Export Import Price Index Manual

24. Measuring the Effects of Changes in the Terms of Trade

July 26, 2008 draft.

A. Introduction

A.1 Chapter Overview

A terms of trade index is generally defined as an economy's index of export prices divided by an index of import prices. The implementation of this definition would not warrant a chapter in a textbook or in a Manual: it is more or less straightforward. However, many economists over the years have observed that an improvement in an economy's terms of trade has effects that are very similar to an improvement in Total Factor Productivity or Multifactor Productivity.[1] Economists have also been interested in quantifying the effects of changing international prices on the real income generated by an economy. Once the discussion of changes in the terms of trade is broadened to include these topics, the original simplicity of the terms of trade index vanishes. Thus the purpose of this chapter is to address the effects of changing international prices on the real income of an economy or a sector of an economy. In order to narrow the topic, only an approach to these measurement problems that is based on economic approaches to producer and consumer theory will be considered.

In section A.2 of this chapter, a technical introduction to the effects of changing international prices on the growth of an economy's real income will be undertaken. A production theory framework is laid out and some preliminary definitions are made.[2]

Section B considers the effects of a change in the real export price facing the economy on the real income generated by the market oriented production sector of the economy while section C considers the effects of a change in the real import price. Various theoretical definitions for these effects are considered and empirical approximations to these theoretical indexes are defined and analyzed. In section D the combined effects of changes in real import and export prices on the real income generated by the production sector are considered. These combined effects indexes are then related to the partial indexes defined in sections B and C.

Some goods and services are imported directly into the household sector. An important example of such expenditures is tourism expenditures abroad. The production theory approach developed in sections B through D is not applicable for these classes of

[1] For materials on these productivity concepts, see the pioneering articles by Jorgenson and Griliches (1967) (1972) and the excellent OECD Manuals written by Schreyer (2001) (2007).

[2] Background material on producer theory approaches to production theory can be found in Caves, Christensen and Diewert (1982), Diewert (1983), Balk (1998), Alterman, Diewert and Feenstra (1999) and chapter 18 of the present Manual.

household imported goods and services so in section E, a consumer theory approach is developed. It turns out that the structure of the producer theory methodology can readily be adapted to deal with this situation with a few key changes.

There are also certain goods and services that are directly exported by households. For example, self employed consultants can directly export business services to customers around the world. Also small scale household manufacturers of clothing and other goods can advertise on the internet and sell their products abroad rather easily. Thus there is a need to model household exports, as well as goods and services that are directly imported by households. However, in principle, household exports can be treated using the production theory methodology developed in sections B through D: all that needs to be done is to create a set of household production accounts.[3] Thus these household production units will use various capital inputs (machines, parts of the structures that they inhabit), intermediate inputs and their own labor in order to produce commodities for sale in their domestic and foreign markets. This household production sector is much the same as "regular" incorporated production units except that it will usually be difficult to get accurate measures of the capital employed and the labor used by these household production units. However, as the reader will note, when the producer theory approach to exports and imports is developed in sections B-D, it is not necessary to know what inputs of labour and capital are actually used by the production units in order to implement the terms of trade adjustment factors that are developed in these sections. Thus there is no need to develop a separate theory for directly exported goods and services by households.

Section E concludes.

A.2 Technical Introduction

Let $P_X{}^t$ be the price index for exports in an economy in period t and let $P_M{}^t$ be the corresponding import price index. Then the period t *terms of trade index*, T^t, is defined as an export price index divided by an import price index:

$$(24.1)\quad T^t \equiv P_X{}^t/P_M{}^t\,; \qquad\qquad t = 0,1.$$

A country's terms of trade is said to have *improved* going from period 0 to 1 if T^1/T^0 is greater than one and to have *deteriorated* if T^1/T^0 is less than one. For an improvement, the export price index has increased more rapidly than the import price index.

Thus the definition of a terms of trade index is very straightforward and relatively easy to implement: only the exact form of the export and import price index needs to be determined. Presumably, preliminary versions of a terms of trade index would use Laspeyres type indexes while a retrospective, historical version, compiled when current period weights become available, would use a superlative index. However, the definition of a terms of trade index is not the end of the story as will be explained below.

[3] In practice, this is not an easy task!

It has been well known for a long time that an improvement in a country's terms of trade is beneficial for a country and has effects that are similar to an improvement in the country's Total Factor Productivity or Multifactor Productivity.[4] However, determining how to measure *precisely* the degree of improvement due to a change in a country's terms of trade has proven to be a difficult question.

The measurement question addressed in this chapter is the following one: can the effects of changes in the price of exports and imports on the growth of *real income* in the economy be determined? Thus at the outset, the focus is on the measurement of the real income generated by the economy and then the effects of changes in international prices on the chosen real income measure will be considered.

To begin the analysis, consider the following definition for the *net domestic product* of a country in period t, NDP^t, as the sum of the usual macroeconomic aggregates:

$$(24.2)\ NDP^t = P_C{}^t C^t + P_I{}^t I^t + P_G{}^t G^t + P_X{}^t X^t - P_M{}^t M^t\ ; \qquad t = 0,1$$

where NDP^t is the net domestic product produced by the economy in period t, C^t, I^t, G^t, X^t and M^t are the period t quantities of consumption, net investment[5], government final consumption, exports and imports respectively and $P_C{}^t$, $P_I{}^t$, $P_G{}^t$, $P_X{}^t$ and $P_M{}^t$ are the corresponding period t final demand prices. Using the usual circular flow arguments used by national income accountants, net domestic product is produced by the production sector in the economy and the value of this production generates a flow of income received by primary inputs used in the economy. It is growth in this flow of income (which is also equal to NDP^t) that is to be analyzed in this chapter.[6]

The rate of growth in the flow of nominal net product going from period 0 to 1, NDP^1/NDP^0, (or more accurately, one plus this rate of growth), is of limited interest to policy analysts and the public as an indicator of welfare growth because it includes the effects of general inflation. Thus it is necessary to deflate the nominal net domestic product in period t, NDP^t, by a "reasonable" period t deflator or price index, say $P_D{}^t$. The first problem that needs to be addressed is: what is a "reasonable" deflator?

Three choices have been suggested in the literature:

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[4] See Diewert and Morrison (1986), Morrison and Diewert (1990) and Kohli (1990).

[5] Note that when the focus is on income flows generated by an economy, it is necessary to deduct depreciation of capital from gross investment since depreciation is not a sustainable income flow. Thus in this chapter, the target macroeconomic aggregate is (deflated) net domestic product rather than gross domestic product.

[6] Note that the flow of income of concern here is the income received by primary inputs used in the market sector of the economy and thus excludes the difference between real primary incomes and current transfers receivable and payable from abroad. Indeed the framework used by the *2008 SNA* and outlined in Silver and Mahdavy (1989) defines real net disposable national income as the volume of GDP, plus the trading gain or loss resulting from changes in the terms of trade, plus difference between real primary incomes and current transfers receivable and payable from abroad. The formulas for the terms of trade effect given in the *2008 SNA* is, unlike the formal framework outlined here, heuristic in nature.

- The price of consumption, $P_C{}^t$;
- The price of domestic goods or the price of absorption, $P_A{}^t$ (an aggregate of $P_C{}^t$, $P_I{}^t$ and $P_G{}^t$) or
- The net domestic product deflator, $P_N{}^t$ (an aggregate of $P_C{}^t$, $P_I{}^t$, $P_G{}^t$, $P_X{}^t$ and $P_M{}^t$ where $P_M{}^t$ has negative weights).

The consumption price deflator, $P_C{}^t$, and the absorption deflator, $P_A{}^t$, can be justified. Diewert and Lawrence (2006) and Diewert (2008) preferred the first deflator while Kohli (2006) preferred the second one. However, these authors do not recommend the use of either the GDP deflator or the NDP deflator in the present context because they maintain that since virtually all internationally traded goods are intermediate goods and hence are not directly consumed by households, the prices of these goods are not needed to deflate nominal income flows into real income flows.[7] The case for using the price of consumption as a deflator for the nominal income that is generated by the production side of the economy is very simple: the deflated amount, $NDP^t/P_C{}^t$, is the potential amount of consumption that could be purchased by the owners of primary inputs in period t if they chose to buy zero units of net investment and government outputs. If the price of domestic absorption is used as the deflator, then $NDP^t/P_A{}^t$ is the number of units of a (constant utility) aggregate of C, I and G that could be purchased by the suppliers of primary inputs to the production sector of the economy in period t.

Suppose that a choice of the nominal income deflator, $P_D{}^t$, has been made. It is now desired to look at the *growth of the real income generated by the production sector in the economy*; i.e., look at the growth of $NDP^t/P_D{}^t$:

$$(24.3)\ NDP^t/P_D{}^t = [P_C{}^t C^t + P_I{}^t I^t + P_G{}^t G^t + P_X{}^t X^t - P_M{}^t M^t]/P_D{}^t; \qquad t = 0,1$$
$$= p_C{}^t C^t + p_I{}^t I^t + p_G{}^t G^t + p_X{}^t X^t - p_M{}^t M^t$$

where the *real prices* of consumption, net investment, government consumption, exports and imports are defined as the nominal prices divided by the chosen income deflator $P_D{}^t$:[8]

$$(24.4)\ p_C{}^t \equiv P_C{}^t/P_D{}^t\ ;\ p_I{}^t \equiv P_I{}^t/P_D{}^t\ ;\ p_G{}^t \equiv P_G{}^t/P_D{}^t;\ p_X{}^t \equiv P_X{}^t/P_D{}^t\ ;\ p_M{}^t \equiv P_M{}^t/P_D{}^t.$$

Using equations (24.3) and definitions (24.4), (one plus) *the rate of growth of real income* over the two periods under consideration can be defined as follows:

$$(24.5)\ [NDP^1/P_D{}^1]/[NDP^0/P_D{}^0]$$
$$= [p_C{}^1C^1 + p_I{}^1I^1 + p_G{}^1G^1 + p_X{}^1X^1 - p_M{}^1M^1]/[p_C{}^0C^0 + p_I{}^0I^0 + p_G{}^0G^0 + p_X{}^0X^0 - p_M{}^0M^0].$$

Looking at equation (24.5), it can be seen that, holding all else constant, an increase in the period 1 real price of exports $p_X{}^1$ will *increase* real income growth generated by the

[7] There are other reasons for not using the GDP or NDP deflators as measures of general inflation; see Kohli (1982; 211) (1983; 142), Hill (1996; 95) and Diewert (2002; 556-560) for additional discussion.

[8] If it is desired to explain nominal income growth generated by the production sector, then it is not necessary to deflate the period t data by $P_D{}^t$. In this case, it can be assumed that $P_D{}^t$ equals one so that $P_C{}^0 = p_C{}^0$, and so forth.

production sector of the economy. Conversely, an increase in the period 1 real price of imports $p_M{}^1$ will *decrease* real income growth.

Equation (24.5) indicates the complexity of trying to determine the effects of changes in real import and export prices on the growth of real income: p_X and p_M change but so do the real prices of consumption, net investment and government consumption. In addition, the quantities of C, I, G, X and M are changing and in the background, there are also changes in the amount of labour L and capital K that is being utilized by the economy's production sector. It is evident that some measure of the effect on real income growth of the changes in the real prices of exports and imports is desired, holding constant the rest of the economic environment. But if export and import prices change, producers will be induced to change the composition of their exports and imports. Thus a careful specification of what is exogenous and what is endogenous is needed in order to isolate the effects of changes in real export and import prices.

In the following section, a production theory framework will be used in order to specify more precisely exactly what is being held fixed and what is being allowed to vary as real export and import prices change. Other approaches to modeling the effects on production and welfare of changes in the prices of exports and imports are reviewed in Diewert and Morrison (1986), Silver and Mahdavy (1989) and Kohli (2006).[9]

B. The Effects of Changes in the Real Price of Exports

B.1 Theoretical Measures of the Effects of Changes in the Real Price of Exports

Kohli (1978) (1991) has long argued that since most internationally traded goods are intermediate products and services, it is natural to model the effects of international trade using production theory.[10] Kohli's example will be followed in this section and in subsequent sections and a production theory framework will be used with exports as outputs of the production sector and imports as intermediate inputs into the production sector.

For simplicity, it is assumed that C, I, G and X (consumption, net investment, government consumption and exports) are outputs of the production sector and M, L and K (imports, labour and capital) are inputs into the production sector.[11] In period t, there is a feasible set of (C,I,G,X,M,L,K) outputs and inputs, which is denoted by the set S^t for periods t equal to 0 and 1. It will prove useful to define the economy's period t *real net domestic product function*, $n^t(p_C,p_I,p_X,p_M,L,K)$ for t = 0,1:

(24.6) $n^t(p_C,p_I,p_G,p_X,p_M,L,K)$

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[9] Ulrich Kohli, the chief economist for the Swiss National Bank, has long had an interest in adjusting income measures for changes in a country's terms of trade using production theory; see Kohli (1990) (2003) (2004a) (2004b) (2006) and Fox and Kohli (1998). Kohli's methodology is compared with the Diewert and Lawrence methodology that is used in this chapter in Diewert (2008).

[10] The recent textbook by Feenstra (2004) also takes this point of view.

[11] These scalar quantities could be replaced by vectors but this extension is left to the reader.

$$\equiv \max_{C,I,G,X,M} \{ p_C C + p_I I + p_G G + p_X X - p_M M : (C,I,G,X,M,L,K) \text{ belongs to } S^t\}.$$

Thus the real net product $n^t(p_C,p_I,p_G,p_X,p_M,L,K)$ is the maximum amount of (net) real value added that the economy can produce if producers face the real price p_C for consumption, the real price p_I for net investment, the real price p_G for government consumption, the real price p_X for exports and the real price p_M for imports and given that producers have at their disposal the period t production possibilities set S^t as well as the amount L of labour services and the amount K of capital (waiting) services.[12]

It is reasonable to assume that the actual period t amounts of outputs produced and inputs used in period t, $C^t,I^t,G^t,X^t,M^t,L^t,K^t$, belong to the corresponding period t production possibilities set, S^t, for t = 0,1. It is a stronger assumption to assume that producers are competitively profit maximizing in periods 0 and 1 so that the following equalities are valid:

$$(24.7)\quad n^t(p_C{}^t,p_I{}^t,p_G{}^t,p_X{}^t,p_M{}^t,L^t,K^t) = p_C{}^t C^t + p_I{}^t I^t + p_G{}^t G^t + p_X{}^t X^t - p_M{}^t M^t\,; \qquad t = 0,1$$

where $p_C{}^t,p_I{}^t,p_G{}^t,p_X{}^t,p_M{}^t$ are the real prices for consumption, net investment, government consumption, exports and imports that producers face in period t [13] and L^t and K^t are the amounts of labour and capital used by producers in period t. In what follows, it will be assumed that equations (24.7) hold. Basically, these equations rest on the assumption that producers in the economy are competitively maximizing net domestic product in periods 0 and 1 subject to the technological constraints on the economy for each period.

In a first attempt to measure the effects of changing real export prices over the two periods under consideration, a hypothetical net domestic product maximization problem is considered where producers have at their disposal the period 0 technology set S^0, the period 0 actual labour and capital inputs, L^0 and K^0 respectively, and they face the period 0 real prices for consumption, net investment, government consumption and imports, $p_C{}^0$, $p_I{}^0$, $p_G{}^0$ and $p_M{}^0$ respectively, but they face the period 1 real export price, $p_X{}^1$. The solution to this hypothetical net product maximization problem is $n^0(p_C{}^0,p_I{}^0,p_G{}^0,p_X{}^1,p_M{}^0,L^0,K^0)$. Using this hypothetical net product or net income, *a theoretical Laspeyres type measure* α_{LX} *of the effects on real income growth of changes in real export prices* from the period 0 level, $p_X{}^0$, to the period 1 level, $p_X{}^1$, can be

[12] Depreciation has been subtracted from gross investment so the user cost of capital in the present model excludes depreciation so the price of capital services is basically Rymes' (1968) (1983) waiting services; see also Cas and Rymes (1991).

[13] Producers actually face the prices $P_C{}^t,P_I{}^t,P_G{}^t,P_X{}^t,P_M{}^t$ rather than the deflated (by $P_D{}^t$) prices $p_C{}^t,p_I{}^t,p_G{}^t,p_X{}^t,p_M{}^t$. However, if producers maximize net product facing the prices $P_C{}^t,P_I{}^t,P_X{}^t.P_M{}^t$, they will also maximize net product facing the real prices $p_C{}^t,p_I{}^t,p_G{}^t,p_X{}^t,p_M{}^t$. There is one additional difficulty: the prices that producers face are different than the prices that consumers and other final demanders face because of commodity taxes. Thus strictly speaking, the theory that is developed in this section and subsequent sections that relies on production theory applies to producer prices (or basic prices) rather than final demand prices.

defined as the ratio of the hypothetical net real income $n^0(p_C^0,p_I^0,p_G^0,p_X^1,p_M^0,L^0,K^0)$ to the actual period 0 net real income $n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0)$:[14]

$$(24.8)\ \alpha_{LX} \equiv n^0(p_C^0,p_I^0,p_G^0,p_X^1,p_M^0,L^0,K^0)/n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0).$$

The index α_{LX} of the effects of the change in the real price of exports is termed a *Laspeyres type index* because it holds constant all exogenous prices and quantities at their period 0 levels except for the two real export prices, p_X^0 and p_X^1, and the index also holds technology constant at the base period level.

Using assumption (24.7) for t = 0, the denominator on the right hand side of (24.8) is equal to period 0 observed real net product, $p_C^0 C^0 + p_I^0 I^0 + p_G^0 G^0 + p_X^0 X^0 - p_M^0 M^0$. Using definition (24.6), it can be seen that C^0, I^0, G^0, X^0 and M^0 is a feasible solution for the net product maximization problem defined by the numerator on the right hand side of (24.8), $n^0(p_C^0,p_I^0,p_G^0,p_X^1,p_M^0,L^0,K^0)$. These facts mean that there is the following *observable lower bound* to the theoretical index α_{LX} defined by (24.8):[15]

$$\begin{aligned}(24.9)\ \alpha_{LX} &\geq [p_C^0 C^0+p_I^0 I^0+p_G^0 G^0+p_X^1 X^0-p_M^0 M^0]/[p_C^0 C^0+p_I^0 I^0+p_G^0 G^0+p_X^0 X^0-p_M^0 M^0]\\ &\equiv P_{LX}\end{aligned}$$

where P_{LX} is an *observable Laspeyres type index* of the effects on real income of a change in real export prices going from period 0 to 1. P_{LX} generally understates the hypothetical change in the real income generated by the economy which is defined by the theoretical index α_{LX} due to *substitution bias*; i.e., the change in the real price of exports will induce producers to substitute away from their base period production decisions in order to take advantage of the change in real export prices from p_X^0 to p_X^1. Note that the numerator and denominator on the right hand side of (24.9) are identical except that p_X^1 appears in the numerator and p_X^0 appears in the denominator.

It is possible to show that the Laspeyres type observable index P_{LX} is a first order Taylor series approximation to the theoretical Laspeyres type index α_{LX} as will be shown below. A first order Taylor series approximation to the hypothetical net real income defined by $n^0(p_C^0,p_I^0,p_G^0,p_X^1,p_M^0,L^0,K^0)$ is given by the first line of (24.10) below:[16]

$$\begin{aligned}(24.10)\ n^0(p_C^0,p_I^0,p_G^0,p_X^1,p_M^0,L^0,K^0) &\approx n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0)\\ &\quad + [\partial n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0)/\partial p_X][p_X^1 - p_X^0]\end{aligned}$$

[14] Definition (24.8) is similar to the Laspeyres output price effect defined by Diewert and Morrison (1986; 666) except that they used a GDP function instead of a net product function and they did not deflate their aggregate by a price index. Diewert, Mizobuchi and Nomura (2005; 19-20) and Diewert and Lawrence (2006; 12-17) developed much of the theory used in this chapter.

[15] The inequality (24.9) rests on a feasibility argument and this type of argument was first used by Konüs (1924) in the consumer price context.

[16] This type of approximation was used by Diewert (1983; 1095-1096) and Morrison and Diewert (1990; 211-212) in the producer theory context but the basic technique (in the consumer theory context) is due to Hicks (1942; 127-134) (1946; 331).

$= n^0(p_C{}^0,p_I{}^0,p_G{}^0,p_X{}^0,p_M{}^0,L^0,K^0) + X^0[p_X{}^1 - p_X{}^0]$ using Hotelling's Lemma[17]
$= p_C{}^0 C^0+p_I{}^0 I^0+p_G{}^0 G^0+p_X{}^0 X^0-p_M{}^0 M^0 + X^0[p_X{}^1 - p_X{}^0]$ using (7) for t = 0
$= p_C{}^0 C^0+p_I{}^0 I^0+p_G{}^0 G^0+p_X{}^1 X^0-p_M{}^0 M^0$

where $p_C{}^0 C^0+p_I{}^0 I^0+p_G{}^0 G^0+p_X{}^1 X^0-p_M{}^0 M^0$ is the numerator on the right hand side of (24.9). Since the denominator on the right hand side of (24.9) is equal to $p_C{}^0 C^0+p_I{}^0 I^0+p_G{}^0 G^0+p_X{}^0 X^0-p_M{}^0 M^0$ which in turn is equal to $n^0(p_C{}^0,p_I{}^0,p_G{}^0,p_X{}^0,p_M{}^0,L^0,K^0)$, it can be seen that P_{LX} is indeed a first order approximation to the theoretical index α_{LX} defined by (24.8).[18]

In a second attempt to measure the effects of changing real export prices over the two periods under consideration, a hypothetical net domestic product maximization problem is considered where producers have at their disposal the period 1 technology set S^1, the period 1 actual labour and capital inputs, L^1 and K^1 respectively, and they face the period 1 real prices for consumption, net investment, government consumption and imports, $p_C{}^1$, $p_I{}^1$, $p_G{}^1$ and $p_M{}^1$ respectively, but they face the period 0 real export price, $p_X{}^0$. The solution to this hypothetical (real) net product maximization problem is $n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^0,p_M{}^1,L^1,K^1)$. Using this hypothetical net product or net income, *a theoretical Paasche type measure* α_{PX} *of the effects on real income growth of changes in real export prices* from the period 0 level, $p_X{}^0$, to the period 1 level, $p_X{}^1$, can be defined as the ratio of the actual period 1 net real income $n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^1,p_M{}^1,L^1,K^1)$ to the hypothetical net real income $n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^0,p_M{}^1,L^1,K^1)$:[19]

(24.11) $\alpha_{PX} \equiv n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^1,p_M{}^1,L^1,K^1)/n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^0,p_M{}^1,L^1,K^1)$.

The index α_{PX} of the effects of the change in the real price of exports is termed a *Paasche type index* because it holds constant all exogenous prices and quantities at their period 1 levels except for the two real export prices, $p_X{}^0$ and $p_X{}^1$, and the index also holds technology constant at the period 1 level.

Using assumption (24.7) for t = 1, the numerator on the right hand side of (24.11) is equal to period 1 observed real net product, $p_C{}^1 C^1 + p_I{}^1 I^1 + p_G{}^1 G^1 + p_X{}^1 X^1 - p_M{}^1 M^1$. Using definition (24.6), it can be seen that C^1, I^1, G^1, X^1 and M^1 is a feasible solution for the net product maximization problem defined by the denominator on the right hand side of (24.11), $n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^0,p_M{}^1,L^1,K^1)$. These facts mean that there is the following *observable upper bound* to the theoretical index α_{LX} defined by (24.11):

(24.12) $\alpha_{PX} \leq [p_C{}^1C^1+p_I{}^1 I^1+p_G{}^1 G^1+p_X{}^1 X^1-p_M{}^1 M^1]/[p_C{}^1C^1+p_I{}^1 I^1+p_G{}^1 G^1+p_X{}^0 X^1-p_M{}^1 M^1]$
$\equiv P_{PX}$

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[17] Hotelling's Lemma (1932; 594) says that the first order partial derivatives of the net product function $n^t(p_C{}^t,p_I{}^t,p_G{}^t,p_X{}^t,p_M{}^t,L^t,K^t)$ with respect to the prices p_C,p_I,p_G,p_X,p_M are equal to C^t,I^t,G^t,X^t and $-M^t$ respectively for t = 0,1.

[18] This result was established by Diewert and Lawrence (2006; 16).

[19] Definition (24.11) is analogous to the Paasche output price effect defined by Diewert and Morrison (1986; 666) in the nominal GDP context. Diewert, Mizobuchi and Nomura (2005; 19) and Diewert and Lawrence (2006; 13) used definitions (24.8), (24.11) and (24.14).

where P_{PX} is an *observable Paasche type index* of the effects on real income of a change in real export prices going from period 0 to 1. P_{PX} generally overstates the hypothetical change in the real income generated by the economy which is defined by the theoretical index α_{PX} due to *substitution bias*; i.e., the change in the real price of exports from p_X^1 to p_X^0 will induce producers to substitute away from their period 1 production decisions so that $n^1(p_C^1,p_I^1,p_G^1,p_X^0,p_M^1,L^1,K^1)$ will generally be greater than $[p_C^1C^1+p_I^1 I^1+p_G^1 G^1+p_X^0 X^1-p_M^1 M^1]$ so that $1/n^1(p_C^1,p_I^1,p_G^1,p_X^0,p_M^1,L^1,K^1)$ will generally be less than $1/[p_C^1C^1+p_I^1 I^1+p_G^1 G^1+p_X^0 X^1-p_M^1 M^1]$ and the inequality in (24.12) follows. Note that the numerator and denominator on the right hand side of (24.12) are identical except that p_X^1 appears in the numerator and p_X^0 appears in the denominator.

It is possible to show that the Paasche type observable index P_{PX} is a first order Taylor series approximation to the theoretical Paasche type index α_{PX} as will be shown below. The proof is entirely analogous to the derivation of (24.10). A first order Taylor series approximation to the hypothetical net real income defined by $n^1(p_C^1,p_I^1,p_G^1,p_X^0,p_M^1,L^1,K^1)$ is given by the first line of (24.13) below:

$$
\begin{aligned}
(24.13)\ n^1(p_C^1,p_I^1,p_G^1,p_X^0,p_M^1,L^1,K^1) &\approx n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^1,L^1,K^1) \\
&\quad + [\partial n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^1,L^1,K^1)/\partial p_X][p_X^0 - p_X^1] \\
&= n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^1,L^1,K^1) + X^1[p_X^0 - p_X^1] \qquad \text{using Hotelling's Lemma} \\
&= p_C^1 C^1+p_I^1 I^1+p_G^1 G^1+p_X^1 X^1-p_M^1 M^1 + X^1[p_X^0 - p_X^1] \qquad \text{using (7) for } t = 1 \\
&= p_C^1 C^1+p_I^1 I^1+p_G^1 G^1+p_X^0 X^1-p_M^1 M^1
\end{aligned}
$$

where $p_C^1 C^1+p_I^1 I^1+p_G^1 G^1+p_X^0 X^1-p_M^1 M^1$ is the denominator on the right hand side of (24.12). Since the numerator on the right hand side of (24.12) is equal to $p_C^1C^1+p_I^1 I^1+p_G^1 G^1+p_X^1 X^1-p_M^1 M^1$ which in turn is equal to $n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^1,L^1,K^1)$, it can be seen that P_{PX} is indeed a first order approximation to the theoretical index α_{PX} defined by (24.11).[20]

Note that both the Laspeyres and Paasche theoretical indexes of the effects on real income generated by the production sector of a change in the (real) price of exports are equally plausible and there is no reason to use one or the other of these two indexes. Thus if it is desired to have a single theoretical measure of the effects of a change in real export prices, α_{LX} and α_{PX} should be averaged in a symmetric fashion to form a single target index that would summarize the effects on real income growth of a change in real export prices. Two obvious choices for the symmetric average are the arithmetic or geometric means of α_{LX} and α_{PX}. Following Diewert (1997) and Chapter 16, it seems preferable to use the geometric mean of α_{LX} and α_{PX} as the "best" single theoretical estimator of the effects of a change in real export prices on real income growth, since the resulting Fisher (1922) like theoretical index satisfies the time reversal test so that if the ordering of the two periods is switched, the resulting index is the reciprocal of the

[20] This result was established by Diewert and Lawrence (2006; 16) and is closely related to similar results derived by Morrison and Diewert (1990; 211-213).

original index.[21] Thus define the *theoretical Fisher type measure* α_{FX} *of the effects on real income growth of changes in real export prices* as the geometric mean of the Laspeyres and Paasche type theoretical measures:

(24.14) $\alpha_{FX} \equiv [\alpha_{LX}\alpha_{PX}]^{1/2}$.

With the target index defined by (24.14) in mind, in the following section, the problem of finding empirical approximations to this theoretical index will now be considered.

B.2 Empirical Measures of the Effects of Changes in the Real Price of Exports on the Growth of Real Income Generated by the Production Sector

Two empirical indexes that provide estimates of the effects on the growth of real income of a change in real export prices have already been defined in section B.1 above: the Laspeyres type index P_{LX} defined on the right hand side of (24.9) and the Paasche type index P_{PX} defined on the right hand side of (24.12). It was noted that P_{LX} was a lower bound to the theoretical index α_{LX} and P_{PX} was an upper bound to the theoretical index α_{PX}. Thus P_{LX} will generally have a downward bias compared to its theoretical counterpart while P_{PX} will generally have a upward bias compared to its theoretical counterpart These inequalities suggest that the geometric mean of P_{LX} and P_{PX} is likely to be a reasonably good approximation to the target Fisher type index α_{FX} defined as the geometric mean of α_{LX} and α_{PX}. Thus define the *Diewert Lawrence index of the effects on real income of a change in real export prices going from period 0 to 1* as follows:[22]

(24.15) $P_{DLX} \equiv [P_{LX}\, P_{PX}]^{1/2}$.

It will be useful to develop some alternative expressions for the indexes P_{LX}, P_{PX} and P_{DLX}.

As a preliminary step in developing these alternative expressions, recall definitions (24.7) which defined the production sector's period t real net product, $n^t(p_C^t,p_I^t,p_G^t,p_X^t,p_M^t,L^t,K^t)$ for t = 0,1, which will be abbreviated to n^t. The period t shares of net product of C, I, G, X and M are defined in the usual way as follows:

(24.16) $s_C^t \equiv p_C^tC^t/n^t$; $s_I^t \equiv p_I^tI^t/n^t$; $s_G^t \equiv p_G^tG^t/n^t$; $s_X^t \equiv p_X^tX^t/n^t$; $s_M^t \equiv -p_M^tM^t/n^t$; t = 0,1.

It can be seen that the shares defined by (24.16) sum up to unity for each period t but note that the period t "share" for imports, s_M^t, is negative whereas the other shares are positive.

Now consider the definition of P_{LX} which occurred in (24.9) and subtract 1 from this expression:

---

[21] The arithmetic average of the Laspeyres and Paasche theoretical indexes does not satisfy this time reversal test.

[22] Diewert and Lawrence (2006; 14-17) seem to have been the first to define and empirically estimate the indexes defined by P_{LX}, P_{PX} and (24.15) but the closely related work of Morrison and Diewert (1990; 211-212) should also be noted.

$$
\begin{aligned}
(24.17)\ P_{LX} - 1 \\
&= [p_C{}^0 C^0+p_I{}^0 I^0+p_G{}^0 G^0+p_X{}^1 X^0-p_M{}^0 M^0]/[p_C{}^0 C^0+p_I{}^0 I^0+p_G{}^0 G^0+p_X{}^0 X^0-p_M{}^0 M^0] - 1 \\
&= [p_X{}^1 - p_X{}^0]\, X^0/[p_C{}^0 C^0+p_I{}^0 I^0+p_G{}^0 G^0+p_X{}^0 X^0-p_M{}^0 M^0] \\
&= [(p_X{}^1/p_X{}^0) - 1][p_X{}^0 X^0]/[p_C{}^0 C^0+p_I{}^0 I^0+p_G{}^0 G^0+p_X{}^0 X^0-p_M{}^0 M^0] \\
&= s_X{}^0\, [r_X - 1]
\end{aligned}
$$

where r_X is (one plus) the rate of growth in the real price of exports going from 0 to 1; i.e.,

$$(24.18)\ r_X \equiv p_X{}^1/p_X{}^0.$$

Thus P_{LX} depends on only $(r_X - 1)$, the growth rate in the price of real exports going from period 0 to 1 and $s_X{}^0$, the share of exports in period 0 real net product; i.e.,

$$(24.19)\ P_{LX} = 1 + s_X{}^0\, (r_X - 1).$$

Using similar techniques, it can be shown that P_{PX} depends only on $s_X{}^1$, the share of exports in period 1, and the real export price relative, r_X defined by (24.18):

$$(24.20)\ P_{PX} = [1 + s_X{}^1(r_X{}^{-1} - 1)]^{-1}.$$

Comparing (24.19) and (24.20), it can be seen that both P_{LX} and P_{PX} are increasing functions of r_X so that as the real price of exports increases, both indexes of growth in real income also increase as expected. It can also be seen that P_{LX} is increasing (decreasing) in $s_X{}^0$ and P_{PX} is increasing (decreasing) in $s_X{}^1$ if r_X is more (less) than one. These properties are also intuitively sensible.

Substituting expressions (24.19) and (24.20) into (24.15) leads to the following expression for the Diewert Lawrence export index:

$$(24.21)\ P_{DLX} = \{[1 + s_X{}^0\, (r_X - 1)]/[1 + s_X{}^1(r_X{}^{-1} - 1)]\}^{1/2}.$$

As indicated above, the Diewert Lawrence index P_{DLX} defined by (24.21) is likely to be closer to the target Fisher index α_{FX} defined by (24.14) than the Laspeyres and Paasche type indexes P_{LX} and P_{PX} defined by (24.19) and (24.20).

There is one additional empirically defined index that attempts to measure the effects of a change in real export prices on the growth of real income generated by the production sector and that is based on the work of Diewert and Morrison (1986). Using the same notation that is used in (24.21) above, the logarithm of the *Diewert Morrison index,* P_{DMX}*, of the effects on real income of a change in real export prices going from period 0 to 1* is defined as follows:[23]

[23] Strictly speaking, Diewert and Morrison (1986; 666) defined their index in the context of a GDP function rather than a net product function and did not deflate prices by a price index. The first applications of

(24.22) $\ln P_{DMX} \equiv (1/2)(s_X^0 + s_X^1)\ln r_X$.

It can be verified that P_{DMX} satisfies the time reversal property that was mentioned earlier; i.e., if the two time periods are switched, then the new P_{DMX} index is equal to the reciprocal of the original P_{DMX} index.

The interest in the Diewert Morrison index stems from the fact that it has a very direct connection with production theory; in fact this index is *exactly* equal to the target index α_{FX} provided that the technology of the production sector can be represented by a general translog functional form in each period. This sentence will be explained in more detail below.

In order to explain the above result, it is necessary to establish a general mathematical result. Thus let $x \equiv [x_1,...,x_N]$ and $y \equiv [y_1,...,y_M]$ be N and M dimensional vectors respectively and let f^0 and f^1 be two general quadratic functions defined as follows:

$$(24.23)\quad f^0(x,y) \equiv a_0^{\,0} + \sum_{n=1}^{N} a_n^{\,0} x_n + \sum_{m=1}^{M} b_m^{\,0} y_m + (1/2) \sum_{n=1}^{N}\sum_{j=1}^{N} a_{nj}^{\,0} x_n x_j + + (1/2) \sum_{m=1}^{M}\sum_{k=1}^{M} b_{mk}^{\,0} y_m y_k + \sum_{n=1}^{N}\sum_{j=1}^{N} c_{nm}^{\,0} x_n y_m ;$$

$$(24.24)\quad f^1(x,y) \equiv a_0^{\,1} + \sum_{n=1}^{N} a_n^{\,1} x_n + \sum_{m=1}^{M} b_m^{\,1} y_m + (1/2) \sum_{n=1}^{N}\sum_{j=1}^{N} a_{nj}^{\,1} x_n x_j + + (1/2) \sum_{m=1}^{M}\sum_{k=1}^{M} b_{mk}^{\,1} y_m y_k + \sum_{n=1}^{N}\sum_{j=1}^{N} c_{nm}^{\,1} x_n y_m ;$$

where the parameters $a_{nj}^{\,t}$ satisfy the symmetry restrictions $a_{nj}^{\,t} = a_{jn}^{\,t}$ for n,j = 1,...,N and t = 0,1 and the parameters $b_{mk}^{\,t}$ satisfy the symmetry restrictions $b_{mk}^{\,t} = b_{mk}^{\,t}$ for m,k = 1,...,M and t = 0,1. It can be shown that if

$$(24.25)\quad a_{nj}^{\,0} = a_{nj}^{\,1} \text{ for } n,j = 1,...,N ,$$

then the following equation holds for all vectors x^0, x^1, y^0 and y^1:

$$(24.26)\quad f^0(x^1,y^0) - f^0(x^0,y^0) + f^1(x^1,y^1) - f^0(x^0,y^1) = \sum_{n=1}^{N} [\partial f^0(x^0,y^0)/\partial x_n + \partial f^1(x^1,y^1)/\partial x_n][x_n^{\,1} - x_n^{\,0}].$$

The proof of the above proposition is very simple: just use definitions (24.23) and (24.24), do the differentiation on the right hand side of (24.26) and the result will emerge. The above result is a generalization of Diewert's (1976; 118) *quadratic identity*. A logarithmic version of the above identity corresponds to the *translog identity* which was established in the Appendix to Caves, Christensen and Diewert (1982; 1412-1413).

Recall the definition of the period t real net product function $n^t(p_C,p_I,p_G,p_X,p_M,L,K)$ defined by (24.6). The notation will now be changed a bit. Let $p \equiv [p_1,...,p_5]$ denote the vector of real output prices $[p_C,p_I,p_G,p_X,p_M]$ and let $z \equiv [z_1,z_2]$ denote the vector of

---

formula (24.22) were made by Diewert, Mizobuchi and Nomura (2005) and Diewert and Lawrence (2006) but the basic methodology is due to Diewert and Morrison. Kohli (1990) independently developed the same methodology as Diewert and Morrison.

primary input quantities [L,K]. The example of Diewert and Morrison (1986; 663) is now followed and it is assumed that the log of the period t real net product function, $n^t(p,z)$, has the following translog functional form:[24]

$$(24.27)\ \ln n^t(p,z) \equiv a_0^t + \sum_{n=1}^5 a_n^t \ln p_n + (1/2) \sum_{n=1}^5\sum_{j=1}^5 a_{nj} \ln p_n \ln p_j + \sum_{m=1}^2 b_m^t \ln z_m + (1/2)\sum_{m=1}^2\sum_{k=1}^2 b_{mk}^t \ln z_m \ln z_k + \sum_{n=1}^5\sum_{m=1}^2 c_{nm}^t \ln p_n \ln z_m\ ; \qquad t = 0,1.$$

Note that the coefficients for the quadratic terms in the logarithms of prices are assumed to be constant over time; i.e., it is assumed that $a_{nj}^0 = a_{nj}^1 = a_{nj}$. The coefficients must satisfy the following restrictions in order for n^t to satisfy the linear homogeneity properties that are consistent with a constant returns to scale technology:[25]

(24.28) $\sum_{n=1}^5 a_n^t = 1$ for $t = 0,1$;
(24.29) $\sum_{m=1}^2 b_m^t = 1$ for $t = 0,1$;
(24.30) $a_{nj} = a_{jn}$ for all n,j ;
(24.31) $b_{mk}^t = b_{km}^t$ for all m,k and $t = 0,1$.
(24.32) $\sum_{k=1}^M a_{mk} = 0$ for $m = 1,2$;
(24.33) $\sum_{j=1}^N b_{nj}^t = 0$ for $n = 1,...,5$ and $t = 0,1$;
(24.34) $\sum_{n=1}^N c_{mn}^t = 0$ for $m = 1,2$ and $t = 0,1$;
(24.35) $\sum_{m=1}^M c_{mn}^t = 0$ for $n = 1,...,5$ and $t = 0,1$.

Note that using Hotelling's Lemma, the logarithmic derivatives of $n^t(p_C^t,p_I^t,p_G^t,p_X^t,p_M^t,L^t,K^t)$ with respect to the logarithm of the export price are equal to the following expressions for $t = 0,1$:

$$\begin{aligned}(24.36)\ \partial\ln n^t(p_C^t,p_I^t,p_G^t,p_X^t,p_M^t,L^t,K^t)/\partial\ln p_X &= [p_X^t/n^t]\ \partial n^t(p_C^t,p_I^t,p_G^t,p_X^t,p_M^t,L^t,K^t)/\partial p_X \\ &= [p_X^t/n^t]\ X^t \\ &= s_X^t \qquad \text{using (24.16).}\end{aligned}$$

Noting that assumptions (24.27) imply that the logarithms of the net product functions are quadratic in the logarithms of prices and quantities, the result given by (24.26) can be applied to definitions (24.7), (24.8), (24.11) and (24.14) to imply the following result:

$$\begin{aligned}(24.37)\ 2 \ln \alpha_{FX} &= \ln \alpha_{LX} + \ln \alpha_{PX} \\ &= [\partial\ln n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0)/\partial\ln p_X\end{aligned}$$

[24] This functional form was first suggested by Diewert (1974; 139) as a generalization of the translog functional form introduced by Christensen, Jorgenson and Lau (1971). Diewert (1974; 139) indicated that this functional form was flexible. Flexible functional forms can approximate arbitrary functions to the second order at any given point and hence it is desirable to assume that the technological production possibilities can be represented by a flexible functional form in each period. Flexible functional forms are discussed in more detail in Diewert (1974).

[25] There are additional restrictions on the parameters which are necessary to ensure that $n^t(p,z)$ is convex in p and concave in z. The restrictions (24.29), (24.33) and (24.34) are not required for the results in this chapter. However, they impose constant returns to scale on the technology which is useful if a complete decomposition of real income growth into explanatory factors is attempted as in Diewert and Lawrence (2006).

$$+ \partial \ln n^1(p_C^{\,1},p_I^{\,1},p_G^{\,1},p_X^{\,1},p_M^{\,1},L^1,K^1)/\partial \ln p_X][\ln p_X^{\,1} - \ln p_X^{\,0}]$$
$$= [s_X^{\,0} + s_X^{\,1}] \ln (p_X^{\,1}/p_X^{\,0}) \qquad \text{using (24.36).}$$

Thus using (24.22) and (24.37), it can be seen that under the assumptions made on the technology, the following exact equality holds:[26]

(24.38) $\alpha_{FX} = P_{DMX}$.

Thus the Diewert Morrison index P_{DMX} defined by (24.22) is exactly equal to the target theoretical index, α_{FX}, under very weak assumptions on the technology.

Although the Diewert Morrison index gets a strong endorsement from the above result, the Diewert Lawrence index also had a reasonably strong justification and so the question arises: which index should be used in empirical applications? In section B.3 below, it is shown that numerically these two indexes will be quite close and so empirically, it will usually not matter which of these two alternative indexes is chosen.

B.3 The Numerical Equivalence of the Diewert Lawrence and Diewert Morrison Measures of the Effects of Changes in the Real Price of Exports

Let $p \equiv [p_1,...,p_5]$ denote the vector of real output prices $[p_C,p_I,p_G,p_X,p_M]$ and let $q \equiv [q_1,...qp_5]$ denote the corresponding vector of quantities $[C,I,G,X,-M]$. Thus the data pertaining to period t can be denoted by the vectors $p^t \equiv [p_C^{\,t},p_I^{\,t},p_G^{\,t},p_X^{\,t},p_M^{\,t}]$ and $q^t \equiv [C^t,I^t,G^t,X^t,-M^t]$ for t = 0,1. Note that each of the four empirical indexes P_{LX}, P_{PX}, P_{DLX} and P_{DMX} defined in the previous section can be regarded as functions of the data pertaining to the two periods under consideration. Thus P_{LX} should be more precisely be written as the function $P_{LX}(p^0,p^1,q^0,q^1)$, P_{PX} should be written as $P_{PX}(p^0,p^1,q^0,q^1)$ and so on. In this section, it is desired to compare the numerical properties of the four indexes P_{LX}, P_{PX}, P_{DLX} and P_{DMX}.

Diewert (1978) undertook a similar comparison of all superlative indexes that were known at that time. He showed that all known superlative indexes approximated each other to the second order around when the derivatives where evaluated at a point where the period 0 price vector p^0 was equal to the period 1 price vector p^1 and where the period 0 quantity vector was equal to the period 1 quantity vector.[27]

A somewhat similar result holds in the present context; i.e., it can be shown that the following equalities hold for the four indexes P_{LX}, P_{PX}, P_{DLX} and P_{DMX}:[28]

(24.39) $P_{LX}(p,p,q,q) = P_{PX}(p,p,q,q) = P_{DLX}(p,p,q,q) = P_{DMX}(p,p,q,q) = 1$;
(24.40) $\nabla P_{LX}(p,p,q,q) = \nabla P_{PX}(p,p,q,q) = \nabla P_{DLX}(p,p,q,q) = \nabla P_{DMX}(p,p,q,q)$

[26] This result is a straightforward adaptation of the results of Diewert and Morrison (1986; 666).
[27] Subsequent research by Robert Hill (2006) has shown that Diewert's approximation results break down for the quadratic mean of order r superlative indexes as r becomes large in magnitude.
[28] The proof is a series of straightforward computations.

where $\nabla P_{LX}(p,p,q,q)$ is the 20 dimensional vector of first order partial derivatives of $P_{LX}(p^0,p^1,q^0,q^1)$ with respect to the components of p^0, p^1, q^0 and q^1 but evaluated at a point where $p^0 = p^1 \equiv p$ and $q^0 = q^1 \equiv q$. The meaning of (24.39) and (24.40) is that the four indexes approximate each other to the accuracy of a first order Taylor series approximation around a data point where the real prices are equal in each period and the net output quantities are also equal to each other across periods.

The second order derivatives of the Laspeyres and Paasche type indexes, P_{LX} and P_{PX}, are *not* equal to each other when evaluated at an equal price and quantity point; i.e.,

(24.41) $\nabla^2 P_{LX}(p,p,q,q) \neq \nabla^2 P_{PX}(p,p,q,q)$

where $\nabla^2 P_{LX}(p,p,q,q)$ is the 20 by 20 dimensional matrix of second order partial derivatives of $P_{LX}(p^0,p^1,q^0,q^1)$ with respect to the components of p^0, p^1, q^0 and q^1 but evaluated at a point where $p^0 = p^1 \equiv p$ and $q^0 = q^1 \equiv q$. Thus as might be expected, P_{LX} and P_{PX} do not approximate each other to the accuracy of a second order Taylor series approximation around an equal price and quantity point.

However, the second order derivatives of the Diewert Lawrence and Diewert Morrison indexes, P_{DLX} and P_{DMX}, are equal to each other when evaluated at an equal price and quantity point; i.e.,[29]

(24.42) $\nabla^2 P_{DLX}(p,p,q,q) = \nabla^2 P_{DMX}(p,p,q,q)$.

Thus P_{DLX} and P_{DMX} approximate each other to the accuracy of a second order Taylor series approximation around a data point where the real prices are equal in each period and the net output quantities are also equal to each other across periods. The practical significance of this result is that for normal time series data where adjacent periods are compared, the Diewert Lawrence and Diewert Morrison indexes will give virtually identical results.[30]

B.4 Real Time Approximations to the Preferred Measures

The Diewert Lawrence index of the effects on real income growth of a change in the real export price, P_{DLX} defined by (24.21), depends on the real export price relative, r_X, the period 0 real export share in net product, s_X^0, and the corresponding period 1 real export share, s_X^1. Our other preferred measure of the effects of a change in the real export price, P_{DMX} defined by (24.22) also depends on these same three variables, r_X, s_X^0 and s_X^1. However, the current period export share s_X^1 is unlikely to be available to analysts until some time later than the current period. Thus the question arises: how can

[29] Again a long series of routine computations establishes this result. Note that these second derivative matrices are not equal to $\nabla^2 P_{LX}(p,p,q,q)$ or to $\nabla^2 P_{PX}(p,p,q,q)$.

[30] See Tables 5 and 9 in Diewert and Lawrence (2006) which establish the approximate equality of these indexes (to four significant figures) using Australian data in a gross product framework and Tables 12 and 14 which establish the approximate equality of these indexes in a net product framework for Australia.

approximations be formed to the preferred indexes defined by (24.21) and (24.22)? An answer to this question is as follows:

- Suppose that it is suspected that quantities are relatively unresponsive to changes in relative prices so that the period 1 quantity vector $[C^1,I^1,G^1,X^1,-M^1]$ will be approximately proportional to the corresponding period 0 quantity vector $[C^{10},I^0,G^0,X^0,-M^0]$. Under these conditions α_{LX} will be close to the Laspeyres type index defined by (24.19), which is $P_{LX} = 1 + s_X^0\,(r_X - 1)$, and a close approximation to α_{PX} can be obtained by using the formula $[p_C^1C^0+p_I^1\,I^0+p_G^1\,G^0+p_X^1\,X^0-p_M^1\,M^0]/[p_C^1C^0+p_I^1\,I^0+p_G^1\,G^0+p_X^0\,X^0-p_M^1\,M^0]$. Now multiply this last formula by P_{LX} and take the positive square root in order to obtain a good approximation to the theoretical export price effects index α_{PX}.
- Suppose that the share of exports in net product in period 1, s_X^1, is expected to be approximately equal to the corresponding period 0 share, s_X^0. Then simply use formula (24.22) with s_X^1 set equal to s_X^0.
- If neither of the above conditions is expected to hold for the period 1 data, simply make an approximate forecast for the period 1 export share s_X^1 and use (24.22).

C. The Effects of Changes in the Real Price of Imports

The theory that was outlined in section B can be repeated in the present section in order to measure the effects on real income generated by the production sector of a change in real import prices. Basically, all that needs to be done is to replace p_X by p_M and note that the import shares s_M^t defined in (24.16) are negative whereas the export shares s_X^t used in section B were positive.

Some of the definitions will be listed here without much explanation. The reader should be able to work out the analogies with the export indexes.

A theoretical Laspeyres type measure α_{LM} *of the effects on real income growth of changes in real import prices* from the period 0 level, p_M^0, to the period 1 level, p_M^1, can be defined as the ratio of the hypothetical net real income $n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^1,L^0,K^0)$ to the actual period 0 net real income $n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0)$:

$$(24.43)\quad \alpha_{LM} \equiv n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^1,L^0,K^0)/n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0).$$

The index α_{LM} of the effects of the change in the real price of imports is termed a *Laspeyres type index* because it holds constant all exogenous prices and quantities at their period 0 levels except for the two real import prices, p_M^0 and p_M^1, and the index also holds technology constant at the base period level.

There is the following *observable lower bound* to the theoretical index α_{LM} defined by (24.43):

$$(24.44)\quad \alpha_{LM} \geq [p_C^0C^0+p_I^0I^0+p_G^0G^0+p_X^0\,X^0-p_M^1\,M^0]/[p_C^0\,C^0+p_I^0\,I^0+p_G^0\,G^0+p_X^0\,X^0-p_M^0\,M^0] \equiv P_{LM}$$

where P_{LM} is an *observable Laspeyres type index* of the effects on real income of a change in real import prices going from period 0 to 1. Note that the numerator and denominator on the right hand side of (24.44) are identical except that p_M^1 appears in the numerator and p_M^0 appears in the denominator.

It is possible to show that the Laspeyres type observable index P_{MX} is a first order Taylor series approximation to the theoretical Laspeyres type index α_{MX}; i.e., it is possible to derive a counterpart to the approximation (24.10).

A theoretical Paasche type measure α_{PM} *of the effects on real income growth of changes in real import prices* from the period 0 level, p_M^0, to the period 1 level, p_M^1, can be defined as the ratio of the actual period 1 net real income $n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^1,L^1,K^1)$ to the hypothetical net real income $n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^0,L^1,K^1)$:

(24.45) $\alpha_{PM} \equiv n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^1,L^1,K^1)/n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^0,L^1,K^1)$.

The index α_{PM} of the effects of the change in the real price of imports is termed a *Paasche type index* because it holds constant all exogenous prices and quantities at their period 1 levels except for the two real import prices, p_M^0 and p_M^1, and the index also holds technology constant at the period 1 level.

Using assumption (24.7) for t = 1, the numerator on the right hand side of (24.45) is equal to period 1 observed real net product, $p_C^1 C^1 + p_I^1 I^1 + p_G^1 G^1 + p_X^1 X^1 - p_M^1 M^1$. Using definition (24.6), it can be seen that C^1, I^1, G^1, X^1 and M^1 is a feasible solution for the net product maximization problem defined by the denominator on the right hand side of (24.45), $n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^0,L^1,K^1)$. These facts mean that there is the following *observable upper bound* to the theoretical index α_{LM} defined by (24.45):

$$(24.46)\ \alpha_{PM} \leq [p_C^1C^1+p_I^1I^1+p_G^1G^1+p_X^1X^1-p_M^1M^1]/[p_C^1C^1+p_I^1I^1+p_G^1G^1+p_X^1X^1-p_M^0M^1] \equiv P_{PM}$$

where P_{PM} is an *observable Paasche type index* of the effects on real income of a change in real import prices going from period 0 to 1. Note that the numerator and denominator on the right hand side of (24.46) are identical except that p_M^1 appears in the numerator and p_M^0 appears in the denominator.

It is possible to show that the Paasche type observable index P_{PM} is a first order Taylor series approximation to the theoretical Paasche type index α_{PM}; i.e., a counterpart to the approximation (24.13) can be derived.

Note that both the Laspeyres and Paasche theoretical indexes of the effects on real income generated by the production sector of a change in the (real) price of imports are equally plausible and there is no reason to use one or the other of these two indexes. Thus if it is desired to have a single theoretical measure of the effects of a change in real import prices, α_{LM} and α_{PM} should be geometrically averaged. Thus define the *theoretical*

Fisher type measure α_{FM} *of the effects on real income growth of changes in real import prices* as the geometric mean of the Laspeyres and Paasche type theoretical measures:

$$(24.47)\ \alpha_{FM} \equiv [\alpha_{LM}\alpha_{PM}]^{1/2}.$$

With the target import index defined by (24.47) in hand, the problem of finding empirical approximations to this theoretical index will now be considered.

Two empirical indexes that provide estimates of the effects on the growth of real income of a change in real import prices have already been defined above: the Laspeyres type index P_{LM} defined on the right hand side of (24.44) and the Paasche type index P_{PM} defined on the right hand side of (24.46). It was noted that P_{LM} was a lower bound to the theoretical index α_{LM} and P_{PM} was an upper bound to the theoretical index α_{PM}. Thus P_{LM} will generally have a downward bias compared to its theoretical counterpart while P_{PM} will generally have a upward bias compared to its theoretical counterpart These inequalities suggest that the geometric mean of P_{LM} and P_{PM} is likely to be a reasonably good approximation to the target Fisher type index α_{FM} defined as the geometric mean of α_{LM} and α_{PM}. Thus define the *Diewert Lawrence index of the effects on real income of a change in real import prices going from period 0 to 1* as follows:[31]

$$(24.48)\ P_{DLM} \equiv [P_{LM}\, P_{PM}]^{1/2}.$$

As in section B, it will be useful to develop some alternative expressions for the indexes P_{LM}, P_{PM} and P_{DLM}. Define the *price relative* r_M *for real import prices* as

$$(24.49)\ r_M \equiv p_M^{1}/p_M^{0}.$$

Using the techniques described in section B, the following alternative formulae for P_{LM}, P_{PM} and P_{DLM} can be derived:[32]

$$(24.50)\ P_{LM} = 1 + s_M^{0}\,(r_M - 1)\ ;$$
$$(24.51)\ P_{PM} = [1 + s_M^{1}(r_M^{-1} - 1)]^{-1}.$$

Noting that s_M^0 and s_M^1 are negative, it can be seen that both P_{LM} and P_{PM} are decreasing functions of r_M so that as the real price of imports increases, both indexes of growth in real income also decrease as expected.

Substituting expressions (24.50) and (24.51) into (24.48) leads to the following expression for the Diewert Lawrence import index:

$$(24.52)\ P_{DLM} = \{[1 + s_M^{0}\,(r_M - 1)]/[1 + s_M^{1}(r_M^{-1} - 1)]\}^{1/2}.$$

[31] Diewert and Lawrence (2006; 14-17) seem to have been the first to define and empirically estimate the indexes defined by P_{LM}, P_{PM} and (24.48).

[32] Remember that the period 0 “share” of imports in net real product, s_M^0, is negative whereas the period 0 share of exports which appeared in the counterpart result (24.19), s_X^0, was positive.

The Diewert Lawrence index P_{DLM} defined by (24.52) is likely to be closer to the target Fisher index α_{FM} defined by (24.47) than the Laspeyres and Paasche type indexes P_{LM} and P_{PM} defined by (24.50) and (24.51).

Using the same notation that is used in (24.52) above, the logarithm of the *Diewert Morrison index,* P_{DMM}*, of the effects on real income of a change in real import prices going from period 0 to 1* is defined as follows:[33]

(24.53) $\ln P_{DMM} \equiv (1/2)(s_M{}^0 + s_M{}^1)\ln r_M$.

It can be verified that P_{DMM} satisfies the time reversal; i.e., if the two time periods are switched, then the new P_{DMM} index is equal to the reciprocal of the original P_{DMM} index.

As in section B, the interest in the Diewert Morrison index stems from the fact that it has a very direct connection with production theory; in fact this index is *exactly* equal to the target index α_{FM} provided that the technology of the production sector can be represented by a general translog functional form in each period. Again make the translog assumptions (24.27)-(24.35).

Using Hotelling's Lemma, the logarithmic derivatives of $n^t(p_C{}^t,p_I{}^t,p_G{}^t,p_X{}^t,p_M{}^t,L^t,K^t)$ with respect to the logarithm of the import price are equal to the following expressions for t = 0,1:

$$
\begin{aligned}
(24.54)\ \partial\ln n^t(p_C{}^t,p_I{}^t,p_G{}^t,p_X{}^t,p_M{}^t,L^t,K^t)/\partial\ln p_M &= [p_M{}^t/n^t]\ \partial n^t(p_C{}^t,p_I{}^t,p_G{}^t,p_X{}^t,p_M{}^t,L^t,K^t)/\partial p_M \\
&= [p_M{}^t/n^t][-M^t] \\
&= s_M{}^t \qquad \text{using (24.16).}
\end{aligned}
$$

Noting that assumptions (24.27) imply that the logarithms of the net product functions are quadratic in the logarithms of prices and quantities, the result given by (24.26) can be applied to definitions (24.43), (24.45) and (24.47) to imply the following result:

$$
\begin{aligned}
(24.55)\ 2\ln\alpha_{FM} &= \ln\alpha_{LM} + \ln\alpha_{PM} \\
&= [\partial\ln n^0(p_C{}^0,p_I{}^0,p_G{}^0,p_X{}^0,p_M{}^0,L^0,K^0)/\partial\ln p_M \\
&\qquad + \partial\ln n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^1,p_M{}^1,L^1,K^1)/\partial\ln p_M][\ln p_M{}^1 - \ln p_M{}^0] \\
&= [s_M{}^0 + s_M{}^1]\ln(p_M{}^1/p_M{}^0) \qquad \text{using (24.54).}
\end{aligned}
$$

Thus using (24.53) and (24.55), it can be seen that under the assumptions made on the technology, the following exact equality holds:[34]

(24.56) $\alpha_{FM} = P_{DMM}$.

[33] Strictly speaking, Diewert and Morrison (1986; 666) defined their index in the context of a GDP function rather than a net product function and did not deflate prices by a price index. The first applications of formula (24.22) were made by Diewert, Mizobuchi and Nomura (2005) and Diewert and Lawrence (2006).
[34] This result is a straightforward adaptation of the results of Diewert and Morrison (1986; 666).

Thus the Diewert Morrison import price effects on real income growth index P_{DMM} defined by (24.53) is *exactly* equal to the target theoretical index, α_{FM}, under very weak assumptions on the technology.

It can be shown that the following equalities hold for the four empirical indexes P_{LM}, P_{PM}, P_{DLM} and P_{DMM}:[35]

(24.57) $P_{LM}(p,p,q,q) = P_{PM}(p,p,q,q) = P_{DLM}(p,p,q,q) = P_{DMM}(p,p,q,q) = 1$;
(24.58) $\nabla P_{LM}(p,p,q,q) = \nabla P_{PM}(p,p,q,q) = \nabla P_{DLM}(p,p,q,q) = \nabla P_{DMM}(p,p,q,q)$

where $\nabla P_{LM}(p,p,q,q)$ is the 20 dimensional vector of first order partial derivatives of $P_{LM}(p^0,p^1,q^0,q^1)$ with respect to the components of p^0, p^1, q^0 and q^1 but evaluated at a point where $p^0 = p^1 \equiv p$ and $q^0 = q^1 \equiv q$. As usual, the meaning of (24.57) and (24.58) is that the four indexes approximate each other to the accuracy of a first order Taylor series approximation around an equal prices and quantities data point.

The second order derivatives of the Laspeyres and Paasche type indexes, P_{LM} and P_{PM}, are *not* equal to each other when evaluated at an equal price and quantity point; i.e.,

(24.59) $\nabla^2 P_{LM}(p,p,q,q) \neq \nabla^2 P_{PM}(p,p,q,q)$

where $\nabla^2 P_{LM}(p,p,q,q)$ is the 20 by 20 dimensional matrix of second order partial derivatives of $P_{LM}(p^0,p^1,q^0,q^1)$ with respect to the components of p^0, p^1, q^0 and q^1 but evaluated at a point where $p^0 = p^1 \equiv p$ and $q^0 = q^1 \equiv q$. Thus as might be expected, P_{LM} and P_{PM} do *not* approximate each other to the accuracy of a second order Taylor series approximation around an equal price and quantity point.

However, the second order derivatives of the Diewert Lawrence and Diewert Morrison import indexes, P_{DLM} and P_{DMM}, are equal to each other when evaluated at an equal price and quantity point; i.e.,[36]

(24.60) $\nabla^2 P_{DLM}(p,p,q,q) = \nabla^2 P_{DMM}(p,p,q,q)$.

Thus P_{DLM} and P_{DMM} approximate each other to the accuracy of a second order Taylor series approximation around a data point where the real prices are equal in each period and the net output quantities are also equal to each other across periods. The practical significance of this result is that for normal time series data where adjacent periods are compared, the Diewert Lawrence and Diewert Morrison indexes will give virtually identical results.[37]

D. The Combined Effects of Changes in the Real Prices of Exports and Imports

[35] The proof is a series of straightforward computations.
[36] Again a long series of routine computations establishes this result.
[37] See Tables 5 and 9 and 12 and 14 in Diewert and Lawrence (2006) for a numerical illustration of this result.

In this section, instead of *separately* considering the effects of a change in real export or real import prices on the real income generated by the production sector, the effects of a *combined* change in real export and import prices will be considered. It will turn out that the same type of analysis that was used in the previous two sections can be used in the present section.

In a first attempt to measure the effects of changing real import and export prices over the two periods under consideration, a hypothetical period 1 net domestic product maximization problem is considered where producers have at their disposal the period 0 technology set S^0, the period 0 actual labour and capital inputs, L^0 and K^0 respectively, and they face the period 0 real prices for consumption, net investment and government consumption, p_C^0, p_I^0 and p_G^0 respectively, but they face the period 1 real export and import prices, p_X^1 and p_M^1. The solution to this hypothetical net product maximization problem is $n^0(p_C^0,p_I^0,p_G^0,p_X^1,p_M^1,L^0,K^0)$. A *theoretical Laspeyres type measure* α_{LXM} *of the effects on real income growth of the combined changes in real export and import prices* from their period 0 levels, p_X^0 and p_M^0, to their period 1 levels, $p_X^1$and p_M^1, can be defined as the ratio of the hypothetical net real income $n^0(p_C^0,p_I^0,p_G^0,p_X^1,p_M^1,L^0,K^0)$ to the actual period 0 net real income $n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0)$:

$$(24.61)\ \alpha_{LXM} \equiv n^0(p_C^0,p_I^0,p_G^0,p_X^1,p_M^1,L^0,K^0)/n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0).$$

The index α_{LXM} of the effects of the change in the real prices of exports and imports is termed a *Laspeyres type index* because it holds constant all exogenous prices and quantities at their period 0 levels except for the four real export and import prices, p_X^0, p_X^1, p_M^0 and p_M^1, and the index also holds technology constant at the base period level.

As usual, a feasibility argument leads to the following *observable lower bound* to the theoretical index α_{LXM} defined by (24.61):

$$\begin{aligned}(24.62)\ \alpha_{LXM} &\geq [p_C^0C^0+p_I^0I^0+p_G^0G^0+p_X^1X^0-p_M^1M^0]/[p_C^0C^0+p_I^0I^0+p_G^0G^0+p_X^0X^0-p_M^0M^0] \\ &\equiv P_{LXM}\end{aligned}$$

where P_{LXM} is an *observable Laspeyres type index* of the effects on real income of a change in real export and import prices going from period 0 to 1. Note that the numerator and denominator on the right hand side of (24.62) are identical except that p_X^1 and p_M^1 appear in the numerator while p_X^0 and p_M^0 appear in the denominator.

It is possible to show that the Laspeyres type observable index P_{LMX} is a first order Taylor series approximation to the theoretical Laspeyres type index α_{LMX}; i.e., it is possible to derive a counterpart to the approximation (24.10).

A theoretical Paasche type measure α_{PM} *of the effects on real income growth of changes in real export and import prices* from the period 0 levels, p_X^0 and p_M^0, to the period 1 levels, p_X^1 and p_M^1, can be defined as the ratio of the actual period 1 net real income

$n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^1,p_M{}^1,L^1,K^1)$ to the hypothetical net real income defined by $n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^0,p_M{}^0,L^1,K^1)$:

$$(24.63)\ \alpha_{PXM} \equiv n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^1,p_M{}^1,L^1,K^1)/n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^0,p_M{}^0,L^1,K^1).$$

The index α_{PXM} of the effects of the changes in the real prices of exports and imports is termed a *Paasche type index* because it holds constant all exogenous prices and quantities at their period 1 levels except for the real export and import prices, $p_X{}^0$, $p_X{}^1$, $p_M{}^0$ and $p_M{}^1$, and the index also holds technology constant at the period 1 level.

Using assumption (24.7) for $t = 1$, the numerator on the right hand side of (24.63) is equal to period 1 observed real net product, $p_C{}^1\,C^1 + p_I{}^1\,I^1 + p_G{}^1\,G^1 + p_X{}^1\,X^1 - p_M{}^1\,M^1$. Using definition (24.6), it can be seen that C^1, I^1, G^1, X^1 and M^1 is a feasible solution for the net product maximization problem defined by the denominator on the right hand side of (24.63), $n^1(p_C{}^1,p_I{}^1,p_G{}^1,p_X{}^0,p_M{}^0,L^1,K^1)$. These facts mean that there is the following *observable upper bound* to the theoretical index α_{LXM} defined by (24.63):

$$(24.64)\ \alpha_{PXM} \leq [p_C{}^1C^1+p_I{}^1I^1+p_G{}^1G^1+p_X{}^1X^1-p_M{}^1M^1]/[p_C{}^1C^1+p_I{}^1\,I^1+p_G{}^1G^1+p_X{}^0X^1-p_M{}^0M^1] \equiv P_{PXM}$$

where P_{PXM} is an *observable Paasche type index* of the effects on real income of a change in real export and import prices going from period 0 to 1. Note that the numerator and denominator on the right hand side of (24.64) are identical except that $p_X{}^1$ and $p_M{}^1$ appear in the numerator while $p_X{}^0$ and $p_M{}^0$ appear in the denominator.

As usual, it is possible to show that the Paasche type observable index P_{PXM} is a first order Taylor series approximation to the theoretical Paasche type index α_{PXM}; i.e., a counterpart to the approximation (24.13) can be derived.

Note that both the Laspeyres and Paasche theoretical indexes of the effects on real income generated by the production sector of a change in the (real) prices of exports and imports are equally plausible and there is no reason to use one or the other of these two indexes. Thus as usual, α_{LXM} and α_{PXM} should be geometrically averaged. Hence define the *theoretical Fisher type measure* α_{FXM} *of the effects on real income growth of changes in real export and import prices* as the geometric mean of the Laspeyres and Paasche type theoretical measures:

$$(24.65)\ \alpha_{FXM} \equiv [\alpha_{LXM}\alpha_{PXM}]^{1/2}.$$

Having defined a target export and import index defined by (24.65), the problem of finding empirical approximations to this theoretical index will now be considered.

Two empirical indexes that provide estimates of the effects on the growth of real income of a change in real export prices have already been defined above: the Laspeyres type index P_{LXM} defined on the right hand side of (24.44) and the Paasche type index P_{PXM} defined on the right hand side of (24.46). It was noted that P_{LXM} was a lower bound to the theoretical index α_{LXM} and P_{PXM} was an upper bound to the theoretical index α_{PXM}. Thus P_{LXM} will generally have a downward bias compared to its theoretical counterpart while P_{PXM} will generally have a upward bias compared to its theoretical counterpart These inequalities suggest that the geometric mean of P_{LXM} and P_{PXM} is likely to be a reasonably good approximation to the target Fisher type index α_{FXM} defined as the geometric mean of α_{LXM} and α_{PXM}. Thus define the *Diewert Lawrence index of the effects on real income of a change in real export and import prices going from period 0 to 1* as follows:[38]

(24.66) $P_{DLXM} \equiv [P_{LXM}\, P_{PXM}]^{1/2}$.

As in section B, it will be useful to develop some alternative expressions for the indexes P_{LXM}, P_{PXM} and P_{DLXM}. Using the techniques described in section B, the following alternative formulae for P_{LXM}, P_{PXM} and P_{DLXM} can be derived:

(24.67) $P_{LXM} = 1 + s_X^0\,(r_X - 1) + s_M^0\,(r_M - 1)$;
(24.68) $P_{PXM} = [1 + s_X^1(r_X^{-1} - 1) + s_M^1(r_M^{-1} - 1)]^{-1}$

where the export shares s_X^t (positive) and import shares s_M^t (negative) are defined in (24.16), $r_X \equiv p_X^1/p_X^0$ is the real export price relative and $r_M \equiv p_M^1/p_M^0$ is the real import price relative. It can be seen that both P_{LXM} and P_{PXM} are increasing functions of r_X and decreasing functions of r_M, since the s_M^t are negative as defined in (24.16), so that as the real price of exports increases, both indexes of growth in real income increase and as the real price of imports increases, both indexes of growth in real income decrease.

Substituting expressions (24.67) and (24.68) into (24.48) leads to the following expression for the Diewert Lawrence export and import index:

(24.69) $P_{DLXM} = \{[1 + s_X^0\,(r_X - 1) + s_M^0\,(r_M - 1)]/[1 + s_X^1(r_X^{-1} - 1) + s_M^1(r_M^{-1} - 1)]\}^{1/2}$.

The Diewert Lawrence index P_{DLXM} defined by (24.69) is likely to be closer to the target Fisher index α_{FXM} defined by (24.65) than the Laspeyres and Paasche type indexes P_{LXM} and P_{PXM} defined by (24.67) and (24.68).

As in sections B and C, there is an alternative to the Diewert Lawrence index P_{DLXM} defined by (24.69), namely the Diewert Morrison index P_{DMMX}. Using the same notation that is used in (24.69) above, the logarithm of the *Diewert Morrison index,* P_{DMXM}*, of the effects on real income of changes in real export and import prices going from period 0 to 1* is defined as follows:[39]

[38] Diewert and Lawrence (2006; 15) did not actually define the index on the right hand side of (24.66); instead they defined a counterpart index that looked at the effects on real income growth of a change in all real output prices (rather than the effects of a change in just real export and import prices). However, the basic idea behind (24.48) is due to Diewert and Lawrence.
[39] Strictly speaking, Diewert and Morrison (1986; 666) defined their index in the context of a GDP function rather than a net product function and did not deflate prices by a price index. The first applications of formula (24.70) were made by Diewert, Mizobuchi and Nomura (2005) and Diewert and Lawrence (2006).

(24.70) $\ln P_{DMXM} \equiv (1/2)(s_X^0 + s_X^1)\ln r_X + (1/2)(s_M^0 + s_M^1)\ln r_M$.

It can be verified that P_{DMXM} satisfies the time reversal; i.e., if the two time periods are switched, then the new P_{DMXM} index is equal to the reciprocal of the original P_{DMXM} index.

As in section B, the Diewert Morrison index is *exactly* equal to the target index α_{FXM} provided that the technology of the production sector can be represented by a general translog functional form in each period. Thus again make the translog assumptions (24.27)-(24.35). Using the Hotelling's Lemma results (24.36) and (24.54) and noting that assumptions (24.27) imply that the logarithms of the net product functions are quadratic in the logarithms of prices and quantities, the result given by (24.26) can be applied to definitions (24.61), (24.63) and (24.65) to imply the following result:

$$
\begin{aligned}
(24.71)\ 2 \ln \alpha_{FXM} &= \ln \alpha_{LXM} + \ln \alpha_{PXM} \\
&= [\partial \ln n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0)/\partial \ln p_X + \partial \ln n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^1,L^1,K^1)/\partial \ln p_X] \\
&\quad \times[\ln p_X^1 - \ln p_X^0] \\
&\quad +[\partial \ln n^0(p_C^0,p_I^0,p_G^0,p_X^0,p_M^0,L^0,K^0)/\partial \ln p_M + \partial \ln n^1(p_C^1,p_I^1,p_G^1,p_X^1,p_M^1,L^1,K^1)/\partial \ln p_M] \\
&\quad \times[\ln p_M^1 - \ln p_M^0] \\
&= [s_X^0 + s_X^1] \ln(p_X^1/p_X^0) + [s_M^0 + s_M^1] \ln(p_M^1/p_M^0)
\end{aligned}
$$

Thus using (24.70) and (24.71), it can be seen that under the assumptions made on the technology, the following exact equality holds:[40]

(24.72) $\alpha_{FM} = P_{DMM}$.

Thus the Diewert Morrison combined export and import price effects on real income growth index P_{DMXM} defined by (24.70) is *exactly* equal to the target theoretical index α_{FXM} defined by (24.65) under very weak assumptions on the technology.

As in sections B and C above, it can be shown that the four empirical indexes P_{LXM}, P_{PXM}, P_{DLXM} and P_{DMXM} numerically approximate each other to the first order around an equal price and quantity point:

(24.73) $P_{LXM}(p,p,q,q) = P_{PXM}(p,p,q,q) = P_{DLXM}(p,p,q,q) = P_{DMXM}(p,p,q,q) = 1$;
(24.74) $\nabla P_{LXM}(p,p,q,q) = \nabla P_{PXM}(p,p,q,q) = \nabla P_{DLXM}(p,p,q,q) = \nabla P_{DMXM}(p,p,q,q)$

where $\nabla P_{LXM}(p,p,q,q)$ is the 20 dimensional vector of first order partial derivatives of $P_{LXM}(p^0,p^1,q^0,q^1)$ with respect to the components of p^0, p^1, q^0 and q^1 but evaluated at a point where $p^0 = p^1 \equiv p$ and $q^0 = q^1 \equiv q$.

The second order derivatives of the Laspeyres and Paasche type indexes, P_{LXM} and P_{PxM}, are *not* equal to each other when evaluated at an equal price and quantity point; i.e.,

[40] This result is again a straightforward adaptation of the results of Diewert and Morrison (1986; 666).

(24.75) $\nabla^2 P_{LXM}(p,p,q,q) \neq \nabla^2 P_{PXM}(p,p,q,q)$.

However, the second order derivatives of the Diewert Lawrence and Diewert Morrison combined export and import indexes, P_{DLXM} and P_{DMXM}, are equal to each other when evaluated at an equal price and quantity point; i.e.,[41]

(24.76) $\nabla^2 P_{DLXM}(p,p,q,q) = \nabla^2 P_{DMXM}(p,p,q,q)$.

Thus P_{DLXM} and P_{DMXM} approximate each other to the accuracy of a second order Taylor series approximation around a data point where the real prices are equal in each period and the net output quantities are also equal to each other across periods. The practical significance of this result is that for normal time series data where adjacent periods are compared, the Diewert Lawrence and Diewert Morrison combined effects indexes will give virtually identical results.

The above material is very similar to the results derived in sections B and C above. But at this point, some new results can be derived. In section B, measures of the effects on real income growth of a change in real export prices were derived; in section C, measures of the effects on real income growth of a change in real import prices were derived and finally, in this section, measures of the combined effects on real income growth of a change in both real export prices and in real import prices were derived. A natural question to ask at this point is: how do the partial measures considered in sections B and C compare to the combined effects measures considered in the present section?

Using the Diewert Morrison measures, the answer to the above question is very simple. Recalling the expressions (24.22), (24.53) and (24.70) which defined the Diewert Morrison index of the effects on real income growth of a change in real export prices P_{DMX}, of a change in real import prices P_{DMM} and the combined effects of changes in real export and import prices P_{DMXM} respectively, it can be seen that there is the following *simple multiplicative relationship* between these three indexes:

(24.77) $P_{DMXM} = P_{DMX} P_{DMM}$;

i.e., the combined price effects index P_{DMXM} is exactly equal to the product of the export price effect index P_{DMX} and the import price effect index P_{DMM}.[42] Thus when using the Diewert Morrison indexes, the product of the partial effects is equal to the combined effect.[43]

The exact decomposition given by (24.77) for the Diewert Morrison indexes translates into the following approximate decomposition for the Diewert Lawrence indexes:

[41] Again a series of routine computations establishes this result.
[42] This is a counterpart to a result obtained by Diewert and Morrison (1986; 666) and Kohli (1990).
[43] This result generalizes to the case where there is a finer classification of exports and imports. In this case, the Diewert Morrison disaggregated effect indexes can be multiplied together to obtain the overall effect of changes in all real export and import prices.

(24.78) $P_{DLXM} \approx P_{DLX}\, P_{DLM}$.

The meaning of the approximate equality is this: from the approximation results derived in this section and the previous sections, it is known that the Diewert Morrison combined effects index P_{DMXM} approximates the Diewert Lawrence combined effects index P_{DLXM} to the second order around an equal price and quantity point and the Diewert Morrison separate effects indexes P_{DMX} and P_{DMM} similarly approximate the corresponding Diewert Lawrence separate effects indexes P_{DMX} and P_{DMM}. Using these approximation results and the exact identity (24.77) means that the right hand side of (24.78) will approximate the left hand side of (24.78) to the second order around an equal price and quantity point.

The decomposition results derived above should be useful when dealing with disaggregated export and import data. The reader should be able to use the techniques explained in this chapter to extend the analysis to the case where there are a large number of export and import categories. The corresponding analogues to (24.77) and (24.78) will enable the analyst to decompose the overall effects on real income growth due to changes in the prices of internationally traded goods into separate effects in each category. These separate effects multiply together to give the overall effect on real income growth of changes in the real prices of exports and imports.

E. The Effects on Household Cost of Living Indexes of Changes in the Prices of Directly Imported Goods and Services

E.1 The Case of a Single Household: The Basic Framework

As was mentioned in the introduction to this chapter, households frequently directly import consumer goods and services from abroad without these goods and services passing through the production sector of the economy. Examples of such commodities are tourist expenditures abroad and the direct importation of automobiles. Thus it would be useful to have a framework for modeling the effects of changes in the prices of these directly imported products on household welfare.

The case of a single household that imports a product will be considered. Let C^t and M^t denote the quantities of a domestic and foreign commodity consumed by the household in period t and let $P_C{}^t$ and $P_M{}^t$ denote the corresponding period t nominal prices for t = 0,1.[44] The *period t household nominal expenditure on all goods and services* or period t household "*income*", Y^t, is defined as the total value of consumer expenditures on consumption products provided by the domestic production sector and by directly imported products:

$$(24.79)\quad Y^t \equiv P_C{}^tC^t + P_M{}^tM^t\,; \qquad t = 0,1.$$

[44] These price and quantity scalars can be replaced by vectors but for simplicity, only the scalar case is considered here. Note that the C^t in the present section matches up with the C^t that appeared in previous sections but that the M^t in this section is not equal to the production theoretic M^t that appeared in previous sections.

As opposed to the generation of real income approach taken in previous sections in this chapter, in this section, a more traditional *cost of living approach* to changes in the prices of directly imported goods and services will be taken. Thus the objective of the present section is to derive measures of the effects on the household's cost of living index of a change in import prices from the period 0 nominal level, P_M^0, to the period 1 nominal level, P_M^1.

At this point, household preferences over different combinations of C and M are brought into the picture. It is assumed that in period t = 0,1, the household's preferences are defined by the period t utility function, $U^t(C,M)$, where the function U^t is increasing, continuous and quasiconcave in its two variables C and M. It will prove useful to define the household's period t *expenditure function*, $e^t(P_C,P_M,u)$ for periods t = 0,1, positive (nominal) prices P_C and P_M and for utility level u belonging to the range of U^t:

(24.80) $e^t(P_C,P_M,u) \equiv \min_{C,M} \{ P_C C + P_M M : U^t(C,M) \geq u\}$; $t = 0,1.$

Thus $e^t(P_C,P_M,u)$ is the minimum income that the household needs in period t in order to attain the utility level u, given that it faces the prices P_C and P_M for domestically and foreign supplied goods and services respectively.[45]

In what follows, it will be assumed that the household minimizes the cost of achieving its utility level $u^t \equiv U^t(C^t,M^t)$ in each period t so that the following equalities hold:

(24.81) $e^t(P_C{}^t,P_M{}^t,u^t) = P_C{}^t C^t + P_M{}^t M^t$; $t = 0,1.$

Assumptions (24.81) are the household counterparts to the producer equalities (24.7).

The household expenditure functions defined in this subsection will play a key role in the remainder of section E.

E.2 Theoretical Measures of the Effects on Income of Changes in Household Import Prices

Note that $e^0(P_C{}^0,P_M{}^1,u^0)$ is the amount of income that the household would need, using the household preferences of period 0, to be able to attain the same level of utility that it attained in period 0 (which is u^0) if it faced the period 0 domestic consumption price $P_C{}^0$ but the period 0 household import price was changed from $P_M{}^0$ to the period 1 import price $P_M{}^1$. This hypothetical amount of expenditure could be compared to the period 0 actual expenditure level, $e^0(P_C{}^0,P_M{}^0,u^0)$. Thus a *theoretical Konüs (1924) Laspeyres partial cost of living index that measures the effects of changes in the price of imports that the household faces* going from the period 0 level, $P_M{}^0$, to the period 1 level, $P_M{}^1$, can

[45] The expenditure function e^t can be used to represent consumer preferences under some assumptions on the utility function U^t; see Diewert (1974) for the properties of e^t and references to the literature. An important property for present purposes is that $e^t(P_C,P_M,u)$ is increasing in P_C and P_M.

be defined as the ratio of the hypothetical expenditure $e^0(P_C^0,P_M^1,u^0)$ to the actual period 0 expenditure:

(24.82) $\kappa_{LM} \equiv e^0(P_C^0,P_M^1,u^0)/e^0(P_C^0,P_M^0,u^0)$.

The index κ_{LM} of the effects of the change in the price of imports is termed a (partial) *Laspeyres type index* because it holds constant all exogenous prices and utility levels at their period 0 levels except for the import price, P_M^1.[46]

Note that as P_M^1 *increases*, κ_{LM} defined by (24.82) also *increases*. This is quite different from the properties of the corresponding producer index α_{LM} defined by (24.43) where α_{LM} *decreased* as the (real) price of imports *increased*. But there is a difference in perspective between the previous sections and the present one: in the previous sections, growth of real income due to changes in international prices was positive for the providers of primary input services whereas in the present section, growth in cost due to changes in international prices is negative for households.

There is the following *observable upper bound* to the theoretical index κ_{LM} defined by (24.82):[47]

(24.83) $\kappa_{LM} \leq [P_C^0C^0 + Pp_M^1 M^0]/[P_C^0 C^0 + P_M^0 M^0]$
$\equiv P_{LM}^*$

where P_{LM}^* is an *observable Laspeyres partial import price index* of the effects on the cost of living of a change in import prices going from period 0 to 1, holding the price of consumption constant at the period 0 level. Note that the numerator and denominator on the right hand side of (24.83) are identical except that P_M^1 appears in the numerator and P_M^0 appears in the denominator.

It is possible to show that the Laspeyres type observable index P_{LM}^* is a first order Taylor series approximation to the theoretical index κ_{LM}; i.e., it is possible to derive a counterpart to the approximation (24.10).[48]

The above theoretical measure of the effects of a change in the price of imports used the period 0 preferences for the consumer. It is possible to develop a parallel measure of price change using the consumer's period 1 preferences. Note that $e^1(P_C^1,P_M^0,u^1)$ is the

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[46] The index defined by (24.82) is a partial price change counterpart to a cost of living index concept defined by Balk (1989; 159) in the context of changing consumer preferences. Balk used the idea of holding tastes constant to work out the effects of price changes. The measure defined by (24.84) is also related to various *price compensating variations* defined by Hicks (1945-46; 68) (1946; 331-332) except that Hicks used differences rather than ratios.

[47] Note that assumption (24.81) for t = 0 implies that $e^0(P_C^0,P_M^0,u^0) = p_C^0 C^0+p_M^0 M^0$. Since $[C^0,M^0]$ is a feasible solution for the cost minimization problem defined by $e(P_C^0,P_M^1,u^0)$, it must be the case that $e^0(P_C^0,P_M^1,u^0)$ is equal or less than $P_C^0 C^0+P_M^1 M^0$, which establishes the inequality in (24.83).

[48] A first order Taylor series approximation to $e^0(P_C^0,P_M^1,u^0)$ is $e^0(P_C^0,P_M^0,u^0) + [\partial e^0(P_C^0,P_M^0,u^0)/\partial P_M][P_M^1 - P_M^0] = [p_C^0 C^0+p_M^0 M^0] + M^0[P_M^1 - P_M^0] = P_C^0 C^0+P_M^1 M^0$ using $M^0 = \partial e^0(P_C^0,P_M^0,u^0)/\partial P_M$ which is implied by Shephard's (1953; 11) Lemma. This type of approximation is due to Hicks (1942) (1946; 331).

amount of income that the household would need, using the household preferences of period 1, to be able to attain the same level of utility that it attained in period 1 (which is u^1) if it faced the period 1 domestic consumption price P_C^1 and the period 0 import price P_M^0. This hypothetical amount of expenditure could be compared to the period 1 actual expenditure level, $e^1(P_C^1,P_M^1,u^1)$. Thus a *theoretical Konüs Paasche partial cost of living index that measures the effects of changes in the price of imports that the household faces* going from the period 0 level, P_M^0, to the period 1 level, P_M^1, can be defined as the ratio of the actual period 1 expenditure $e^1(P_C^1,P_M^1,u^1)$ to the hypothetical expenditure $e^1(P_C^1,P_M^0,u^1)$:

(24.84) $\kappa_{PM} \equiv e^1(P_C^1,P_M^1,u^1)/e^1(P_C^1,P_M^0,u^1)$.

The index κ_{PM} of the effects of the change in the price of imports is termed a (partial) *Paasche type index* because it holds constant all exogenous prices and utility levels at their period 1 levels except for the two import prices, P_M^0 and P_M^1.[49]

There is the following *observable lower bound* to the theoretical index κ_{PM} defined by (24.84):

(24.85) $\kappa_{PM} \geq [P_C^1C^1+P_M^1 M^1]/[P_C^1 C^1+P_M^0 M^1]$
$\equiv P_{PM}^*$

where P_{PM}^* is an *observable Paasche partial import price index* of the effects on the cost of living of a change in real import prices going from period 0 to 1, holding the price of consumption constant at the period 1 level. Note that as usual, the numerator and denominator on the right hand side of (24.85) are identical except that P_M^1 appears in the numerator and P_M^0 appears in the denominator.

It is possible to show that the Paasche type observable index P_{PM}^* is a first order Taylor series approximation to the theoretical index κ_{LM}; i.e., it is possible to derive a counterpart to the approximation (24.13).

Note that both the Konüs Laspeyres and Paasche theoretical partial cost of living indexes of the effects generated by a change in the price of imports are equally plausible and there is no reason to use one or the other of these two indexes. Thus if it is desired to have a single theoretical measure of the effects of a change in import prices on the household's cost of living, κ_{LM} and κ_{PM} should be geometrically averaged. Hence define the *theoretical Fisher type partial cost of living index* κ_{FM} *of the effects of changes in household import prices* as the geometric mean of the Konüs Laspeyres and Paasche theoretical measures:

(24.86) $\kappa_{FM} \equiv [\kappa_{LM}\, \kappa_{PM}]^{1/2}$.

[49] The index defined by (24.85) is also a partial price change counterpart to a cost of living index concept defined by Balk (1989; 159) in the context of changing consumer preferences.

With the target import index (24.86) defined, the problem of finding empirical approximations to this theoretical index will now be considered.

E.3 Empirical Measures of the Effects on Income of Changes in Household Import Prices

Two empirical indexes that provide estimates of the effects on the cost of living of a household have been defined above: the Laspeyres partial import price index P_{LM}^* defined on the right hand side of (24.83) and the Paasche partial import price index P_{PM}^* defined on the right hand side of (24.85). It was noted that P_{LM}^* was an upper bound to the theoretical index κ_{LM} and P_{PM}^* was a lower bound to the theoretical index κ_{PM}. Thus P_{LM}^* will generally have a upward bias compared to its theoretical counterpart while P_{PM}^* will generally have a downward bias compared to its theoretical counterpart These inequalities suggest that the geometric mean of P_{LM}^* and P_{PM}^* is likely to be a reasonably good approximation to the target Fisher type index κ_{FM} defined as the geometric mean of κ_{LM} and κ_{PM}. Thus define the *Diewert Lawrence partial cost of living index of the effects of a change in import prices going from period 0 to 1* as follows:[50]

$$(24.87)\ P_{DLM}^* \equiv [P_{LM}^* P_{PM}^*]^{1/2}.$$

As in section B, it will be useful to develop some alternative expressions for the indexes P_{LM}^*, P_{PM}^* and P_{DLM}^*. Define the *household's period t share of directly imported commodities* S_M^t for t = 0,1 and the *price relative for nominal import prices* R_M as follows:

$$(24.88)\ S_M^t \equiv P_M^t M^t/[P_C^t C^t + P_M^t M^t],\ t = 0,1\ ;\ R_M \equiv P_M^1/P_M^0.$$

Using the techniques described in section B, the following alternative formulae for P_{LM}^*, P_{PM}^* and P_{DLM}^* can be derived:

$$(24.89)\ P_{LM}^* = 1 + S_M^0 (R_M - 1)\ ;$$
$$(24.90)\ P_{PM}^* = [1 + S_M^1 (R_M^{-1} - 1)]^{-1}.$$

Since S_M^0 and S_M^1 are positive, it can be seen that both P_{LM}^* and P_{PM}^* are increasing functions of R_M so that as the nominal price of imports increases, both partial cost of living indexes increase as expected.

Substituting expressions (24.89) and (24.90) into (24.87) leads to the following expression for the Diewert Lawrence import index:

$$(24.91)\ P_{DLM}^* = \{[1 + S_M^0 (R_M - 1)]/[1 + S_M^1(R_M^{-1} - 1)]\}^{1/2}.$$

[50] Diewert and Lawrence did not actually suggest this index in the consumer context but it is the consumer theory counterpart to their producer context index discussed earlier.

The Diewert Lawrence index P_{DLM}^* defined by (24.91) is likely to be closer to the target Fisher index κ_{FM} defined by (24.86) than the Laspeyres and Paasche type indexes P_{LM}^* and P_{PM}^* defined by (24.89) and (24.90).

Using the same notation that is defined in (24.88) above, the logarithm of the *Diewert Morrison partial cost of living index,* P_{DMM}^**, of the effects of a change in import prices going from period 0 to 1* is defined as follows:[51]

$$(24.92)\ \ln P_{DMM}^* \equiv (1/2)(S_M^0 + S_M^1)\ln R_M.$$

It can be verified that P_{DMM}^* satisfies the time reversal; i.e., if the two time periods are switched, then the new P_{DMM}^* index is equal to the reciprocal of the original P_{DMM}^* index.[52]

As in section B, the interest in the Diewert Morrison index stems from the fact that it has a very direct connection with consumer theory; in fact this index is *exactly* equal to the target index κ_{FM} provided that the preferences of the consumer are translog in each period with certain quadratic coefficients equal to each other. The assumptions made on the consumer's expenditure functions e^t for each period are the following general translog counterparts to the translog assumptions (24.27)-(24.35) that were made in earlier sections, letting $p \equiv [P_C,P_M]$:

$$(24.93)\ \ln e^t(p,u) \equiv a_0^t + \sum_{n=1}^2 a_n^t \ln p_n + (1/2) \sum_{n=1}^2\sum_{j=1}^2 a_{nj} \ln p_n \ln p_j + b_1^t \ln u + (1/2)\, b_{11}^t (\ln u)^2 + \sum_{n=1}^2 c_n^t \ln p_n \ln u\ ; \qquad t = 0,1.$$

Note that as before, the coefficients for the quadratic terms in the logarithms of prices are assumed to be constant over time; i.e., it is assumed that $a_{nj}^0 = a_{nj}^1 = a_{nj}$. The coefficients must satisfy the following restrictions in order for e^t to be linearly homogeneous in the prices p:

$$(24.94)\ \sum_{n=1}^2 a_n^t = 1 \text{ for } t = 0,1;$$
$$(24.95)\ a_{nj} = a_{jn} \text{ for all } n,j\ ;$$
$$(24.96)\ \sum_{k=1}^2 a_{nk} = 0 \text{ for } n = 1,2\ ;$$
$$(24.97)\ \sum_{n=1}^5 c_n^t = 0 \text{ for } t = 0,1.$$

Note that using Shephard's Lemma, the logarithmic derivatives of $e^t(P_C^t,P_M^t,u^t)$ with respect to the logarithm of the import price are equal to the following expressions:

$$\begin{aligned}(24.98)\ \partial\ln e^t(P_C^t,P_M^t,u^t)/\partial\ln P_M &= [P_M^t/e^t]\ \partial e^t(P_C^t,P_M^t,u^t)/\partial P_M\ ; && t = 0,1\\ &= [P_M^t/e^t]\ M^t\\ &= S_M^t && \text{using (24.88).}\end{aligned}$$

[51] Strictly speaking, Diewert and Morrison did not define this index in the consumer context but it obviously has the same structure as their partial index which was defined in the producer context.
[52] The Diewert Lawrence index P_{DLM}^* also satisfies this time reversal property.

Noting that assumptions (24.93) imply that the logarithms of the expenditure functions are quadratic in the logarithms of prices and utility, the result given by (24.26) can be applied to definitions (24.82), (24.84) and (24.86) to imply the following result:

$$
\begin{aligned}
(24.99)\ 2 \ln \kappa_{FM} &= \ln \kappa_{LM} + \ln \kappa_{PM} \\
&= [\partial \ln e^0(P_C^0,P_M^0,u^0)/\partial \ln P_M + \partial \ln e^1(P_C^1,P_M^1,u^1)/\partial \ln P_M][\ln P_M^1 - \ln P_M^0] \\
&= [S_M^0 + S_M^1] \ln(P_M^1/P_M^0) \qquad \text{using (24.98).}
\end{aligned}
$$

Thus using (24.92) and (24.99), it can be seen that under the assumptions made on the technology, the following exact equality holds:[53]

$$(24.100)\ \kappa_{FM} = P_{DMM}^*.$$

Thus the Diewert Morrison partial cost of living P_{DMM}^* defined by (24.92) is *exactly* equal to the target theoretical index, κ_{FM}, under very weak assumptions on the technology.

It can be shown that counterparts to the equalities (24.57), (24.58) and (24.60) hold for the four empirical partial cost of living indexes P_{LM}^*, P_{PM}^*, P_{DLM}^* and P_{DMM}^*.[54] Thus P_{DLM}^* and P_{DMM}^* approximate each other to the accuracy of a second order Taylor series approximation around a data point where the prices are equal in each period and the consumer demands are also equal to each other across periods. The practical significance of this result is that for normal time series data where adjacent periods are compared, the Diewert Lawrence and Diewert Morrison indexes will give virtually identical results.

Obviously, the above analysis can be repeated to develop theoretical and empirical partial indexes that measure the effects on the cost of living of a change in domestic consumer prices P_C^t. Thus define the period t share of consumer expenditures on domestic goods as S_C^t and the consumption price relative R_C as follows:

$$(24.101)\ S_C^t \equiv P_C^tC^t/[P_C^tC^t + P_M^tM^t],\ t = 0,1\ ;\ R_C \equiv P_C^1/P_C^0.$$

Using this notation, the logarithm of the *Diewert Morrison partial cost of living index,* P_{DMC}^**, of the effects of a change in consumption prices going from period 0 to 1* is defined as follows:

$$(24.102)\ \ln P_{DMC}^* \equiv (1/2)(S_C^0 + S_C^1)\ln R_C.$$

The Diewert Morrison complete cost of living index can also be defined and this index, P_{DMCM}^*, is simply the usual Törnqvist Theil price index which is defined as follows:

$$(24.103)\ \ln P_{DMCM}^* \equiv (1/2)(S_C^0 + S_C^1)\ln R_C + (1/2)(S_M^0 + S_M^1)\ln R_M.$$

[53] This result is a partial counterpart to results obtained by Caves, Christensen and Diewert (1982; 1410) and Balk (1989; 165-166).

[54] As usual, the proof is a series of straightforward computations.

Using (24.92), (24.102) and (24.103), we have the following counterpart to the earlier production theory multiplicative result (24.77):

(24.104) $P_{DMCM}^{*} = P_{DMC}^{*} P_{DMM}^{*}$;

i.e., the overall cost of living index P_{DMCM}^{*} is exactly equal to the product of the partial domestic consumption price cost of living index P_{DMC}^{*} and the partial import price cost of living index P_{DMM}^{*}. Thus when using the Diewert Morrison indexes, the product of the partial effects is equal to the combined effect.[55]

The exact decomposition given by (24.104) for the Diewert Morrison indexes translates into the following approximate decomposition for the counterpart Diewert Lawrence indexes:

(24.105) $P_{DLCM}^{*} \approx P_{DLC}^{*} P_{DLM}^{*}$.

F. Conclusion

There are a large number of approaches that have been suggested over the years that attempt to determine the welfare effects of changes in the prices of exports and imports. The approach taken in this chapter is rather narrow in scope in that only approaches to the measurement of the effects on real income of changes in international prices that are based on producer theory (sections B-D) or consumer theory (section E) have been considered. However, the approaches outlined in this chapter should prove to be useful in an environment where large fluctuations in food and energy prices are taking place.

The production theory approach outlined in this chapter can be extended to provide a more complete description of the factors that determine the growth in the real income generated by a production sector. In addition to changes in the real prices of exports and imports, other determinants include changes in real domestic prices, changes in the utilization of primary inputs and changes in productivity.[56] For additional materials on how these additional explanatory factors can be added to the export and import price change factors, see Diewert, Mizobuchi and Nomura (2005), Diewert and Lawrence (2006) and Kohli (2006).[57]

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[55] This result generalizes to the case where there is a finer classification of domestic consumption and directly imported household imports.

[56] Another factor which is important in explaining real income growth is tax and tariff policy; see Diewert (2001) and Feenstra, Reinsdorf and Slaughter (2008) on this topic.

[57] Diewert (2008) provides a reconciliation of the approaches of these authors.

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