A Note on Schreyer's Model for Depreciation and Welfare

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1. Introduction

The Canberra II Group on capital measurement has been debating the merits of alternative approaches to the measurement of depreciation.[1] Paul Schreyer (2005) has made a new contribution to the debate in a recent note where he developed a simple continuous time model of an economy along the lines pioneered by Weitzman (1976) in order to study the links between sustainable income, depreciation and expected price changes. We will develop a discrete time version of Schreyer's model in this note and show how the usual user cost formula can be derived in this framework.

2. The Production Side of the Model

In each time period t, the technology of the economy is characterized by a sequence of simple one output, two input constant returns to scale *production functions*, F^t, for t = 0,1,2, … .[2] Period t output Y^t equals $F^t(L^t,K^t)$, where L^t is the aggregate amount of *fixed factors* (labour and land) that is available to the economy in period t and K^t is the amount of *reproducible capital* that is available at the start of period t. Period t *output* Y^t is allocated to *domestic consumption*, C^t, and to *exports*, X^t. Thus the first set of equations that characterize the production side of the economy are equations (1):

$$(1)\ C^t + X^t = F^t(L^t,K^t)\ ; \qquad t = 0,1,2,\ldots .$$

Domestic goods are exported in order to purchase units of the capital good. Thus in period t, I^t units of the *investment good* are purchased from abroad at the price p_I^t. We assume that investment goods that are purchased in the current period do not become productive until the following period.[3] The spot price of the domestic commodity in period t is set equal to unity. We assume that international trade is balanced in each period, so that the value of period t exports is equal to the value of the economy's *capital imports*:

$$(2)\ X^t = p_I^t\, I^t\ ; \qquad t = 0,1,2,\ldots .$$

[1] See Diewert (2003) (2005), Ahmad, Aspden and Schreyer (2005), Hill (2004) (2005) and Schreyer, Diewert and Harrison (2005). The papers by Hill (2000) and Hill and Hill (2003) are also relevant to these discussions.
[2] We assume that each $F^t(L,K)$ is a once differentiable, concave, increasing and linearly homogeneous function of L,K.
[3] This means that the length of the accounting period cannot be chosen to be too long.

The *capital input* that is available to the economy at the beginning of period t is denoted by K^t. The period t+1 capital input, K^{t+1}, is related to the period t capital input, K^t, and the period t investment purchases from abroad, I^t, by the following equation:

$$(3)\ K^{t+1} = (1-\delta)K^t + I^t\ ; \qquad t = 0,1,2,\ldots$$

where δ is a constant geometric *depreciation rate* between 0 and 1.

The period t *price of the fixed factor* (a combination of rents and wages), w^t, and the period t *user cost or rental price of capital services*, u^t, are defined as the partial derivatives of the period t production function with respect to the fixed factor and capital respectively:

$$(4)\ w^t \equiv \partial F^t(L^t,K^t)/\partial L\ ; \qquad t = 0,1,2,\ \ldots\ ;$$
$$(5)\ u^t \equiv \partial F^t(L^t,K^t)/\partial K\ ; \qquad t = 0,1,2,\ \ldots\ .$$

Since the period t production function is assumed to exhibit constant returns to scale, Euler's theorem on homogeneous functions implies the following restrictions:

$$(6)\ F^t(L^t,K^t) = w^t L^t + u^t K^t\ ; \qquad t = 0,1,2,\ldots\ .$$

The above assumptions are sufficient for us to be able to construct the economy's *intertemporal consumption possibilities set* S. First, substitute equations (2) into (1) in order to obtain the following system of equations:

$$(7)\ C^t = F^t(L^t,K^t) - p_I^t\, I^t\ ; \qquad t = 0,1,2,\ldots\ .$$

Next, iterate equations (3), starting with t = 0 in order to obtain the following sequence of equations, which relate the beginning of period capital stocks to the sequence of investment decisions made by the economy, starting at period 0:[4]

$$\begin{aligned}(8)\ K^1 &= (1-\delta)K^0 + I^0\ ;\\ K^2 &= (1-\delta)^2K^0 + (1-\delta)I^0 + I^1\ ;\\ K^3 &= (1-\delta)^3K^0 + (1-\delta)^2I^0 + (1-\delta)I^1 + I^2\ ;\\ \ldots &\qquad \ldots\end{aligned}$$

Finally, substitute equations (8) into (7) in order to obtain the economy's sequence of feasible consumption amounts, C^0, C^1, C^2, as functions of the decision variables, I^0, I^1, I^2, … :

[4] For more general models that use this iteration technique, see Dorfman, Samuelson and Solow (1958; 305-345). Because their model is so general (they do not assume geometric depreciation as in (8) above), they do not obtain the familiar user cost formula (22) below.

$$(9)\ C^0 = F^0(L^0,K^0) - p_I^0\, I^0 ;$$
$$C^1 = F^1(L^1,(1-\delta)K^0 + I^0) - p_I^1\, I^1 ;$$
$$C^2 = F^2(L^2,(1-\delta)^2K^0 + (1-\delta)I^0 + I^1) - p_I^2\, I^2 ;$$
$$C^3 = F^3(L^3,(1-\delta)^3K^0 + (1-\delta)^2I^0 + (1-\delta)I^1 + I^2) - p_I^3\, I^3 ;$$
$$\ldots \quad \ldots$$

Equations (9) show that the economy's intertemporal consumption possibilities set S is a function of the economy's *investment decision variables*, I^0, I^1, I^2, … and the following *exogenous variables*: K^0, the economy's initial capital stock; δ, the economy's depreciation rate; L^0, L^1, L^2, …, the economy's sequence of (expected) endowments of fixed factors; p_I^0, p_I^1, p_I^2, …, the economy's sequence of (expected) import prices for the investment good and F^0, F^1, F^2, …, the economy's sequence of (expected) atemporal production functions.

Assume that producers expect the price of output to change over time at the constant *general inflation rate* ρ and that they face the constant period to period *nominal interest rate* r. Then the producer's intertemporal profit maximization problem is:

$$(10)\ \max_{C^0,C^1,\ldots;\, I^0,I^1,\ldots} \{C^0 + (1+\rho)(1+r)^{-1}C^1 + (1+\rho)^2(1+r)^{-2}C^2 + \ldots : (C^0,C^1,C^2,\ldots) \text{ belongs to } S(I^0,I^1,I^2,\ldots)\}$$

where the intertemporal consumption possibilities set $S(I^0,I^1,I^2,\ldots)$ is described by equations (9) above. Thus problem (10) is the problem of maximizing the economy's discounted value of consumption where the producer chooses the sequence of period by period consumption amounts, C^t, as well as the sequence of period by period investments I^t, subject to the constraints given by equations (9).

The intertemporal prices in the producer's maximization problem (10) are expressed in nominal terms and are discounted by nominal interest rates. The structure of the problem can be simplified if we define the *real interest rate* r^* as follows:[5]

$$(11)\ (1+r^*) \equiv (1+r)/(1+\rho).$$

If we substitute (11) into (10), the producer's intertemporal profit maximization problem becomes:

$$(12)\ \max_{C^0,C^1,\ldots;\, I^0,I^1,\ldots} \{C^0 + (1+r^*)^{-1}C^1 + (1+r^*)^{-2}C^2 + \ldots : (C^0,C^1,C^2,\ldots) \in S(I^0,I^1,I^2,\ldots)\}.$$

We turn now to the household intertemporal optimization problem.

3. The Household Side of the Model

[5] In a more general model, the sequence of short term real interest rates would not necessarily be constant as in our present model.

We assume that the aggregate household sector chooses its sequence of consumptions, C^0, C^1, C^2, in order to maximize the following *intertemporal utility function*:

$$(13)\ U(C^0, C^1, C^2, \ldots) \equiv C^0 + \beta C^1 + \beta^2 C^2 + \beta^3 C^3 + \ldots$$

where β is a *taste parameter* between 0 and 1 that reflects the consumer's preference for current consumption over future consumption; the smaller the β is, the more the consumer prefers current period consumption over future period consumption.

The *household intertemporal utility maximization problem* is the following one where the intertemporal utility function defined by (13) is maximized subject to the sequence of consumption choices belongs to the economy's intertemporal consumption possibility set, $S(I^0,I^1,I^2,\ldots)$:

$$(14)\ \max_{C^0,C^1,\ldots;\,I^0,I^1,\ldots} \{C^0 + \beta C^1 + \beta^2 C^2 + \beta^3 C^3 + \ldots : (C^0,C^1,C^2,\ldots) \in S(I^0,I^1,I^2,\ldots)\}.$$

We can use the household's preference parameter β in order to define a real interest rate r^* as follows:

$$(15)\ 1+r^* \equiv \beta^{-1}.$$

Using definition (15), it can be seen that the producer's intertemporal profit maximization problem (12) is in fact identical to the consumer's intertemporal utility maximization problem. Thus by choosing a somewhat restrictive functional form for the consumer's intertemporal utility function, we have reduced the economy's intertemporal general equilibrium model into the following unconstrained maximization problem involving just the investment variables:[6]

$$\begin{aligned}(16)\ \max_{I^0,I^1,\ldots} \{&F^0(L^0,K^0) - p_I^0 I^0 + \beta[F^1(L^1,(1-\delta)K^0 + I^0) - p_I^1 I^1] \\ &+ \beta^2[F^2(L^2,(1-\delta)^2K^0 + (1-\delta)I^0 + I^1) - p_I^2 I^2] \\ &+ \beta^3[F^3(L^3,(1-\delta)^3K^0 + (1-\delta)^2I^0 + (1-\delta)I^1 + I^2) - p_I^3 I^3] + \ldots\}.\end{aligned}$$

4. User Costs

Our concavity assumptions on the atemporal production functions F^t imply that (16) is an infinite dimensional concave programming problem. Assuming that β is small enough, that technical progress is not too great, that the endowments L^t do not grow too rapidly over time, and that the prices of the investment good p_I^t do not fall too quickly, a solution $I^{0*}, I^{1*}, I^{2*}, \ldots$ to (16) will exist. In addition, if the equilibrium I^{t*} are all positive, then the following first order conditions for (16) will be satisfied:

$$\begin{aligned}(17)\ 0 &= -p_I^0 + \beta F_K^1[L^1,(1-\delta)K^0+I^{0*}] + \beta^2(1-\delta)F_K^2[L^2,(1-\delta)^2K^0+(1-\delta)I^{0*}+I^{1*}] + \ldots \\ 0 &= -\beta p_I^1 + \beta^2 F_K^2[L^2,(1-\delta)^2K^0+(1-\delta)I^{0*}+I^{1*}]\end{aligned}$$

[6] We have substituted equations (9) into (14).

$$+ \beta^3(1-\delta)F_K{}^3[L^3,(1-\delta)^3K^0+(1-\delta)^2I^{0*}+(1-\delta)I^{1*}+I^{2*}] + \ldots$$
$$\ldots \quad \ldots$$

where $F_K{}^t$ is the partial derivative, $\partial F^t(L^t,K^{t*})/\partial K$.

Using (5), the *period t anticipated user cost of capital* is

$$(18)\ u^{t*} \equiv \partial F^t(L^t,K^{t*})/\partial K\ ; \qquad\qquad t = 0,1,2,\ldots\ .$$

Substituting (18) into (17) and rearranging slightly gives us the following system of equations relating the anticipated stock prices for the investment good, $p_I{}^t$, to the anticipated equilibrium user costs, u^{t*}:

$$\begin{aligned}(19)\ p_I{}^0 &= \beta u^{1*} + \beta^2(1-\delta)u^{2*} + \beta^3(1-\delta)^2u^{3*} + \ldots\\ p_I{}^1 &= \beta u^{2*} + \beta^2(1-\delta)u^{3*} + \beta^3(1-\delta)^2u^{4*} + \ldots\\ p_I{}^2 &= \beta u^{3*} + \beta^2(1-\delta)u^{4*} + \beta^3(1-\delta)^2u^{5*} + \ldots\\ &\ldots \quad \ldots\end{aligned}$$

Recall from (15) that β equals $1/(1+r^*)$. Thus the first equation in (19) has a familiar interpretation; namely, that a unit of capital purchased in period 0 at the price $p_I{}^0$, has a value that is equal to its discounted (by the real interest rate) stream of future anticipated rental values, taking into account depreciation.[7] The other equations in (19) have similar interpretations.

It is straightforward to verify that equations (19) imply the following sequence of equations:

$$\begin{aligned}(20)\ p_I{}^0 &= \beta u^{1*} + \beta(1-\delta)p_I{}^1\ ;\\ p_I{}^1 &= \beta u^{2*} + \beta(1-\delta)p_I{}^2\ ;\\ &\ldots \quad \ldots\end{aligned}$$

Now use equations (20) to solve for the anticipated user costs in terms of the anticipated future prices of the investment good:

$$\begin{aligned}(21)\ u^{1*} &= \beta^{-1}p_I{}^0 - (1-\delta)p_I{}^1\ ;\\ u^{2*} &= \beta^{-1}p_I{}^1 - (1-\delta)p_I{}^2\ ;\\ &\ldots \quad \ldots\end{aligned}$$

Using (15), the first equation in (21) becomes the following equation:

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[7] Thus u^{1*} is the rental price for a new unit of capital that was purchased in period 0 and is newly installed at the beginning of period 1. Similarly, u^{2*} is the rental price for a new unit of capital that is installed at the beginning of period 1 but of course, the unit of capital that was purchased in period 0 will be equivalent to only $(1-\delta)$ units of a unit of capital that was purchased in period 1 (due to deterioration) and so on.

$$(22)\ u^{1*} = (1+r^*)p_I^0 - (1-\delta)p_I^1$$
$$= r^* p_I^0 + \delta p_I^1 - [p_I^1 - p_I^0].$$

Thus the period 1 anticipated user cost of capital, u^{1*}, is equal to the real interest cost of the capital tied up in period 0, $r^* p_I^0$, plus the period 1 cross sectional depreciation at period 1 prices, δp_I^1, less the anticipated real holding gain, $[p_I^1 - p_I^0]$. In other words, we have derived a very familiar user cost of capital formula in this intertemporal optimization model.

Another instructive way to view equation (22) is as follows. The consumer decides to purchase I^{0*} units of an investment good in period 0. In order to “justify” this purchase, the consumer has to earn a real rate of return r^* on this investment. Thus in period 1, the investment made in period 0 of $p_I^0 I^{0*}$ must be worth:

$$(23)\ (1+r^*)\, p_I^0\, I^{0*} = [u^{1*} + (1-\delta)p_I^1]\, I^{0*}$$

where the equality in (23) follows by rearranging (22). To interpret the right hand side of (23), note that in period 1, the investment will earn the amount $u^{1*}I^{0*}$ in rents and at the end of period 1, the depreciated investment could be sold at the going opportunity cost price, leading to an additional value of $(1-\delta)p_I^1 I^{0*}$. Thus the period 0 investment will indeed earn the real rate of return of r^* if all expectations are realized.

The important thing to note is that anticipated real holding gains enter into the user cost formula (22). Thus they should not be omitted in the production side of the SNA; this anticipated revaluation term is just as much a part of user cost as is depreciation and the opportunity cost of capital. If $p_I^1 > p_I^0$, so that the price of the investment good is expected to increase going from period 0 to 1, then the user cost of buying the investment good in period 0 will fall relative to a situation where the price is expected to remain constant. Why is this? The user cost falls in order to encourage investment (and importation) in period 0 before the price increases. In the opposite situation where the investment price is expected to fall going from period 0 to 1, then the user cost increases in order to discourage purchases in period 0; consumers should wait until they can purchase the investment good at the anticipated lower price that will prevail in period 1, other things being equal.

5. Welfare Aspects

The question now arises: can we use the above model to develop relationships between welfare, depreciation and expected price change for the investment good? The answer to this question is: in general, no.

Suppose that a solution to the consumer’s intertemporal welfare maximization problem (16) exists and denote the resulting optimized objective function as $W(K^0; L^0,L^1,\ldots; p_I^0,p_I^1,\ldots; F^0,F^1,\ldots)$. Thus expected welfare at the start of period 0 depends on the economy’s initial capital stock K^0, the sequence of expected period by period quantities of fixed factors $L^0,L^1,\ldots$, the sequence of expected import prices for the capital good

$p_I^0, p_I^1, \ldots$, and the sequence of atemporal production functions $F^0, F^1, \ldots$, which in turn reflect anticipated technical progress. Now consider expected welfare if we move forward one period, which can be represented by $W(K^1; L^1, L^2, \ldots; p_I^1, p_I^2, \ldots; F^1, F^2, \ldots)$. We can now ask: which of these two expected welfares is higher?

In general, it is not possible to answer this question at our present level of generality. In order to answer it, we would have to make specific assumptions about the exogenous variables, K^0; $L^0, L^1, \ldots$; $p_I^0, p_I^1, \ldots$; $F^0, F^1, \ldots$, in the period 0 model and then solve the intertemporal utility maximization problems (16) that correspond to the period 0 and 1 situations, an extremely difficult task in general. However, by looking at the size of the feasible consumption possibility sets in the two welfare maximization problems, we can obtain some conditions for a welfare improvement or worsening.

Suppose that the following conditions hold:

(24) $K^1 = (1-\delta)K^0 + I^{0*} \geq K^0$ or equivalently, $I^{0*} \geq \delta K^0$;
(25) $p_I^{t+1} \leq p_I^t$; $t = 0,1,2,\ldots$;
(26) $F^{t+1}(L,K) \geq F^t(L,K)$ for all $L \geq 0$ and $K \geq 0$ for $t = 0,1,2,\ldots$.

Assumption (24) means that investment in period 0 is equal to or greater than depreciation; assumption (25) means that the price of the investment good decreases or remains constant over time and assumption (26) means that there is no technological regress over time; i.e., the technology remains constant over time or there is technological progress over time. Under these conditions, it is straightforward to show that the intertemporal consumption possibilities set that starts with the period 1 data is equal to or greater than the intertemporal consumption possibilities set that starts with the period 0 data and thus

(27) $W(K^1; L^1, L^2, \ldots; p_I^1, p_I^2, \ldots; F^1, F^2, \ldots) \geq W(K^0; L^0, L^1, \ldots; p_I^0, p_I^1, \ldots; F^0, F^1, \ldots)$;

i.e., expected welfare starting at period 1 is equal to or greater than expected welfare starting at period 0 under assumptions (24)-(26). Similarly, it can be shown that the inequality (27) reverses if we reverse the inequalities (24)-(26).

Our conclusion at this point is that *the welfare basis of net product is not very easy to establish*. That is, it is generally thought that some concept of net national product is a measure that can be used to measure welfare change or at least, Net National Product will be closer to a welfare measure than Gross National Product. This is probably true in very simple models but the above algebra indicates that finding an appropriate depreciation adjusted measure of output that can be used to predict welfare change is a very difficult task in more general models.

6. Conclusion

There are two main points that emerge from this note:

- In a Schreyer like intertemporal general equilibrium model, a "traditional" user cost or rental price formula for capital services, (22), emerges that consists of three terms (real opportunity cost of capital, cross sectional depreciation and expected holding gains). Thus there does not appear to be any good theoretical reason for excluding expected real holding gains from the user cost of capital and hence in the production accounts of the System of National Accounts.[8]
- The status of Net National Product as a measure of output that can be given a welfare interpretation is difficult to establish in models that involve international trade and technological change.

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[8] There may be practical reasons for excluding anticipated real holding gains; i.e., they may be difficult to estimate in a reproducible way.

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