0/\sum_{h=1}^H p_h^0\cdot q_h^0 \\
& = p^1\cdot\textstyle\sum_{h=1}^H q_h^0/p^0\cdot\sum_{h=1}^H q_h^0 \\
& = p^1\cdot q^0/p^0\cdot q^0 \\
& \equiv P_L
\end{aligned}
$$

where the total quantity vector in period t is defined as

$$(94)\quad q^t \equiv \textstyle\sum_{h=1}^H q_h^t ; \qquad t = 0,1.$$

The inequality (91) says that the theoretical Laspeyres plutocratic conditional cost of living index, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e^0)$, is bounded from above by the observable (in principle) plutocratic or disaggregated Laspeyres price index, P_{PL}. The special case of inequality (91) when the equal prices assumption (92) holds was first obtained by Pollak (1989; 182) for the case of one household with environmental variables and by Pollak (1980; 276) for the many household case but where the environmental variables are absent from the household utility and cost functions.

In a similar manner, specializing definition (90), *the Paasche conditional plutocratic cost of living index*, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e^1)$, may be written as follows:

$$
\begin{aligned}
(95)\quad & P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e_1^1,e_2^1,\ldots,e_H^1) \\
& \equiv \textstyle\sum_{h=1}^H C^h(u_h^1,e_h^1,p_h^1)/\sum_{h=1}^H C^h(u_h^1,e_h^1,p_h^0) \\
& = \textstyle\sum_{h=1}^H p_h^1\cdot q_h^1/\sum_{h=1}^H C^h(u_h^1,e_h^1,p_h^0) && \text{using (89) for } t = 1 \\
& \geq \textstyle\sum_{h=1}^H p_h^1\cdot q_h^1/\sum_{h=1}^H p_h^0\cdot q_h^1 && \text{using a feasibility argument} \\
& \equiv P_{PP}
\end{aligned}
$$

[56] Thus the plutocratic Laspeyres index can be regarded as an ordinary Laspeyres index except that each commodity in each region is regarded as a separate commodity.

where P_{PP} is defined to be the *plutocratic or disaggregated (over households) Paasche price index*, $\sum_{h=1}^{H} p_h^1 \cdot q_h^1 / \sum_{h=1}^{H} p_h^0 \cdot q_h^1$, which uses the individual vectors of household quantities for period 1, $(q_1^1,\ldots,q_H^1)$, as quantity weights.

If prices are equal across households (or regions), so that assumptions (92) hold, then the disaggregated Paasche price index P_{PP} collapses down to the usual aggregate Paasche index, P_P; i.e., if (92) holds, then P_{PP} in (95) becomes

$$(96)\ P_{PP} \equiv \sum_{h=1}^{H} p_h^1 \cdot q_h^1 / \sum_{h=1}^{H} p_h^0 \cdot q_h^1$$
$$= p^1 \cdot \sum_{h=1}^{H} q_h^1 / p^0 \cdot \sum_{h=1}^{H} q_h^1$$
$$= p^1 \cdot q^1 / p^0 \cdot q^1$$
$$\equiv P_P.$$

Returning to the inequality (95), we see that the theoretical Paasche conditional plutocratic cost of living index, $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e^1)$, is bounded from below by the observable plutocratic or disaggregated Paasche price index P_{PP}. Diewert (1983a; 191) first obtained the inequality (95) for the case where the environmental variables were absent from the household utility and cost functions and prices were equal across households.

In the following section, we shall show how to obtain a theoretical plutocratic cost of living index that is bounded from above and below rather than the theoretical indexes that just have the one sided bounds in (91) and (95).

11. The Fisher Plutocratic Price Index

Using the inequalities (91) and (95) and the continuity properties of the conditional plutocratic cost of living $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u,e)$ defined by (90), it is possible to modify the method of proof used by Konüs (1924) and Diewert (1983a; 191) and establish the following result:[57]

Proposition 4: Under our assumptions, there exists a reference utility vector $u^* \equiv (u_1^*,u_2^*,\ldots,u_H^*)$ such that the household h reference utility level u_h^* lies between the household h period 0 and 1 utility levels, u_h^0 and u_h^1 respectively for h = 1,…,H, and there exist household environmental vectors $e_h^* \equiv (e_{h1}^*,e_{h2}^*,\ldots,e_{hM}^*)$ such that the household h reference mth environmental variable e_{hm}^* lies between the household h period 0 and 1 levels for the mth environmental variable, e_{hm}^0 and e_{hm}^1 respectively for m = 1,2,…,M and h = 1,…,H, and the conditional plutocratic cost of living index $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^*,e^*)$ evaluated at this intermediate reference utility vector u^* and the intermediate reference vector of household environmental variables $e^* \equiv$

[57] Note that the household cost functions must be continuous in the environmental variables which is a real restriction on the types of environmental variables which can be accommodated by the result. Thus if household preferences change discontinuously as the season of the year changes, then Proposition 1 would not be valid.

$(e_1^*,e_2^*,\ldots,e_H^*)$ lies between the observable (in principle) plutocratic Laspeyres and Paasche price indexes, P_{PL} and P_{PP}, defined above by the last equalities in (91) and (95).

Proof: Utilize the method of proof used in Proposition 1 above by defining $g(\lambda)$ for $0 \le \lambda \le 1$ by $g(\lambda) \equiv P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1, (1-\lambda)u^0+\lambda u^1, (1-\lambda)e_1^0+\lambda e_1^1,\ldots, (1-\lambda)e_H^0+\lambda e_H^1)$. Note that $g(0) = P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^0,e_1^0,e_2^0,\ldots,e_H^0)$ and $g(1) = P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^1,e_1^1,e_2^1,\ldots, e_H^1)$. There are 24 = (4)(3)(2)(1) possible a priori inequality relations that are possible between the four numbers $g(0)$, $g(1)$, P_{PL} and P_{PP}. However, the inequalities (91) and (95) above imply that $g(0) \le P_L$ and $P_P \le g(1)$. This means that there are only six possible inequalities between the four numbers:

$$
\begin{aligned}
(97)\quad & g(0) \le P_{PL} \le P_{PP} \le g(1)\ ; \\
& g(0) \le P_{PP} \le P_{PL} \le g(1)\ ; \\
& g(0) \le P_{PP} \le g(1) \le P_{PL}\ ; \\
& P_{PP} \le g(0) \le P_{PL} \le g(1)\ ; \\
& P_{PP} \le g(1) \le g(0) \le P_{PL}; \\
& P_{PP} \le g(0) \le g(1) \le P_{PL}.
\end{aligned}
$$

Using the assumptions that: (a) the consumer's utility function f is continuous over its domain of definition; (b) the utility function is increasing in the components of q and hence is subject to local nonsatiation and (c) the price vectors p^t have strictly positive components, it is possible to use Debreu's (1959; 19) Maximum Theorem to show that the household cost functions $C^h(u_h,e_h,p_h)$ will be continuous in the components of u_h and e_h. Thus using definition (90), it can be seen that $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u,e_1,e_2,\ldots,e_H)$ will also be continuous in the components of the vectors $u,e_1,e_2,\ldots,e_H$. Hence $g(\lambda)$ is a continuous function of λ and assumes all intermediate values between $g(0)$ and $g(1)$. By inspecting the inequalities (97) above, it can be seen that we can choose λ between 0 and 1, λ^* say, such that $P_{PL} \le g(\lambda^*) \le P_{PP}$ or such that $P_{PP} \le g(\lambda^*) \le P_{PL}$. Q.E.D.

The above result tells us that *the theoretical national plutocratic conditional consumer price index* $P^*(p_1^0,\ldots,p_H^0,p_1^1,\ldots,p_H^1,u^*,e^*)$ lies between the plutocratic or disaggregated Laspeyres index P_{PL} and the plutocratic or disaggregated Paasche index P_{PP}. Hence if P_{PL} and P_{PP} are not too different, a good point approximation[58] to the theoretical national plutocratic consumer price index will be the *plutocratic or disaggregated Fisher index* P_{PF} defined as $P_{PF} \equiv [P_{PL}\, P_{PP}]^{1/2}$.

The plutocratic Fisher price index P_{PF} is computed just like the usual Fisher price index, except that each commodity in each region (or for each household) is regarded as a separate commodity. Of course, this index will satisfy the time reversal test.

Since statistical agencies do not calculate Laspeyres, Paasche and Fisher price indexes by taking inner products of price and quantity vectors, it will be useful to obtain formulae for the Laspeyres and Paasche indices that depend only on price relatives and expenditure

[58] Recall the material in section 4 of Chapter 1 for a more formal justification for the use of the Fisher index as an appropriate average of the Paasche and Laspeyres indexes.

shares. In order to do this, we need to introduce some notation. Define the *expenditure share of household h on commodity i in period t* as

$$(98)\quad s_{hi}^t \equiv p_{hi}^t q_{hi}^t/\textstyle\sum_{k=1}^N p_{hk}^t q_{hk}^t ; \qquad t = 0,1 ;\ h = 1,2,\ldots,H ;\ i = 1,2,\ldots,N.$$

Define the *expenditure share of household h in total period t consumption* as:

$$\begin{aligned}(99)\quad S_h^t &\equiv \textstyle\sum_{i=1}^N p_{hi}^t q_{hi}^t/ \sum_{k=1}^H \sum_{i=1}^N p_{ik}^t q_{ik}^t \\ &= p_h^t\cdot q_h^t/\textstyle\sum_{k=1}^H p_k^t\cdot q_k^t \qquad t = 0,1 ;\ h = 1,2,\ldots,H.\end{aligned}$$

Finally, define the *national expenditure share of commodity i in period t* as:

$$\begin{aligned}(100)\quad \sigma_i^t &\equiv \textstyle\sum_{h=1}^H p_{hi}^t q_{hi}^t/\sum_{k=1}^H p_k^t\cdot q_k^t && t = 0,1 ;\ i = 1,2,\ldots,N \\ &= \textstyle\sum_{h=1}^H [p_{hi}^t q_{hi}^t/p_h^t\cdot q_h^t][p_h^t\cdot q_h^t/\sum_{k=1}^H p_k^t\cdot q_k^t] \\ &= \textstyle\sum_{h=1}^H s_{hi}^t\, p_h^t\cdot q_h^t/\sum_{k=1}^H p_k^t\cdot q_k^t && \text{using definitions (10)} \\ &= \textstyle\sum_{h=1}^H s_{hi}^t S_h^t && \text{using definitions (11).}\end{aligned}$$

The *Laspeyres price index for household h* (or region h) is defined as:

$$\begin{aligned}(101)\quad P_{Lh} &\equiv p_h^1\cdot q_h^0/p_h^0\cdot q_h^0 && h = 1,2,\ldots,H \\ &= \textstyle\sum_{i=1}^N (p_{hi}^1/p_{hi}^0)p_{hi}^0 q_{hi}^0/p_h^0\cdot q_h^0 \\ &= \textstyle\sum_{i=1}^N s_{hi}^0(p_{hi}^1/p_{hi}^0) && \text{using definitions (98).}\end{aligned}$$

Referring back to (91), the *plutocratic national Laspeyres price index* P_{PL} can be rewritten as follows:

$$\begin{aligned}(102)\quad P_{PL} &\equiv (\textstyle\sum_{h=1}^H p_h^1\cdot q_h^0)/(\sum_{h=1}^H p_h^0\cdot q_h^0) \\ &= \textstyle\sum_{h=1}^H [p_h^1\cdot q_h^0/p_h^0\cdot q_h^0][p_h^0\cdot q_h^0/\sum_{h=1}^H p_h^0\cdot q_h^0] \\ &= \textstyle\sum_{h=1}^H [p_h^1\cdot q_h^0/p_h^0\cdot q_h^0]\, S_h^0 && \text{using definitions (99) with } t = 0 \\ (103)\quad &= \textstyle\sum_{h=1}^H S_h^0 P_{Lh} && \text{using definitions (101)} \\ &= \textstyle\sum_{h=1}^H S_h^0 \sum_{i=1}^N s_{hi}^0(p_{hi}^1/p_{hi}^0) && \text{using the last line of (101)} \\ (104)\quad &= \textstyle\sum_{h=1}^H \sum_{i=1}^N S_h^0 s_{hi}^0(p_{hi}^1/p_{hi}^0) && \text{rearranging terms.}\end{aligned}$$

Equation (103) shows that the plutocratic national Laspeyres price index is equal to a (period 0) regional expenditure share weighted average of the regional Laspeyres price indexes. Equation (104) shows that the national Laspeyres price index is equal to a period 0 expenditure share weighted average of the regional price relatives, (p_{hi}^1/p_{hi}^0), where the corresponding weight, $S_h^0 s_{hi}^0$, is the period 0 national expenditure share of commodity i for household h.

The *Paasche price index for region h* (or household h) is defined as:

$$\begin{aligned}(105)\quad P_{Ph} &\equiv p_h^1\cdot q_h^1/p_h^0\cdot q_h^1 && h = 1,2,\ldots,H \\ &= 1/\textstyle\sum_{i=1}^N (p_{hi}^0/ p_{hi}^1)p_{hi}^1 q_{hi}^1/p_h^1\cdot q_h^1 \\ &= 1/\textstyle\sum_{i=1}^N s_{hi}^1(p_{hi}^1/p_{hi}^0)^{-1} && \text{using definitions (10)}\end{aligned}$$

$$= [\sum_{i=1}^{N} s_{hi}^{1}(p_{hi}^{1}/p_{hi}^{0})^{-1}]^{-1} .$$

Referring back to (95), the *plutocratic national Paasche price index* P_{PP} can be rewritten as follows:

$$\begin{aligned}
(106)\ P_{PP} &\equiv (\sum_{k=1}^{H} p_k^1\cdot q_k^1)/(\sum_{h=1}^{H} p_h^0\cdot q_h^1) \\
&= 1/\{\sum_{h=1}^{H} [p_h^0\cdot q_h^1/p_h^1\cdot q_h^1][p_h^1\cdot q_h^1/\sum_{k=1}^{H} p_k^1\cdot q_k^1]\} \\
&= 1/\sum_{h=1}^{H} [p_h^1\cdot q_h^0/p_h^0\cdot q_h^0]^{-1}S_h^1 && \text{using definitions (99) with } t = 1 \\
(107)\quad &= [\sum_{h=1}^{H} S_h^1\, P_{Ph}^{-1}]^{-1} && \text{using definitions (105)} \\
&= [\sum_{h=1}^{H} S_h^1 \sum_{i=1}^{N} s_{hi}^1(p_{hi}^1/p_{hi}^0)^{-1}]^{-1} && \text{using the last line of (105)} \\
(108)\quad &= [\sum_{h=1}^{H} \sum_{i=1}^{N} S_h^1 s_{hi}^1(p_{hi}^1/p_{hi}^0)^{-1}]^{-1} && \text{rearranging terms.}
\end{aligned}$$

Equation (107) shows that the national plutocratic Paasche price index is equal to a (period 1) regional expenditure share *weighted harmonic mean* of the regional Paasche price indexes. Equation (108) shows that the national Paasche price index is equal to a period 1 expenditure share *weighted harmonic average* of the regional price relatives, (p_{hi}^1/p_{hi}^0), where the weight for this price relative, $S_h^1 s_{hi}^1$, is the period 1 national expenditure share of commodity i in region h.

Of course, the share formulae for the plutocratic Paasche and Laspeyres indexes, P_{PP} and P_{PL}, given by (106) and (102) can now be used to calculate the plutocratic Fisher index, $P_{PF} \equiv [P_{PP}\, P_{PL}]^{1/2}$.

If prices are equal across regions, the formulae (104) and (108) simplify. The formula for the plutocratic Laspeyres index (104) becomes:

$$\begin{aligned}
(109)\ P_{PL} &= \sum_{h=1}^{H} \sum_{i=1}^{N} S_h^0 s_{hi}^0(p_{hi}^1/p_{hi}^0) \\
&= \sum_{h=1}^{H} \sum_{i=1}^{N} S_h^0 s_{hi}^0(p_i^1/p_i^0) && \text{using assumptions (4)} \\
&= \sum_{i=1}^{N} \sigma_i^0(p_i^1/p_i^0) && \text{using (12) for } t = 0 \\
&= P_L
\end{aligned}$$

where P_L is the usual aggregate Laspeyres price index based on the assumption that each household faces the same vector of commodity prices; see (93) for the definition of P_L. Under the equal prices across households assumption (92), the formula for the plutocratic Paasche index (108) becomes:

$$\begin{aligned}
(110)\ P_{PP} &= [\sum_{h=1}^{H} \sum_{i=1}^{N} S_h^1 s_{hi}^1(p_{hi}^1/p_{hi}^0)^{-1}]^{-1} \\
&= [\sum_{h=1}^{H} \sum_{i=1}^{N} S_h^1 s_{hi}^1(p_i^1/p_i^0)^{-1}]^{-1} && \text{using assumptions (4)} \\
&= [\sum_{i=1}^{N} \sigma_i^1(p_i^1/p_i^0)^{-1}]^{-1} && \text{using (100) for } t = 1 \\
&= P_P
\end{aligned}$$

where P_P is the usual aggregate Paasche price index based on the assumption that each household faces the same vector of commodity prices; see (96) for the definition of P_P.

Thus with the assumption that commodity prices are the same across regions, in order to calculate national Laspeyres and Paasche indexes, we require only "national" price relatives and national commodity expenditure shares for the two periods under consideration. However, if there is regional variation in prices, then the simplified formulae (109) and (110) are not valid and we must use our earlier formulae, (104) and (108).

In the final sections of this Chapter on the economic approach to index number theory, we will adapt the above consumer price index theory to the producer context.[59]

12. The Fisher Shell Output Price Index and Observable Bounds

In this section, we present an outline of the theory of the output price index for a single establishment that was developed by Fisher and Shell (1972) and Archibald (1977). This theory is the producer theory counterpart to the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, A. A. Konüs (1924). These economic approaches to price indexes rely on the assumption of (competitive) *optimizing behavior* on the part of economic agents (consumers or producers). Thus in the case of the output price index, given a vector of output prices p^t that the agent faces in a given time period t, it is assumed that the corresponding hypothetical quantity vector q^t is the solution to a revenue maximization problem that involves the producer's production function f or production possibilities set. (Hereafter the terms value of output and revenue are used interchangeably, inventory changes being ignored).

As noted earlier, in contrast to the *axiomatic approach* to index number theory, the *economic approach* does *not* assume that the two quantity vectors q^0 and q^1 are independent of the two price vectors p^0 and p^1. In the economic approach, the period 0 quantity vector q^0 is determined by the producer's period 0 production function and the period 0 vector of prices p^0 that the producer faces and the period 1 quantity vector q^1 is determined by the producer's period 1 production function f and the period 1 vector of prices p^1.

Before the output price index is defined for an establishment, it is first necessary to describe the establishment's technology in period t. In the economics literature, it is traditional to describe the technology of a firm or industry in terms of a production function, which tells us what the maximum amount of output that can be produced using a given vector of inputs. However, since most establishments produce more than one output, it is more convenient to describe the establishment's technology in period t by means of a *production possibilities set*, S^t. The set S^t describes what output vectors q can be produced in period t if the establishment has at its disposal the vector of inputs $v \equiv [x,z]$, where x is a vector of intermediate inputs and z is a vector of primary inputs. Thus

[59] The following material on Producer Price Indexes is adapted from Chapter 2 in Alterman, Diewert and Feenstra (1999) and Chapter 17 in the *Producer Price Index Manual: Theory and Practice*; see the IMF/ILO/OECD/UNECE/Eurostat/World Bank (2004).

if [q,v] belongs to S^t,[60] then the nonnegative output vector q can be produced by the establishment in period t if it can utilize the nonnegative vector v of inputs.

Let $p \equiv (p_1,\ldots p_N)$ denote a vector of positive output prices that the establishment might face in period t and let $v \equiv [x,z]$ be a nonnegative vector of inputs that the establishment might have available for use during period t. We assume that the v vector has dimension M. Then the establishment's *revenue function* using period t technology is defined as the solution to the following revenue maximization problem:

$$(111)\quad R^t(p,v) \equiv \max_q \{p\cdot q : (q,v)\in S^t\}.$$

Thus $R^t(p,v)$ is the maximum value of output, $p\cdot q \equiv \sum_{n=1}^N p_n q_n$, that the establishment can produce, given that it faces the vector of output prices p and given that the vector of inputs v is available for use, using the period t technology.[61]

The period t revenue function R^t can be used to define the establishment's *period t technology output price index* P^t between any two periods, say period 0 and period 1, as follows:

$$(112)\quad P^t(p^0,p^1,v) = R^t(p^1,v)/R^t(p^0,v)$$

where p^0 and p^1 are the vectors of output prices that the establishment faces in periods 0 and 1 respectively and v is a reference vector of intermediate and primary inputs.[62] If N = 1 so that there is only one output that the establishment produces, then it can be shown that the output price index collapses down to the single output price relative between periods 0 and 1, p_1^1/p_1^0. In the general case, note that the output price index defined by (112) is a ratio of hypothetical revenues that the establishment could realize, given that it has the period t technology and the vector of inputs v to work with. The numerator in (112) is the maximum revenue that the establishment could attain if it faced the output prices of period 1, p^1, while the denominator in (112) is the maximum revenue that the establishment could attain if it faced the output prices of period 0, p^0. Note that all of the variables in the numerator and denominator functions *are exactly the same*, *except that the output price vectors differ*. This is a defining characteristic of an *economic price index*: all environmental variables are held constant with the exception of the prices in the domain of definition of the price index.

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[60] We write this as $[q,v]\in S^t$ in what follows.

[61] The function R^t is known as the *GDP function* or the *national product function* in the international trade literature; see Kohli (1978) (1991) or Woodland (1982). It was introduced into the economics literature by Samuelson (1953). Alternative terms for this function include: (i) the *gross profit function*; see Gorman (1968); (ii) the *restricted profit function*; see Lau (1976) and McFadden (1978) and (iii) the *variable profit function*; see Diewert (1973) (1974) (1993). The mathematical properties of the revenue function are laid out in these references.

[62] This concept of the output price index (or a closely related variant) was defined by Fisher and Shell (1972; 56-58), Samuelson and Swamy (1974; 588-592), Archibald (1977; 60-61), Diewert (1980; 460-461) (1983b; 1055) and Balk (1998; 83-89). Note that the output price index defined by (112) is analogous to the *true cost of living index* which is a ratio of cost functions, $C(u,p^1)/C(u,p^0)$ where u is a reference utility level: R replaces C and the reference utility level u is replaced by the vector of reference variables (t,v).

Note that there are a wide variety of price indexes of the form (112) depending on which reference technology t and reference input vector v that is chosen. Thus there is not a single economic price index of the type defined by (112): there is an entire *family* of indexes.

Usually, interest lies in two special cases of the general definition of the output price index (112): (i) $P^0(p^0,p^1,v^0)$ which uses the period 0 technology set and the input vector v^0 that was actually used in period 0 and (ii) $P^1(p^0,p^1,v^1)$ which uses the period 1 technology set and the input vector v^1 that was actually used in period 1. Let q^0 and q^1 be the observed output vectors for the establishment in periods 0 and 1 respectively. If there is competitive (or price taking) revenue maximizing behavior on the part of the establishment in periods 0 and 1, then observed revenue in periods 0 and 1 should be equal to $R^0(p^0,v^0)$ and $R^1(p^1,v^1)$ respectively; i.e., the following equalities should hold:

$$(113)\quad R^0(p^0,v^0) = p^0\cdot q^0\ ;\ R^1(p^1,v^1) = p^1\cdot q^1.$$

Under these revenue maximizing assumptions, Fisher and Shell (1972; 57-58) and Archibald (1977; 66) showed that the two theoretical indexes, $P^0(p^0,p^1,v^0)$ and $P^1(p^0,p^1,v^1)$ described in (i) and (ii) above, satisfy the following inequalities (114) and (115):

$$\begin{aligned}(114)\quad P^0(p^0,p^1,v^0) &\equiv R^0(p^1,v^0)/R^0(p^0,v^0) && \text{using definition (112)}\\ &= R^0(p^1,v^0)/ \textstyle\sum_{n=1}^N p_n^0 q_n^0 && \text{using (113)}\\ &\geq \textstyle\sum_{n=1}^N p_n^1 q_n^0/\sum_{n=1}^N p_n^0 q_n^0 && \text{since } q^0 \text{ is feasible for the maximization}\\ & && \text{problem which defines } R^0(p^1,v^0) \text{ and so } R^0(p^1,v^0) \geq p^1\cdot q^0\\ &\equiv P_L(p^0,p^1,q^0,q^1)\end{aligned}$$

where P_L is the Laspeyres price index. Similarly,

$$\begin{aligned}(115)\quad P^1(p^0,p^1,v^1) &\equiv R^1(p^1,v^1)/R^1(p^0,v^1) && \text{using definition (112)}\\ &= \textstyle\sum_{n=1}^N p_n^1 q_n^1 /R^1(p^0,v^1) && \text{using (113)}\\ &\leq \textstyle\sum_{n=1}^N p_n^1 q_n^1/\sum_{n=1}^N p_n^0 q_n^1 && \text{since } q^1 \text{ is feasible for the maximization}\\ & && \text{problem which defines } R^1(p^0,v^1) \text{ and so } R^1(p^0,v^1) \geq p^0\cdot q^1\\ &\equiv P_P(p^0,p^1,q^0,q^1)\end{aligned}$$

where P_P is the Paasche price index. Thus the inequality (114) says that the observable Laspeyres index of output prices P_L is a *lower bound* to the theoretical output price index $P^0(p^0,p^1,v^0)$ and the inequality (115) says that the observable Paasche index of output prices P_P is an *upper bound* to the theoretical output price index $P^1(p^0,p^1,v^1)$. Note that these inequalities are in the *opposite direction* compared to their counterparts in the theory of the true cost of living index.[63]

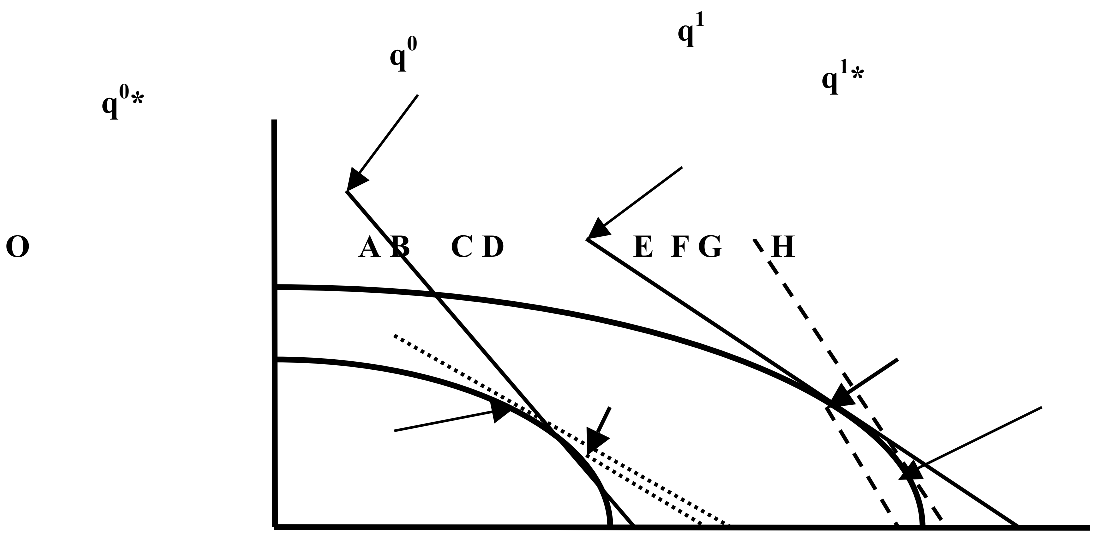
[63] This is due to the fact that the optimization problem in the cost of living theory is a cost *minimization* problem as opposed to our present revenue *maximization* problem. The method of proof used to derive (114) and (115) dates back to Konüs (1924) and Hicks (1940).

Figure 2: Bounds to the Paasche and Laspeyres Output Price Indexes

q_2 $\{q: p^{0T}q = p^{0T}q^0\}$

It is possible to illustrate the two inequalities (114) and (115) if there are only two commodities; see Figure 2 below, which is based on diagrams due to Hicks (1940; 120) and Fisher and Shell (1972; 57).

$\{q: p^{1T}q = p^{1T}q^1\}$



In Figure 2, the inequality (114) is illustrated for the case of two outputs that are both produced in both periods. The solution to the period 0 revenue maximization problem is the vector q^0 and the straight line through B represents the revenue line that is just tangent to the period 0 output production possibilities set, $S^0(v^0) \equiv \{(q_1,q_2,v^0)\in S^0\}$. The curved line through q^0 and A is the frontier to the producer's period 0 output production possibilities set $S^0(v^0)$. The solution to the period 1 revenue maximization problem is the vector q^1 and the straight line through H represents the revenue line that is just tangent to the period 1 output production possibilities set, $S^1(v^1) \equiv \{(q_1,q_2,v^1)\in S^1\}$. The curved line through q^1 and F is the frontier to the producer's period 1 output production possibilities set $S^1(v^1)$. The point q^{0*} solves the hypothetical maximization problem of maximizing revenue when facing the period 1 price vector $p^1 = (p_1^{\,1},p_2^{\,1})$ but using the period 0 technology and input vector. This is given by $R^0(p^1,v^0) = p_1^{\,1}q_1^{\,0*}+p_2^{\,1}q_2^{\,0*}$ and the dotted line through D is the corresponding isorevenue line $p_1^{\,1}q_1+p_2^{\,1}q_2 = R^0(p^1,v^0)$. Note that the hypothetical revenue line through D is parallel to the actual period 1 revenue line through H. From (4), the hypothetical Fisher Shell output price index, $P^0(p^0,p^1,v^0)$, is $R^0(p^1,v^0)/[p_1^{\,0}q_1^{\,0}+p_2^{\,0}q_2^{\,0}]$ while the ordinary Laspeyres output price index is $[p_1^{\,1}q_1^{\,0}+ p_2^{\,1}q_2^{\,0}]/[p_1^{\,0}q_1^{\,0}+p_2^{\,0}q_2^{\,0}]$. Since the denominators for these two indexes are the same, the difference between the indexes is due to the differences in their numerators. In Figure 2, this difference in the numerators is expressed by the fact that the dotted revenue line through C lies *below* the parallel revenue line through D. Now if the producer's period 0 output production possibilities set were block shaped with vertex at q^0, then the producer would not change his or her production pattern in response to a change in the relative prices of the two commodities while using the period 0 technology and inputs. In this case, the hypothetical vector q^{0*} would coincide with q^0, the dotted line through D would coincide with the dotted line through C and the true output price index $P^0(p^0,p^1,v^0)$, would *coincide* with the ordinary Laspeyres price index. However, block shaped production possibilities sets are not generally consistent with producer behavior; i.e., when the price of a commodity increases, producers generally supply more of it. Thus in the general case, there will be a gap between the points C and D. The magnitude of this gap represents the amount of *substitution bias* between the true index and the corresponding Laspeyres index; i.e., the Laspeyres index will generally be *less* than the corresponding true output price index, $P^0(p^0,p^1,v^0)$.

Figure 2 can also be used to illustrate the inequality (115) for the two output case. Note that technical progress or increases in input availability have caused the period 1 output production possibilities set $S^1(v^1) \equiv \{(q_1,q_2) : [q_1,q_2,v^1]$ belongs to $S^1\}$ to be much bigger than the corresponding period 0 output production possibilities set $S^0(v^0) \equiv \{(q_1,q_2) : [q_1,q_2,v^0]$ belongs to $S^0\}$.[64] Secondly, note that the dashed lines through E and G are parallel to the period 0 isorevenue line through B. The point q^{1*} solves the hypothetical revenue maximization problem of maximizing revenue using the period 1 technology and inputs when facing the period 0 price vector $p^0 = (p_1^0,p_2^0)$. This is given by $R^1(p^0,v^1) = p_1^0q_1^{1*}+p_2^0q_2^{1*}$ and the dashed line through G is the corresponding isorevenue line $p_1^1q_1+ p_2^1q_2 = R^1(p^0,v^1)$. From (115), the theoretical output price index using the period 1 technology and inputs is $[p_1^1q_1^1+p_2^1q_2^1]/R^1(p^0,v^1)$ while the ordinary Paasche price index is $[p_1^1q_1^1+p_2^1q_2^1]/[p_1^0q_1^1 + p_2^0q_2^1]$. Since the numerators for these two indexes are the same, the difference between the indexes is due to the differences in their denominators. In Figure 2, this difference in the denominators is expressed by the fact that the revenue line through E lies *below* the parallel cost line through G. The magnitude of this difference represents the amount of *substitution bias* between the true index and the corresponding Paasche index; i.e., the Paasche index will generally be *greater* than the corresponding true output price index using current period technology and inputs, $P^1(p^0,p^1,v^1)$. Note that this inequality goes in the opposite direction to the previous inequality, (114). The reason for this change in direction is due to the fact that one set of differences between the two indexes takes place in the numerators of the two indexes (the Laspeyres inequalities) while the other set takes place in the denominators of the two indexes (the Paasche inequalities).

There are two problems with the inequalities (114) and (115):

- There are *two* equally valid economic price indexes, $P^0(p^0,p^1,v^0)$ and $P^1(p^0,p^1,v^1)$, that could be used to describe the amount of price change that took place between periods 0 and 1 whereas the public will demand that the statistical agency produce a *single* estimate of price change between the two periods.

[64] However, validity of the inequality (115) does not depend on the relative position of the two output production possibilities sets. To obtain the strict inequality version of (115), we need two things: (i) we need the frontier of the period 1 output production possibilities set to be "curved" and (ii) we need relative output prices to change going from period 0 to 1 so that the two price lines through G and H in Figure 2 are tangent to *different* points on the frontier of the period 1 output production possibilities set.

- Only *one sided* observable bounds to these two theoretical price indexes[65] result from this analysis and what are required for most practical purposes are *two sided* bounds.

In the following section, it will be shown how a possible solution to these two problems can be found.

13. The Fisher Ideal Index as an Approximation to an Economic Output Price Index

It is possible to define a theoretical output price index that falls *between* the observable Paasche and Laspeyres price indices. To do this, first define a hypothetical revenue function, $R(p,\alpha)$, that corresponds to the use of an α *weighted average* of the technology sets S^0 and S^1 for periods 0 and 1 as the reference technology and that uses an α weighted average of the period 0 and period 1 input vectors v^0 and v^1 as the reference input vector:

(116) $R(p,\alpha) \equiv \max_q \{p\cdot q : [q,(1-\alpha)v^0 + \alpha v^1] \in [(1-\alpha)S^0 + \alpha S^1]\}$.

Thus the revenue maximization problem in (116) corresponds to the use of a weighted average of the period 0 and 1 input vectors v^0 and v^1 where the period 0 vector gets the weight $1-\alpha$ and the period 1 vector gets the weight α and an "average" of the period 0 and period 1 technology sets is used as the reference technology where the period 0 set gets the weight $1-\alpha$ and the period 1 set gets the weight α , where α is a number between 0 and 1.[66] The meaning of the weighted average technology set in definition (116) can be explained in terms of Figure 2 as follows. As α changes continuously from 0 to 1, the output production possibilities set changes in a continuous manner from the set $S^0(v^0)$ (whose frontier is the curve which ends in the point A) to the set $S^1(v^1)$ (whose frontier is the curve which ends in the point F). Thus for any α between 0 and 1, a hypothetical establishment output production possibilities set is obtained which lies between the base period set $S^0(v^0)$ and the current period set $S^1(v^1)$. For each α, this hypothetical output production possibilities set can be used as the constraint set for a theoretical output price index.

The new revenue function defined by (116) is now used in order to define the following *family* (indexed by α) *of theoretical output price indexes*:

(117) $P(p^0,p^1,\alpha) \equiv R(p^1,\alpha)/R(p^0,\alpha)$.

The important advantage that theoretical output price indexes of the form defined by (112) or (117) have over the traditional Laspeyres and Paasche output price indexes P_L and P_P is that these theoretical indexes deal adequately with *substitution effects*; i.e., when an output price increases, the producer supply should increase, holding inputs and the technology constant.[67]

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[65] As we have shown, the Laspeyres output price index is a lower bound to the theoretical index $P^0(p^0,p^1,v^0)$ while the Paasche output price index is an upper bound to the theoretical index $P^1(p^0,p^1,v^1)$.

[66] When α=0, $R(p,0) = R^0(p,v^0)$ and when $\alpha = 1$, $R(p,1) = R^1(p,v^1)$.

[67] This is a normal output substitution effect. However, empirically, it will often happen that observed period to period decreases in price are not accompanied by corresponding decreases in supply. However,

Diewert (1983b; 1060-1061) showed that, under certain conditions[68], there exists an α between 0 and 1 such that the theoretical output price index defined by (117) lies between the observable (in principle) Paasche and Laspeyres output indexes, P_P and P_L ; i.e., there exists an α such that

(118) $P_L \le P(p^0,p^1,\alpha) \le P_P$ or $P_P \le P(p^0,p^1,\alpha) \le P_L$.

The fact that the Paasche and Laspeyres output price indexes provide upper and lower bounds to a "true" output price $P(p^0,p^1,\alpha)$ in (118) is a more useful and important result than the one sided bounds on the "true" indexes that were derived in (114) and (115) above. If the observable (in principle) Paasche and Laspeyres indexes are not too far apart, then taking a symmetric average of these indexes should provide a good approximation to an economic output price index where the reference technology is somewhere between the base and current period technologies. Using an axiomatic approach, a result in section 3 of Chapter 1 suggested that the geometric average was "best" and this result led to the geometric mean, the Fisher price index, P_F. Thus the Fisher ideal price index receives a fairly strong justification as a good approximation to an unobservable theoretical output price index.[69]

The bounds given by (114), (115) and (118) are the best bounds that can be obtained on economic output price indexes without making further assumptions. In the next section, further assumptions are made on the two technology sets S^0 and S^1 or equivalently, on the two revenue functions, $R^0(p,v)$ and $R^1(p,v)$. With these extra assumptions, it is possible to determine the geometric average of the two theoretical output price indexes that are of primary interest, $P^0(p^0,p^1,v^0)$ and $P^1(p^0,p^1,v^1)$.

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these abnormal "substitution" effects can be rationalized as the effects of *technological progress*. For example, suppose the price of computer chips decreases substantially going from period 0 to 1. If the technology were constant over these two periods, we would expect domestic producers to decrease their supply of chips going from period 0 to 1. In actual fact, the opposite happens but what has happened is that technological progress has led to a sharp reduction in the cost of producing chips which is passed on to demanders of chips. Thus the effects of technological progress cannot be ignored in the theory of the output price index. The counterpart to technological change in the theory of the cost of living index is taste change, which is usually ignored.

[68] The proof is essentially the same as the method used in the proof of Proposition 1 above. Sufficient conditions on the period 0 and 1 technology sets for the result to hold are given in Diewert (1983b; 1105). As noted earlier, our exposition of the material on output price indexes in this chapter also draws on Chapter 2 in Alterman, Diewert and Feenstra (1999).

[69] It should be noted that Fisher (1922) constructed Laspeyres, Paasche and Fisher output price indexes for his U.S. data set. Fisher also adopted the view that the product of the price and quantity index should equal the value ratio between the two periods under consideration, an idea that he already formulated in Fisher (1911; 403). He did not consider explicitly the problem of deflating value added but by 1930, his ideas on deflation and the measurement of quantity growth being essentially the same problem had spread to the problem of deflating nominal value added; see Burns (1930). For additional material on the problems associated with deflating value added, see Chapter 17 of the IMF/ILO/OECD/UNECE/Eurostat/World Bank (2004).

14. The Törnqvist Theil Index as an Approximation to an Economic Output Price Index

An alternative to the Laspeyres and Paasche indexes defined above in (114) and (115) or the Fisher index is to use the Törnqvist (1936)(1937) Theil (1967) price index P_T, whose natural logarithm is defined as follows:

$$(119)\quad \ln P_T(p^0,p^1,q^0,q^1) = \sum_{n=1}^{N} (1/2)(s_n^0 + s_n^1) \ln (p_n^1/p_n^0)$$

where $s_n^t = p_n^t q_n^t / \sum_{k=1}^{N} p_k^t q_k^t$ is the revenue share of commodity n in the total value of sales in period t.

Recall the definition of the period t revenue function, $R^t(p,v)$, defined earlier by (111) above. Now assume that the period t revenue function has the following *translog functional form*[70] ;i.e., for t = 0,1, assume that:

$$\begin{aligned}(120)\quad \ln R^t(p,v) = {}& \alpha_0^t + \sum_{n=1}^{N} \alpha_n^t \ln p_n + \sum_{m=1}^{M} \beta_m^t \ln v_m + (1/2) \sum_{n=1}^{N} \sum_{j=1}^{N} \alpha_{nj}^t \ln p_n \ln p_j \\ & + \sum_{n=1}^{N} \sum_{m=1}^{M} \beta_{nm}^t \ln p_n \ln v_m + (1/2) \sum_{m=1}^{M} \sum_{k=1}^{M} \gamma_{mk}^t \ln v_m \ln v_k\end{aligned}$$

where the coefficients on the right hand side of (120) satisfy the following restrictions:

$$\begin{aligned}
&(121)\quad \textstyle\sum_{n=1}^{N} \alpha_n^t = 1; && t = 0,1;\\
&(122)\quad \alpha_{nj}^t = \alpha_{jn}^t; && t = 0,1;\ 1 \le n < j \le N;\\
&(123)\quad \gamma_{mk}^t = \gamma_{km}^t; && t = 0,1;\ 1 \le m < k \le M;\\
&(124)\quad \textstyle\sum_{n=1}^{N} \alpha_{nj}^t = 0; && t = 0,1;\ j = 1,2,\ldots,N;\\
&(125)\quad \textstyle\sum_{n=1}^{N} \beta_{nm}^t = 0; && t = 0,1;\ m = 1,2,\ldots,M.
\end{aligned}$$

The above restrictions are necessary to ensure that $R^t(p,v)$ is linearly homogeneous in the components of the output price vector p, which is a property that a revenue function must satisfy.[71] Note that at this stage of our argument the coefficients that characterize the technology in each period (the α's, β's and γ's) are allowed to be completely different in each period. It should also be noted that the translog functional form is an example of a *flexible* functional form; i.e., it can approximate an arbitrary technology to the second order.

We can now derive a production theory counterpart to the consumer theory result in Proposition 3 above; i.e., the quadratic identity result in Caves, Christensen and Diewert (1982; 1410) can now be adapted to prove the following result: if the quadratic price coefficients in (120) are equal across the two periods in the index number comparison (i.e., $\alpha_{nj}^0 = \alpha_{nj}^1$ for all n,j), then the geometric mean of the economic output price index that uses period 0 technology and the period 0 input vector v^0, $P^0(p^0,p^1,v^0)$, and the economic output price index that uses period 1 technology and the period 1 input vector

[70] As noted earlier, this functional form was introduced and named by Christensen, Jorgenson and Lau (1971). It was adapted to the revenue function or profit function context by Diewert (1974a).

[71] See Diewert (1973) (1974) for the regularity conditions that a revenue or profit function must satisfy.

v^1, $P^1(p^0,p^1,v^1)$, is *exactly equal* to the Törnqvist output price index P_T defined by (119) above; i.e.,

$$(126)\quad P_T(p^0,p^1,q^0,q^1) = [P^0(p^0,p^1,v^0)\, P^1(p^0,p^1,v^1)]^{1/2}.$$

The assumptions required for this result seem rather weak; in particular, there is no requirement that the technologies exhibit constant returns to scale in either period and our assumptions are consistent with (biased) technological progress occurring between the two periods being compared. Because the index number formula P_T is *exactly* equal to the geometric mean of two theoretical economic output price indexes and it corresponds to a flexible functional form, the Törnqvist output price index number formula is said to be *superlative,* following the terminology used by Diewert (1976).

A few words of caution on the applicability of the economic approach to Producer Price Indexes is warranted.

The above economic approaches to the theory of output price indexes have been based on the assumption that producers take the prices of their outputs as given fixed parameters that they cannot affect by their actions. However, a *monopolistic supplier* of a commodity will be well aware that the average price that can be obtained in the market for their commodity will depend on the number of units supplied during the period. Thus under noncompetitive conditions when outputs are monopolistically supplied (or when intermediate inputs are monopsonistically demanded), the economic approach to producer price indexes breaks down. The problem of modeling noncompetitive behavior typically does not arise in the economic approach to consumer price indexes because, usually, a single household does not have much control over the prices it faces in the marketplace.

The economic approach to producer output price indexes can be modified to deal with certain monopolistic situations. The basic idea is due to Frisch (1936; 14-15) and it involves linearizing the demand functions a producer faces in each period around the observed equilibrium points in each period and then calculating shadow prices which replace market prices. Alternatively, one can assume that the producer is a markup monopolist and simply adds a markup or premium to the relevant marginal cost of production.[72] However, in order to implement these techniques, econometric methods will usually have to be employed. Hence, these methods that rely on econometrics are not really suitable for use by statistical agencies, except in very special circumstances when the problem of noncompetitive behavior is thought to be very significant and the agency has access to econometric resources.

The problems associated with aggregating over producers can be addressed by adapting the analysis presented in section 11 above.

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[72] See Diewert (1993; 584-590) for a more detailed description of these techniques for modeling monopolistic behavior and for additional references to the literature. Diewert and Fox (2008) combine index number techniques with econometric estimation to estimate monopolistic markups.

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