

# The Economic Approach to Index Number Theory: The Single Household Case

## Chapter 17 of *The Consumer Price Index Manual*

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### A. Introduction

1. In this chapter and the following one, the fifth and last approach to index number theory will be studied: the *economic approach*. In this chapter, the case of a *single* household will be considered while in the following chapter, the economic approach to index number theory will be studied when there are *many* households. A brief outline of the contents of the present chapter follows below.

2. In section B, the theory of the cost of living index for a single consumer or household is presented. This theory was originally developed by the Russian economist, Konüs (1924). The relationship between the (unobservable) true cost of living index and the observable Laspeyres and Paasche indices will be explained. It should be noted that in the economic approach to index number theory, it is assumed that households regard the observed price data as given and the quantity data are regarded as solutions to various economic optimization problems. Many price statisticians find the assumptions made in the economic approach to be somewhat implausible. However, perhaps the best way to regard the assumptions made in the economic approach is that these assumptions simply formalize the fact that consumers tend to purchase more of a commodity if its price falls relative to other prices.

3. In section C, the preferences of the consumer are restricted compared to the completely general case treated in section B. In section C, it is assumed that the function that represents the consumer's preferences over alternative combinations of commodities is homogeneous of degree one. This assumption means that each indifference surface (the set of commodity bundles that give the consumer the same satisfaction or utility) is a radial blowup of a single indifference surface. With this extra assumption, the theory of the true cost of living simplifies, as will be seen.

4. In sections D, E and F, it will be shown that the Fisher, Walsh and Törnqvist price indices (which emerged as being "best" in the various noneconomic approaches) are also among the "best" in the economic approach to index number theory. In these sections, the preference function of the single household will be further restricted compared to the assumptions on preferences made in sections B and C. In sections D, E and F, specific functional forms for the consumer's utility function will be assumed and it will turn out that with each of these specific assumptions, the consumer's true cost of living index can be *exactly* calculated using observable price and quantity data. Each of the three specific functional forms for the consumer's utility function has the property that it can approximate an arbitrary linearly homogeneous function to the second order; i.e., in economics terminology, each of these three functional forms is *flexible*. Hence, using the terminology

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introduced by Diewert (1976), the Fisher, Walsh and Törnqvist price indices are examples of *superlative* index number formulae.

5. In section G, it will be shown that the Fisher, Walsh and Törnqvist price indices approximate each other very closely using “normal” time series data. This is a very convenient result since these three index number formulae repeatedly show up as being “best” in all of the approaches to index number theory. Hence this approximation result implies that it normally will not matter which of these three indices is chosen as the preferred target index for a CPI.

6. The Paasche and Laspeyres price indices have a very convenient mathematical property: they are *consistent in aggregation*. For example, if the Laspeyres formula is used to construct subindices for say food, clothing, etc., then these subindex values can be treated as subaggregate price relatives and using the expenditure shares on these subaggregates, the Laspeyres formula can be applied again to form a two stage Laspeyres price index. Consistency in aggregation means that this two stage index is equal to the corresponding single stage index. In section H, it is shown that the superlative indices derived in the earlier sections are not exactly consistent in aggregation but they are approximately consistent in aggregation.

7. In section I, a very interesting index number formula will be derived: the Lloyd (1975) Moulton (1996) price index. This index number formula makes use of the same information that is required in order to calculate a Laspeyres index (namely, base period expenditure shares, base period prices and current period prices) plus one other parameter (the elasticity of substitution between commodities). If information on this extra parameter can be obtained, then the resulting index can largely eliminate substitution bias and it can be calculated using basically the same information that is required to evaluate the Laspeyres index.

8. In section J, the problem of defining a true cost of living index when the consumer has annual preferences over commodities but faces monthly (or quarterly) prices will be studied. This section attempts to provide an economic foundation for the Lowe index studied in Chapter 15 above. It also provides an introduction to the problems associated with the existence of seasonal commodities, which will be considered at more length in Chapter 22. A final note in section K deals with situations where there may be a zero price for a commodity in one period but it is nonzero in the other period.

## **B. The Konüs cost of living index and observable bounds**

9. In this section, the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, A. A. Konüs (1924), will be developed. This theory relies on the assumption of *optimizing behavior* on the part of economic agents (consumers or producers). Thus given a vector of commodity prices  $\mathbf{p}^t$  that the household faces in a given time period  $t$ , it is assumed that the corresponding observed quantity vector  $\mathbf{q}^t$  is the solution to a cost minimization problem that involves the consumer's preference or utility function  $f$ .<sup>2</sup> Thus in contrast to the axiomatic approach to index number theory, the economic approach does *not* assume that the two quantity vectors  $\mathbf{q}^0$  and  $\mathbf{q}^1$  are independent of the two price vectors  $\mathbf{p}^0$  and  $\mathbf{p}^1$ . In the economic approach, the period 0 quantity vector  $\mathbf{q}^0$  is determined by the consumer's preference function  $f$  and the period 0 vector of prices  $\mathbf{p}^0$  that the consumer faces and the period 1 quantity vector  $\mathbf{q}^1$  is determined by the consumer's preference function  $f$  and the period 1 vector of prices  $\mathbf{p}^1$ .

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<sup>2</sup> For a description of the economic theory of the input and output price indices, see Balk (1998). In the economic theory of the output price index,  $\mathbf{q}^t$  is assumed to be the solution to a revenue maximization problem involving the output price vector  $\mathbf{p}^t$ .

10. The economic approach to index number theory assumes that “the” consumer has well defined *preferences* over different combinations of the  $n$  consumer commodities or items.<sup>3</sup> Each combination of items can be represented by a positive quantity vector  $q \equiv [q_1, \dots, q_n]$ . The consumer’s preferences over alternative possible consumption vectors  $q$  are assumed to be representable by a continuous, nondecreasing and concave<sup>4</sup> utility function  $f$ . Thus if  $f(q^1) > f(q^0)$ , then the consumer prefers the consumption vector  $q^1$  to  $q^0$ . It is further assumed that the consumer minimizes the cost of achieving the period  $t$  utility level  $u^t \equiv f(q^t)$  for periods  $t = 0, 1$ . Thus we assume that the observed period  $t$  consumption vector  $q^t$  solves the following *period  $t$  cost minimization problem*:

$$(17.1) C(u^t, p^t) \equiv \min_q \left\{ \sum_{i=1}^n p_i^t q_i : f(q) = u^t \equiv f(q^t) \right\} = \sum_{i=1}^n p_i^t q_i^t; \quad t = 0, 1.$$

The period  $t$  price vector for the  $n$  commodities under consideration that the consumer faces is  $p^t$ . Note that the solution to the cost or expenditure minimization problem (17.1) for a general utility level  $u$  and general vector of commodity prices  $p$  defines the *consumer’s cost function*,  $C(u, p)$ . The cost function will be used below in order to define the consumer’s *cost of living price index*.

11. The Konüs (1924) family of *true cost of living indices* pertaining to two periods where the consumer faces the strictly positive price vectors  $p^0 \equiv (p_1^0, \dots, p_n^0)$  and  $p^1 \equiv (p_1^1, \dots, p_n^1)$  in periods 0 and 1 respectively is defined as the ratio of the minimum costs of achieving the same utility level  $u \equiv f(q)$  where  $q \equiv (q_1, \dots, q_n)$  is a positive reference quantity vector:

$$(17.2) P_K(p^0, p^1, q) \equiv \frac{C(f(q), p^1)}{C(f(q), p^0)}.$$

Note that definition (17.2) defines a *family* of price indices because there is one such index for each reference quantity vector  $q$  chosen.

12. It is natural to choose two specific reference quantity vectors  $q$  in definition (17.2): the observed base period quantity vector  $q^0$  and the current period quantity vector  $q^1$ . The first of these two choices leads to the *following Laspeyres-Konüs true cost of living index*:

<sup>3</sup> In this chapter, these preferences are assumed to be invariant over time. In the following chapter, this assumption will be relaxed (one of the environmental variables could be a time variable that shifts tastes).

<sup>4</sup>  $f$  is concave if and only if  $f(\lambda q^1 + (1-\lambda)q^2) \geq \lambda f(q^1) + (1-\lambda)f(q^2)$  for all  $0 \leq \lambda \leq 1$  and all  $q^1 \gg 0_n$  and  $q^2 \gg 0_n$ . Note that  $q \geq 0_n$  means that each component of the  $N$  dimensional vector  $q$  is nonnegative,  $q \gg 0_n$  means that each component of  $q$  is positive and  $q > 0_n$  means that  $q \geq 0_n$  but  $q \neq 0_n$ ; i.e.,  $q$  is nonnegative but at least one component is positive.

$$\begin{aligned}
(17.3) \quad P_K(p^0, p^1, q^0) &\equiv \frac{C(f(q^0), p^1)}{C(f(q^0), p^0)} \\
&= \frac{C(f(q^0), p^1)}{\sum_{i=1}^n p_i^0 q_i^0} \quad \text{using (17.1) for } t = 0 \\
&= \frac{\min_q \left\{ \sum_{i=1}^n p_i^1 q_i : f(q) = f(q^0) \right\}}{\sum_{i=1}^n p_i^0 q_i^0}
\end{aligned}$$

using the definition of the cost minimization problem that defines  $C(f(q^0), p^1)$

$$\begin{aligned}
&\leq \frac{\sum_{i=1}^n p_i^1 q_i^0}{\sum_{i=1}^n p_i^0 q_i^0} \quad \text{since } q^0 \equiv (q_1^0, \dots, q_n^0) \text{ is feasible for the minimization problem} \\
&= P_L(p^0, p^1, q^0, q^1)
\end{aligned}$$

where  $P_L$  is the Laspeyres price index. Thus the (unobservable) Laspeyres-Konüs true cost of living index is bounded from above by the observable Laspeyres price index.<sup>5</sup>

13. The second of the two natural choices for a reference quantity vector  $q$  in definition (17.2) leads to the following Paasche-Konüs true cost of living index:

$$\begin{aligned}
(17.4) \quad P_K(p^0, p^1, q^1) &\equiv \frac{C(f(q^1), p^1)}{C(f(q^1), p^0)} \\
&= \frac{\sum_{i=1}^n p_i^1 q_i^1}{C(f(q^1), p^0)} \quad \text{using (17.1) for } t = 1 \\
&= \frac{\sum_{i=1}^n p_i^1 q_i^1}{\min_q \left\{ \sum_{i=1}^n p_i^0 q_i : f(q) = f(q^1) \right\}}
\end{aligned}$$

using the definition of the cost minimization problem that defines  $C(f(q^0), p^0)$

$$\begin{aligned}
&\geq \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^1} \quad \text{since } q^1 \equiv (q_1^1, \dots, q_n^1) \text{ is feasible for the minimization problem and thus} \\
C(f(q^1), p^0) &\leq \sum_{i=1}^n p_i^0 q_i^1 \quad \text{and hence } \frac{1}{C(f(q^1), p^0)} \geq \frac{1}{\sum_{i=1}^n p_i^0 q_i^1} \\
&= P_P(p^0, p^1, q^0, q^1)
\end{aligned}$$

<sup>5</sup> This inequality was first obtained by Konüs (1924) (1939; 17). See also Pollak (1983).

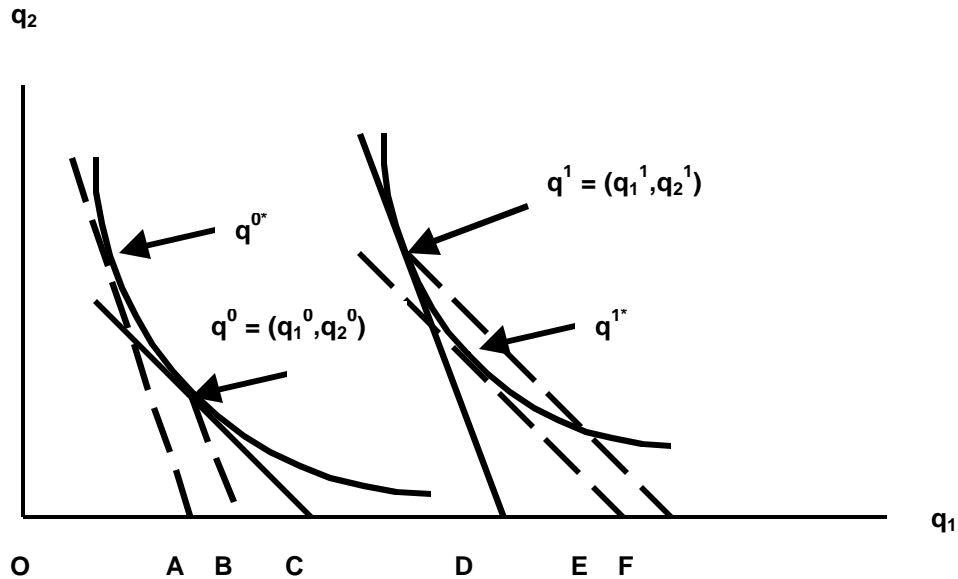
where  $P_p$  is the Paasche price index.. *Thus the (unobservable) Paasche-Konüs true cost of living index is bounded from below by the observable Paasche price index.*<sup>6</sup>

14. It is possible to illustrate the two inequalities (17.3) and (17.4) if there are only two commodities; see Figure 1 below.

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<sup>6</sup> This inequality is also due to Konüs (1924) (1939; 19). See also Pollak (1983).

Figure 17.1: The Laspeyres and Paasche bounds to the true cost of living



The solution to the period 0 cost minimization problem is the vector  $q^0$  and the straight line through C represents the consumer's period 0 budget constraint, the set of quantity points  $q_1, q_2$  such that  $p_1^0 q_1 + p_2^0 q_2 = p_1^0 q_1^0 + p_2^0 q_2^0$ . The curved line through  $q^0$  is the consumer's period 0 indifference curve, the set of points  $q_1, q_2$  such that  $f(q_1, q_2) = f(q_1^0, q_2^0)$ ; i.e., it is the set of consumption vectors that give the same utility as the observed period 0 consumption vector  $q^0$ . The solution to the period 1 cost minimization problem is the vector  $q^1$  and the straight line through D represents the consumer's period 1 budget constraint, the set of quantity points  $q_1, q_2$  such that  $p_1^1 q_1 + p_2^1 q_2 = p_1^1 q_1^1 + p_2^1 q_2^1$ . The curved line through  $q^1$  is the consumer's period 1 indifference curve, the set of points  $q_1, q_2$  such that  $f(q_1, q_2) = f(q_1^1, q_2^1)$ ; i.e., it is the set of consumption vectors that give the same utility as the observed period 1 consumption vector  $q^1$ . The point  $q^{0*}$  solves the hypothetical cost minimization problem of minimizing the cost of achieving the base period utility level  $u^0 \equiv f(q^0)$  when facing the period 1 price vector  $p^1 = (p_1^1, p_2^1)$ . Thus we have  $C[u^0, p^1] = p_1^1 q_1^{0*} + p_2^1 q_2^{0*}$  and the dashed line through A is the corresponding isocost line  $p_1^1 q_1 + p_2^1 q_2 = C[u^0, p^1]$ . Note that the hypothetical cost line through A is parallel to the actual period 1 cost line through D. From (17.3), the Laspeyres-Konüs true index is  $C[u^0, p^1] / [p_1^0 q_1^0 + p_2^0 q_2^0]$  while the ordinary Laspeyres index is  $[p_1^1 q_1^0 + p_2^1 q_2^0] / [p_1^0 q_1^0 + p_2^0 q_2^0]$ . Since the denominators for these two indices are the same, the difference between the indices is due to the differences in their numerators. In Figure 1, this difference in the numerators is expressed by the fact that the cost line through A lies *below* the parallel cost line through B. Now if the consumer's indifference curve through the observed period 0 consumption vector  $q^0$  were L shaped with vertex at  $q^0$ , then the consumer would not change his or her consumption pattern in response to a change in the relative prices of the two commodities while keeping a fixed standard of living. In this case, the hypothetical vector  $q^{0*}$  would coincide with  $q^0$ , the dashed line through A would coincide with the dashed line through B and the true Laspeyres-Konüs index would *coincide* with the ordinary Laspeyres index. However, L shaped indifference curves are not generally consistent with consumer behavior; i.e., when the price of a commodity decreases, consumers generally demand more of it. Thus in the general case, there will be a gap between the points A and B. The magnitude of this gap represents the amount of *substitution bias* between the

true index and the corresponding Laspeyres index; i.e., the Laspeyres index will generally be *greater* than the corresponding true cost of living index,  $P_K(p^0, p^1, q^0)$ .

15. Figure 1 can also be used to illustrate the inequality (17.4). First note that the dashed lines through E and F are parallel to the period 0 isocost line through C. The point  $q^{1*}$  solves the hypothetical cost minimization problem of minimizing the cost of achieving the current period utility level  $u^1 \equiv f(q^1)$  when facing the period 0 price vector  $p^0 = (p_1^0, p_2^0)$ . Thus we have  $C[u^1, p^0] = p_1^0 q_1^{1*} + p_2^0 q_2^{1*}$  and the dashed line through E is the corresponding isocost line  $p_1^1 q_1 + p_2^1 q_2 = C[u^1, p^1]$ . From (17.4), the Paasche-Konüs true index is  $[p_1^1 q_1^1 + p_2^1 q_2^1] / C[u^1, p^0]$  while the ordinary Paasche index is  $[p_1^1 q_1^1 + p_2^1 q_2^1] / [p_1^0 q_1^1 + p_2^0 q_2^1]$ . Since the numerators for these two indices are the same, the difference between the indices is due to the differences in their denominators. In Figure 1, this difference in the denominators is expressed by the fact that the cost line through E lies *below* the parallel cost line through F. The magnitude of this difference represents the amount of *substitution bias* between the true index and the corresponding Paasche index; i.e., the Paasche index will generally be *less* than the corresponding true cost of living index,  $P_K(p^0, p^1, q^1)$ . Note that this inequality goes in the opposite direction to the previous inequality between the two Laspeyres indices. The reason for this change in direction is due to the fact that one set of differences between the two indices takes place in the numerators of the indices (the Laspeyres inequalities) while the other set takes place in the denominators of the indices (the Paasche inequalities).

16. The bound (17.3) on the Laspeyres-Konüs true cost of living  $P_K(p^0, p^1, q^0)$  using the base period level of utility as the living standard is *one sided* as is the bound (17.4) on the Paasche-Konüs true cost of living  $P_K(p^0, p^1, q^1)$  using the *current period* level of utility as the living standard. In a remarkable result, Konüs (1939; 20) showed that there exists an intermediate consumption vector  $q^*$  that is on the straight line joining the base period consumption vector  $q^0$  and the current period consumption vector  $q^1$  such that the corresponding (unobservable) true cost of living index  $P_K(p^0, p^1, q^*)$  is between the observable Laspeyres and Paasche indices,  $P_L$  and  $P_p$ .<sup>7</sup> Thus we have the existence of a number  $\lambda^*$  between 0 and 1 such that

$$(17.5) P_L \leq P_K(p^0, p^1, \mathbf{I}^* q^0 + (1 - \mathbf{I}^*) q^1) \leq P_p \text{ or } P_p \leq P_K(p^0, p^1, \mathbf{I}^* q^0 + (1 - \mathbf{I}^*) q^1) \leq P_L.$$

The above inequalities are of some practical importance. If the observable (in principle) Paasche and Laspeyres indices are not too far apart, then taking a symmetric average of these indices should provide a good approximation to a true cost of living index where the reference standard of living is somewhere between the base and current period living standards. To determine the precise symmetric average of the Paasche and Laspeyres indices, appeal can be made to the results in section C in Chapter 15 above and the geometric mean of the Paasche and Laspeyres indices can be justified as being the “best” average, which is the Fisher price index. Thus the Fisher ideal price index receives a fairly strong justification as a good approximation to an unobservable theoretical cost of living index.

17. The bounds (17.3)-(17.5) are the best bounds that can be obtained on true cost of living indices without making further assumptions. In subsequent subsections, further assumptions are made on the class of utility functions that describe the consumer’s tastes for the  $n$  commodities under consideration. With these extra assumptions, the consumer’s true cost of living can be determined exactly.

### C. The true cost of living index when preferences are homothetic

<sup>7</sup> For more recent applications of the Konüs method of proof, see Diewert (1983a;191) for an application to the consumer context and Diewert (1983b; 1059-1061) for an application to the producer context.

18. Up to now, the consumer's preference function  $f$  did not have to satisfy any particular homogeneity assumption. For the remainder of this section, it is assumed that  $f$  is (positively) *linearly homogeneous*<sup>8</sup>. In the economics literature, this is known as the assumption of *homothetic preferences*.<sup>9</sup> This assumption is not strictly justified from the viewpoint of actual economic behavior, but it leads to economic price indices that are independent from the consumer's standard of living.<sup>10</sup> Under this assumption, the consumer's expenditure or cost function,  $C(u,p)$  defined by (17.1) above, decomposes as follows. For positive commodity prices  $p \gg 0_N$  and a positive utility level  $u$ , then using the definition of  $C$  as the minimum cost of achieving the given utility level  $u$ , the following equalities can be derived:

$$\begin{aligned}
 (17.6) \quad C(u,p) &\equiv \min_q \left\{ \sum_{i=1}^n p_i q_i : f(q_1, \dots, q_n) \geq u \right\} \\
 &= \min_q \left\{ \sum_{i=1}^n p_i q_i : \frac{1}{u} f(q_1, \dots, q_n) \geq 1 \right\} \text{ dividing by } u > 0 \\
 &= \min_q \left\{ \sum_{i=1}^n p_i q_i : f\left(\frac{q_1}{u}, \dots, \frac{q_n}{u}\right) \geq 1 \right\} \text{ using the linear homogeneity of } f \\
 &= u \min_q \left\{ \sum_{i=1}^n \frac{p_i q_i}{u} : f\left(\frac{q_1}{u}, \dots, \frac{q_n}{u}\right) \geq 1 \right\} \\
 &= u \min_z \left\{ \sum_{i=1}^n p_i z_i : f(z_1, \dots, z_n) \geq 1 \right\} \text{ letting } z_i = \frac{q_i}{u} \\
 &= uC(1, p) \qquad \qquad \qquad \text{using definition (17.1)} \\
 &= uc(p)
 \end{aligned}$$

where  $c(p) \equiv C(1,p)$  is the *unit cost function* that corresponds to  $f$ .<sup>11</sup> It can be shown that the unit cost function  $c(p)$  satisfies the same regularity conditions that  $f$  satisfies; i.e.,  $c(p)$  is

<sup>8</sup> The linear homogeneity property means that  $f$  satisfies the following property:  $f(\lambda q) = \lambda f(q)$  for all  $\lambda > 0$  and all  $q \gg 0_n$ . This assumption is fairly restrictive in the consumer context. It implies that each indifference curve is a radial projection of the unit utility indifference curve. It also implies that all income elasticities of demand are unity, which is contradicted by empirical evidence.

<sup>9</sup> More precisely, Shephard (1953) defined a homothetic function to be a monotonic transformation of a linearly homogeneous function. However, if a consumer's utility function is homothetic, it can always be rescaled to be linearly homogeneous without changing consumer behavior. Hence, the homothetic preferences assumption can simply be identified with the linear homogeneity assumption.

<sup>10</sup> This particular branch of the economic approach to index number theory is due to Shephard (1953) (1970) and Samuelson and Swamy (1974). Shephard in particular realized the importance of the homotheticity assumption in conjunction with separability assumptions in justifying the existence of subindices of the overall cost of living index. It should be noted that if the consumer's change in real income or utility between the two periods under consideration is not too large, then assuming that the consumer has homothetic preferences will lead to a true cost of living index which is very close to Laspeyres-Konüs and Paasche-Konüs true cost of living indices defined above by (17.3) and (17.4). Another way of justifying the homothetic preferences assumption is to use (17.49) below, which justifies the use of the superlative Törnqvist-Theil index  $P_T$  in the context of nonhomothetic preferences. Since  $P_T$  is usually numerically close to other superlative indices that are derived using the homothetic preferences assumption, it can be seen that the assumption of homotheticity will usually not be empirically misleading in the index number context.

<sup>11</sup> Economists will recognize the producer theory counterpart to the result  $C(u,p) = uc(p)$ : if a producer's production function  $f$  is subject to constant returns to scale, then the corresponding total cost function  $C(u,p)$  is equal to the product of the output level  $u$  times the unit cost  $c(p)$ .



positive, concave and (positively) linearly homogeneous for positive price vectors.<sup>12</sup> Substituting (17.6) into (17.1) and using  $u^t = f(q^t)$  leads to the following equations:

$$(17.7) \sum_{i=1}^n p_i^t q_i^t = c(p^t) f(q^t) \quad \text{for } t = 0, 1.$$

Thus under the linear homogeneity assumption on the utility function  $f$ , observed period  $t$  expenditure on the  $n$  commodities is equal to the period  $t$  unit cost  $c(p^t)$  of achieving one unit of utility times the period  $t$  utility level,  $f(q^t)$ . Obviously, the period  $t$  unit cost,  $c(p^t)$ , can be identified as the period  $t$  price level  $P^t$  and the period  $t$  level of utility,  $f(q^t)$ , as the period  $t$  quantity level  $Q^t$ .<sup>13</sup>

19. The linear homogeneity assumption on the consumer's preference function  $f$  leads to a simplification for the family of Konüs true cost of living indices,  $P_K(p^0, p^1, q)$ , defined by (17.2) above. Using this definition for an arbitrary reference quantity vector  $q$ :

$$\begin{aligned} (17.8) \quad P_K(p^0, p^1, q) &\equiv \frac{C(f(q), p^1)}{C(f(q), p^0)} \\ &= \frac{c(p^1) f(q)}{c(p^0) f(q)} \quad \text{using (17.6) twice} \\ &= \frac{c(p^1)}{c(p^0)}. \end{aligned}$$

Thus under the homothetic preferences assumption, the entire family of Konüs true cost of living indices collapses to a single index,  $c(p^1)/c(p^0)$ , the ratio of the minimum costs of achieving unit utility level when the consumer faces period 1 and 0 prices respectively. Put another way, under the homothetic preferences assumption,  $P_K(p^0, p^1, q)$  is independent of the reference quantity vector  $q$ .

20. If the Konüs true cost of living index defined by the right hand side of (17.8) is used as the price index concept, then the corresponding implicit quantity index defined using the product test (i.e., the product of the price index times the quantity index is equal to the value ratio) has the following form:

<sup>12</sup> Obviously, the utility function  $f$  determines the consumer's cost function  $C(u, p)$  as the solution to the cost minimization problem in the first line of (17.6). Then the unit cost function  $c(p)$  is defined as  $C(1, p)$ . Thus  $f$  determines  $c$ . But we can also use  $c$  to determine  $f$  under appropriate regularity conditions. In the economics literature, this is known as *duality theory*. For additional material on duality theory and the properties of  $f$  and  $c$ , see Samuelson (1953), Shephard (1953) and Diewert (1974) (1993b; 107-123).

<sup>13</sup> There is also a producer theory interpretation of the above theory; i.e., let  $f$  be the producer's (constant returns to scale) production function, let  $p$  be a vector of input prices that the producer faces, let  $q$  be an input vector and let  $u = f(q)$  be the maximum output that can be produced using the input vector  $q$ .  $C(u, p) \equiv \min_q \{ \sum_{i=1}^n p_i q_i : f(q) \geq u \}$  is the producer's cost function in this case and  $c(p^t)$  can be identified as the period  $t$  input price level while  $f(q^t)$  is the period  $t$  aggregate input level.

$$\begin{aligned}
(17.9) \quad Q(p^0, p^1, q^0, q^1) &\equiv \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^t q_i^t P_K(p^0, p^1, q)} \\
&= \frac{c(p^1) f(q^1)}{c(p^0) f(q^0) P_K(p^0, p^1, q)} \quad \text{using (17.7) twice} \\
&= \frac{c(p^1) f(q^1)}{c(p^0) f(q^0) \{c(p^1)/c(p^0)\}} \quad \text{using (17.8)} \\
&= \frac{f(q^1)}{f(q^0)}.
\end{aligned}$$

Thus under the homothetic preferences assumption, the *implicit quantity index* that corresponds to the true cost of living price index  $c(p^1)/c(p^0)$  is the *utility ratio*  $f(q^1)/f(q^0)$ . Since the utility function is assumed to be homogeneous of degree one, this is the natural definition for a quantity index.

21. In subsequent material, two additional results from economic theory will be needed: Wold's Identity and Shephard's Lemma. *Wold's* (1944; 69-71) (1953; 145) *Identity* is the following result. Assuming that the consumer satisfies the cost minimization assumptions (17.1) for periods 0 and 1 and that the utility function  $f$  is differentiable at the observed quantity vectors  $q^0$  and  $q^1$ , it can be shown<sup>14</sup> that the following equations hold:

$$(17.10) \quad \frac{p_i^t}{\sum_{k=1}^n p_k^t q_k^t} = \frac{\frac{\partial f(q^t)}{\partial q_i}}{\sum_{k=1}^n q_k^t \frac{\partial f(q^t)}{\partial q_k}}; \quad t = 0, 1; \quad k = 1, \dots, n$$

where  $\partial f(q^t)/\partial q_i$  denotes the partial derivative of the utility function  $f$  with respect to the  $i$ th quantity  $q_i$  evaluated at the period  $t$  quantity vector  $q^t$ .

22. If the homothetic preferences assumption is made and it is assumed that the utility function is linearly homogeneous, then *Wold's Identity* simplifies into the following equations which will prove to be very useful:<sup>15</sup>

$$(17.11) \quad \frac{p_i^t}{\sum_{k=1}^n p_k^t q_k^t} = \frac{\partial f(q^t)/\partial q_i}{f(q^t)}; \quad t = 0, 1; \quad k = 1, \dots, n.$$

<sup>14</sup> To prove this, consider the first order necessary conditions for the strictly positive vector  $q^t$  to solve the period  $t$  cost minimization problem. The conditions of Lagrange with respect to the vector of  $q$  variables are:  $p^t = \lambda^t \nabla f(q^t)$  where  $\lambda^t$  is the optimal Lagrange multiplier and  $\nabla f(q^t)$  is the vector of first order partial derivatives of  $f$  evaluated at  $q^t$ . Note that this system of equations is the price equals a constant times marginal utility equations that are familiar to economists. Now take the inner product of both sides of this equation with respect to the period  $t$  quantity vector  $q^t$  and solve the resulting equation for  $\lambda^t$ . Substitute this solution back into the vector equation  $p^t = \lambda^t \nabla f(q^t)$  and (17.10) is obtained.

<sup>15</sup> Differentiate both sides of the equation  $f(\lambda q) = \lambda f(q)$  with respect to  $\lambda$  and then evaluate the resulting equation at  $\lambda=1$ . The equation  $\sum_{i=1}^n f_i(q) q_i = f(q)$  is obtained where  $f_i(q) \equiv \partial f(q)/\partial q_i$ .

23. *Shephard's* (1953; 11) *Lemma* is the following result. Consider the period  $t$  cost minimization problem defined by (17.1) above. If the cost function  $C(u, p)$  is differentiable with respect to the components of the price vector  $p$ , then the period  $t$  quantity vector  $q^t$  is equal to the vector of first order partial derivatives of the cost function with respect to the components of  $p$ :

$$(17.12) \quad q_i^t = \frac{\partial C(u^t, p^t)}{\partial p_i}; \quad i = 1, \dots, n; \quad t = 0, 1.$$

24. To explain why equations (17.12) hold, consider the following argument. Because it is assumed that the observed period  $t$  quantity vector  $q^t$  solves the cost minimization problem defined by  $C(u^t, p^t)$ , then  $q^t$  must be feasible for this problem so it must be the case that  $f(q^t) = u^t$ . Thus  $q^t$  is a feasible solution for the following cost minimization problem where the general price vector  $p$  has replaced the specific period  $t$  price vector  $p^t$ :

$$(17.13) \quad C(u^t, p) \equiv \min_q \left\{ \sum_{i=1}^n p_i q_i : f(q_1, \dots, q_n) \geq u^t \right\} \leq \sum_{i=1}^n p_i q_i^t$$

where the inequality follows from the fact that  $q^t \equiv (q_1^t, \dots, q_n^t)$  is a feasible (but usually not optimal) solution for the cost minimization problem in (17.13). Now define for each strictly positive price vector  $p$  the function  $g(p)$  as follows:

$$(17.14) \quad g(p) \equiv \sum_{i=1}^n p_i q_i^t - C(u^t, p)$$

where as usual,  $p \equiv (p_1, \dots, p_n)$ . Using (17.13) and (17.1), it can be seen that  $g(p)$  is minimized (over all strictly positive price vectors  $p$ ) at  $p = p^t$ . Thus the first order necessary conditions for minimizing a differentiable function of  $n$  variables hold, which simplify to equations (17.12).

25. If the homothetic preferences assumption is made and it is assumed that the utility function is linearly homogeneous, then using (17.6), *Shephard's Lemma* (17.12) becomes:

$$(17.15) \quad q_i^t = u^t \frac{\partial c(p^t)}{\partial p_i}; \quad i = 1, \dots, n; \quad t = 0, 1.$$

Combining equations (17.15) and (17.7), the following system of equations is obtained:

$$(17.16) \quad \frac{q_i^t}{\sum_{k=1}^n p_k^t q_k^t} = \frac{\partial c(p^t)}{\partial p_i} / c(p^t); \quad i = 1, \dots, n; \quad t = 0, 1.$$

26. Note the symmetry of equations (17.16) with equations (17.11). It is these two sets of equations that will be used in subsequent material in this chapter.

#### **D. Superlative indices: the Fisher ideal index**

27. Suppose the consumer has the following utility function:

$$(17.17) f(q_1, \dots, q_n) \equiv \sqrt{\sum_{i=1}^n \sum_{k=1}^n a_{ik} q_i q_k}; a_{ik} = a_{ki} \text{ for all } i \text{ and } k.$$

Differentiating  $f(q)$  defined by (17.17) with respect to  $q_i$  yields the following equations:

$$(17.18) f_i(q) = \frac{1}{2} \frac{2 \sum_{k=1}^n a_{ik} q_k}{\sqrt{\sum_{j=1}^n \sum_{k=1}^n a_{jk} q_j q_k}}; i = 1, \dots, n$$

$$= \frac{\sum_{k=1}^n a_{ik} q_k}{f(q)}$$

where  $f_i(q) \equiv \partial f(q) / \partial q_i$ . In order to obtain the first equation in (17.18), it is necessary to use the symmetry conditions,  $a_{ik} = a_{ki}$ . Now evaluate the second equation in (17.18) at the observed period  $t$  quantity vector  $\hat{q} \equiv (q_1^t, \dots, q_n^t)$  and divide both sides of the resulting equation by  $f(q^t)$ . The following equations are obtained:

$$(17.19) \frac{f_i(q^t)}{f(q^t)} = \frac{\sum_{k=1}^n a_{ik} q_k^t}{\{f(q^t)\}^2} \quad t = 0, 1; \quad i = 1, \dots, n.$$

Assume cost minimizing behavior for the consumer in periods 0 and 1. Since the utility function  $f$  defined by (17.17) is linearly homogeneous and differentiable, equations (17.11) will hold. Now recall the definition of the Fisher ideal quantity index,  $Q_F$  defined earlier in Chapter 15:

$$\begin{aligned}
(17.20) \quad Q_F(p^0, p^1, q^0, q^1) &= \sqrt{\frac{\sum_{i=1}^n p_i^0 q_i^1}{\sum_{k=1}^n p_k^0 q_k^0}} \sqrt{\frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{k=1}^n p_k^1 q_k^0}} \\
&= \sqrt{\sum_{i=1}^n f_i(q^0) \frac{q_i^1}{f(q^0)}} \sqrt{\frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{k=1}^n p_k^1 q_k^0}} \quad \text{using (17.11) for } t=0 \\
&= \sqrt{\sum_{i=1}^n f_i(q^0) \frac{q_i^1}{f(q^0)}} / \sqrt{\frac{\sum_{k=1}^n p_k^1 q_k^0}{\sum_{i=1}^n p_i^1 q_i^1}} \\
&= \sqrt{\sum_{i=1}^n f_i(q^0) \frac{q_i^1}{f(q^0)}} / \sqrt{\sum_{i=1}^n f_i(q^1) \frac{q_i^0}{f(q^1)}} \quad \text{using (17.11) for } t=1 \\
&= \sqrt{\sum_{i=1}^n \sum_{k=1}^n a_{ik} q_k^0 \frac{q_i^1}{\{f(q^0)\}^2}} / \sqrt{\sum_{i=1}^n \sum_{k=1}^n a_{ik} q_k^1 \frac{q_i^0}{\{f(q^1)\}^2}} \quad \text{using (17.19)} \\
&= \sqrt{\frac{1}{\{f(q^0)\}^2}} / \sqrt{\frac{1}{\{f(q^1)\}^2}} \quad \text{using (17.17) and canceling terms} \\
&= \frac{f(q^1)}{f(q^0)}.
\end{aligned}$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the  $n$  commodities that correspond to the utility function defined by (17.17), the Fisher ideal quantity index  $Q_F$  is *exactly* equal to the true quantity index,  $f(q^1)/f(q^0)$ .<sup>16</sup>

28. As was noted in section C.1 of Chapter 15 above, the price index that corresponds to the Fisher quantity index  $Q_F$  using the product test (15.3) is the Fisher price index  $P_F$  defined by (15.12). Let  $c(p)$  be the unit cost function that corresponds to the homogeneous quadratic utility function  $f$  defined by (17.17). Then using (17.16) and (17.20), it can be seen that

$$(17.21) \quad P_F(p^0, p^1, q^0, q^1) = \frac{c(p^1)}{c(p^0)}.$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the  $n$  commodities that correspond to the utility function defined by (17.17), the Fisher ideal price index  $P_F$  is exactly equal to the true price index,  $c(p^1)/c(p^0)$ .

29. A twice continuously differentiable function  $f(q)$  of  $n$  variables  $q \equiv (q_1, \dots, q_n)$  can provide a *second order approximation* to another such function  $f^*(q)$  around the point  $q^*$  if

<sup>16</sup> For the early history of this result, see Diewert (1976; 184).

the level and all of the first and second order partial derivatives of the two functions coincide at  $q^*$ . It can be shown<sup>17</sup> that the homogeneous quadratic function  $f$  defined by (17.17) can provide a second order approximation to an arbitrary  $f^*$  around any (strictly positive) point  $q^*$  in the class of linearly homogeneous functions. Thus the homogeneous quadratic functional form defined by (17.17) is a *flexible functional form*.<sup>18</sup> Diewert (1976; 117) termed an index number formula  $Q(p^0, p^1, q^0, q^1)$  that was *exactly* equal to the true quantity index  $f(q^1)/f(q^0)$  (where  $f$  is a flexible functional form) a *superlative index number formula*.<sup>19</sup> Equation (17.20) and the fact that the homogeneous quadratic function  $f$  defined by (17.17) is a flexible functional form shows that the Fisher ideal quantity index  $Q_F$  defined by (15.14) is a superlative index number formula. Since the Fisher ideal price index  $P_F$  satisfies (17.21) where  $c(p)$  is the unit cost function that is generated by the homogeneous quadratic utility function,  $P_F$  is also called a superlative index number formula.

30. It is possible to show that the Fisher ideal price index is a superlative index number formula by a different route. Instead of starting with the assumption that the consumer's utility function is the homogeneous quadratic function defined by (17.17), it is possible to start with the assumption that the consumer's unit cost function is a homogeneous quadratic.<sup>20</sup> Thus suppose that the consumer has the following unit cost function:

$$(17.22) \quad c(p_1, \dots, p_n) \equiv \sqrt{\sum_{i=1}^n \sum_{k=1}^n b_{ik} p_i p_k}; \quad b_{ik} = b_{ki} \text{ for all } i \text{ and } k.$$

Differentiating  $c(p)$  defined by (17.22) with respect to  $p_i$  yields the following equations:

$$(17.23) \quad c_i(p) = \frac{1}{2} \frac{2 \sum_{k=1}^n b_{ik} p_k}{\sqrt{\sum_{j=1}^n \sum_{k=1}^n b_{jk} p_j p_k}}; \quad i = 1, \dots, n$$

$$= \frac{\sum_{k=1}^n b_{ik} p_k}{c(p)}$$

where  $c_i(p) \equiv \partial c(p) / \partial p_i$ . In order to obtain the first equation in (17.23), it is necessary to use the symmetry conditions. Now evaluate the second equation in (17.23) at the observed period  $t$  price vector  $p^t \equiv (p_1^t, \dots, p_n^t)$  and divide both sides of the resulting equation by  $c(p^t)$ . The following equations are obtained:

$$(17.24) \quad \frac{c_i(p^t)}{c(p^t)} = \frac{\sum_{k=1}^n b_{ik} p_k^t}{\{c(p^t)\}^2} \quad t = 0, 1; \quad i = 1, \dots, n.$$

<sup>17</sup> See Diewert (1976; 130) and let the parameter  $r$  equal 2.

<sup>18</sup> Diewert (1974; 133) introduced this term to the economics literature.

<sup>19</sup> Fisher (1922; 247) used the term superlative to describe the Fisher ideal price index. Thus Diewert adopted Fisher's terminology but attempted to give some precision to Fisher's definition of superlativeness. Fisher defined an index number formula to be superlative if it approximated the corresponding Fisher ideal results using his data set.

<sup>20</sup> Given the consumer's unit cost function  $c(p)$ , Diewert (1974; 112) showed that the corresponding utility function  $f(q)$  can be defined as follows: for a strictly positive quantity vector  $q$ ,  $f(q) \equiv 1 / \max_p \{ \sum_{i=1}^n p_i q_i : c(p) = 1 \}$ .

As cost minimizing behavior for the consumer in periods 0 and 1 is being assumed and since the unit cost function  $c$  defined by (17.22) is differentiable, equations (17.16) will hold. Now recall the definition of the Fisher ideal price index,  $P_F$  given by (15.12) in Chapter 15:

$$\begin{aligned}
 (17.25) \quad P_F(p^0, p^1, q^0, q^1) &= \sqrt{\frac{\sum_{i=1}^n p_i^1 q_i^0}{\sum_{k=1}^n p_k^0 q_k^0}} \sqrt{\frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{k=1}^n p_k^0 q_k^1}} \\
 &= \sqrt{\sum_{i=1}^n p_i^1 \frac{c_i(p^0)}{c(p^0)}} \sqrt{\frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{k=1}^n p_k^0 q_k^1}} \quad \text{using (17.16) for } t = 0 \\
 &= \sqrt{\sum_{i=1}^n p_i^1 \frac{c_i(p^0)}{c(p^0)}} / \sqrt{\frac{\sum_{k=1}^n p_k^0 q_k^1}{\sum_{i=1}^n p_i^1 q_i^1}} \\
 &= \sqrt{\sum_{i=1}^n p_i^1 \frac{c_i(p^0)}{c(p^0)}} / \sqrt{\sum_{i=1}^n p_i^0 \frac{c_i(p^1)}{c(p^1)}} \quad \text{using (17.16) for } t = 1 \\
 &= \sqrt{\sum_{i=1}^n \sum_{k=1}^n b_{ik} p_k^0 \frac{p_i^1}{\{c(p^0)\}^2}} / \sqrt{\sum_{i=1}^n \sum_{k=1}^n b_{ik} p_k^1 \frac{p_i^0}{\{c(p^1)\}^2}} \quad \text{using (17.24)} \\
 &= \sqrt{\frac{1}{\{c(p^0)\}^2}} / \sqrt{\frac{1}{\{c(p^1)\}^2}} \quad \text{using (17.22) and canceling terms} \\
 &= \frac{c(p^1)}{c(p^0)}.
 \end{aligned}$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the  $n$  commodities that correspond to the unit cost function defined by (17.22), the Fisher ideal price index  $P_F$  is *exactly* equal to the true price index,  $c(p^1)/c(p^0)$ .<sup>21</sup>

31. Since the homogeneous quadratic unit cost function  $c(p)$  defined by (17.22) is also a flexible functional form, the fact that the Fisher ideal price index  $P_F$  exactly equals the true price index  $c(p^1)/c(p^0)$  means that  $P_F$  is a *superlative index number formula*.<sup>22</sup>

32. Suppose that the  $b_{ik}$  coefficients in (17.22) satisfy the following restrictions:

$$(17.26) \quad b_{ik} = b_i b_k \quad \text{for } i, k = 1, \dots, n$$

<sup>21</sup> This result was obtained by Diewert (1976; 133-134).

<sup>22</sup> Note that it has been shown that the Fisher index  $P_F$  is exact for the preferences defined by (17.17) as well as the preferences that are dual to the unit cost function defined by (17.22). These two classes of preferences do not coincide in general. However, if the  $n$  by  $n$  symmetric matrix  $A$  of the  $a_{ik}$  has an inverse, then it can be shown that the  $n$  by  $n$  matrix  $B$  of the  $b_{ik}$  will equal  $A^{-1}$ .

where the  $n$  numbers  $b_i$  are nonnegative. In this special case of (17.22), it can be seen that the unit cost function simplifies as follows:

$$(17.27) \quad c(p_1, \dots, p_n) \equiv \sqrt{\sum_{i=1}^n \sum_{k=1}^n b_i b_k p_i p_k} = \sqrt{\sum_{i=1}^n b_i p_i \sum_{k=1}^n b_k p_k} = \sum_{i=1}^n b_i p_i.$$

Substituting (17.27) into Shephard's Lemma (17.15) yields the following expressions for the period  $t$  quantity vectors,  $q^t$ :

$$(17.28) \quad q_i^t = u^t \frac{\partial c(p^t)}{\partial p_i} = b_i u^t \quad i = 1, \dots, n; t = 0, 1.$$

Thus if the consumer has the preferences that correspond to the unit cost function defined by (17.22) where the  $b_k$  satisfy the restrictions (17.26), then the period 0 and 1 quantity vectors are equal to a multiple of the vector  $b \equiv (b_1, \dots, b_n)$ ; i.e.,  $q^0 = b u^0$  and  $q^1 = b u^1$ . Under these assumptions, the Fisher, Paasche and Laspeyres indices,  $P_F$ ,  $P_P$  and  $P_L$ , *all coincide*. However, the preferences which correspond to the unit cost function defined by (17.27) are not consistent with normal consumer behavior since they imply that the consumer will not substitute away from more expensive commodities to cheaper commodities if relative prices change going from period 0 to 1.

### E. Quadratic mean of order $r$ superlative indices

33. It turns out that there are many other superlative index number formulae; i.e., there exist many quantity indices  $Q(p^0, p^1, q^0, q^1)$  that are exactly equal to  $f(q^1)/f(q^0)$  and many price indices  $P(p^0, p^1, q^0, q^1)$  that are exactly equal to  $c(p^1)/c(p^0)$  where the aggregator function  $f$  or the unit cost function  $c$  is a flexible functional form. Two families of superlative indices will be defined below.

34. Suppose the consumer has the *following quadratic mean of order  $r$  utility function*:<sup>23</sup>

$$(17.29) \quad f^r(q_1, \dots, q_n) \equiv \sqrt[r]{\sum_{i=1}^n \sum_{k=1}^n a_{ik} q_i^{r/2} q_k^{r/2}}$$

where the parameters  $a_{ik}$  satisfy the symmetry conditions  $a_{ik} = a_{ki}$  for all  $i$  and  $k$  and the parameter  $r$  satisfies the restriction  $r \neq 0$ . Diewert (1976; 130) showed that the utility function  $f^r$  defined by (17.29) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when  $r = 2$ ,  $f^r$  equals the homogeneous quadratic function defined by (17.17) above.

35. Define the *quadratic mean of order  $r$  quantity index*  $Q^r$  by:

$$(17.30) \quad Q^r(p^0, p^1, q^0, q^1) \equiv \frac{\sqrt[r]{\sum_{i=1}^n s_i^0 (q_i^1 / q_i^0)^{r/2}}}{\sqrt[r]{\sum_{i=1}^n s_i^1 (q_i^1 / q_i^0)^{r/2}}}$$

where  $s_i^t \equiv p_i^t q_i^t / \sum_{k=1}^n p_k^t q_k^t$  is the period  $t$  expenditure share for commodity  $i$  as usual.

<sup>23</sup> This terminology is due to Diewert (1976; 129).



36. Using exactly the same techniques as were used in section D above, it can be shown that  $Q^r$  is exact for the aggregator function  $f^r$  defined by (17.29); i.e., the following exact relationship between the quantity index  $Q^r$  and the utility function  $f^r$  holds:

$$(17.31) \quad Q^r(p^0, p^1, q^0, q^1) = \frac{f^r(q^1)}{f^r(q^0)}.$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the  $n$  commodities that correspond to the utility function defined by (17.29), the quadratic mean of order  $r$  quantity index  $Q^r$  is *exactly* equal to the true quantity index,  $f^r(q^1)/f^r(q^0)$ .<sup>24</sup> Since  $Q^r$  is exact for  $f^r$  and  $f^r$  is a flexible functional form, it can be seen that the quadratic mean of order  $r$  quantity index  $Q^r$  is a *superlative index* for each  $r \neq 0$ . Thus there are an infinite number of superlative quantity indices.

37. For each quantity index  $Q^r$ , the product test (15.3) in Chapter 15 can be used in order to define the corresponding *implicit quadratic mean of order  $r$  price index*  $P^{r*}$ :

$$(17.32) \quad P^{r*}(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0 Q^r(p^0, p^1, q^0, q^1)} = \frac{c^{r*}(p^1)}{c^{r*}(p^0)}$$

where  $c^{r*}$  is the unit cost function that corresponds to the aggregator function  $f^r$  defined by (17.29) above. For each  $r \neq 0$ , the implicit quadratic mean of order  $r$  price index  $P^{r*}$  is also a superlative index.

38. When  $r = 2$ ,  $Q^r$  defined by (17.30) simplifies to  $Q_F$ , the Fisher ideal quantity index and  $P^{r*}$  defined by (17.32) simplifies to  $P_F$ , the Fisher ideal price index. When  $r = 1$ ,  $Q^r$  defined by (17.30) simplifies to:

$$(17.33) \quad Q^1(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^n s_i^0 \sqrt{\frac{q_i^1}{q_i^0}}}{\sum_{i=1}^n s_i^1 \sqrt{\frac{q_i^0}{q_i^1}}} = \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} \frac{\sum_{i=1}^n p_i^0 q_i^0 \sqrt{\frac{q_i^1}{q_i^0}}}{\sum_{i=1}^n p_i^1 q_i^1 \sqrt{\frac{q_i^0}{q_i^1}}}$$

$$= \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} \frac{\sum_{i=1}^n p_i^0 \sqrt{q_i^0 q_i^1}}{\sum_{i=1}^n p_i^1 \sqrt{q_i^0 q_i^1}} \bigg/ \frac{\sum_{i=1}^n p_i^1 \sqrt{q_i^0 q_i^1}}{\sum_{i=1}^n p_i^0 \sqrt{q_i^0 q_i^1}} =$$

$$= \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} / P_W(p^0, p^1, q^0, q^1)$$

where  $P_W$  is the *Walsh price index* defined previously by (15.19) in Chapter 15. Thus  $P^{1*}$  is equal to  $P_W$ , the *Walsh price index*, and hence it is also a superlative price index.

<sup>24</sup> See Diewert (1976; 130).

39. Suppose the consumer has the *following quadratic mean of order r unit cost function*:<sup>25</sup>

$$(17.34) c^r(p_1, \dots, p_n) \equiv \sqrt[r]{\sum_{i=1}^n \sum_{k=1}^n b_{ik} p_i^{r/2} p_k^{r/2}}$$

where the parameters  $b_{ik}$  satisfy the symmetry conditions  $b_{ik} = b_{ki}$  for all  $i$  and  $k$  and the parameter  $r$  satisfies the restriction  $r \neq 0$ . Diewert (1976; 130) showed that the unit cost function  $c^r$  defined by (17.34) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when  $r = 2$ ,  $c^r$  equals the homogeneous quadratic function defined by (17.22) above.

40. Define the *quadratic mean of order r price index*  $P^r$  by:

$$(17.35) P^r(p^0, p^1, q^0, q^1) \equiv \frac{\sqrt[r]{\sum_{i=1}^n s_i^0 \left(\frac{p_i^1}{p_i^0}\right)^{r/2}}}{\sqrt[r]{\sum_{i=1}^n s_i^1 \left(\frac{p_i^1}{p_i^0}\right)^{r/2}}}$$

where  $s_i^t \equiv p_i^t q_i^t / \sum_{k=1}^n p_k^t q_k^t$  is the period  $t$  expenditure share for commodity  $i$  as usual.

41. Using exactly the same techniques as were used in section D above, it can be shown that  $P^r$  is exact for the aggregator function  $c^r$  defined by (17.34); i.e., the following exact relationship between the index number formula  $P^r$  and the unit cost function  $c^r$  holds:

$$(17.36) P^r(p^0, p^1, q^0, q^1) = \frac{c^r(p^1)}{c^r(p^0)}.$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the  $n$  commodities that correspond to the unit cost function defined by (17.34), the quadratic mean of order  $r$  price index  $P^r$  is *exactly* equal to the true price index,  $c^r(p^1)/c^r(p^0)$ .<sup>26</sup> Since  $P^r$  is exact for  $c^r$  and  $c^r$  is a flexible functional form, it can be seen that the quadratic mean of order  $r$  price index  $P^r$  is a *superlative index* for each  $r \neq 0$ . Thus there are an infinite number of superlative price indices.

42. For each price index  $P^r$ , the product test (15.3) in Chapter 15 can be used in order to define the corresponding *implicit quadratic mean of order r quantity index*  $Q^{r*}$ :

$$(17.37) Q^{r*}(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0 P^r(p^0, p^1, q^0, q^1)} = \frac{f^{r*}(p^1)}{f^{r*}(p^0)}$$

<sup>25</sup> This terminology is due to Diewert (1976; 130). This unit cost function was first defined by Denny (1974).

<sup>26</sup> See Diewert (1976; 133-134).

where  $f^{r*}$  is the aggregator function that corresponds to the unit cost function  $c^r$  defined by (5.34) above.<sup>27</sup> For each  $r \neq 0$ , the implicit quadratic mean of order  $r$  quantity index  $Q^{r*}$  is also a superlative index.

43. When  $r = 2$ ,  $P^r$  defined by (17.35) simplifies to  $P_F$ , the Fisher ideal price index and  $Q^{r*}$  defined by (17.37) simplifies to  $Q_F$ , the Fisher ideal quantity index. When  $r = 1$ ,  $P^r$  defined by (17.35) simplifies to:

$$\begin{aligned}
 (17.38) \quad P^1(p^0, p^1, q^0, q^1) &\equiv \frac{\sum_{i=1}^n s_i^0 \sqrt{\frac{p_i^1}{p_i^0}}}{\sum_{i=1}^n s_i^1 \sqrt{\frac{p_i^0}{p_i^1}}} = \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} \frac{\sum_{i=1}^n p_i^0 q_i^0 \sqrt{\frac{p_i^1}{p_i^0}}}{\sum_{i=1}^n p_i^1 q_i^1 \sqrt{\frac{p_i^0}{p_i^1}}} \\
 &= \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} \frac{\sum_{i=1}^n q_i^0 \sqrt{p_i^0 p_i^1}}{\sum_{i=1}^n q_i^1 \sqrt{p_i^0 p_i^1}} = \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} \left/ \frac{\sum_{i=1}^n q_i^1 \sqrt{p_i^0 p_i^1}}{\sum_{i=1}^n q_i^0 \sqrt{p_i^0 p_i^1}} \right. = \\
 &= \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0} / Q_W(p^0, p^1, q^0, q^1)
 \end{aligned}$$

where  $Q_W$  is the *Walsh quantity index* defined previously in footnote 30 of Chapter 15. Thus  $Q^{1*}$  is equal to  $Q_W$ , the *Walsh quantity index*, and hence it is also a superlative quantity index.

## F. Superlative indices: The Törnqvist index

44. In this section, the same assumptions that were made on the consumer in section B above are made here. In particular, it is *not* assumed that the consumer's utility function  $f$  is *necessarily* linearly homogeneous as in sections C, D and E above.

45. Before the main result is derived, a preliminary result is required. Suppose the function of  $n$  variables,  $f(z_1, \dots, z_n) \equiv f(z)$ , is quadratic; i.e.,

$$(17.39) \quad f(z_1, \dots, z_n) \equiv a_0 + \sum_{i=1}^n a_i z_i + \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n a_{ik} z_i z_k; \quad a_{ik} = a_{ki} \text{ for all } i \text{ and } k,$$

where the  $a_i$  and the  $a_{ik}$  are constants. Let  $f_i(z)$  denote the first order partial derivative of  $f$  evaluated at  $z$  with respect to the  $i$ th component of  $z$ ,  $z_i$ . Let  $f_{ik}(z)$  denote the second order partial derivative of  $f$  with respect to  $z_i$  and  $z_k$ . Then it is well known that the second order Taylor series approximation to a quadratic function is *exact*; i.e., if  $f$  is defined by (17.39) above, then for any two points,  $z^0$  and  $z^1$ , the following equation holds:

$$(17.40) \quad f(z^1) - f(z^0) = \sum_{i=1}^n f_i(z^0) \{z_i^1 - z_i^0\} + \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n f_{ik}(z^0) \{z_i^1 - z_i^0\} \{z_k^1 - z_k^0\}$$

<sup>27</sup> The function  $f^{r*}$  can be defined by using  $c^r$  as follows:  $f^{r*}(q) \equiv 1 / \max_p \{ \sum_{i=1}^n p_i q_i : c^r(p) = 1 \}$ .

It is less well known that *an average of two first order Taylor series approximations* to a quadratic function is also *exact*, i.e., if  $f$  is defined by (17.39) above, then for any two points,  $z^0$  and  $z^1$ , the following equation holds<sup>28</sup>

$$(17.41) \quad f(z^1) - f(z^0) = \frac{1}{2} \sum_{i=1}^n \{f_i(z^0) + f_i(z^1)\} \{z_i^1 - z_i^0\}$$

Diewert (1976; 118) and Lau (1979) showed that equation (17.41) characterized a quadratic function and called the equation the *quadratic approximation lemma*. In this Chapter, (17.41) will be called the *quadratic identity*.

46. Suppose that the consumer's *cost function*,<sup>29</sup>  $C(u,p)$ , has the following *translog functional form*:<sup>30</sup>

$$(17.42) \quad \ln C(u, p) \equiv a_0 + \sum_{i=1}^n a_i \ln p_i + \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^n a_{ik} \ln p_i \ln p_k + b_0 \ln u + \sum_{i=1}^n b_i \ln p_i \ln u + \frac{1}{2} b_{00} (\ln u)^2$$

where  $\ln$  is the natural logarithm function and the parameters  $a_i$ ,  $a_{ik}$ , and  $b_i$  satisfy the following restrictions:

$$(17.43) \quad a_{ik} = a_{ki}; i, k = 1, \dots, n; \sum_{i=1}^n a_i = 1; \sum_{i=1}^n b_i = 0; \sum_{k=1}^n a_{ik} = 0; i = 1, \dots, n.$$

These parameter restrictions ensure that  $C(u,p)$  defined by (17.42) is linearly homogeneous in  $p$ , a property that a cost function must have. It can be shown that the translog cost function defined by (17.42) can provide a second order Taylor series approximation to an arbitrary cost function.<sup>31</sup>

47. Assume that the consumer has preferences that correspond to the translog cost function and that the consumer engages in cost minimizing behavior during periods 0 and 1. Let  $p^0$  and  $p^1$  be the period 0 and 1 observed price vectors and let  $q^0$  and  $q^1$  be the period 0 and 1 observed quantity vectors. These assumptions imply:

$$(17.44) \quad C(u^0, p^0) = \sum_{i=1}^n p_i^0 q_i^0 \quad \text{and} \quad C(u^1, p^1) = \sum_{i=1}^n p_i^1 q_i^1$$

where  $C$  is the translog cost function defined above. Now apply Shephard's lemma, (17.12) above and the following equations result:

$$(17.45) \quad q_i^t = \frac{\partial C(u^t, p^t)}{\partial p_i}; i = 1, \dots, n; t = 0, 1$$

$$= \frac{C(u^t, p^t)}{p_i^t} \frac{\partial \ln C(u^t, p^t)}{\partial \ln p_i}.$$

<sup>28</sup> The proof of this and the foregoing relation is by straightforward verification.

<sup>29</sup> The consumer's cost function was defined by (17.6) above.

<sup>30</sup> Christensen, Jorgenson and Lau (1971) introduced this function into the economics literature.

<sup>31</sup> It can also be shown that if all of the  $b_i = 0$  and  $b_{00} = 0$ , then  $C(u,p) = uC(1,p) \equiv uc(p)$ ; i.e., with these additional restrictions on the parameters of the general translog cost function, homothetic preferences are the result of these restrictions. Note that it is also assumed that utility  $u$  is scaled so that  $u$  is always positive.

Now use (17.44) to replace  $C(u^t, p^t)$  in (17.45). After some cross multiplication, this becomes the following system of equations:

$$(17.46) \frac{p_i^t q_i^t}{\sum_{k=1}^n p_k^t q_k^t} \equiv s_i^t = \frac{\partial \ln C(u^t, p^t)}{\partial \ln p_i}; i = 1, \dots, n; t = 0, 1$$

or

$$(17.47) s_i^t = a_i + \sum_{k=1}^n a_{ik} \ln p_k^t + b_i \ln u^t; i = 1, \dots, n; t = 0, 1$$

where  $s_i^t$  is the period  $t$  expenditure share on commodity  $i$ .

48. Define the geometric average of the period 0 and 1 utility levels as  $u^*$ ; i.e., define

$$(17.48) u^* \equiv \sqrt{u^0 u^1}.$$

Now observe that the right hand side of the equation that defines the natural logarithm of the translog cost function, equation (17.42), is a quadratic function of the variables  $z_i \equiv \ln p_i$  if utility is held constant at the level  $u^*$ . Hence the quadratic identity, (17.41), can be applied and the following equation is obtained:

$$\begin{aligned} (17.49) & \ln C(u^*, p^1) - \ln C(u^*, p^0) \\ &= \frac{1}{2} \sum_{i=1}^n \left\{ \frac{\partial \ln C(u^*, p^0)}{\partial \ln p_i} + \frac{\partial \ln C(u^*, p^1)}{\partial \ln p_i} \right\} \{ \ln p_i^1 - \ln p_i^0 \} \\ &= \frac{1}{2} \sum_{i=1}^n \left( a_i + \sum_{k=1}^n a_{ik} \ln p_k^0 + b_i \ln u^* + a_i + \sum_{k=1}^n a_{ik} \ln p_k^1 + b_i \ln u^* \right) (\ln p_i^1 - \ln p_i^0) \\ &= \frac{1}{2} \sum_{i=1}^n \left( a_i + \sum_{k=1}^n a_{ik} \ln p_k^0 + b_i \ln \sqrt{u^0 u^1} + a_i + \sum_{k=1}^n a_{ik} \ln p_k^1 + b_i \ln \sqrt{u^0 u^1} \right) (\ln p_i^1 - \ln p_i^0) \\ &= \frac{1}{2} \sum_{i=1}^n \left( a_i + \sum_{k=1}^n a_{ik} \ln p_k^0 + b_i \ln u^0 + a_i + \sum_{k=1}^n a_{ik} \ln p_k^1 + b_i \ln u^1 \right) (\ln p_i^1 - \ln p_i^0) \\ &= \frac{1}{2} \sum_{i=1}^n \left\{ \frac{\partial \ln C(u^0, p^0)}{\partial \ln p_i} + \frac{\partial \ln C(u^1, p^1)}{\partial \ln p_i} \right\} (\ln p_i^1 - \ln p_i^0) \\ &= \frac{1}{2} \sum_{i=1}^n (s_i^0 + s_i^1) (\ln p_i^1 - \ln p_i^0) \quad \text{using (17.47)}. \end{aligned}$$

The last equation in (17.49) can be recognized as the logarithm of the Törnqvist-Theil index number formula  $P_T$  defined earlier by (15.81) in Chapter 15. Hence exponentiating both sides of (17.49) yields the following equality between the true cost of living between periods 0 and 1, evaluated at the intermediate utility level  $u^*$  and the observable Törnqvist-Theil index  $P_T$ :<sup>32</sup>

<sup>32</sup> This result is due to Diewert (1976; 122).

$$(17.50) \frac{C(u^*, p^1)}{C(u^*, p^0)} = P_T(p^0, p^1, q^0, q^1).$$

Since the translog cost function which appears on the left hand side of (17.49) is a flexible functional form, the Törnqvist-Theil price index  $P_T$  is also a *superlative index*.

49. It is somewhat mysterious how a ratio of *unobservable* cost functions of the form appearing on the left hand side of the above equation can be *exactly* estimated by an *observable* index number formula but the key to this mystery is the assumption of cost minimizing behavior and the quadratic identity (17.41) along with the fact that derivatives of cost functions are equal to quantities, as specified by Shephard's Lemma. In fact, all of the exact index number results derived in sections D and E can be derived using transformations of the quadratic identity along with Shephard's Lemma (or Wold's Identity).<sup>33</sup> Fortunately, for most empirical applications, assuming that the consumer has (transformed) quadratic preferences will be an adequate assumption so the results presented in section D-F are quite useful to index number practitioners who are willing to adopt the economic approach to index number theory.<sup>34</sup> Essentially, the economic approach to index number theory provides a strong justification for the use of the Fisher price index  $P_F$  defined by (15.12), the Törnqvist-Theil price index  $P_T$  defined by (15.81), the implicit quadratic mean of order  $r$  price indices  $P^{r*}$  defined by (17.32) (when  $r = 1$ , this index is the Walsh price index defined by (15.19) in Chapter 15) and the quadratic mean of order  $r$  price indices  $P^r$  defined by (17.35). In the next section, it is asked if it matters which one of these formula is chosen as "best".

### G. The approximation properties of superlative indices

50. The results of sections D-F provide price statisticians with a large number of index number formulae which appear to be equally good from the viewpoint of the economic approach to index number theory. Two questions arise as a consequence of these results:

- Does it matter which of these formulae is chosen?
- If it does matter, which formula should be chosen?

51. With respect to the first question, Diewert (1978; 888) showed that all of the superlative index number formulae listed above in sections D-F approximate each other to the second order around any point where the two price vectors,  $p^0$  and  $p^1$ , are equal and where the two quantity vectors,  $q^0$  and  $q^1$ , are equal. In particular, this means that the following equalities are valid for all  $r$  and  $s$  not equal to 0 provided that  $p^0 = p^1$  and  $q^0 = q^1$ .<sup>35</sup>

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<sup>33</sup> See Diewert (2002a).

<sup>34</sup> However, if consumer preferences are nonhomothetic and the change in utility is substantial between the two situations being compared, then it may be desirable to compute separately the Laspeyres-Konüs and Paasche-Konüs true cost of living indices defined above by (17.3) and (17.4),  $C(u^0, p^1)/C(u^0, p^0)$  and  $C(u^1, p^1)/C(u^1, p^0)$  respectively. In order to do this, it would be necessary to use econometrics and estimate empirically the consumer's cost or expenditure function.

<sup>35</sup> To prove the equalities in (17.51)-(17.56), simply differentiate the various index number formulae and evaluate the derivatives at  $p^0 = p^1$  and  $q^0 = q^1$ . Actually, equations (17.51)-(17.56) are still true provided that  $p^1 = \lambda p^0$  and  $q^1 = \mu q^0$  for any numbers  $\lambda > 0$  and  $\mu > 0$ ; i.e., provided that the period 1 price vector is proportional to the period 0 price vector and that the period 1 quantity vector is proportional to the period 0 quantity vector.

$$(17.51) P_T(p^0, p^1, q^0, q^1) = P^r(p^0, p^1, q^0, q^1) = P^{s*}(p^0, p^1, q^0, q^1);$$

$$(17.52) \frac{\partial P_T(p^0, p^1, q^0, q^1)}{\partial p_i^t} = \frac{\partial P^r(p^0, p^1, q^0, q^1)}{\partial p_i^t} = \frac{\partial P^{s*}(p^0, p^1, q^0, q^1)}{\partial p_i^t}; i=1, \dots, n; t=0,1;$$

$$(17.53) \frac{\partial P_T(p^0, p^1, q^0, q^1)}{\partial q_i^t} = \frac{\partial P^r(p^0, p^1, q^0, q^1)}{\partial q_i^t} = \frac{\partial P^{s*}(p^0, p^1, q^0, q^1)}{\partial q_i^t}; i=1, \dots, n; t=0,1;$$

$$(17.54) \frac{\partial^2 P_T(p^0, p^1, q^0, q^1)}{\partial p_i^t \partial p_k^t} = \frac{\partial^2 P^r(p^0, p^1, q^0, q^1)}{\partial p_i^t \partial p_k^t} = \frac{\partial^2 P^{s*}(p^0, p^1, q^0, q^1)}{\partial p_i^t \partial p_k^t}; i, k=1, \dots, n; t=0,1;$$

$$(17.55) \frac{\partial^2 P_T(p^0, p^1, q^0, q^1)}{\partial p_i^t \partial q_k^t} = \frac{\partial^2 P^r(p^0, p^1, q^0, q^1)}{\partial p_i^t \partial q_k^t} = \frac{\partial^2 P^{s*}(p^0, p^1, q^0, q^1)}{\partial p_i^t \partial q_k^t}; i, k=1, \dots, n; t=0,1;$$

$$(17.56) \frac{\partial^2 P_T(p^0, p^1, q^0, q^1)}{\partial q_i^t \partial q_k^t} = \frac{\partial^2 P^r(p^0, p^1, q^0, q^1)}{\partial q_i^t \partial q_k^t} = \frac{\partial^2 P^{s*}(p^0, p^1, q^0, q^1)}{\partial q_i^t \partial q_k^t}; i, k=1, \dots, n; t=0,1;$$

where the Törnqvist-Theil price index  $P_T$  is defined by (15.81), the implicit quadratic mean of order  $r$  price indices  $P^{s*}$  are defined by (17.32) and the quadratic mean of order  $r$  price indices  $P^r$  are defined by (17.35). Using the above results, Diewert (1978; 884) concluded that “all superlative indices closely approximate each other”.

52. However, the above conclusion is not true even though the equations (17.51)-(17.56) are true. The problem is that the quadratic mean of order  $r$  price indices  $P^r$  and the implicit quadratic mean of order  $s$  price indices  $P^{s*}$  are (continuous) functions of the parameters  $r$  and  $s$  respectively. Hence as  $r$  and  $s$  become very large in magnitude, the indices  $P^r$  and  $P^{s*}$  can differ substantially from say  $P^2 = P_F$ , the Fisher ideal index. In fact, using definition (17.35) and the limiting properties of means of order  $r$ <sup>36</sup>, Robert Hill (2000;7) showed that  $P^r$  has the following limit as  $r$  approaches plus or minus infinity:

$$(17.57) \lim_{r \rightarrow +\infty} P^r(p^0, p^1, q^0, q^1) = \lim_{r \rightarrow -\infty} P^r(p^0, p^1, q^0, q^1) = \sqrt{\min_i \left( \frac{p_i^1}{p_i^0} \right) \max_i \left( \frac{p_i^1}{p_i^0} \right)}.$$

Using Hill's method of analysis, it can be shown that the implicit quadratic mean of order  $r$  price index has the following limit as  $r$  approaches plus or minus infinity:

$$(17.58) \lim_{r \rightarrow +\infty} P^{r*}(p^0, p^1, q^0, q^1) = \lim_{r \rightarrow -\infty} P^{r*}(p^0, p^1, q^0, q^1) \\ = \frac{\sum_{i=1}^n p_i^1 q_i^1}{\sum_{i=1}^n p_i^0 q_i^0 \sqrt{\min_i \left( \frac{p_i^1}{p_i^0} \right) \max_i \left( \frac{p_i^1}{p_i^0} \right)}}.$$

Thus for  $r$  large in magnitude,  $P^r$  and  $P^{r*}$  can differ substantially from  $P_T$ ,  $P^1$ ,  $P^{1*} = P_W$  (the Walsh price index) and  $P^2 = P^{2*} = P_F$  (the Fisher ideal index).<sup>37</sup>

<sup>36</sup> See Hardy, Littlewood and Polya (1934).

<sup>37</sup> Robert Hill (2000) documents this for two data sets. His time series data consists of annual expenditure and quantity data for 64 components of U.S. GDP from 1977 to 1994. For this data set, Hill (2000; 16) found that “superlative indexes can differ by more than a factor of two (i.e., by more than 100 percent), even though Fisher and Törnqvist never differ by more than 0.6 percent.”

53. Although Robert Hill's theoretical and empirical results demonstrate conclusively that all superlative indices will not necessarily closely approximate each other, there is still the question of how well the more commonly used superlative indices will approximate each other. All of the commonly used superlative indices,  $P^r$  and  $P^{r*}$ , fall into the interval  $0 \leq r \leq 2$ .<sup>38</sup> Hill (2000; 16) summarized how far apart the Törnqvist and Fisher indices were, making all possible bilateral comparisons between any two data points for his time series data set as follows:

“The superlative spread  $S(0,2)$  is also of interest since, in practice, Törnqvist ( $r = 0$ ) and Fisher ( $r = 2$ ) are by far the two most widely used superlative indexes. In all 153 bilateral comparisons,  $S(0,2)$  is less than the Paasche-Laspeyres spread and on average, the superlative spread is only 0.1 percent. It is because attention, until now, has focussed almost exclusively on superlative indexes in the range  $0 \leq r \leq 2$  that a general misperception has persisted in the index number literature that all superlative indexes approximate each other closely.”

Thus for Hill's time series data set covering 64 components of U.S. GDP from 1977 to 1994 and making all possible bilateral comparisons between any two years, the Fisher and Törnqvist price indices differed by only 0.1 percent on average. This close correspondence is consistent with the results of other empirical studies using annual time series data.<sup>39</sup> Additional evidence on this topic may be found in chapter 19.

54. In the earlier Chapters of this Manual, it has been found that several index number formulae seem “best” when viewed from various perspectives. Thus it was found that the Fisher ideal index  $P_F = P^2 = P^{2*}$  defined by (15.12) seemed to be best from one axiomatic viewpoint, the Törnqvist-Theil price index  $P_T$  defined by (15.81) seemed to be best from another axiomatic perspective as well as from the stochastic viewpoint, and the Walsh index  $P_W$  defined by (15.19) (which is equal to the implicit quadratic mean of order  $r$  price indices  $P^{r*}$  defined by (17.32) when  $r = 1$ ) seemed to be best from the viewpoint of the “pure” price index. The results presented in this section indicate that for “normal” time series data, these 3 indices will give virtually the same answer. To determine *precisely* which one of these three alternative indices to use as a theoretical target or actual index, the statistical agency will have to decide which approach to bilateral index number theory is most consistent with its goals. However, for most practical purposes, it will not matter which of these three indices is chosen as a theoretical target index for making price comparisons between two periods.

## H. Superlative indices and two stage aggregation

55. Most statistical agencies use the Laspeyres formula to aggregate prices in two stages. At the first stage of aggregation, the Laspeyres formula is used to aggregate components of the overall index (e.g., food, clothing, services, etc.) and then at the second stage of aggregation, these component subindices are further combined into the overall index. The following question then naturally arises: does the index computed in two stages coincide with the index computed in a single stage? Initially, this question is addressed in the context of the Laspeyres formula.<sup>40</sup>

56. Suppose that the price and quantity data for period  $t$ ,  $p^t$  and  $q^t$ , can be written in terms of  $M$  subvectors as follows:

<sup>38</sup> Diewert (1980; 451) showed that the Törnqvist index  $P_T$  is a limiting case of  $P^r$  as  $r$  tends to 0.

<sup>39</sup> See for example Diewert (1978; 894) or Fisher (1922), which is reproduced in Diewert (1976; 135).

<sup>40</sup> Much of the material in this section is adapted from Diewert (1978) and Alterman, Diewert and Feenstra (1999). See also Balk (1996b) for a discussion of alternative definitions for the two stage aggregation concept and references to the literature on this topic.



$$(17.59) p^t = (p^{t_1}, p^{t_2}, \dots, p^{t_M}); q^t = (q^{t_1}, q^{t_2}, \dots, q^{t_M}); t = 0, 1$$

where the dimensionality of the subvectors  $p^{tm}$  and  $q^{tm}$  is  $N_m$  for  $m = 1, 2, \dots, M$  with the sum of the dimensions  $N_m$  equal to  $n$ . These subvectors correspond to the price and quantity data for subcomponents of the consumer price index for period  $t$ . Now construct subindices for each of these components going from period 0 to 1. For the base period, set the price for each of these subcomponents, say  $P_m^0$  for  $m = 1, 2, \dots, M$ , equal to 1 and set the corresponding base period subcomponent quantities, say  $Q_m^0$  for  $m = 1, 2, \dots, M$ , equal to the base period value of consumption for that subcomponent for  $m = 1, 2, \dots, M$ :

$$(17.60) P_m^0 \equiv 1; Q_m^0 \equiv \sum_{i=1}^{N_m} p_i^{0m} q_i^{0m} \text{ for } m = 1, 2, \dots, M.$$

Now use the Laspeyres formula in order to construct a period 1 price for each subcomponent, say  $P_m^1$  for  $m = 1, 2, \dots, M$ , of the consumer price index. Since the dimensionality of the subcomponent vectors,  $p^{tm}$  and  $q^{tm}$ , differs from the dimensionality of the complete period  $t$  vectors of prices and quantities,  $p^t$  and  $q^t$ , it is necessary to use different symbols for these subcomponent Laspeyres indices, say  $P_L^m$  for  $m = 1, 2, \dots, M$ . Thus the period 1 subcomponent prices are defined as follows:

$$(17.61) P_m^1 \equiv P_L^m(p^{0m}, p^{1m}, q^{0m}, q^{1m}) \equiv \frac{\sum_{i=1}^{N_m} p_i^{1m} q_i^{0m}}{\sum_{i=1}^{N_m} p_i^{0m} q_i^{0m}} \text{ for } m = 1, 2, \dots, M.$$

Once the period 1 prices for the  $M$  subindices have been defined by (17.61), then corresponding subcomponent period 1 quantities  $Q_m^1$  for  $m = 1, 2, \dots, M$  can be defined by deflating the period 1 subcomponent values  $\sum_{i=1}^{N_m} p_i^{1m} q_i^{1m}$  by the prices  $P_m^1$ :

$$(17.62) Q_m^1 \equiv \frac{\sum_{i=1}^{N_m} p_i^{1m} q_i^{1m}}{P_m^1} \text{ for } m = 1, 2, \dots, M.$$

Now define subcomponent price and quantity vectors for each period  $t = 0, 1$  using equations (17.60) to (17.62) above. Thus define the period 0 and 1 subcomponent price vectors  $P^0$  and  $P^1$  as follows:

$$(17.63) P^0 = (P_1^0, P_2^0, \dots, P_M^0) \equiv 1_M; P^1 = (P_1^1, P_2^1, \dots, P_M^1)$$

where  $1_M$  denotes a vector of ones of dimension  $M$  and the components of  $P^1$  are defined by (17.61). The period 0 and 1 subcomponent quantity vectors  $Q^0$  and  $Q^1$  are defined as follows:

$$(17.64) Q^0 = (Q_1^0, Q_2^0, \dots, Q_M^0); Q^1 = (Q_1^1, Q_2^1, \dots, Q_M^1)$$

where the components of  $Q^0$  are defined in (17.60) and the components of  $Q^1$  are defined by (17.62). The price and quantity vectors in (17.63) and (17.64) represent the results of the first stage aggregation. Now use these vectors as inputs into the second stage aggregation problem; i.e., apply the Laspeyres price index formula using the information in (17.63) and (17.64) as inputs into the index number formula. Since the price and quantity vectors that are inputs into this second stage aggregation problem have dimension  $M$  instead of the single stage formula

which utilized vectors of dimension  $n$ , a different symbol is required for the new Laspeyres index which is chosen to be  $P_L^*$ . Thus the Laspeyres price index computed in two stages can be denoted as  $P_L^*(P^0, P^1, Q^0, Q^1)$ . Now ask whether this two stage Laspeyres index equals the corresponding single stage index  $P_L$  that was studied in the previous sections of this chapter; i.e., ask whether

$$(17.65) P_L^*(P^0, P^1, Q^0, Q^1) = P_L(p^0, p^1, q^0, q^1).$$

If the Laspeyres formula is used at each stage of each aggregation, the answer to the above question is yes: straightforward calculations show that the Laspeyres index calculated in two stages equals the Laspeyres index calculated in one stage.

57. Now suppose that the Fisher or Törnqvist formula is used at each stage of the aggregation; i.e., in equations (17.61), suppose that the Laspeyres formula  $P_L^m(p^{0m}, p^{1m}, q^{0m}, q^{1m})$  is replaced by the Fisher formula  $P_F^m(p^{0m}, p^{1m}, q^{0m}, q^{1m})$  (or by the Törnqvist formula  $P_T^m(p^{0m}, p^{1m}, q^{0m}, q^{1m})$ ) and in equation (17.65),  $P_L^*(P^0, P^1, Q^0, Q^1)$  is replaced by  $P_F^*$  (or by  $P_T^*$ ) and  $P_L(p^0, p^1, q^0, q^1)$  is replaced by  $P_F$  (or by  $P_T$ ). Then is it the case that counterparts are obtained to the two stage aggregation result for the Laspeyres formula, (17.65)? The answer is no; it can be shown that, in general,

$$(17.66) P_F^*(P^0, P^1, Q^0, Q^1) \neq P_F(p^0, p^1, q^0, q^1) \text{ and } P_T^*(P^0, P^1, Q^0, Q^1) \neq P_T(p^0, p^1, q^0, q^1).$$

Similarly, it can be shown that the quadratic mean of order  $r$  index number formula  $P^r$  defined by (17.35) and the implicit quadratic mean of order  $r$  index number formula  $P^{r*}$  defined by (17.32) are also not consistent in aggregation.

58. However, even though the Fisher and Törnqvist formulae are not *exactly* consistent in aggregation, it can be shown that these formulae are *approximately* consistent in aggregation. More specifically, it can be shown that the two stage Fisher formula  $P_F^*$  and the single stage Fisher formula  $P_F$  in (17.66), both regarded as functions of the  $4n$  variables in the vectors  $p^0, p^1, q^0, q^1$ , approximate each other to the second order around a point where the two price vectors are equal (so that  $p^0 = p^1$ ) and where the two quantity vectors are equal (so that  $q^0 = q^1$ ) and a similar result holds for the two stage and single stage Törnqvist indices in (17.66).<sup>41</sup> As was seen in the previous section, the single stage Fisher and Törnqvist indices have a similar approximation property so all four indices in (17.66) approximate each other to the second order around an equal (or proportional) price and quantity point. Thus for normal time series data, single stage and two stage Fisher and Törnqvist indices will usually be numerically very close.<sup>42</sup> This result will be illustrated in chapter 19 for an artificial data set.

59. Similar approximate consistency in aggregation results (to the results for the Fisher and Törnqvist formulae explained in the previous paragraph) can be derived for the *quadratic mean of order  $r$  indices*,  $P^r$ , and for the implicit quadratic mean of order  $r$  indices,  $P^{r*}$ ; see Diewert (1978; 889). However, the results of Hill (2000) again imply that *the second order approximation property of the single stage quadratic mean of order  $r$  index  $P^r$  to its two stage counterpart will break down as  $r$  approaches either plus or minus infinity*. To see this, consider a simple example where there are only four commodities in total. Let the first price

<sup>41</sup> See Diewert (1978; 889). In other words, a string of equalities similar to (17.50)-(17.55) hold between the two stage indices and their single stage counterparts. In fact, these equalities are still true provided that  $p^1 = \lambda p^0$  and  $q^1 = \mu q^0$  for any numbers  $\lambda > 0$  and  $\mu > 0$ .

<sup>42</sup> For an empirical comparison of the four indices, see Diewert (1978; 894-895). For the Canadian consumer data considered there, the chained two stage Fisher in 1971 was 2.3228 and the corresponding chained two stage Törnqvist was 2.3230, the same values as for the corresponding single stage indices.

ratio  $p_1^1/p_1^0$  be equal to the positive number  $a$ , let the second two price ratios  $p^1/p_i^0$  equal the  $b$  and let the last price ratio  $p_4^1/p_4^0$  equal  $c$  where we assume  $a < c$  and  $a \leq b \leq c$ . Using Hill's result (17.57), the limiting value of the single stage index is:

$$(17.67) \lim_{r \rightarrow +\infty} P^r(p^0, p^1, q^0, q^1) = \lim_{r \rightarrow -\infty} P^r(p^0, p^1, q^0, q^1) = \sqrt{\min_i \left( \frac{p_i^1}{p_i^0} \right) \max_i \left( \frac{p_i^1}{p_i^0} \right)} = \sqrt{ac}.$$

Now aggregate commodities 1 and 2 into a subaggregate and commodities 3 and 4 into another subaggregate. Using Hill's result (17.57) again, it is found that the limiting price index for the first subaggregate is  $[ab]^{1/2}$  and the limiting price index for the second subaggregate is  $[bc]^{1/2}$ . Now apply the second stage of aggregation and use Hill's result once again to conclude that the limiting value of the two stage aggregation using  $P^r$  as the index number formula is  $[ab^2c]^{1/4}$ . Thus the limiting value as  $r$  tends to plus or minus infinity of the single stage aggregate over the two stage aggregate is  $[ac]^{1/2}/[ab^2c]^{1/4} = [ac/b^2]^{1/4}$ . Now  $b$  can take on any value between  $a$  and  $c$  and so the ratio of the single stage limiting  $P^r$  to its two stage counterpart can take on any value between  $[c/a]^{1/4}$  and  $[a/c]^{1/4}$ . Since  $c/a$  is less than 1 and  $a/c$  is greater than 1, it can be seen that the ratio of the single stage to the two stage index can be arbitrarily far from 1 as  $r$  becomes large in magnitude with an appropriate choice of the numbers  $a$ ,  $b$  and  $c$ .

60. The results in the previous paragraph show that some caution is required in assuming that *all* superlative indices will be approximately consistent in aggregation. However, for the three most commonly used superlative indices (the Fisher ideal  $P_F$ , the Törnqvist-Theil  $P_T$  and the Walsh  $P_W$ ), the available empirical evidence indicates that these indices satisfy the consistency in aggregation property to a sufficiently high enough degree of approximation that users will not be unduly troubled by any inconsistencies.<sup>43</sup>

## I. The Lloyd Moulton index number formula

61. The index number formula that will be discussed in this section on the single household economic approach to index number theory is a potentially very useful one for statistical agencies that are faced with the problem of producing a CPI in a timely manner. The Lloyd Moulton formula that will be discussed in this section makes use of the same information that is required in order to implement a Laspeyres index except that one additional piece of information is required.

62. In this section, the same assumptions on the consumer are made that were made in section C above. In particular, it is assumed that the consumer's utility function  $f(q)$  is linearly homogeneous<sup>44</sup> and the corresponding unit cost function is  $c(p)$ . It is supposed that the unit cost function has the following functional form:

$$(17.68) c(p) \equiv \mathbf{a}_0 \left( \sum_{i=1}^n \mathbf{a}_i p_i^{1-s} \right)^{1/(1-s)} \quad \text{if } s \neq 1 \text{ or } \ln c(p) \equiv \mathbf{a}_0 + \sum_{i=1}^n \mathbf{a}_i \ln p_i \quad \text{if } s = 1$$

where the  $\alpha_i$  and  $\sigma$  are nonnegative parameters with  $\sum_{i=1}^n \alpha_i = 1$ . The unit cost function defined by (17.68) corresponds to a *Constant Elasticity of Substitution (CES) aggregator function* which was introduced into the economics literature by Arrow, Chenery, Minhas and Solow (1961)<sup>45</sup>. The parameter  $\sigma$  is the *elasticity of substitution*; when  $\sigma = 0$ , the unit cost

<sup>43</sup> See chapter 19 for some additional evidence on this topic.

<sup>44</sup> Thus homothetic preferences are assumed in this section.

<sup>45</sup> In the mathematics literature, this aggregator function or utility function is known as a mean of order  $r$ ; see Hardy, Littlewood and Polyá (1934; 12-13).

function defined by (17.68) becomes linear in prices and hence corresponds to a fixed coefficients aggregator function which exhibits 0 substitutability between all commodities. When  $\sigma = 1$ , the corresponding aggregator or utility function is a Cobb-Douglas function. When  $\sigma$  approaches  $+\infty$ , the corresponding aggregator function  $f$  approaches a linear aggregator function which exhibits infinite substitutability between each pair of inputs. The CES unit cost function defined by (17.68) is *not* a fully flexible functional form (unless the number of commodities  $n$  being aggregated is 2) but it is considerably more flexible than the zero substitutability aggregator function (this is the special case of (17.68) where  $\sigma$  is set equal to zero) that is exact for the Laspeyres and Paasche price indices.

63. Under the assumption of cost minimizing behavior in period 0, Shephard's Lemma, (17.12) above, tells us that the observed first period consumption of commodity  $i$ ,  $q_i^0$ , will be equal to  $u^0 \partial c(p^0)/\partial p_i$  where  $\partial c(p^0)/\partial p_i$  is the first order partial derivative of the unit cost function with respect to the  $i$ th commodity price evaluated at the period 0 prices and  $u^0 = f(q^0)$  is the aggregate (unobservable) level of period 0 utility. Using the CES functional form defined by (17.68) and assuming that  $\sigma \neq 1$ , the following equations are obtained:

$$(17.69) \quad q_i^0 = u^0 \mathbf{a}_0 \left\{ \sum_{k=1}^n \mathbf{a}_k (p_k^0)^r \right\}^{(1/r)-1} \mathbf{a}_i (p_i^0)^{r-1}; r \equiv 1 - \sigma \neq 0; i = 1, 2, \dots, n$$

$$= \frac{u^0 c(p^0) \mathbf{a}_i (p_i^0)^{r-1}}{\sum_{k=1}^n \mathbf{a}_k (p_k^0)^r}.$$

These equations can be rewritten as

$$(17.70) \quad \frac{p_i^0 q_i^0}{u^0 c(p^0)} = \frac{\mathbf{a}_i (p_i^0)^r}{\sum_{k=1}^n \mathbf{a}_k (p_k^0)^r}; i = 1, 2, \dots, n$$

where  $r \equiv 1 - \sigma$ . Now consider the following *Lloyd (1975) Moulton (1996) index number formula*:

$$(17.71) \quad P_{LM}(p^0, p^1, q^0, q^1) \equiv \left\{ \sum_{i=1}^n s_i^0 \left( \frac{p_i^1}{p_i^0} \right)^{1-\sigma} \right\}^{1/(1-\sigma)}; \sigma \neq 1$$

where  $s_i^0$  is the period 0 expenditure share of commodity  $i$  as usual:

$$(17.72) \quad s_i^0 \equiv \frac{p_i^0 q_i^0}{\sum_{k=1}^n p_k^0 q_k^0}; i = 1, 2, \dots, n$$

$$= \frac{p_i^0 q_i^0}{u^0 c(p^0)} \text{ using the assumption of cost minimizing behavior}$$

$$= \frac{\mathbf{a}_i (p_i^0)^r}{\sum_{k=1}^n \mathbf{a}_k (p_k^0)^r} \text{ using (17.70).}$$

If (17.72) is substituted into (17.71), it is found that:

$$\begin{aligned}
 (17.73) \quad P_{LM}(p^0, p^1, q^0, q^1) &= \left\{ \sum_{i=1}^n s_i^0 \left( \frac{p_i^1}{p_i^0} \right)^r \right\}^{1/r} \\
 &= \left\{ \frac{\sum_{i=1}^n \mathbf{a}_i (p_i^0)^r \left( \frac{p_i^1}{p_i^0} \right)^r}{\sum_{k=1}^n \mathbf{a}_k (p_k^0)^r} \right\}^{1/r} \\
 &= \left\{ \frac{\sum_{i=1}^n \mathbf{a}_i (p_i^1)^r}{\sum_{k=1}^n \mathbf{a}_k (p_k^0)^r} \right\}^{1/r} \\
 &= \frac{\mathbf{a}_0 \left\{ \sum_{i=1}^n \mathbf{a}_i (p_i^1)^r \right\}^{1/r}}{\mathbf{a}_0 \left\{ \sum_{k=1}^n \mathbf{a}_k (p_k^0)^r \right\}^{1/r}} \\
 &= \frac{c(p^1)}{c(p^0)} \text{ using } r \equiv 1 - \sigma \text{ and definition (17.68).}
 \end{aligned}$$

64. Equation (17.73) shows that the Lloyd Moulton index number formula  $P_{LM}$  is *exact* for CES preferences. Lloyd (1975) and Moulton (1996) independently derived this result but it was Moulton who appreciated the significance of the formula (17.71) for statistical agency purposes. Note that in order to evaluate (17.71) numerically, it is necessary to have information on:

- base period expenditure shares  $s_i^0$ ;
- the price relatives  $p_i^1/p_i^0$  between the base period and the current period and
- an estimate of the elasticity of substitution between the commodities in the aggregate,  $\sigma$ .

The first two pieces of information are the standard information sets that statistical agencies use to evaluate the Laspeyres price index  $P_L$  (note that  $P_{LM}$  reduces to  $P_L$  if  $\sigma = 0$ ). Hence, if the statistical agency is able to estimate the elasticity of substitution  $\sigma$  based on past experience<sup>46</sup>, *then the Lloyd Moulton price index can be evaluated using essentially the same information set that is used in order to evaluate the traditional Laspeyres index*. Moreover, the resulting consumer price index will be free of *substitution bias* to a reasonable degree of approximation.<sup>47</sup> Of course, the practical problem with implementing this methodology is

<sup>46</sup> For the first application of this methodology (in the context of the consumer price index), see Shapiro and Wilcox (1997; 121-123). They calculated superlative Törnqvist indices for the U.S. for the years 1986-1995 and then calculated the Lloyd Moulton CES index for the same period using various values of  $\sigma$ . They then chose the value of  $\sigma$  (which was .7) which caused the CES index to most closely approximate the Törnqvist index. Essentially the same methodology was used by Alterman, Diewert and Feenstra (1999) in their study of U.S. import and export price indices. For alternative methods for estimating  $\sigma$ , see Balk (2000b).

<sup>47</sup> What is a "reasonable" degree of approximation depends on the context. Assuming that consumers have CES preferences is not a reasonable assumption in the context of estimating elasticities of demand: at least a second order approximation to the consumer's preferences is required in this

that estimates of the elasticity of substitution parameter  $\sigma$  are bound to be somewhat uncertain and hence the resulting Lloyd Moulton index may be subject to charges that it is not *objective* or *reproducible*. The statistical agency will have to balance the benefits of reducing substitution bias with these possible costs.

## J. Annual preferences and monthly prices

65. Recall the definition of the Lowe index,  $P_{Lo}(p^0, p^1, q)$  defined by (15.15) in Chapter 15. In section D of Chapter 15, it was noted that this formula is frequently used by statistical agencies as a target index for a CPI. It was also noted that while the price vectors  $p^0$  (the base period price vector) and  $p^1$  (the current period price vector) were *monthly or quarterly* price vectors, the quantity vector  $q \equiv (q_1, q_2, \dots, q_n)$  which appeared in this basket type formula was usually taken to be an *annual* quantity vector that referred to a *base year*,  $b$  say, that is prior to the base period for the prices, month 0. Thus, typically, a statistical agency will produce a Consumer Price Index at a monthly frequency that has the form  $P_{Lo}(p^0, p^t, q^b)$ , where  $p^0$  is the price vector pertaining to the base period month for prices, month 0,  $p^t$  is the price vector pertaining to the current period month for prices, month  $t$  say, and  $q^b$  is a reference basket quantity vector that refers to the base year  $b$ , which is equal to or prior to month 0.<sup>48</sup> The question to be addressed in the present section is: can this index be related to one based on the economic approach to index number theory?

### J.1 The Lowe index as an approximation to a true cost of living index

66. Assume that the consumer has preferences defined over consumption vectors  $q \equiv [q_1, \dots, q_n]$  that can be represented by the continuous increasing utility function  $f(q)$ . Thus if  $f(q^1) > f(q^0)$ , then the consumer prefers the consumption vector  $q^1$  to  $q^0$ . Let  $q^b$  be the annual consumption vector for the consumer in the base year  $b$ . Define the base year utility level  $u^b$  as the utility level that corresponds to  $f(q)$  evaluated at  $q^b$ :

$$(17.74) \quad u^b \equiv f(q^b).$$

67. For any vector of positive commodity prices  $p \equiv [p_1, \dots, p_n]$  and for any feasible utility level  $u$ , the consumer's *cost function*,  $C(u, p)$ , can be defined in the usual way as the minimum expenditure required to achieve the utility level  $u$  when facing the prices  $p$ :

$$(17.75) \quad C(u, p) \equiv \min_q \left\{ \sum_{i=1}^n p_i q_i : f(q_1, \dots, q_n) = u \right\}.$$

Let  $p^b \equiv [p_1^b, \dots, p_n^b]$  be the vector of annual prices that the consumer faced in the base year  $b$ . Assume that the observed base year consumption vector  $q^b \equiv [q_1^b, \dots, q_n^b]$  solves the following base year cost minimization problem:

$$(17.76) \quad C(u^b, p^b) \equiv \min_q \left\{ \sum_{i=1}^n p_i^b q_i : f(q_1, \dots, q_n) = u^b \right\} = \sum_{i=1}^n p_i^b q_i^b.$$

The cost function will be used below in order to define the consumer's *cost of living price index*.

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context. However, in the context of approximating changes in a consumer's expenditures on the  $n$  commodities under consideration, it is usually adequate to assume a CES approximation.

<sup>48</sup> As noted in Chapter 15, month 0 is called the price reference period and year  $b$  is called the weight reference period.

68. Let  $p^0$  and  $p^t$  be the *monthly* price vectors that the consumer faces in months 0 and  $t$ . Then the *Konüs true cost of living index*,  $P_K(p^0, p^t, q^b)$ , between months 0 and  $t$ , using the base year utility level  $u^b = f(q^b)$  as the reference standard of living, is defined as the following ratio of minimum monthly costs of achieving the utility level  $u^b$ :

$$(17.77) \quad P_K(p^0, p^t, q^b) \equiv \frac{C(f(q^b), p^t)}{C(f(q^b), p^0)}.$$

69. Using the definition of the monthly cost minimization problem that corresponds to the cost  $C(f(q^b), p^t)$ , it can be seen that the following inequality holds:

$$(17.78) \quad \begin{aligned} C(f(q^b), p^t) &\equiv \min_q \left\{ \sum_{i=1}^n p_i^t q_i : f(q_1, \dots, q_n) = f(q_1^b, \dots, q_n^b) \right\} \\ &\leq \sum_{i=1}^n p_i^t q_i^b \end{aligned}$$

since the base year quantity vector  $q^b$  is feasible for the cost minimization problem. Similarly, using the definition of the monthly cost minimization problem that corresponds to the month 0 cost  $C(f(q^b), p^0)$ , it can be seen that the following inequality holds:

$$(17.79) \quad \begin{aligned} C(f(q^b), p^0) &\equiv \min_q \left\{ \sum_{i=1}^n p_i^0 q_i : f(q_1, \dots, q_n) = f(q_1^b, \dots, q_n^b) \right\} \\ &\leq \sum_{i=1}^n p_i^0 q_i^b \end{aligned}$$

since the base year quantity vector  $q^b$  is feasible for the cost minimization problem.

70. It will prove useful to rewrite the two inequalities (17.78) and (17.79) as equalities. This can be done if nonnegative *substitution bias terms*,  $e^t$  and  $e^0$ , are subtracted from the right hand sides of these two inequalities. Thus (17.78) and (17.79) can be rewritten as follows:

$$(17.80) \quad C(u^b, p^t) = \sum_{i=1}^n p_i^t q_i^b - e^t ;$$

$$(17.81) \quad C(u^b, p^0) = \sum_{i=1}^n p_i^0 q_i^b - e^0 .$$

71. Using (17.80) and (17.81) and the definition (15.15) in Chapter 15 of the Lowe index, the following approximate equality for the Lowe index results:

$$(17.82) \quad \begin{aligned} P_{Lo}(p^0, p^t, q^b) &\equiv \frac{\sum_{i=1}^n p_i^t q_i^b}{\sum_{i=1}^n p_i^0 q_i^b} = \frac{\{C(u^b, p^t) + e^t\}}{\{C(u^b, p^0) + e^0\}} \\ &\approx \frac{C(u^b, p^t)}{C(u^b, p^0)} = P_K(p^0, p^t, q^b). \end{aligned}$$

Thus if the nonnegative substitution bias terms  $e^0$  and  $e^t$  are small, then the Lowe index between months 0 and t,  $P_{Lo}(p^0, p^t, q^b)$ , will be an adequate approximation to the true cost of living index between months 0 and t,  $P_K(p^0, p^t, q^b)$ .

72. A bit of algebraic manipulation shows that the Lowe index will be *exactly* equal to its cost of living counterpart if the substitution bias terms satisfy the following relationship:<sup>49</sup>

$$(17.83) \quad \frac{e^t}{e^0} = \frac{C(u^b, p^t)}{C(u^b, p^0)} = P_K(p^0, p^t, q^b).$$

Equations (17.82) and (17.83) can be interpreted as follows: if the rate of growth in the amount of substitution bias between months 0 and t is equal to the rate of growth in the minimum cost of achieving the base year utility level  $u^b$  between months 0 and t, then the observable Lowe index,  $P_{Lo}(p^0, p^t, q^b)$ , will be exactly equal to its true cost of living index counterpart,  $P_K(p^0, p^t, q^b)$ .<sup>50</sup>

73. It is difficult to know whether condition (17.83) will hold or whether the substitution bias terms  $e^0$  and  $e^t$  will be small. Thus in the following two subsections, first and second order Taylor series approximations to these substitution bias terms will be developed.

## J.2 A first order approximation to the bias of the Lowe index

74. The true cost of living index between months 0 and t, using the base year utility level  $u^b$  as the reference utility level, is the ratio of two unobservable costs,  $C(u^b, p^t)/C(u^b, p^0)$ . However, both of these hypothetical costs can be approximated by first order Taylor series approximations that can be evaluated using observable information on prices and base year quantities. The first order Taylor series approximation to  $C(u^b, p^t)$  around the annual base year price vector  $p^b$  is given by the following approximate equation:<sup>51</sup>

(17.84)

$$\begin{aligned} C(u^b, p^t) &\approx C(u^b, p^b) + \sum_{i=1}^n \left[ \frac{\partial C(u^b, p^b)}{\partial p_i} \right] [p_i^t - p_i^b] \\ &= C(u^b, p^b) + \sum_{i=1}^n q_i^b [p_i^t - p_i^b] \end{aligned}$$

using assumption (17.76) and Shephard's Lemma (17.12)

$$\begin{aligned} &= \sum_{i=1}^n p_i^b q_i^b + \sum_{i=1}^n q_i^b [p_i^t - p_i^b] && \text{using (17.76)} \\ &= \sum_{i=1}^n p_i^t q_i^b \end{aligned}$$

Similarly, the first order Taylor series approximation to  $C(u^b, p^0)$  around the annual base year price vector  $p^b$  is given by the following approximate equation:

<sup>49</sup> This assumes that  $e^0$  is greater than zero. If  $e^0$  is equal to zero, then to have equality of  $P_K$  and  $P_{Lo}$ , it must be the case that  $e^t$  is also equal to zero.

<sup>50</sup> It can be seen that when month t is set equal to month 0,  $e^t = e^0$  and  $C(u^b, p^t) = C(u^b, p^0)$  and thus (17.83) is satisfied and  $P_{Lo} = P_K$ . This is not surprising since both indices are equal to unity when  $t = 0$ .

<sup>51</sup> This type of Taylor series approximation was used in Schultze and Mackie (2002; 91) in the cost of living index context but it essentially dates back to Hicks (1941-42; 134) in the consumer surplus context. See also Diewert (1992b; 568) and Hausman (2002; 8).



$$\begin{aligned}
C(u^b, p^0) &\approx C(u^b, p^b) + \sum_{i=1}^n \left[ \frac{\partial C(u^b, p^b)}{\partial p_i} \right] [p_i^0 - p_i^b] \\
(17.85) \quad &= C(u^b, p^b) + \sum_{i=1}^n q_i^b [p_i^0 - p_i^b] \\
&= \sum_{i=1}^n p_i^b q_i^b + \sum_{i=1}^n q_i^b [p_i^0 - p_i^b] \\
&= \sum_{i=1}^n p_i^0 q_i^b.
\end{aligned}$$

75. Comparing (17.84) with (17.80) and comparing (17.85) with (17.81), it can be seen that to the accuracy of the first order approximations used in (17.84) and (17.85), the substitution bias terms  $e^t$  and  $e^0$  will be zero. Using these results to reinterpret (17.82), it can be seen that if the month 0 and month t price vectors,  $p^0$  and  $p^t$ , are not too different from the base year vector of prices  $p^b$ , then the Lowe index  $P_{Lo}(p^0, p^t, q^b)$  will approximate the true cost of living index  $P_K(p^0, p^t, q^b)$  to the accuracy of a first order approximation. *This result is quite useful, since it indicates that if the monthly price vectors  $p^0$  and  $p^t$  are just randomly fluctuating around the base year prices  $p^b$  (with modest variances), then the Lowe index will serve as an adequate approximation to a theoretical cost of living index.* However, if there are systematic long term trends in prices and month t is fairly distant from month 0 (or the end of year b is quite distant from month 0), then the first order approximations given by (17.84) and (17.85) may no longer be adequate and the Lowe index may have a considerable bias relative to its cost of living counterpart. The hypothesis of long run trends in prices will be explored in the following subsection.

### J.3 A second order approximation to the substitution bias of the Lowe index

76. A second order Taylor series approximation to  $C(u^b, p^t)$  around the base year price vector  $p^b$  is given by the following approximate equation:

$$\begin{aligned}
C(u^b, p^t) &\approx C(u^b, p^b) + \sum_{i=1}^n \left[ \frac{\partial C(u^b, p^b)}{\partial p_i} \right] [p_i^t - p_i^b] \\
(17.86) \quad &+ \left( \frac{1}{2} \right) \sum_{i=1}^n \sum_{j=1}^n \left[ \frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j} \right] [p_i^t - p_i^b] [p_j^t - p_j^b] \\
&= \sum_{i=1}^n p_i^b q_i^b + \left( \frac{1}{2} \right) \sum_{i=1}^n \sum_{j=1}^n \left[ \frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j} \right] [p_i^t - p_i^b] [p_j^t - p_j^b]
\end{aligned}$$

where the last equality follows using (17.84).<sup>52</sup> Similarly, a second order Taylor series approximation to  $C(u^b, p^0)$  around the base year price vector  $p^b$  is given by the following approximate equation:

<sup>52</sup> This type of second order approximation is due to Hicks (1941-42; 133-134) (1946; 331). See also Diewert (1992b; 568), Hausman (2002; 18) and Schultze and Mackie (2002; 91). For alternative approaches to modeling substitution bias, see Diewert (1998a) (2002c; 598-603) and Hausman (2002).

$$\begin{aligned}
(17.87) \quad C(u^b, p^0) &\approx C(u^b, p^b) + \sum_{i=1}^n \left[ \frac{\partial C(u^b, p^b)}{\partial p_i} \right] [p_i^0 - p_i^b] \\
&+ \left( \frac{1}{2} \right) \sum_{i=1}^n \sum_{j=1}^n \left[ \frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j} \right] [p_i^0 - p_i^b] [p_j^0 - p_j^b] \\
&= \sum_{i=1}^n p_i^0 q_i^b + \left( \frac{1}{2} \right) \sum_{i=1}^n \sum_{j=1}^n \left[ \frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j} \right] [p_i^0 - p_i^b] [p_j^0 - p_j^b]
\end{aligned}$$

where the last equality follows using (17.85).

77. Comparing (17.86) to (17.80), and (17.87) to (17.81), it can be seen that to the accuracy of a second order approximation, the month 0 and month t substitution bias terms,  $e^0$  and  $e^t$ , will be equal to the following expressions involving the second order partial derivatives of the consumer's cost function  $\partial^2 C(u^b, p^b) / \partial p_i \partial p_j$  evaluated at the base year standard of living  $u^b$  and at the base year prices  $p^b$ :

$$(17.88) \quad e^0 \approx - \left( \frac{1}{2} \right) \sum_{i=1}^n \sum_{j=1}^n \left[ \frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j} \right] [p_i^0 - p_i^b] [p_j^0 - p_j^b];$$

$$(17.89) \quad e^t \approx - \left( \frac{1}{2} \right) \sum_{i=1}^n \sum_{j=1}^n \left[ \frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j} \right] [p_i^t - p_i^b] [p_j^t - p_j^b].$$

Since the consumer's cost function  $C(u, p)$  is a concave function in the components of the price vector  $p^b$ , it is known<sup>54</sup> that the  $n$  by  $n$  (symmetric) matrix of second order partial derivatives  $[\partial^2 C(u^b, p^b) / \partial p_i \partial p_j]$  is negative semidefinite.<sup>55</sup> Hence, for arbitrary price vectors  $p^b$ ,  $p^0$  and  $p^t$ , the right hand sides of (17.88) and (17.89) will be nonnegative. Thus to the accuracy of a second order approximation, the substitution bias terms  $e^0$  and  $e^t$  will be nonnegative.

78. Now assume that there are *long run systematic trends in prices*. Assume that the last month of the base year for quantities occurs  $M$  months prior to month 0, the base month for prices, and assume that prices trend *linearly* with time, starting with the last month of the base year for quantities. Thus assume the existence of constants  $\alpha_j$  for  $j = 1, \dots, n$  such that the price of commodity  $j$  in month  $t$  is given by:

$$(17.90) \quad p_i^t = p_i^b + \alpha_j (M + t) \quad j = 1, \dots, n; \quad t = 0, 1, \dots, T.$$

Substituting (17.90) into (17.88) and (17.89) leads to the following second order approximations to the two substitution bias terms,  $e^0$  and  $e^t$ :<sup>56</sup>

$$(17.91) \quad e^0 \approx \mathbf{g} M^2;$$

<sup>53</sup> See Diewert (1993b; 109-110).

<sup>54</sup> See Diewert (1993b; 149).

<sup>55</sup> A symmetric  $n$  by  $n$  matrix  $A$  with  $ij$ th element equal to  $a_{ij}$  is negative semidefinite if and only if for every vector  $z \equiv [z_1, \dots, z_n]$ , it is the case that  $\sum_{i=1}^n \sum_{j=1}^n a_{ij} z_i z_j \leq 0$ .

<sup>56</sup> Note that the period 0 approximate bias defined by the right hand side of (17.91) is fixed while the period  $t$  approximate bias defined by the right hand side of (17.92) increases quadratically with time  $t$ . Hence the period  $t$  approximate bias term will eventually overwhelm the period 0 approximate bias in this linear time trends case if  $t$  is allowed to become large enough.

$$(17.92) \quad e^t \approx \mathbf{g}(M + t)^2$$

where  $\gamma$  is defined as follows:

$$(17.93) \quad \mathbf{g} \equiv -\left(\frac{1}{2}\right) \sum_{i=1}^n \sum_{j=1}^n \left[ \frac{\partial^2 C(u^b, p^b)}{\partial p_i \partial p_j} \right] \mathbf{a}_i \mathbf{a}_j \geq 0.$$

79. It should be noted that the parameter  $\gamma$  will be zero under two sets of conditions:<sup>57</sup>

- All of the second order partial derivatives of the consumer's cost function  $\partial^2 C(u^b, p^b) / \partial p_i \partial p_j$  are equal to zero;
- Each commodity price change parameter  $\alpha_j$  is proportional to the corresponding commodity  $j$  base year price  $p_j^b$ .<sup>58</sup>

The first condition is empirically unlikely since it implies that the consumer will not substitute away from commodities whose relative price has increased. The second condition is also empirically unlikely, since it implies that the structure of relative prices remains unchanged over time. Thus in what follows, it will be assumed that  $\gamma$  is a positive number.

80. In order to simplify the notation in what follows, define the denominator and numerator of the month  $t$  Lowe index,  $P_{L_o}(p^0, p^t, q^b)$ , as  $a$  and  $b$  respectively; i.e.; define:

$$(17.94) \quad a \equiv \sum_{i=1}^n p_i^0 q_i^b ;$$

$$(17.95) \quad b \equiv \sum_{i=1}^n p_i^t q_i^b .$$

Using equations (17.90) to eliminate the month 0 prices  $p^0$  from (17.94) and the month  $t$  prices  $p^t$  from (17.95) leads to the following expressions for  $a$  and  $b$ :

$$(17.96) \quad a = \sum_{i=1}^n p_i^b q_i^b + \sum_{i=1}^n \mathbf{a}_i q_i^b M ;$$

$$(17.97) \quad b = \sum_{i=1}^n p_i^b q_i^b + \sum_{i=1}^n \mathbf{a}_i q_i^b (M + t) .$$

It is assumed that  $a$  and  $b$  are positive<sup>59</sup> and that

$$(17.98) \quad \sum_{i=1}^n \mathbf{a}_i q_i^b \geq 0 .$$

<sup>57</sup> A more general condition that ensures the positivity of  $\gamma$  is that the vector  $[\alpha_1, \dots, \alpha_n]$  is not an eigenvector of the matrix of second order partial derivatives  $\partial^2 C(u, p) / \partial p_i \partial p_j$  that corresponds to a zero eigenvalue.

<sup>58</sup> It is known that  $C(u, p)$  is linearly homogeneous in the components of the price vector  $p$ ; see Diewert (1993b; 109) for example. Hence using Euler's Theorem on homogeneous functions, it can be shown that  $p^b$  is an eigenvector of the matrix of second order partial derivatives  $\partial^2 C(u, p) / \partial p_i \partial p_j$  that corresponds to a zero eigenvalue and thus  $\sum_{i=1}^n \sum_{j=1}^n [\partial^2 C(u, p) / \partial p_i \partial p_j] p_i^b p_j^b = 0$ ; see Diewert (1993b; 149) for a detailed proof of this result.

<sup>59</sup> It is also assumed that  $a - \gamma M^2$  is positive.

Assumption (17.98) rules out a general deflation in prices.

81. Define the *bias* in the month  $t$  Lowe index,  $B^t$ , as the difference between the true cost of living index  $P_K(p^0, p^t, q^b)$  defined by (17.77) and the corresponding Lowe index  $P_{Lo}(p^0, p^t, q^b)$ :

$$\begin{aligned}
 (17.99) \quad B^t &\equiv P_K(p^0, p^t, q^b) - P_{Lo}(p^0, p^t, q^b) \\
 &= \left\{ \frac{C(u^b, p^t)}{C(u^b, p^0)} \right\} - \left( \frac{b}{a} \right) && \text{using (17.94) and (17.95)} \\
 &= \left\{ \frac{[b - e^t]}{[a - e^0]} \right\} - \left( \frac{b}{a} \right) && \text{using (17.80) and (17.81)} \\
 &\approx \left\{ \frac{[b - \mathbf{g}(M+t)^2]}{a - \mathbf{g}M^2} \right\} - \left( \frac{b}{a} \right) && \text{using (17.91) and (17.92)} \\
 &= \mathbf{g} \frac{\{(b-a)M^2 - 2aMt - at^2\}}{\{a[a - \mathbf{g}M^2]\}} && \text{simplifyin g terms} \\
 &= \mathbf{g} \frac{\left\{ \left[ \sum_{i=1}^n \mathbf{a}_i q_i^b t \right] M^2 - 2 \left[ \sum_{i=1}^n p_i^b q_i^b + \sum_{i=1}^n \mathbf{a}_i q_i^b M \right] Mt - at^2 \right\}}{\{a[a - \mathbf{g}M^2]\}} && \text{using (17.96) and (17.97)} \\
 &= -\mathbf{g} \frac{\left\{ \left[ \sum_{i=1}^n \mathbf{a}_i q_i^b t \right] M^2 - 2 \left[ \sum_{i=1}^n p_i^b q_i^b \right] Mt + at^2 \right\}}{\{a[a - \mathbf{g}M^2]\}} \\
 &< 0 && \text{using (17.98).}
 \end{aligned}$$

Thus for  $t \geq 1$ , *the Lowe index will have an upward bias* (to the accuracy of a second order Taylor series approximation) compared to the corresponding true cost of living index  $P_K(p^0, p^t, q^b)$ , since the approximate bias defined by the last expression in (17.99) is the sum of one nonpositive and two negative terms. *Moreover this approximate bias will grow quadratically in time  $t$ .*<sup>60</sup>

82. In order to give the reader some idea of the magnitude of the approximate bias  $B^t$  defined by the last line of (17.99), a simple special case will be considered at this point. Suppose there are only 2 commodities and at the base year, all prices and quantities are equal to 1. Thus  $p_i^b = q_i^b = 1$  for  $i = 1, 2$  and  $\sum_{i=1}^n p_i^b q_i^b = 2$ . Assume that  $M = 24$  so that the base year data on quantities take 2 years to process before the Lowe index can be implemented. Assume that the monthly rate of growth in price for commodity 1 is  $\alpha_1 = 0.002$  so that after 1 year, the price of commodity 1 rises 0.024 or 2.4 %. Assume that commodity 2 falls in price each month with  $\alpha_2 = -0.002$  so that the price of commodity 2 falls 2.4 % in the first year after the base year for quantities. Thus the relative price of the two commodities is steadily diverging by about 5 percent per year. Finally, assume that  $\partial^2 C(u^b, p^b) / \partial p_1 \partial p_1 = \partial^2 C(u^b, p^b) / \partial p_2 \partial p_2 = -1$  and  $\partial^2 C(u^b, p^b) / \partial p_1 \partial p_2 = \partial^2 C(u^b, p^b) / \partial p_2 \partial p_1 = 1$ . These assumptions imply that the own elasticity of demand for each commodity is  $-1$  at the base year consumer equilibrium. Making all of these assumptions means that:

<sup>60</sup> If  $M$  is large relative to  $t$ , then it can be seen that the first two terms in the last equation of (17.99) can dominate the last term, which is the quadratic in  $t$  term.

$$(17.100) \quad 2 = \sum_{i=1}^n p_i^b q_i^b = a = b; \sum_{i=1}^n a_i q_i^b = 0; M = 24; \mathbf{g} = 0.000008.$$

Substituting the parameter values defined in (17.100) into (17.99) leads to the following formula for the approximate amount that the Lowe index will exceed the corresponding true cost of living index at month  $t$ :

$$(17.101) \quad -B^t = 0.000008 \frac{(96t + 2t^2)}{2(2 - 0.004608)}.$$

Evaluating (17.101) at  $t = 12$ ,  $t = 24$ ,  $t = 36$ ,  $t = 48$  and  $t = 60$  leads to the following estimates for  $-B^t$ : 0.0029 (the approximate bias in the Lowe index at the end of the first year of operation for the index); 0.0069 (the bias after 2 years); 0.0121 (3 years); 0.0185 (4 years); 0.0260 (5 years). Thus at the end of the first year of the operation of the Lowe index, it will only be above the corresponding true cost of living index by approximately a third of a percentage point but by the end of the fifth year of operation, it will exceed the corresponding cost of living index by about 2.6 percentage points, which is no longer a negligible amount.<sup>61</sup>

83. The numerical results in the previous paragraph are only indicative of the approximate magnitude of the difference between a cost of living index and the corresponding Lowe index. The important point to note is that to the accuracy of a second order approximation, the Lowe index will generally exceed its cost of living counterpart. However, the results also indicate that this difference can be reduced to a negligible amount if:

- the lag in obtaining the base year quantity weights is minimized and
- the base year is changed as frequently as possible.

It also should be noted that the numerical results depend on the assumption that long run trends in prices exist, which may not be true,<sup>62</sup> and on elasticity assumptions that may not be justified.<sup>63</sup> Thus statistical agencies should prepare their own carefully constructed estimates of the differences between a Lowe index and a cost of living index in the light of their own particular circumstances.

#### J.4 The problem of seasonal commodities

84. The assumption that the consumer has annual preferences over commodities purchased in the base year for the quantity weights and that these annual preferences can be used in the context of making monthly purchases of the same commodities was a key one in relating the economic approach to index number theory to the Lowe index. However, this assumption that annual preferences can be used in a monthly context is somewhat questionable due to the *seasonal* nature of some commodity purchases. The problem is that it is very likely that consumers' preference functions *systematically change* as the season of the year changes. Thus national customs and weather changes cause households to purchase certain goods and services during some months and not at all for other months. For example, Christmas trees are purchased only in December and ski jackets are not usually purchased during summer

<sup>61</sup> Note that the relatively large magnitude of  $M$  compared to  $t$  leads to a bias that grows approximately linearly with  $t$  rather than quadratically.

<sup>62</sup> For mathematical convenience, the trends in prices were assumed to be linear rather than the more natural assumption of geometric trends in prices.

<sup>63</sup> Another key assumption that was used to derive the numerical results is the magnitude of the divergent trends in prices. If the price divergence vector is doubled to  $\alpha_1 = 0.004$  and  $\alpha_2 = -0.004$ , then the parameter  $\gamma$  quadruples and the approximate bias will also quadruple.

months. Thus the assumption that annual preferences are applicable during each month of the year is only acceptable as a very rough approximation to economic reality.

85. The economic approach to index number theory can be adapted to deal with seasonal preferences. The simplest economic approach is to assume that the consumer has annual preferences over commodities classified not only by their characteristics but also by the month of purchase.<sup>64</sup> Thus instead of assuming that the consumer's annual utility function is  $f(q)$  where  $q$  is an  $n$  dimensional vector, assume that the consumer's annual utility function is  $F[f^1(q^1), f^2(q^2), \dots, f^{12}(q^{12})]$  where  $q^1$  is an  $n$  dimensional vector of commodity purchases made in January,  $q^2$  is an  $n$  dimensional vector of commodity purchases made in February, ..., and  $q^{12}$  is an  $n$  dimensional vector of commodity purchases made in December.<sup>65</sup> The subutility functions  $f^1, f^2, \dots, f^{12}$  represent the consumer's preferences when making purchases in January, February, ..., and December respectively. These monthly subutilities can then be aggregated up using the macro utility function  $F$  in order to define overall annual utility. It can be seen that these assumptions on preferences can be used to justify *two types of cost of living index*:

- An annual cost of living index that compares the prices in all months of a current year with the corresponding monthly prices in a base year;<sup>66</sup>
- Twelve monthly cost of living indices where the index for month  $m$  compares the prices of month  $m$  in the current year with the prices of month  $m$  in the base year for  $m = 1, 2, \dots, 12$ .<sup>67</sup>

86. The annual Mudgett Stone indices compare costs in a current calendar year with the corresponding costs in a base year. However, any month could be chosen as the year ending month of the current year and the prices and quantities of this new noncalendar year could be compared to the prices and quantities of the base year, where the January prices of the noncalendar year are matched to the January prices of the base year, the February prices of the noncalendar year are matched to the February prices of the base year and so on. If further assumptions are made on the macro utility function  $F$ , then this framework can be used in order to justify a third type of cost of living index: a *moving year annual index*.<sup>68</sup> This index compares the cost over the last 12 months of achieving the annual utility achieved in the base year with the base year cost, where the January costs in the current moving year are matched to January costs in the base year, the February costs in the current moving year are matched to February costs in the base year, and so on. These moving year indices can be calculated for each month of the current year and the resulting series can be interpreted as (uncentered) seasonally adjusted (annual) price indices.<sup>69</sup>

87. It should be noted that *none* of the three types of indices described in the previous two paragraphs are suitable for describing the movements of prices going from one month to the following month; i.e., *they are not suitable for describing short run movements in inflation*. This is obvious for the first two types of index. To see the problem with the moving year indices, consider a special case where the bundle of commodities purchased in each month is entirely specific to each month. Then it is obvious that even though all of the above three types of index are well defined, *none of them can describe anything useful about month to*

<sup>64</sup> This assumption and the resulting annual indices were first proposed by Mudgett (1955; 97) and Stone (1956; 74-75).

<sup>65</sup> If some commodities are not available in certain months  $m$ , then those commodities can be dropped from the corresponding monthly quantity vectors  $q^m$ .

<sup>66</sup> For further details on how to implement this framework, see Mudgett (1955; 97), Stone (1956; 74-75) and Diewert (1998b; 459-460).

<sup>67</sup> For further details on how to implement this framework, see Diewert (1999a; 50-51).

<sup>68</sup> See Diewert (1999a; 56-61) for the details of this economic approach.

<sup>69</sup> See Diewert (1999a; 67-68) for an empirical example of this approach applied to quantity indices. An empirical example of this moving year approach to price indices will be presented in Chapter 22.

*month changes in prices*, since it is impossible to compare like with like, going from one month to the next, under the hypotheses of this special case. It is impossible to compare the incomparable!

88. Fortunately, it is not the case that household purchases in each month are entirely specific to the month of purchase. Thus month to month price comparisons can be made if the commodity space is restricted to commodities that are purchased in each month of the year. This observation leads to a *fourth type of cost of living index*, a *month to month index*, defined over commodities that are available in every month of the year.<sup>70</sup> This model can be used to justify the economic approach described in sections J.1-J.3 above. However, commodities that are purchased only in certain months of the year must be dropped from the scope of the index. Unfortunately, it is likely that consumers have varying monthly preferences over the always available commodities and if this is the case, the month to month cost of living index (and the corresponding Lowe index) defined over always available commodities will generally be subject to *seasonal fluctuations*. This will limit the usefulness of the index as a short run indicator of general inflation since it will be difficult to distinguish a seasonal movement in the index from a systematic general movement in prices.<sup>71</sup> Note also that if the scope of the index is restricted to always available commodities, then the resulting month to month index will not be *comprehensive*, whereas the moving year indices will be comprehensive in the sense of using all of the available price information.

89. The above considerations lead to the conclusion that it may be useful for statistical agencies to produce at least two consumer price indices:

- A moving year index which is comprehensive and seasonally adjusted but which is not necessarily useful for indicating month to month changes in general inflation and
- A month to month index which is restricted to nonseasonal commodities (and hence is not comprehensive) but which is useful for indicating short run movements in general inflation.

#### **K. The problem of a zero price increasing to a positive price**

90. In a recent paper, Haschka (2003) raised the problem of what to do when a price which was previously zero is increased to a positive level. He gave two examples for Austria where parking and hospital fees were raised from zero to a positive level. In this situation, it turns out that basket type indices have an advantage over indices that are weighted geometric averages of price relatives, since basket type indices are well defined even if some prices are zero.

91. The problem can be considered in the context of evaluating the Laspeyres and Paasche indices. Suppose as usual that the prices  $p^t$  and quantities  $q_i^t$  of the first  $n$  commodities are positive for periods 0 and 1 but that the price of commodity  $n+1$  in period 0 is zero but is positive in period 1. In both periods, the consumption of commodity  $n+1$  is positive. Thus the assumptions on the prices and quantities of commodity  $n+1$  in the two periods under consideration can be summarized as follows:

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<sup>70</sup> See Diewert (1999a; 51-56) for the assumptions on preferences that are required in order to justify this economic approach.

<sup>71</sup> One problem with using annual weights in the context of seasonal movements in prices and quantities is that a change in price when a commodity is “out of season” can be greatly magnified by the use of annual weights. Baldwin (1990; 251) noted this problem with an annual weights price index: “But a price index is adversely affected if any seasonal good has the same basket share for all months of the year; the good will have an inappropriately small basket share in its in season months, an inappropriately large share in its off season months.” Seasonality problems will be considered again from a more pragmatic point of view in Chapter 22.

$$(17.102) \quad p_{n+1}^0 = 0; p_{n+1}^1 > 0; q_{n+1}^0 > 0; q_{n+1}^1 > 0.$$

Typically the increase in price of commodity  $n+1$  from its initial zero level will cause consumption to fall so that  $q_{n+1}^1 < q_{n+1}^0$ , but this inequality is not required for the analysis below.

92. Let the Laspeyres index between periods 0 and 1, restricted to the first  $n$  commodities be denoted as  $P_L^n$  and let the Laspeyres index defined over all  $n+1$  commodities be defined as  $P_L^{n+1}$ . Also let  $v_i^0 \equiv p_i^0 q_i^0$  denote the value of expenditures on commodity  $i$  in period 0. Then by the definition of the Laspeyres index defined over all  $n+1$  commodities:

$$(17.103) \quad P_L^{n+1} \equiv \frac{\sum_{i=1}^{n+1} p_i^1 q_i^0}{\sum_{i=1}^{n+1} p_i^0 q_i^0} \\ = P_L^n + \frac{p_{n+1}^1 q_{n+1}^0}{\sum_{i=1}^n v_i^0}$$

where  $p_{n+1}^0 = 0$  was used in order to derive the second equation above. Thus the complete Laspeyres index  $P_L^{n+1}$  defined over all  $n+1$  commodities is equal to the incomplete Laspeyres index  $P_L^n$  (which can be written in traditional price relative and base period expenditure share form) plus the mixed or hybrid expenditure  $p_{n+1}^1 q_{n+1}^0$  divided by the base period expenditure on the first  $n$  commodities,  $\sum_{i=1}^n v_i^0$ . Thus the complete Laspeyres index can be calculated using the usual information available to the price statistician plus two additional pieces of information: the new nonzero price for commodity  $n+1$  in period 1,  $p_{n+1}^1$ , and an estimate of consumption of commodity  $n+1$  in period 0 (when it was free),  $q_{n+1}^0$ . Since it is often governments that change the previously zero price to a positive price, the decision to do this is usually announced in advance, which will give the price statistician an opportunity to form an estimate for the base period demand,  $q_{n+1}^0$ .

93. Let the Paasche index between periods 0 and 1, restricted to the first  $n$  commodities be denoted as  $P_P^n$  and let the Paasche index defined over all  $n+1$  commodities be defined as  $P_P^{n+1}$ . Also let  $v_i^1 \equiv p_i^1 q_i^1$  denote the value of expenditures on commodity  $i$  in period 1. Then by the definition of the Paasche index defined over all  $n+1$  commodities:

$$(17.104) \quad P_P^{n+1} \equiv \frac{\sum_{i=1}^{n+1} p_i^1 q_i^1}{\sum_{i=1}^{n+1} p_i^0 q_i^1} \\ = P_P^n + \frac{v_{n+1}^1}{\sum_{i=1}^n p_i^0 q_i^1} \\ = P_P^n + \frac{v_{n+1}^1}{\sum_{i=1}^n v_i^1 / (p_i^1 / p_i^0)}$$

where  $p_{n+1}^0 = 0$  was used in order to derive the second equation above. Thus the complete Paasche index  $P_P^{n+1}$  defined over all  $n+1$  commodities is equal to the incomplete Paasche index  $P_P^n$  (which can be written in traditional price relative and current period expenditure share form) plus the current period expenditure on commodity  $n+1$ ,  $v_{n+1}^1$ , divided by a sum of current period expenditures on the first  $n$  commodities,  $v_i^1$ , divided by the  $i$ th price relative for the first  $n$  commodities,  $p_i^1/p_i^0$ . Thus the complete Paasche index can be calculated using the



usual information available to the price statistician plus information on current period expenditures.

94. Once the complete Laspeyres and Paasche indices have been calculated using (17.103) and (17.104), then the complete Fisher index can be calculated as the square root of the product of these two indices:

$$(17.105) \quad P_F^{n+1} = [P_L^{n+1} P_P^{n+1}]^{1/2} .$$

It should be noted that the complete Fisher index defined by (17.105) satisfies the same exact index number results as were demonstrated in section D above; i.e., the Fisher index remains a superlative index even if prices are zero in one period but positive in the other. Thus the Fisher price index remains a suitable target index even in the face of zero prices.

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