Index Number Theory Using Differences Rather Than Ratios

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Abstract

Traditional index number theory decomposes a value ratio into the product of a price index times a quantity index. The price (quantity) index is interpreted as an aggregate price (quantity) ratio. The present paper takes an alternative approach to index number theory, started by Bennet and Montgomery in the 1920's, that decomposes a value difference into the sum of a price difference plus a quantity difference. Axiomatic and economic approaches to this alternative branch of index number theory are considered in the present paper. The analysis presented has some relevance to accounting theory where revenue, cost or profit changes need to be decomposed into price and quantity components or where standard or budgeted performance is compared with actual performance (variance analysis). The methdology presented in the paper is also relevant for consumer surplus analysis.

Key Words: index number theory, consumer surplus analysis; revenue, cost and profit decompositions; variance analysis in accounting; superlative index numbers; indicators of price and quantity change; ideal indexes; symmetric means of order r; logarithmic means; index number tests.

JEL Classification Numbers: B31; C43; D24; D63; E31; M4; O47

1. *Introduction*

> "When, forty-two years ago, I wrote my doctor's thesis on certain mathematical investigations in the theory of value and prices, I was a student of mathematical physics and, with youthful enthusiasm, dreamed dreams of seeing economics, or one branch of it, grow into a true science by the same methods which had long since built up physics into a true and majestic science."
>
> *Irving Fisher*[1933;2]

Rather than review all of the many contributions of Irving Fisher to the index number literature, we will develop an alternative branch of index number theory that had its origins around the time that Fisher [1911] [1921] [1922] developed his test approach. This alternative branch of index number theory was started by T.L. Bennet [1920] and J.K. Montgomery [1929] [1937], but for various reasons, their approach never prospered and has been mostly forgotten by present day index number theorists.[1]

In order to explain this alternative approach to index number theory, we first need to explain the traditional approach. Suppose we have collected price and quantity information on N commodities for a base period 0 and a current period 1. Denote the price and quantity of commodity n in period t as p_n^t and q_n^t respectively for $n = 1, 2, \ldots, N$ and $t = 0, 1$. Define the period t price and quantity vectors as $p^t \equiv [p_1^t, p_2^t, \ldots, p_N^t]$ and $q^t \equiv [q_1^t, q_2^t, \ldots, q_N^t]$ for $t = 0, 1$. Then the *value* of the N commodities in period t is

(1) $\quad p^t \cdot q^t \equiv \Sigma_{n=1}^N p_n^t q_n^t \qquad \text{for} \qquad t = 0, 1.$

Fisher [1911] [1922] framed the now traditional test approach to index number theory as follows: find two functions of the $4N$ price and quantity variables that pertain to the two periods under consideration, say $P(p^0, p^1, q^0, q^1)$ and $Q(p^0, p^1, q^0, q^1)$, such that the value *ratio* for the two periods; $p^1 \cdot q^1/p^0 \cdot q^0$, is equal to the *product* of P and Q; i.e.,

(2) $\quad p^1 \cdot q^1/p^0 \cdot q^0 = P(p^0, p^1, q^0, q^1)Q(p^0, p^1, q^0, q^1)$

and the functions P and Q satisfy certain properties that allow us to identify $P(p^0, p^1, q^0, q^1)$ as an aggregate measure of relative price change and $Q(p^0, p^1, q^0, q^1)$ as an aggregate measure of relative quantity change. Fisher called these properties or axioms, *tests*. The function $P(p^0, p^1, q^0, q^1)$ is called the *price index* and the function $Q(p^0, p^1, q^0, q^1)$ is called the *quantity index*. If the number of commodities is one (i.e., $N = 1$), then the price index P collapses down to the single price ratio p_1^1/p_1^0 and the quantity index Q collapses down to the single quantity ratio q_1^1/q_1^0. The index number problem (i.e., the problem of determining the functional forms for P and Q) is trivial in this $N = 1$ case. However, in the multiple commodity case where $N > 1$, the problem of finding functions P and Q which satisfy (2) and satisfy appropriate tests is far from trivial.

Note that if we have somehow determined the appropriate functional form for P, then equation (2) can be used to define the quantity index Q that will be consistent with it. Thus we can concentrate on finding a functional form for P that satisfies an appropriate sets of tests. This is the traditional test approach to index number theory in a nutshell.

The alternative branch of index number theory that we wish to study is the one which uses a *difference* counterpart to the *ratio* equation (2). Thus we look for two functions of $4N$ variables, $\triangle P(p^0, p^1, q^0, q^1)$ and $\triangle Q(p^0, p^1, q^0, q^1)$, which sum to the value *difference* between the two periods; i.e., we want $\triangle P$ and $\triangle Q$ to satisfy the following equation:

$$(3) \quad p^1 \cdot q^1 - p^0 \cdot q^0 = \triangle P(p^0, p^1, q^0, q^1) + \triangle Q(p^0, p^1, q^0, q^1)$$

and the functions $\triangle P$ and $\triangle Q$ are to satisfy certain properties or tests that will allow us to identify $\triangle P$ as a *measure of aggregate price change* and $\triangle Q$ as a *measure of aggregate quantity or volume change* between the two periods.

As was the case with traditional index number theory where P and Q cannot be determined independently if (2) is to hold, then if (3) is to hold, $\triangle P$ and $\triangle Q$ cannot be determined independently. Thus in what follows, we will postulate axioms or tests for $\triangle P$ and once $\triangle P$ has been determined, we will define $\triangle Q$ using (3).

As the notation $\triangle P$ and $\triangle Q$ is somewhat awkward, we will use the notation $I(p^0, p^1, q^0, q^1)$ (for *indicator* of price change) to replace $\triangle P(p^0, p^1, q^0, q^1)$ and we will use $V(p^0, p^1, q^0, q^1)$ (for indicator of *volume* change) to replace $\triangle Q(p^0, p^1, q^0, q^1)$. Using our new notation, (3) can be rewritten as follows:

$$(4) \quad \Sigma_{n=1}^{N} [p_n^1 q_n^1 - p_n^0 q_n^0] = I(p^0, p^1, q^0, q^1) + V(p^0, p^1, q^0, q^1).$$

In sections 3 and 4 below, we simplify the problem of finding a suitable I and V that satisfy (4) by postulating that the overall indicators of price and volume change, I and V, decompose into a sum of N commodity specific indicators of price and volume change, $\Sigma_{n=1}^{N} I_n(p_n^0, p_n^1, q_n^0, q_n^1)$ and $\Sigma_{n=1}^{N} V_n(p_n^0, p_n^1, q_n^0, q_n^1)$, where each of the functions I_n and V_n depend only on the two prices that pertain to commodity n, p_n^0 and p_n^1, and the corresponding commodity n quantities, q_n^0, and q_n^1. Thus in this simplified *separable* approach, the value change for each commodity is postulated to have the following components of price change (I_n) and quantity change (V_n):

$$(5) \quad p_n^1 q_n^1 - p_n^0 q_n^0 = I_n(p_n^0, p_n^1, q_n^0, q_n^1) + V_n(p_n^0, p_n^1, q_n^0, q_n^1); \quad n = 1, \ldots, N.$$

Once the commodity specific indicators of price change I_n have been determined, the overall indicator of price change I is defined as the sum of the specific indicators:

(6) $\quad I(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^{N} I_n(p_n^0, q_n^0, p_n^1, q_n^1).$

Of course, once $I(p^0, p^1, q^0, q^1)$ has been defined, the corresponding aggregate volume change indicator $V(p^0, p^1, q^0, q^1)$ can be determined using equation (4).

In comparing the value difference decomposition (3) or (4) with the value ratio decomposition (2), it is interesting to note that the value ratio decomposition is trivial in the one commodity case ($N = 1$) but nontrivial in the general case ($N > 1$) while the value change decomposition (5) is nontrivial in the case of one commodity but once the one commodity decompositions of the form (5) have been determined, the many commodity case is a straightforward summation of the one commodity effects; i.e., see (6).

We summarize the above introductory material as follows: we are searching for "reasonable" candidates for the commodity specific indicators of price change $I_n(p_n^0, p_n^1, q_n^0, q_n^1)$ to insert into the commodity specific value difference equations (5) or more generally, we are looking for "reasonable" candidates for the overall indicator of price change $I(p^0, p^1, q^0, q^1)$ that we can insert into equation (4) above. Once the "reasonable" $I(p^0, p^1, q^0, q^1)$ has been found, the corresponding indicator of volume change $V(p^0, p^1, q^0, q^1)$ can be determined by solving equation (4) for V.

In sections 2 and 3 below, we present the solutions of Bennet [1920] and Montgomery [1929] [1937] to this search for a "reasonable" indicator of price change. In section 4, we follow the examples of Fisher [1911] [1922] and Montgomery [1929] and pursue axiomatic approaches to the determination of an indicator of price change. In section 6, we show that many "reasonable" indicators of price change numerically approximate each other to the second order around equal price and quantity vectors. Section 7 concludes. Proofs of various propositions may be found in Appendix 1. The remaining Appendices pursue some more specialized topics.

At this point, it is useful to consider a couple of preliminary problems.

The first preliminary problem that needs to be addressed is concerned with the interpretation of the period t prices and quantities for commodity n, p_n^t and q_n^t. In real life applications of value decompositions into price and quantity parts, during any period t, there will typically be many transactions in commodity n at a number of different prices. Hence, there is a need to provide a more precise definition for the "average" or "representative" price for commodity n in period t, p_n^t. Irving Fisher [1922; 318] addressed this preliminary aggregation problem as follows:

"Essentially the same problem enters, however, whenever, as is usually the case, the data for prices and quantities with which we start are averages instead of being the original market quotations. Throughout this book, 'the price' of any commodity or 'the quantity' of it for any year was assumed given. But what is such a price or such a quantity?... The quantities sold will, of course, vary widely. What is needed is their sum for the year ...

Or, if it is worth while to put any finer point on it, we may take the weighted arithmetic average for the prices, the weights being the quantities sold."

Thus Fisher more or less advocated the use of the unit value (total value transacted divided by total quantity) as the appropriate price p_n^t for commodity n and the total quantity transacted during period t as the appropriate quantity q_n^t. As an aggregation formula at the first stage of aggregation, the unit value and total quantity transacted has been proposed by many economists and statisticians, perhaps starting with Walsh [1901; 96] [1921; 88] and Davies [1924; 183] [1932; 59] and including many other more recent writers.[2] If we want q_n^t to equal the total quantity of commodity n transacted during period t and we also want the product of the price p_n^t times quantity q_n^t to equal the value of period t transactions in commodity n, then we are *forced* to define the aggregate period t price p_n^t to be the total value divided by the total quantity, or the unit value.

The second preliminary problem that needs to be addressed is also a fundamental one: for what *purpose* do we want to decompose a value change into price and quantity components? It is quite possible that a given decomposition with certain properties would be appropriate for one purpose but not another.[3]

There are at least five general areas where it would be useful to be able to decompose a value change into price and quantity components. The first four areas of application pertain to business units while the fifth pertains to households.

(i) *Revenue Change Decompositions.* When a company reports its current period revenue and compares it to the revenue of the previous accounting period, it is useful to be able to decompose this revenue change into a part that is due to the change in prices that the firm faced in the two periods and a part that is due to increased (or decreased) production; i.e., a volume change. While this type of revenue decomposition is not widespread in financial accounting, oil producers frequently make this type of decomposition in their annual reports so that shareholders can determine the effects of changes in the world price for oil on the revenue change.

(ii) *Cost Change Decompositions.* Shareholders are not only interested in revenue decompositions, they are also interested in decomposing a company's change in costs into a change in prices part (an exogenous effect that is mostly beyond the control of the firm) and a change in quantities part (an endogenous effect that is presumably under the control of the firm). This type of decomposition is not yet common in annual reports, presumably due to the difficulties involved in decomposing cost values into identifiable price and quantity components. However, as computerization of all transactions proceeds, in the future it should be possible to construct detailed unit values for cost components.

(iii) *Profit Change Decompositions.* As was the case with (i) and (ii) above, shareholders will be very interested in the decomposition of profit change over two accounting periods into a price change component and a quantity or volume change component. Investor interest will focus on the volume change indicator because this is an indicator of firm efficiency improvement: the more positive is the volume change, the greater is the ex post efficiency gain for the firm. Note that profit change is equal to revenue change less cost change, so if revenue and cost decompositions have been calculated, then the profit change decomposition is simply equal to the revenue decomposition less the cost decomposition. Profit change decompositions can also be used by the firm for internal control and performance evaluation purposes.[4]

(iv) *Variance Analysis.* For this use of the value decomposition, the period 0 prices and quantities are interpreted as period 1 *budgeted* or *forecasted* or *"standard"* prices and quantities that are supposed to prevail in period 1. Then the value difference is the difference between the period 1 actual value, $p^1 \cdot q^1$, and budgeted performance, $p^0 \cdot q^0$. The indicator of price change, $I(p^0, p^1, q^0, q^1)$, is now interpreted as the contribution of price change (between actual and budgeted prices) to the ex post difference between actual and standard values and the indicator of volume change, $V(p^0, p^1, q^0, q^1)$, is similarly interpreted as the contribution of quantity change (between actual and budgeted quantities). Variance analysis can be traced back to the early accounting and industrial engineering literature; see Whitmore [1908] [1931], Harrison [1918][5] and Solomons [1968; 46-47] for the early history.

(v) *Changes in Consumer Surplus.* In this application; the values are the expenditures of a consumer or household on N commodities during two periods. The task is to decompose the change in consumer expenditures over the two periods into a price change component $I(p^0, p^1, q^0, q^1)$ and a quantity change component $V(p^0, p^1, q^0, q^1)$ which can be interpreted as a constant dollar measure of real utility change. This line of research was started by Marshall [1890] and Bennet [1920]. Other early contributors include Hotelling [1938; 253-254] and Hicks [1941-42; 134] [1945-46; 73] [1946; 330-333].[6]

With the above preliminary material disposed of, we turn our attention to the contributions of Bennet.

2. Bennet's Indicator of Price Change

> "The fundamental idea is that in a short period the rate of increase of expenditure of a family can be divided into two parts, x and ℓ, where x measures the increase due to change of prices and ℓ measures the increase due to increase of consumption; ..." *T.L. Bennet*[1920;455]

Bennet [1920; 457] proposed the following decomposition of a value change:

$$
\begin{aligned}
(7) \qquad p^1 \cdot q^1 - p^0 \cdot q^0 &= (1/2)(q^0 + q^1) \cdot (p^1 - p^0) + (1/2)(p^0 + p^1) \cdot (q^1 - q^0) \\
&= \Sigma_{n=1}^{N}(1/2)(q_n^0 + q_n^1)(p_n^1 - p_n^0) + \Sigma_{n=1}^{N}(1/2)(p_n^0 + p_n^1)(q_n^1 - q_n^0).
\end{aligned}
$$

That (7) is true follows simply by multiplication and cancellation of terms. Thus the Bennet indicators of price and volume change are defined as follows:

$$(8) \qquad I^B(p^0, p^1, q^0, q^1) \equiv (1/2)(q^0 + q^1) \cdot (p^1 - p^0);$$

$$(9) \qquad V^B(p^0, p^1, q^0, q^1) \equiv (1/2)(p^0 + p^1) \cdot (q^0 - q^1).$$

Bennet [1920; 456-457] justified his volume indicator as a linear approximation to the area under a demand curve and his price indicator as a linear approximation to an area under an inverse demand curve. Hence, Bennet was following in Marshall's [1890] partial equilibrium consumer surplus footsteps. However, it is possible to derive Bennet's indicators by an alternative line of reasoning, which we now explain.

In the early industrial engineering literature, Harrison [1918; 393] made the following decomposition of a cost change into a price variation plus an efficiency variation:

$$(10) \qquad p^1 \cdot q^1 - p^0 \cdot q^0 = q^1 \cdot (p^1 - p^0) + p^0 \cdot (q^1 - q^0).$$

Again, the proof that (10) is true follows by straightforward arithmetic. The reader familiar with index number theory will recognize that Harrison's indicator of price change, $q^1 \cdot (p^1 - p^0)$, is the difference counterpart to the Paasche price index, $p^1 \cdot q^1/p^0 \cdot q^1$. Similarly, Harrison's indicator of quantity or efficiency change, $p^0 \cdot (q^1 - q^0)$, is the difference counterpart to the Laspeyres quantity index, $p^0 \cdot q^1/p^0 \cdot q^0$. Thus we define the Paasche indicator of price change I^P and the Laspeyres indicator of quantity change V^L as follows:

$$(11) \qquad I^P(p^0, p^1, q^0, q^1) \equiv q^1 \cdot (p^1 - p^0);$$

$$(12) \qquad V^L(p^0, p^1, q^0, q^1) \equiv p^0 \cdot (q^1 - q^0).$$

More recently in the accounting literature[8], it was recognized that the traditional variance analysis decomposition of a value change, (10) above, may not be as appropriate as the following decomposition:

$$(13) \quad p^1 \cdot q^1 - p^0 \cdot q^0 = q^0 \cdot (p^1 - p^0) + p^1 \cdot (q^1 - q^0).$$

The reason why the decomposition (13) may be preferable to (10) in the context of comparing actual performance to "standard" performance is that in the case of exogenous prices, the firm manager will have an incentive to maximize profits $p^1 \cdot q^1$ with respect to q^1 and thus the efficiency change term, $p^1 \cdot (q^1 - q^0)$ in (13), is consistent with profit maximizing behaviour. Again, the reader will recognize that the indicator of price change in (13), $q^0 \cdot (p^1 - p^0)$, is the difference analogue to the Laspeyres price index, $p^1 \cdot q^0 / p^0 \cdot q^0$, and the indicator of quantity change in (13), $p^1 \cdot (q^1 - q^0)$, is the difference counterpart to the Paasche quantity index, $p^1 \cdot q^1 / p^1 \cdot q^0$. Thus we define the Laspeyres and Paasche indicators of price and quantity change respectively as follows:

$$(14) \quad I^L(p^0, p^1, q^0, q^1) \equiv q^0 \cdot (p^1 - p^0);$$

$$(15) \quad V^P(p^0, p^1, q^0, q^1) \equiv p^1 \cdot (q^1 - q^0).$$

Now we can explain our alternative derivation of the Bennet indicators: the Bennet indicator of price (quantity) change is simply the arithmetic average of the Paasche and Laspeyres indicators of price (quantity) change; i.e.,

$$(16) \quad I^B(p^0, p^1, q^0, q^1) = (1/2)I^L(p^0, p^1, q^0, q^1) + (1/2)I^P(p^0, p^1, q^0, q^1);$$

$$(17) \quad V^B(p^0, p^1, q^0, q^1) = (1/2)V^L(p^0, p^1, q^0, q^1) + (1/2)V^P(p^0, p^1, q^0, q^1).$$

We will present a geometric interpretation due to Bennet [1920; 456] of his indicators in the one commodity case in the next section where we discuss the work of Montgomery.

3. Montgomery's Indicator of Price Change

> "One of the reasons why there has been so much controversy on the subject of the price index of a group of commodities is that writers have never agreed on a definition." *J.K. Montgomery*[1937;1]

In a rather obscure paper, Montgomery [1929][9] defined some interesting indicators of price and quantity change. We revert to the separable framework explained in section 1 above where we first find indicators of price and volume change, I_n and V_n, for commodity

n (recall equation (5) above) and then the overall indicators of price and volume change are obtained by summing over the commodity specific indicators (see equation (6) above).

The Paasche, Laspeyres and Bennet indicators of price and quantity change were well defined irrespective of the signs of the individual prices and quantities, p_n^t and q_n^t. However, in the present section, we shall restrict all prices and quantities to be positive since it will be necessary to take natural logarithms of the individual prices and quantities. The restriction that all prices and quantities be positive is not restrictive in the context of computing revenue and cost indicators. Obviously, a profit indicator can be defined as the difference between the revenue and cost indicators and so, even in this context, the positivity restrictions are not too restrictive.[10]

The Montgomery [1929; 5] indicators of price and volume change for the nth commodity are defined as follows:

(18) $$I_n^M(p_n^0, p_n^1, q_n^0, q_n^1) \equiv \{[p_n^1 q_n^1 - p_n^0 q_n^0]/[ln(p_n^1 q_n^1) - ln(p_n^0 q_n^0)]\}\ ln[p_n^1/p_n^0];$$

(19) $$V_n^M(p_n^0, p_n^1, q_n^0, q_n^1) \equiv \{[p_n^1 q_n^1 - p_n^0 q_n^0]/[ln(p_n^1 q_n^1) - ln(p_n^0 q_n^0)]\}\ ln[q_n^1/q_n^0].$$

Note that the functional form for V_n^M is the same as the functional form for I_n^M except that the roles of prices and quantities have been interchanged. Montgomery [1929; 3-9] also showed that

(20) $$p_n^1 q_n^1 - p_n^0 q_n^0 = I_n^M(p_n^0, p_n^1, q_n^0, q_n^1) + V_n^M(p_n^0, p_n^1, q_n^0, q_n^1).$$

In order to understand (18) and (19) better, the reader should note that $L(a, b) \equiv [a - b]/[lna - lnb]$ where $a > 0, b > 0$ is known in the economics literature as the *Vartia* [1976a] [1976b] *mean* and in the mathematics literature as the *logarithmic mean*.[11] It can be shown that $L(a, b)$ is a linearly homogeneous symmetric mean.[12] In (18) and (19), $a \equiv p_n^1 q_n^1$ and $b \equiv p_n^0 q_n^0$.

Montgomery [1929; 7-9] derived his indicators by using a very interesting argument (which parallels that of Bennet [1920]) which we shall repeat since it shows how a large number of "reasonable" price and quantity indicators can be derived. Suppose that q_n is functionally related to p_n by a "supply" function:

(21) $$q_n = s_n(p_n).$$

Note that the "supply" function s_n is a partial equilibrium supply function since only the price of the nth good appears in (21) as an argument of the function. Montgomery [1929; 7] assumed the following functional form for s_n:

(22) $$s_n(p_n) \equiv \alpha_n p_n^{\beta_n}, \quad \alpha_n > 0, \quad \beta_n \neq 0.$$

Now define a theoretical price change indicator as the area under the "supply" curve going from p_n^0 to p_n^1:

$$
\begin{aligned}
(23)\quad P_n^*(p_n^0, p_n^1, s_n) &\equiv \int_{p_n^0}^{p_n^1} s_n(p)dp \\
&= \int_{p_n^0}^{p_n^1} \alpha_n p^{\beta_n} dp && \text{using (22)} \\
&= \alpha_n(1+\beta_n)^{-1} p^{1+\beta_n}]_{p_n^0}^{p_n^1} && \text{assuming } \beta_n \neq -1 \\
(24)\quad &= \alpha_n(1+\beta_n)^{-1}[(p_n^1)^{1+\beta_n} - (p_n^0)^{1+\beta_n}].
\end{aligned}
$$

The unknown parameters α_n and β_n which appear in (24) can be determined by assuming that the two data points (p_n^0, q_n^0) and (p_n^1, q_n^1) are on the "supply" function defined by (22). Thus we have

$$(25)\quad q_n^0 = \alpha_n(p_n^0)^{\beta_n} \quad \text{and} \quad q_n^1 = \alpha_n(p_n^1)^{\beta_n} \quad \text{or}$$

$$(26)\quad q_n^0/\alpha_n = (p_n^0)^{\beta_n} \quad \text{and} \quad q_n^1/\alpha_n = (p_n^1)^{\beta_n} \quad \text{or}$$

$$(27)\quad p_n^0 q_n^0/\alpha_n = (p_n^0)^{1+\beta_n} \quad \text{and} \quad p_n^1 q_n^1/\alpha_n = (p_n^1)^{1+\beta_n}.$$

By taking ratios in (25), we can also deduce that

$$(28)\quad [q_n^1/q_n^0] = [p_n^1/p_n^0]^{\beta_n} \quad \text{or}$$

$$
\begin{aligned}
\beta_n &= ln[q_n^1/q_n^0]/ln[p_n^1/p_n^0] \quad \text{or} \\
1+\beta_n &= \{ln[p_n^1/p_n^0] + ln[q_n^1/q_n^0]\}/ln[p_n^1/p_n^0] \\
(29)\quad &= \{ln[p_n^1 q_n^1] - ln[p_n^0 q_n^0]\}/ln[p_n^1/p_n^0].
\end{aligned}
$$

Now substitute (27) into (24) to get

$$
\begin{aligned}
P_n^*(p_n^0, p_n^1, s_n) &= (1+\beta_n)^{-1}[p_n^1 q_n^1 - p_n^0 q_n^0] \\
&= ln[p_n^1/p_n^0][p_n^1 q_n^1 - p_n^0 q_n^0]/\{ln[p_n^1 q_n^1] - ln[p_n^0 q_n^0]\} \\
&\qquad \text{using (29)} \\
(30)\quad &= I_n^M(p_n^0, p_n^1, q_n^0, q_n^1)
\end{aligned}
$$

where the Montgomery indicator of price change for good n, I_n^M, was defined by (18).

Montgomery (following Bennet [1920; 456]) also defined the theoretical indicator of quantity change going from q_n^0 to q_n^1 as the area under the *inverse* "supply" curve S_n where $p_n = S_n(q_n)$; i.e., define

$$(31) \quad Q_n^*(q_n^0, q_n^1, S_n) \equiv \int_{q_n^0}^{q_n^1} S_n(q)dq.$$

If the "supply" function s_n is defined by (22), then the corresponding inverse "supply" function has the same functional form; i.e., we have

$$(32) \quad p_n = S_n(q_n) = [q_n/\alpha_n]^{1/\beta_n} \equiv \gamma_n q_n^{\delta_n}$$

where $\delta_n \equiv 1/\beta_n$ (hence we require $\beta_n \neq 0$) and $\gamma_n \equiv (1/\alpha_n)^{1/\beta_n}$ (hence we require $\alpha_n \neq 0$). Thus the same argument that we used to derive (30) can be adapted to prove that if S_n is defined by (32) or equivalently if s_n is defined by (22), then

$$(33) \quad Q_n^*(q_n^0, q_n^1, S_n) = V_n^M(p_n^0, p_n^1, q_n^0, q_n^1)$$

where V_n^M is the Montgomery indicator of quantity change defined by (19).

Montgomery [1929; 13] gave a nice geometric interpretation of his method for the case where $0 < p_n^0 < p_n^1$ and $0 < q_n^0 < q_n^1$ which we repeat below.

Figure 1

The line $0AB$ is the curve defined by $q_n = \alpha_n p_n^{\beta_n}$ which passes through the observed points A and B. The period 1 value, $p_n^1 q_n^1$ can be interpreted as the area of the big rectangle $0q_n^1 B p_n^1$, while the period 0 value $p_n^0 q_n^0$ can be interpreted as the area of the small rectangle $0q_n^0 A p_n^0$. The part of the value change that is due to price change, I_n^M, is the shaded area below the curve AB (the area enclosed by $p_n^0 AB p_n^1$) and the part of the value change that is due to quantity change, V_n^M, is the shaded area to the left of AB (the area enclosed by $q_n^0 q_n^1 BA$).

It is easy to see how Montgomery's idea could be generalized: instead of using the constant elasticity functional form defined by (22) or (32) to join the parts A and B, *any* monotonic curve could be used to join A and B and the corresponding indicators of price and quantity change can be defined by (23) and (31).

In fact, Montgomery [1929; 9-11] considered an alternative curve: he repeated his analysis assuming that the "supply" function had the following *linear functional form*:

$$(34) \quad s_n(p_n) \equiv a_n + b_n p_n.$$

For the linear "supply" function defined by (34), the area under the curve definition of the price change indicator yields:

$$\begin{aligned} P_n^*(p_n^0, p_n^1, s_n) &\equiv \int_{p_n^0}^{p_n^1} [a_n + b_n p] dp \\ &= a_n p + \frac{1}{2} b_n p^2]_{p_n^0}^{p_n^1} \\ (35) \qquad &= a_n [p_n^1 - p_n^0] + \frac{1}{2} b_n [(p_n^1)^2 - (p_n^0)^2]. \end{aligned}$$

As usual, we determine the unknown parameters a_n and b_n in (35) by assuming that the observed points (p_n^0, q_n^0) and (p_n^1, q_n^1) lie on the curve defined by (34). Thus we have:

$$(36) \quad q_n^0 = a_n + b_n p_n^0 \quad ; \quad q_n^1 = a_n + b_n p_n^1.$$

Assuming that $p_n^0 \neq p_n^1$, we find that

$$(37) \quad a_n = [p_n^1 q_n^0 - p_n^0 q_n^1]/[p_n^1 - p_n^0] \text{ and}$$

$$(38) \quad b_n = [q_n^1 - q_n^0]/[p_n^1 - p_n^0].$$

Substituting (37) and (38) into (35) yields

$$(39) \quad P_n^*(p_n^0, p_n^1, s_n) = \tfrac{1}{2}(q_n^0 + q_n^1)(p_n^1 - p_n^0) \equiv I_n^B(p_n^0, p_n^1, q_n^0, q_n^1);$$

i.e., the theoretical indicator of price change $P_n^*(p_n^0, p_n^1, s_n)$ under the assumption that s_n is the linear "supply" function defined by (34), turns out to equal the Bennet [1920] indicator of price change, $I_n^B(p_n^0, p_n^1, y_n^0, y_n^1)$, for commodity n.

The inverse "supply" function that corresponds to (34) is:

$$(40) \quad p_n = S_n(q_n) \equiv -(a_n/b_n) + (1/b_n) q_n \equiv c_n + d_n q_n$$

where $c_n \equiv -a_n/b_n$ and $d_n \equiv 1/b_n$ (assuming $b_n \neq 0$). The functional form for the inverse "supply" function defined by (40) is linear and so we can simply adapt the above argument that we used when the direct "supply" function s_n was linear. Thus if (34) or (40) holds, then

$$(41) \quad Q_n^*(q_n^0, q_n^1, S_n) \equiv \int_{q_n^0}^{q_n^1} S_n(q)dq$$

$$= \frac{1}{2}(p_n^0 + p_n^1)(q_n^1 - q_n^0)$$

$$(42) \qquad \equiv V_n^B(p_n^0, p_n^1, q_n^0, q_n^1)$$

where $V_n^B(p_n^0, p_n^1, q_n^0, q_n^1)$ is the Bennet quantity change indicator for commodity n.

The geometry of the linear "supply" function is given below.

Figure 2

Montgomery [1929; 14] preferred the indicators I_n^M and V_n^M over the Bennet [1920] indicators I_n^B and V_n^B because the constant elasticity "supply" curve defined by (22) passes through the origin whereas the linear "supply" curve defined by (34) will not generally pass through the origin (unless $p_n^1/p_n^0 = q_n^1/q_n^0$ so that the price change is equal to the quantity change in proportional terms). However, Bennet [1920; 457] argued that the straight line joining A and B was the simplest hypothesis to use in the absence of other information. Moreover, Montgomery's model which has a "supply" curve going through the origin is not appropriate in the context of a consumer demand for commodities model or a producer's demand for inputs model where a negatively sloped "demand" curve (which would not pass through the origin) would be more appropriate. Bennet's framework is perfectly valid in the demand context whereas Montgomery's model breaks down.[13]

Referring to Figure 2, the Laspeyres I_n^L indicator of price change can be represented as the area of the rectangle $q_n^0(p_n^1 - p_n^0)$ while the Paasche indicator I_n^P can be represented as the area of the larger rectangle $q_n^1(p_n^1 - p_n^0)$. It can be seen that the Bennet indicator I_n^B is the average of these two rectangles.

The above analysis shows that there are a large number of possible measures of price and quantity change, I_n and V_n, that satisfy equation (5). Which measure should we use in empirical applications? In the following section, we suggest an axiomatic or test approach to the determination of the functional form for I_n and V_n while in section 5, we suggest an economic approach.

4. The Test Approach for Indicators of Price and Quantity Change

> "How is this [increase in value] to be apportioned between the increase in price ... and the increase in value due to the increase in quantity?.... Again we note that if there is an increase in price, but no increase in quantity, the whole of the increase in value is due to the increase in price.... Conversely, if there is an increase in quantity, but no increase in price, the whole of the increase in value is due to the increase in quantity." *J.K. Montgomery*[1929;3]

We consider axioms for the commodity n price change indicator, $I_n(p_n^0, p_n^1, q_n^0, q_n^1)$. We assume that the commodity n volume change indicator $V_n(p_n^0, p_n^1, q_n^0, q_n^1)$ is determined by (5) once I_n has been determined. We also assume that all of the scalar prices and quantities $p_n^0, p_n^1, q_n^0, q_n^1$ are positive.

The tests and axioms that we propose are, for the most part, analogues to the tests that Diewert [1992b] used to characterize the Fisher [1922] ideal price index.

We assume that I_n and V_n satisfy the identity (5). Thus if I_n is defined, then the corresponding V_n is implicitly defined by:

$$(43) \quad V_n(p_n^0, p_n^1, q_n^0, q_n^1) = p_n^1 q_n^1 - p_n^0 q_n^0 - I_n(p_n^0, p_n^1, q_n^0, q_n^1).$$

Thus tests or properties of the quantity change indicator V_n can be imposed on the price change indicator I_n using (43).

The first property we wish to consider for our price indicator I_n is the property of continuity.

A1 $\quad I_n(p_n^0, p_n^1, q_n^0, q_n^1)$ is defined for $p_n^0 > 0, p_n^1 > 0, q_n^0 > 0, q_n^1 > 0$ and is a *continuous* function over this domain of definition.

The following two tests were first proposed by Montgomery [1929; 3]. If prices remain unchanged during the two periods, then we want our indicator of price change to equal zero; if quantities remain unchanged during the two periods, then we want our indicator of quantity change to equal 0.

A2 *Identity Test for Prices*: $I_n(p_n, p_n, q_n^0, q_n^1) = 0.$

A3 *Identity Test for Quantities*: $I(p_n^0, p_n^1, q_n, q_n) = (p_n^1 - p_n^0)q_n.$

We derived the test A3 using (43). From (43), we have

$$(44) \quad V_n(p_n^0, p_n^1, q_n, q_n) = p_n^1 q_n - p_n^0 q_n - I(p_n^0, p_n^1, q_n, q_n) = 0$$

and the second equation in (44) is equivalent to the equation in A3.

Recall the Paasche and Laspeyres price indicators defined in the previous section, $I_n^P(p_n^0, p_n^1, q_n^0, q_n^1) \equiv (p_n^1 - p_n^0)q_n^1$ and $I_n^L(p_n^0, p_n^1, q_n^0, q_n^1) \equiv (p_n^1 - p_n^0)q_n^0$. From the perspective of the Montgomery approach presented in the previous section, it can be seen that the Paasche and Laspeyres price indicators give the most extreme results that could be considered acceptable. Hence it seems reasonable to require that $I_n(p_n^0, p_n^1, q_n^0, q_n^1)$ lie between these two "extreme" indicators.

A4 *Bounding Test:* $\min\{(p_n^1 - p_n^0)q_n^0, (p_n^1 - p_n^0)q_n^1\} \leq I(p_n^0, p_n^1, q_n^0, q_n^1) \leq \max\{(p_n^1 - p_n^0)q_n^0, (p_n^1 - p_n^0)q_n^1\}.$

It can be shown that if I_n satisfies A4, then the corresponding V_n defined by (43) satisfies:

$$(45) \quad \min\{(q_n^1 - q_n^0)p_n^0, (q_n^1 - q_n^0)p_n^1\} \leq V_n(p_n^0, p_n^1, q_n^0, q_n^1) \leq \max\{(q_n^1 - q_n^0)p_n^0, (q_n^1 - q_n^0)p_n^1\}.$$

Our next four tests are monotonicity properties for I_n and V_n.

A5 *Monotonicity in Period 1 Prices:* $I_n(p_n^0, p_n^1, q_n^0, q_n^1) < I_n(p_n^0, p_n, q_n^0, q_n^1)$ if $p_n^1 < p_n$.

If the period 1 price increases from p_n^1 to p_n, then the indicator of price change should also increase.

A6 *Monotonicity in Period 0 Prices:* $I_n(p_n^0, p_n^1, q_n^0, q_n^1) > I_n(p_n, p_n^1, q_n^0, q_n^1)$ if $p_n^0 < p_n$.

If the period 0 price increases from p_n^0 to p_n, then the indicator of price change should decrease.

A7 *Monotonicity in Period 1 Quantities*: $p_n^1 q_n^1 - p_n^0 q_n^0 - I_n(p_n^0, p_n^1, q_n^0, q_n^1) < p_n^1 q_n - p_n^0 q_n^0 - I_n(p_n^0, p_n^1, q_n^0, q_n)$ if $q_n^1 < q_n$.

Thus if the period 1 quantity increases from q_n^1 to q_n, the implicit indicator of quantity change V_n defined by (43) should also increase; i.e., A7 is equivalent to

(46) $\quad V_n(p_n^0, p_n^1, q_n^0, q_n^1) < V_n(p_n^0, p_n^1, q_n^0, q_n)$ if $q_n^1 < q_n$.

A8 *Monotonicity in Period 0 Quantities:* $p_n^1 q_n^1 - p_n^0 q_n^0 - I_n(p_n^0, p_n^1, q_n^0, q_n^1) > p_n^1 q_n^1 - p_n^0 q_n - I_n(p_n^0, p_n^1, q_n, q_n^1)$ if $q_n^0 < q_n$.

Thus if period 1 quantity increases from q_n^0 to q_n, the implicit indicator of quantity change V_n defined by (43) should decrease; i.e., A8 is equivalent to

(47) $\quad V_n(p_n^0, p_n^1, q_n^0, q_n^1) > V_n(p_n^0, p_n^1, q_n, q_n^1)$ if $q_n^0 < q_n$.

The following 4 tests are positivity (or negativity) tests that are somewhat weaker than the previous four monotonicity tests: it can be shown that A2 and A5 imply A9, A2 and A6 imply A10, A3 and A7 imply A11 and A3 and A8 imply A12.

A9 *Positivity of Price Change if the Period 1 Price Exceeds the Period 0 Price*: $I_n(p_n^0, p_n^1, q_n^0, q_n^1) > 0$ if $p_n^1 > p_n^0$.

A10 *Negativity of Price Change if the Period 0 Price Exceeds the Period 1 Price:* $I_n(p_n^0, p_n^1, q_n^0, q_n^1) < 0$ if $p_n^0 > p_n^1$.

A11 *Positivity of Quantity Change if the Period 1 Quantity Exceeds the Period 0 Quantity*: $p_n^1 q_n^1 - p_n^0 q_n^0 - I_n(p_n^0, p_n^1, q_n^0, q_n^1) > 0$ if $q_n^1 > q_n^0$.

Of course, using (43), the inequality in A11 is equivalent to

(48) $\quad V_n(p_n^0, p_n^1, q_n^0, q_n^1) > 0$ if $q_n^1 > q_n^0$.

A12 *Negativity of Quantity Change if the Period 0 Quantity Exceeds the Period 1 Quantity:* $p_n^1 q_n^1 - p_n^0 q_n^0 - I_n(p_n^0, p_n^1, q_n^0, q_n^1) < 0$ if $q_n^0 > q_n^1$.

Using (43), the inequality in A12 is equivalent to:

(49) $\quad V_n(p_n^0, p_n^1, q_n^0, q_n^1) < 0$ if $q_n^0 > q_n^1$.

In index number theory, we generally require our price and quantity indexes to be invariant to changes in the units of measurement. It seems reasonable to impose this invariance property in the present context.

A13 *Invariance to Changes in the Units of Measurement:* $I_n(p_n^0, p_n^1, q_n^0, q_n^1) = I_n(\lambda p_n^0, \lambda p_n^1, \lambda^{-1} q_n^0, \lambda^{-1} q_n^1)$ where $\lambda > 0$.

There are also the following two linear homogeneity properties that we can impose on P.

A14 *Linear Homogeneity in Prices:* $I_n(\lambda p_n^0, \lambda p_n^1, q_n^0, q_n^1) = \lambda I_n(p_n^0, p_n^1, q_n^0, q_n^1)$ for all $\lambda > 0$.

A15 *Linear Homogeneity in Quantities:* $I_n(p_n^0, p_n^1, \mu q_n^0, \mu q_n^1) = \mu I(p_n^0, p_n^1, q_n^0, q_n^1)$ for all $\mu > 0$.

Note that the price homogeneity axioms A14 and A15 differ from their index number counterparts; i.e., the prices in *both* periods are scaled up by the multiplicative factor λ in A14 and A15.

Note that if A14 holds, we have $I_n(p_n^0, p_n^1, q_n^0, q_n^1) = p_n^0 I_n(1, p_n^1/p_n^0, q_n^0, q_n^1)$ (let $\lambda = 1/p_n^0$) and if A15 holds, we have $I_n(p_n^0, p_n^1, q_n^0, q_n^1) = q_n^0 I(p_n^0, p_n^1, 1, q_n^1/q_n^0)$ (let $\mu = 1/q_n^0$). If both A14 and A15 hold, then we have A13 and moreover:

$$(50) \quad I_n(p_n^0, p_n^1, q_n^0, q_n^1) = p_n^0 q_n^0 I_n(1, p_n^1/p_n^0, 1, q_n^1/q_n^0).$$

Our final three tests are symmetry tests.

A16 *Time Reversal:* $I_n(p_n^0, p_n^1, q_n^0, q_n^1) = -I_n(p_n^1, p_n^0, q_n^1, q_n^0)$:

i.e., if we interchange prices and quantities for the two periods, then the price indicator function should change sign. Put another way, the price change going from period 0 to 1 plus the price change going from period 1 back to period 0 should sum to zero; i.e., $I_n(p_n^0, p_n^1, q_n^0, q_n^1) + I_n(p_n^1, p_n^0, q_n^1, q_n^0) = 0$.

A17 *Quantity Weights Symmetry:* $I_n(p_n^0, p_n^1, q_n^0, q_n^1) = I_n(p_n^0, p_n^1, q_n^1, q_n^0)$.

Thus if (A17) holds, we can interchange the role of quantities in the two periods and the indicator of price change remains unchanged. This implies that the quantities q_n^0 and q_n^1 enter into the price indicator formula in a symmetric manner.

A18 *Price Weights Symmetry:* $p_n^1 q_n^1 - p_n^0 q_n^0 - I_n(p_n^0, p_n^1, q_n^0, q_n^1) = p_n^0 q_n^1 - p_n^1 q_n^0 - I_n(p_n^1, p_n^0, q_n^0, q_n^1)$.

Making use of (43), A18 is equivalent to:

$$(51) \quad V_n(p_n^0, p_n^1, q_n^0, q_n^1) = V_n(p_n^1, p_n^0, q_n^0, q_n^1):$$

i.e., if we interchange prices in the indicator of quantity change this indicator of remains unchanged. Thus p_n^0 and p_n^1 enter into the quantity indicator formula in a symmetric fashion.

It turns out that the Bennet price and quantity indicators

$$(52)\quad I_n^B(p^0,p^1,q^0,q^1) \equiv \tfrac{1}{2}(q_n^0+q_n^1)(p_n^1-p_n^0);$$

$$(53)\quad V_n^B(p^0,p^1,q^0,q^1) \equiv \tfrac{1}{2}(p_n^0+p_n^1)(q_n^1-q_n^0)$$

satisfy all 18 of the above tests and in fact are uniquely characterized by the above tests.

Proposition 1: If I_n satisfies the three symmetry tests A16, A17 and A18, then $I_n = I_n^B$ where I_n^B is defined by (52).

See Appendix 1 for proofs of Propositions.

Proposition 2: I_n^B defined by (52) satisfies Tests A1-A18.

What tests does the Montgomery price change indicator I_n^M defined by (18) satisfy?

Proposition 3: $I_n^M(p_n^0,p_n^1,q_n^0,q_n^1)$ defined by (18) satisfies all of the tests except the monotonicity tests A5, A6, A7, A8 and the symmetry tests A17 and A18.

It should be mentioned that Montgomery [1929; 3] also proposed the following symmetry test, which is a counterpart to Fisher's [1921; 53] [1922; 72] factor reversal test in index number theory:

A19 $\quad V_n(p_n^0,p_n^1,q_n^0,q_n^1) = I_n(q_n^0,q_n^1,p_n^0,p_n^1)$:

i.e., the volume indicator is equal to the price indicator when the role of prices and quantities is interchanged in the latter function.[14] Montgomery [1929; 11] noted that both the Bennet and Montgomery indicators satisfied the factor reversal axiom (A19) and that he could not find any other simple formula that satisfied this axiom. However, it is easy to adapt Fisher's [1922; 142][15] crossing of factor antitheses approach to rectify any given indicator of price change $I_n(p_n^0,p_n^1,q_n^0,q_n^1)$ into an indicator $I_n^r(p_n^0,p_n^1,q_n^0,q_n^1)$ that satisfies the factor reversal test A19. We show how this can be done.

Let $I_n(p_n^0,p_n^1,q_n^0,q_n^1)$ be an arbitrary indicator of price change that perhaps does not satisfy A19. Define the *factor antithesis* I_n^f to I_n as follows:

$$(54)\quad I_n^f(p_n^0,p_n^1,q_n^0,q_n^1) \equiv p_n^1q_n^1 - p_n^0q_n^0 - I_n(q_n^0,q_n^1,p_n^0,p_n^1).$$

Define the *factor rectified indicator* I_n^{fr} as the arithmetic average of I_n and I_n^f:

(55) $\quad I_n^{fr}(p_n^0, p_n^1, q_n^0, q_n^1) \equiv (1/2)I_n(p_n^0, p_n^1, q_n^0, q_n^1) + (1/2)I_n^f(p_n^0, p_n^1, q_n^0, q_n^1)$.

Proposition 4: The factor rectified indicator of price change I_n^{fr} defined by (55) satisfies the factor reversal test A19; i.e., we have

(56) $\quad I_n^{fr}(p_n^0, p_n^1, q_n^0, q_n^1) + I_n^{fr}(q_n^0, q_n^1, p_n^0, p_n^1) = p_n^1 q_n^1 - p_n^0 q_n^0$.

It is also easy to modify Fisher's [1922; 140] time rectification procedure which will allow us to transform an arbitrary indicator of price change $I_n(p_n^0, p_n^1, q_n^0, q_n^1)$ that perhaps does not satisfy the time reversal test A16 into one that does. Define the *time antithesis* I_n^t to I_n as follows:

(57) $\quad I_n^t(p_n^0, p_n^1, q_n^0, q_n^1) \equiv -I_n(p_n^1, p_n^0, q_n^1, q_n^0)$.

Define the *time rectified indicator* I_n^{tr} as the arithmetic mean of I_n and I_n^t:

(58) $\quad I_n^{tr}(p_n^0, p_n^1, q_n^0, q_n^1) \equiv (1/2)I_n(p_n^0, p_n^1, q_n^0, q_n^1) + (1/2)I_n^t(p_n^0, p_n^1, q_n^0, q_n^1)$.

Proposition 5: The time rectified indicator of price change I_n^{tr} defined by (58) satisfies the time reversal test A16; i.e., we have

(59) $\quad I_n^{tr}(p_n^0, p_n^1, q_n^0, q_n^1) = -I_n^{tr}(p_n^1, p_n^0, q_n^1, q_n^0)$.

The above Propositions show that there are many parallels between the test approach to index numbers and the test approach to indicators of price change. The Fisher [1921] [1922] ideal price index (the geometric mean of the Paasche and Laspeyres price indexes) appears to be "best" from the viewpoint of the test approach[16] while the Bennet indicator of price change (the arithmetic average of the Paasche and Laspeyres price indicators) appears to be "best" from the viewpoint of the test approach to indicators of price change.

We note that the Paasche and Laspeyres indicators of price change, I^P and I^L, defined by (11) and (14) respectively, have rather good axiomatic properties as is indicated in the following result:

Proposition 6: The Paasche and Laspeyres indicators of price change defined by (11) and (14) satisfy all of the above axioms except the symmetry axioms, (A16), (A17), (A18) and (A19).

The reader may have noticed that we did not strongly endorse the factor symmetry test A19; it is not a particularly compelling test.[17] The weight symmetry tests (A17) and (A18) are also not tests that simply must hold. However, the time reversal test is an important test that we definitely will want to impose in any context where the data from the two periods is symmetric. With respect to the five applications (outlined in section 1) where indicators of price and quantity change might be useful, all of these applications would seem to require a symmetric treatment of the data, with the exception of variance analysis, where the data are not symmetric. Thus the failure of the Paasche and Laspeyres price indicators I_n^P and I_n^L to satisfy the time reversal test A16 means that we should be cautious in using these indicators. The failure of the Montgomery indicator of price change I_n^M to satisfy the monotonicity tests A5-A8 means that we should be cautious in using it as well. This leaves the Bennet indicator of price change, I_n^B, as the "best" from the viewpoint of the test approach.

We now turn our attention to economic approaches to the measurement of price and quantity change.

5. The Economic Approach to Indicators of Price and Quantity Change

> "Therefore the problem of constructing a true index of the cost of living is inseparabably bound up with the general problem of establishing a functional relation between consumption and prices." *A.A. Könus*[1939;12]

In the economic approach to index number theory, prices and quantities are no longer regarded as being completely independent variables: prices are regarded as exogenous but quantities are determined as solutions to optimization problems. For example, in the consumer context, it is assumed that a consumer maximizes a utility function $f(q)$ subject to a budget constraint or alternatively, the consumer minimizes the cost of achieving a given utility level. Following Samuelson and Swamy [1974] and Diewert [1976b], we assume that the utility function is homogeneous of degree one, in which case the consumer's cost or expenditure function C, defined as

(60) $\quad C(u,p) \equiv \min_q\{p \cdot q : f(q) \geq u\},$

has the following decomposition:

(61) $\quad C(u,p) \equiv uC(1,p) \equiv uc(p)$

where $c(p) \equiv C(1,p)$ is the consumer's unit cost function.[18]

In this case, where preferences are homothetic, the observed price and quantity data for period t, p^t and q^t, satisfy:

(62) $\quad p^t \cdot q^t = f(q^t)c(p^t); \quad t = 0, 1.$

Thus taking ratios, (62) implies

(63) $p^1 \cdot q^1/p^0 \cdot q^0 = [c(p^1)/c(p^0)][f(q^1)/f(q^0)].$

For certain specific functional forms for the unit cost function c, we can find an index number formula, $P(p^0, p^1, q^0, q^1)$, such that the unit cost ratio is equal to this price index; i.e., for such an index, we have

(64) $P(p^0, p^1, q^0, q^1) = c(p^1)/c(p^0).$

Associated with such an exact price index P is the quantity index Q that satisfies equation (2) above. Using (2), (63) and (64), for such a Q, we have

(65) $Q(p^0, p^1, q^0, q^1) = f(q^1)/f(q^0).$

This is a brief outline of the theory of exact index number formulae that is discussed by Pollak [1989; 22-32], Samuelson and Swamy [1974] and Diewert [1976b].

A particular example of (64) which is used frequently in empirical applications is the Fisher [1922] ideal index P_F defined as:

(66) $P_F(p^0, p^1, q^0, q^1) \equiv [p^1 \cdot q^0 p^1 \cdot q^1/p^0 \cdot q^0 p^0 \cdot q^1]^{1/2}$

which is exact for the homogeneous quadratic unit cost function $c(p) \equiv [p \cdot Bp]^{1/2}$ where B is a symmetric matrix of constants.[19] Another example is the Törnqvist[20] price index P_T defined as

(67) $lnP_T(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^N (1/2)(s_n^0 + s_n^1) ln(p_n^1/p_n^0)$

where the period t expenditure share on commodity n, s_n^t, is defined as $p_n^t q_n^t/p^t \cdot q^t$ for $n = 1, \ldots, N$ and $t = 0, 1$. This index is exact for a translog unit cost function.[21]

Now return to the case of a generic exact price index P satisfying (64) along with its partner Q satisfying (2) and (65). We can write the difference in consumer expenditures between periods 0 and 1 as follows:

$$\begin{aligned} p^1 \cdot q^1 - p^0 \cdot q^0 &= p^0 \cdot q^0[(p^1 \cdot q^1/p^0 \cdot q^0) - 1] \\ &= p^0 \cdot q^0[PQ - 1] \qquad \text{using (2)} \\ &= p^0 \cdot q^0[(1/2)(1 + Q)(P - 1) + (1/2)(1 + P)(Q - 1)] \qquad (68) \end{aligned}$$

where (68) follows from the line above using the Bennet identity:

(69) $PQ - 1 = (1/2)(1 + Q)(P - 1) + (1/2)(1 + P)(Q - 1).$

Now from (68), we see that the value change has been decomposed into an "economic" price change (the term involving $P - 1$) and an "economic" quantity change (the term involving $Q - 1$). Thus for our generic exact economic price and quantity indexes P and Q satisfying (2), we define the following economic indicators of price and volume change:

$$I_E(p^0, p^1, q^0, q^1) \equiv (1/2)p^0 \cdot q^0[1 + Q(p^0, p^1, q^0, q^1)][P(p^0, p^1, q^0, q^1) - 1]; \tag{70}$$

$$V_E(p^0, p^1, q^0, q^1) \equiv (1/2)p^0 \cdot q^0[1 + P(p^0, p^1, q^0, q^1)][Q(p^0, p^1, q^0, q^1) - 1]. \tag{71}$$

Proposition 7: If the exact price and quantity indexes P and Q satisfy (2), (64) and (65) and there is cost minimizing behavior in the two periods under consideration so that (62) holds, then the economic price and quantity indicators defined by (70) and (71) satisfy the following equations:

$$\begin{aligned}
(72)\quad I_E(p^0, p^1, q^0, q^1) &= (1/2)c(p^0)f(q^0)[1 + f(q^1)/f(q^0)][\{c(p^1)/c(p^0)\} - 1] \\
(73)\quad &= (1/2)f(q^0)[c(p^1) - c(p^0)] - (1/2)f(q^1)[c(p^0) - c(p^1)] \\
(74)\quad &= (1/2)[f(q^0) + f(q^1)][c(p^1) - c(p^0)];
\end{aligned}$$

$$\begin{aligned}
(75)\quad V_E(p^0, p^1, q^0, q^1) &= (1/2)c(p^0)f(q^0)[1 + c(p^1)/c(p^0)][\{f(q^1)/f(q^0)\} - 1] \\
(76)\quad &= (1/2)c(p^0)[f(q^1) - f(q^0)] - (1/2)c(p^1)[f(q^0) - f(q^1)] \\
(77)\quad &= (1/2)[c(p^0) + c(p^1)][f(q^1) - f(q^0)].
\end{aligned}$$

Equation (72) is a straightforward translation of (70) using the exactness of the indexes P and Q. Equation (73) is a symmetric rearrangement of (72); it shows that the overall price change can be written as the arithmetic average of the period 0 to 1 price change $c(p^1) - c(p^0)$ weighted by the period 0 aggregate quantity $f(q^0)$ and the negative of the period 1 to 0 price change $c(p^0) - c(p^1)$ weighted by the period 1 aggregate quantity $f(q^1)$. Equation (74) is the familiar Bennet price change decomposition, treating $c(p^0)$ and $c(p^1)$ as aggregate prices for periods 0 and 1 and $f(q^0)$ and $f(q^1)$ as aggregate quantities for periods 0 and 1. Note that the left hand side of (72), (73) or (74) is $I_E(p^0, p^1, q^0, q^1)$ which can be readily calculated if P and Q are given whereas the right hand side of (72), (73) or (74) involves the unobserved aggregates $c(p^t)$ and $f(q^t)$.

It is useful to obtain an alternative formula for the economic indicator of price change I_E defined by (70). Rearranging the right hand side of (70) yields

$$\begin{aligned}
I_E(p^0, p^1, q^0, q^1) &= (1/2)p^0 \cdot q^0[P - 1] + (1/2)p^0 \cdot q^0 Q[P - 1] \\
&= (1/2)p^0 \cdot q^0[P - 1] + (1/2)p^0 \cdot q^0[p^1 \cdot q^1/p^0 \cdot q^0 P][P - 1] \qquad \text{using(2)} \\
(78)\quad &= (1/2)p^0 \cdot q^0[P(p^0, p^1, q^0, q^1) - 1] - (1/2)p^1 \cdot q^1[\{1/P(p^0, p^1, q^0, q^1)\} - 1].
\end{aligned}$$

Equation (78) can be used to establish the following result.

Proposition 8: If P satisfies the time reversal test for price indexes (i.e., $P(p^1, p^0, q^1, q^0) = 1/P(p^0, p^1, q^0, q^1)$), then the corresponding economic indicator of price change I_E defined by (70) or (78) satisfies a version of the time reversal test for price indicators, A16.

Obviously, there are other relationships between the axiomatic properties of the price index P and the axiomatic properties of the resulting economic price indicator I_E defined (78). For example if P satisfies the identity test for price indexes ($P(p, p, q^0, q^1) = 1$), then the corresponding I_E will satisfy a version of the identity test A2; if P satisfies the invariance to changes in the units of measurement test for price indexes, then the corresponding price indicator I_E will satisfy a version of the test A13 and so on.

Which I_E defined by (78) is "best" from the viewpoint of the economic approach? To answer this question, we must pick the "best" formula for the underlying economic price index P from the viewpoint of the economic approach. Unfortunately, there is no single "best" price index P from the viewpoint of the economic approach. However, there is a class of price indexes P which are considered "best" from the viewpoint of the economic approach: namely those indexes P which are exact for a flexible functional form for c or f. Such indexes P were called superlative by Diewert [1976b; 117].[22] Hence if the price index formula P in (78) is superlative, we will call the induced indicator of price change I_E *superlative* as well. Since the Fisher index P_F and the Törnqvist index P_T defined by (66) and (67) above are superlative, the associated economic indicators of price change I_E^F and I_E^T are also superlative indicators. Thus from the viewpoint of the economic approach to price indicators, any superlative price indicator could be considered "best".

6. *Which Indicator of Price Change Should Be Used in Practice?*

> "It is worth emphasizing that these theorems hold *without* the assumption of optimizing behavior on the part of economic agents; i.e., they are theorems in numerical analysis rather than economics." *W.E. Diewert*[1978;889]

From the viewpoint of the test or axiomatic approach to indicators of price change, the results of section 4 above suggest that the Bennet indicator I^B defined by (8) is "best". From the viewpoint of an economic approach to indicators of price change, the results of section 5 suggest that a superlative indicator of price change such as I_E^F (replace P in (78) by P_F defined by (66) above) or I_E^T (replace P in (78) by P_T defined by (67) above) is "best". Which of these "best" indicators should we use in empirical applications? The result below suggest that it will not matter much in practice which of these three indicators of price change is used.

Proposition 9: The Bennet indicator of price change approximates any superlative indicator (such as the Fisher or Törnqvist indicators) to the second order at any point where the two price vectors are equal (i.e., $p^0 = p^1$) and where the two quantity vectors are equal (i.e., $q^0 = q^1$).

The proof of the above Proposition rests on analogous results for superlative price indexes: every known superlative price index $P(p^0, p^1, q^0, q^1)$ has the same levels, vectors of first order partial derivative and matrices of second order partial derivatives when evaluated at an equal price (i.e., $p^0 = p^1$) and quantity (i.e., $q^0 = q^1$) point.[23]

Somewhat surprisingly, the Montgomery indicator of price change, $I^M(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^N I_n^M(p_n^0, p_n^1, q_n^0, q_n^1)$ where the I_n^M are defined by (18), also approximates the indicators described in Proposition 9 to the second order around an equal price and quantity point.

Proposition 10: The Montgomery indicator of price change, $I^M(p^0, p^1, q^0, q^1)$, approximates the Bennet indicator $I^B(p^0, p^1, q^0, q^1)$ to the second order at any point where $p^0 = p^1$ and $q^0 = q^1$.

We require all prices and quantities to be positive in Proposition 10. All prices must be positive in Proposition 9, but the requirement that the quantity vectors be strictly positive can be relaxed provided that the superlative $P(p^0, p^1, q^0, q^1)$ is still well defined.

Proposition 10 appears to be the indicator counterpart to Theorem 3 in Diewert [1978; 887] who showed that the Vartia I [1976a; 124-125] [1976b; 122] price index approximated any superlative price index to the second order around an equal price and quantity point. However, it should be noted that Montgomery[1937; 35] actually came up with the formula for the Vartia I price index many years before Vartia's derivation.[24] The formula for the Montgomery-Vartia I index is:

$$(79) \quad lnP_M(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^N w_n ln(p_n^1/p_n^0)$$

where the weights w_n are defined, using the logarithmic mean L, as follows:

$$(80) \quad w_n \equiv L(p_n^1 q_n^1, p_n^0 q_n^0)/L(p^1 \cdot q^1, p^0 \cdot q^0); \quad n = 1, \ldots, N.$$

Thus the Montgomery price indicator function I^M plays the same role in Proposition 10 above as the Montgomery-Vartia price index P_M played in Theorem 10 of Diewert [1978].

In order to check the accuracy of Propositions 9 and 10 above, we calculated the Paasche and Laspeyres indicators of price change, I^P and I^L defined by (11) and (14) above, the Bennet indicator I^B defined by (8), the superlative Fisher and Törnqvist indicators I_E^F and I_E^T and the Montgomery indicator I^M using the price and quantity data on 36 primary commodities in the U.S. for the years 1913-1918 that are tabled in Fisher [1922; 489-490]. We calculated the chain links for each year; i.e., we calculated

$I(p^{t-1}, p^t, q^{t-1}, q^t)$ for $t = 1914, \ldots, 1918$. We also calculated the Fisher and Törnqvist chain links $P_F(p^{t-1}, p^t, q^{t-1}, q^t)$ and $P_T(p^{t-1}, p^t, q^{t-1}, q^t)$ to show the year over year inflation rates. The results of these computations can be found in Table 1 below.

Table 1: Comparison of Various Indicators of Price Change

As can be seen from the above Table, there does appear to be a fairly close correspondence between the Bennet, Fisher, Törnqvist and Montgomery indicators of price change, I^B, I^F, I^T and I^M respectively. Note also that the Paasche and Laspeyres indicators of price change, I^P and I^L, are rather different from each other and the other superlative and pseudosuperlative indicators.[25]

7. *Conclusion*

> "Today's dollar is, then, a totally different unit from the dollar of 1897. As the general price level fluctuates, the dollar is bound to become a unit of different magnitude. To mix these units is like mixing inches and centimeters or measuring a field with a rubber tape-line."
>
> *Livingston Middleditch*[1918;114-115]

The above quotation alerts us to a potential problem with our treatment of value changes; namely, if there is a great change in the general purchasing power of money between the two periods being compared, then our indicators of volume change may be "excessively" heavily weighted by the prices of the period that has the highest general price level. Put another way, the units that quantities are measured in do not require any comparisons with other quantities but the dollar price of a quantity is the valuation of a unit of a commodity *relative to a numeraire commodity, money.* Thus the indicators of price change that we have discussed in this paper encompass both *general changes* in the purchasing power of money as well as *changes in inflation adjusted* prices. Thus if there is high inflation between periods 0 and 1 and quantities have increased, then the use of symmetric in prices and quantities indicators (like the Bennet and Montgomery indicators) will shift some of the inflationary increase in values over to the indicator of volume change. This problem can be cured as follows: if the general inflation rate going from period 0 to 1 is i, define the period 1 inflation adjusted price vector p^{1*} as $(1+i)^{-1}p^1$. Then the value change going from period 0 to 1 is first decomposed into a value change due to general inflation, $p^1 \cdot q^1 - p^{1*} \cdot q^1$, and then we use our "best" indicator of (inflation adjusted) price change, $I(p^0, p^{1*}, q^0, q^1)$, along with the corresponding indicator of volume change, $V(p^0, p^{1*}, q^0, q^1)$, where the inflation adjusted period 1 prices p^{1*} are used in the indicators of price and volume change rather than the inflation contaminated prices p^1. Using this inflation adjusted methodology, the actual value change has the following decomposition:

$$(81) \quad p^1 \cdot q^1 - p^0 \cdot q^0 = p^1 \cdot q^1 - p^{1*} \cdot q^1 + I(p^0, p^{1*}, q^0, q^1) + V(p^0, p^{1*}, q^0, q^1).$$

Why are decompositions of value changes like (81) not more common in financial and cost accounting? The basic problem is that it is difficult to construct unit values and the corresponding total quantities by commodity when the firm may be producing thousands of outputs and utilizing hundreds or thousands of inputs. However, as the computer revolution spreads through the business world, decompositions of the type (3) or (81) become increasingly feasible.

Another problem which limits the usefulness of value change decompositions is the *new goods problem*. Our methodological approach assumed that the list of N commodities remained fixed over the two periods but in the real world, the list of inputs and outputs does not remain fixed from period to period. New products and new services are constantly being developed and so the problem arises of what price do we use for a new commodity in the period prior to its introduction? Hicks [1940; 114] provided a theoretical solution to this problem but it is difficult to implement. Thus the new goods problems has also limited the use of value decompositions.

A nice feature of the Bennet indicators of price and volume change is their additive over commodities property which gives them a big advantage over the superlative indicators of price and volume change, which are inherently nonadditive over commodities. The Montgomery indicators of price and volume change are also additive over commodities but the axiomatic properties of the Montgomery indicators are not as attractive as those of the Bennet indicators.

Appendix 1: Proofs of Propositions

Proof of Proposition 1: Rearranging A18, we have

$$
\begin{aligned}
(q_n^0 + q_n^1)(p_n^1 - p_n^0) &= I_n(p_n^0, p_n^1, q_n^0, q_n^1) - I_n(p_n^1, p_n^0, q_n^0, q_n^1) \\
&= I_n(p_n^0, p_n^1, q_n^0, q_n^1) - I_n(p_n^1, p_n^0, q_n^1, q_n^0) \quad \text{using A17} \\
&= I_n(p_n^0, p_n^1, q_n^0, q_n^1) + I_n(p_n^0, p_n^1, q_n^0, q_n^1) \quad \text{using A16} \\
&= 2I_n(p_n^0, p_n^1, q_n^0, q_n^1) \\
&= 2I_N^B(p_n^0, p_n^1, q_n^0, q_n^1) \quad \text{using definition (52).}
\end{aligned}
$$

Proof of Proposition 2: Routine Computations.

Proof of Proposition 3: The proof of A4 follows from the definition of the Montgomery indicators of price and volume change as integrals. An algebraic proof procedes as follows. Define $\bar{q}_n \equiv \max\{q_n^0, q_n^1\}$ and assume that $p_n^1 > p_n^0$. Using the definition of the Montgomery indicator (18) and the fact that the logarithmic mean $L(a,b)$ is nondecreasing and homogeneous in its arguments, we have:

$$
\begin{aligned}
I_n^M(p_n^0, p_n^1, q_n^0, q_n^1) &= L(p_n^1 q_n^1, p_n^0 q_n^0) ln(p_n^1/p_n^0) \\
&\leq L(p_n^1 \bar{q}_n, p_n^0 \bar{q}_n) ln(p_n^1/p_n^0) \\
&= \bar{q}_n L(p_n^1, p_n^0) ln(p_n^1/p_n^0) \\
&= \bar{q}_n \{[p_n^1 - p_n^0]/[lnp_n^1 - lnp_n^0]\}\{lnp_n^1 - lnp_n^0\} \\
&= [p_n^1 - p_n^0]\bar{q}_n \\
&= \max\{[p_n^1 - p_n^0]q_n^0, [p_n^1 - p_n^0]q_n^1\}. \qquad (82)
\end{aligned}
$$

If $p_n^1 < p_n^0$, then the inequality (82) is reversed and we deduce:

$$(83) \quad I_n^M(p_n^0, p_n^1, q_n^0, q_n^1) \geq [p_n^1 - p_n^0]\bar{q}_n = \min\{[p_n^1 - p_n^0]q_n^0, [p_n^1 - p_n^0]q_n^1\}.$$

Now rework (82) and (83) letting $\bar{q}_n \equiv \min\{q_n^0, q_n^1\}$.

To show that the monotonicity test A5 does not hold, differentiate $I_n^M(p_n^0, p_n^1, q_n^0, q_n^1)$ defined by (18) with respect to p_n^1 and evaluate the derivative at a point where

$$(84) \quad p_n^0 q_n^0 = p_n^1 q_n^1.$$

L'Hospital's Rule for differentiation can be used to show that $\partial L(a,b)/\partial b = 1/2$ if $a = b$ where $L(a,b)$ is the logarithmic mean of a and b. Using this result, we have

$\partial I_n^M(p_n^0, p_n^1, q_n^0, q_n^1)/\partial p_n^1$

$$= (1/2)\{\partial[p_n^1 q_n^1]/\partial p_n^1\} \ln[p_n^1/p_n^0] + L(p_n^0 q_n^0, p_n^1 q_n^1)(p_n^1)^{-1}$$
$$= (1/2)q_n^1 ln[p_n^1/p_n^0] + p_n^1 q_n^1/p_n^1 \quad \text{using (84)}$$
$$= (1/2)q_n^1\{ln[p_n^1/p_n^0] + 2]$$
$$< 0$$

if $ln[p_n^1/p_n^0] + 2 < 0$. Thus if we choose p_n^1/p_n^0 small enough so that $ln[p_n^1/p_n^0] < -2$ while simultaneously choosing q_n^1/q_n^0 to satisfy (84), then we violate the monotonicity axiom A5. Proofs of violations for the axioms A6, A7 and A8 follow in an analogous manner.

Proof of Proposition 4: Substitute (54) into (55) to obtain the following formula for I_n^{fr} in terms of the original indicator I_n:

(85) $\quad I_n^{fr}(p_n^0, p_n^1, q_n^0, q_n^1) = (1/2)[p_n^1 q_n^1 - p_n^0 q_n^0] - (1/2)I_n(q_n^0, q_n^1, p_n^0, p_n^1) + (1/2)I_n(p_n^0, p_n^1, q_n^0, q_n^1).$

Using formula (85) for I_n^{fr}, a simple calculation shows that I_n^{fr} satisfies equation (56).

Proof of Proposition 5: Substitute (57) into (58) to obtain the following formula for I_n^{tr} in terms of the original indicator I_n:

(86) $\quad I_n^{fr}(p_n^0, p_n^1, q_n^0, q_n^1) = (1/2)I_n(p_n^0, p_n^1, q_n^0, q_n^1) - (1/2)I_n(p_n^1, p_n^0, q_n^1, q_n^0).$

Using formula (86) for I_n^{tr}, a simple calculation shows that I_n^{tr} satisfies (59).

Proof of Proposition 6: Straightfoward computations.

Proof of Proposition 7: Replace P and Q in (70) and (71) by the right hand sides of (64) and (65) respectively and simplify.

Proof of Proposition 8: Replace the term $1/P(p^0, p^1, q^0, q^1)$ in (78) by $P(p^1, p^0, q^1, q^0)$ and we obtain the following formula for the economic indicator I_E:

(87) $\quad I_E(p^0, p^1, q^0, q^1) = (1/2)p^0 \cdot q^0[P(p^0, p^1, q^0, q^1) - 1] - (1/2)p^1 \cdot q^1[P(p^1, p^0, q^1, q^0) - 1].$

Using formula (87) for I_E, verify that I_E satisfies the following time reversal test:

(88) $\quad I_E(p^0, p^1, q^0, q^1) + I_E(p^1, p^0, q^1, q^0) = 0.$

Proof of Proposition 9: Routine computations show that the vectors of first order partial derivatives of the Bennet indicator of price change $I^B(p^0, p^1, q^0, q^1)$ evaluated at a point where $p^0 = p^1 = p$ and $q^0 = q^1 = q$ are:

(89) $\quad \nabla_{p^0} I^B = -q;$

$$(90) \quad \nabla_{p^1} I^B = q;$$

$$(91) \quad \nabla_{q^0} I^B = 0_N;$$

$$(92) \quad \nabla_{q^1} I^B = 0_N.$$

More routine computations show that the matrices of second order partial derivatives of $I^B(p^0, p^1, q^0, q^1)$ evaluated at $p^0 = p^1 = p$ and $q^0 = q^1 = q$ are:

$$(93) \quad \nabla^2_{p^0 p^0} I^B = O_{N\times N};$$

$$(94) \quad \nabla^2_{p^0 p^1} I^B = O_{N\times N};$$

$$(95) \quad \nabla^2_{p^0 q^0} I^B = -(1/2) I_N;$$

$$(96) \quad \nabla^2_{p^0 q^1} I^B = -(1/2) I_N;$$

$$(97) \quad \nabla^2_{p^1 p^1} I^B = O_{N\times N};$$

$$(98) \quad \nabla^2_{p^1 q^0} I^B = \tfrac{1}{2} I_N;$$

$$(99) \quad \nabla^2_{p^1 q^1} I^B = \tfrac{1}{2} I_N;$$

$$(100) \quad \nabla^2_{q^0 q^0} I^B = O_{N\times N};$$

$$(101) \quad \nabla^2_{q^0 q^1} I^B = O_{N\times N};$$

$$(102) \quad \nabla^2_{q^1 q^1} I^B = O_{N\times N}.$$

By differentiating the formula for a superlative price index P (such as the Fisher ideal index), we can calculate the vectors of first order partial derivatives and the matrices of second order partial derivatives of a superlative price index evaluated at $p^0 = p^1$ and $q^0 = q^1$. Now calculate the first and second order partial derivatives of $I_E(p^0, p^1, q^0, q^1)$ defined by (78) and evaluate these derivatives at $p^0 = p^1$ and $q^0 = q^1$.

Proof of Proposition 10. More routine computations. We need to use L'Hospital's Rule to show that $\partial L(a, b)/\partial a = \frac{1}{2}$ when $a = b > 0$.

Appendix 2: Generalized Bennet Indicators of Price Change.

The Bennet indicator of price change has the form

(103) $\quad I^{GB}(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^{N} M(q_n^0, q_n^1)[p_n^1 - p_n^0]$

where $M(q_n^0, q_n^1)$ is the *arithmetic mean* of the period 0 and period 1 quantities for commodity n, $(1/2)q_n^0 + (1/2)q_n^1$. We call an indicator I^{GB} of the type defined by (103) where $M(a,b)$ is a *general homogeneous symmetric mean* of the positive quantities a and b, a *Generalized Bennet indicator* of price change.

An interesting special case of a homogeneous symmetric mean is the mean of order r, M_r, defined as follows:

(104) $\quad M_r(a,b) \equiv [(1/2)a^r + (1/2)b^r]^{1/r}; \quad r \neq 0;$

(105) $\quad M_0(a,b) \equiv a^{1/2}b^{1/2}.$

Means of the rth order (for r a positive or negative integer) were defined by Schlömilch [1858; 303] who showed that $M_r(a,b)$ increased as r increased if $a \neq b$. Dunkel [1909; 27] showed that M_r could be defined by (104) and (105) for all real numbers r and he showed that M_r increased as r increased if $a \neq b$. The term "mean of order r" was used by Darmois [1928; 31], who also allowed for general positive weights p_i summing to one to replace the equal weights 1/2 in (104) and (105). The properties of means of order r are documented in Hardy, Littlewood and Pòlya [134; 12-21].

If we replace the general mean M in definition (103) by the mean of order r M_r defined by (104) and (105), we obtain the *Mean of Order r Generalized Bennet indicator* of price change I^{GBr}:

(106) $\quad I^{GBr}(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^{N} I_n^{GBr}(p_n^0, p_n^1, q_n^0, q_n^1)$

where the indicator of price change for commodity n, I_n^{GBr}, is defined as:

(107) $\quad I_n^{GBr}(p_n^0, p_n^1, q_n^0, q_n^1) \equiv M_r(q_n^0, q_n^1)[p_n^1 - p_n^0].$

What are the axiomatic properties of the indicator of price change I_n^{GBr}?

Proposition 11: $I_n^{GBr}(p_n^0, p_n^1, q_n^0, q_n^1)$ defined by (107) satisfies all of the tests listed in section 4 above if $r = 1$ but fails the monotonicity tests A7 and A8, the price weights symmetry test (A18) and the factor reversal test (A19) if $r \neq 1$.

Proof of A3: Use $M_r(q_n, q_n) = q_n$.

Proof of A4: Use $\min\{q_n^0, q_n^1\} \leq M_R(q_n^0, q_n^1) \leq \max\{q_n^0, q_n^1\}$.
Proof of A5 and A6: Use $M_r(q_n^0, q_n^1) > 0$ if $q_n^0 > 0, q_n^1 > 0$.
Proof of A7: Define the corresponding volume indicator V_n^{BGr} by:

$$(108) \quad V_n^{GBr}(p_n^0, p_n^1, q_n^0, q_n^1) \equiv p_n^1 q_n^1 - p_n^0 q_n^0 - I_n^{GBr}(p_n^0, p_n^1, q_n^0, q_n^1).$$

For A7 to be true, we require that $\partial V_n^{GBr}(p_n^0, p_n^1, q_n^0, q_n^1)/\partial q_n^1 > 0$, except possibly at isolated points. Using (107) and (108), this inequality becomes:

$$(109) \quad p_n^1 - (p_n^1 - p_n^0)\partial M_r(q_n^0, q_n^1)/\partial q_n^1 > 0.$$

Letting p_n^0 tend to zero, (109) becomes

$$(110) \quad p_n^1[1 - \partial M_r(q_n^0, q_n^1)/\partial q_n^1] > 0.$$

Using $p_n^1 > 0$ and performing the differentiation of M_r assuming that $r \neq 0$ and $r \neq 1$, (110) is equivalent to

$$(111) \quad [(1/2)(q_n^0/q_n^1)^r + (1/2)]^{(1-r)/r} < 2$$

which is not true for all positive q_n^0 and q_n^1. For $r = 0$, (110) is equivalent to

$$(112) \quad (q_n^0/q_n^1)^{1/2} < 2$$

which again is not true for all positive q_n^0 and q_n^1. Of course, if $r = 1$, (110) becomes $p_n^1[1 - (1/2)] > 0$, which is true.

Proof of A8: Similar to the proof of A7.
Proof of A9: Follows from A2 and A5.
Proof of A10: Follows from A2 and A6.
Proof of A11: By assumption, $q_n^1 > q_n^0$. Hence, by the properties of M_r:

$$(113) \quad q_n^0 < M_r(q_n^0, q_n^1) < q_n^1 \text{ or}$$

$$(114) \quad -q_n^1 < -M_r(q_n^0, q_n^1) < -q_n^0.$$

If $p_n^1 = p_n^0$, the desired inequality follows immediately. Now consider the case where:

$$(115) \quad p_n^1 > p_n^0.$$

We have:

$$
\begin{aligned}
&p_n^1 q_n^1 - p_n^0 q_n^0 - M_r(q_n^0, q_n^1)(p_n^1 - p_n^0) \\
&\qquad > p_n^1 q_n^1 - p_n^0 q_n^0 - q_n^1(p_n^1 - p_n^0) \quad \text{using (114) and(115)} \\
&\qquad = p_n^0(q_n^1 - q_n^0) \\
&\qquad > 0 \qquad \text{using } p_n^0 > 0 \text{ and } q_n^1 - q_n^0 > 0,
\end{aligned}
$$

which is the desired inequality. Now consider the case where:

(116) $\quad p_n^1 < p_n^0.$

We have:

$$
\begin{aligned}
&p_n^1 q_n^1 - p_n^0 q_n^0 - M_r(q_n^0, q_n^1)(p_n^1 - p_n^0) \\
&\qquad > p_n^1 q_n^1 - p_n^0 q_n^0 - q_n^0(p_n^1 - p_n^0) \quad \text{using (113) and (116)} \\
&\qquad = p_n^1(q_n^1 - q_n^0) \\
&\qquad > 0 \qquad \text{using } p_n^1 > 0 \text{ and } q_n^1 - q_n^0 > 0,
\end{aligned}
$$

which is the desired inequality.

Proof of A12: Similar to the proof of A11.
Proof of A13: Use $M_r(\lambda q_n^0, \lambda q_n^1) = \lambda M_r(q_n^0, q_n^1)$.
Proof of A15: Use $M_r(\lambda q_n^0, \lambda q_n^1) = \lambda M_r(q_n^0, q_n^1)$.
Proof of A16: Use $M_r(q_n^0, q_n^1) = M_r(q_n^1, q_n^0)$.
Proof of A17: Use $M_r(q_n^0, q_n^1) = M_r(q_n^1, q_n^0)$.

Q.E.D.

The above Proposition shows that the Mean of Order r Generalized Bennet indicators of price change, $I^{GBr}(p^0, p^1, q^0, q^1)$ have fairly good axiomatic properties. The failure of the price weights symmetry test A18 and the factor reversal test A19 is not particularly bothersome but the failure of the two quantity monotonicity tests A7 and A8 when $r \neq 1$ is troublesome. However, the failure of I^{GBr} to satisfy A7 and A8 for $r \neq 1$ highlights the superiority of the Bennet indicator $I^B (= I^{GB1})$ which does satisfy these important monotonicity tests.

The following Proposition shows that the Mean of Order r Generalized Bennet indicators of price change are pseudosuperlative.

Proposition 12: The Mean of Order r Generalized Bennet indicators of price change $I^{GBr}(p^0, p^1, q^0, q^1)$ approximate the Bennet Indicator $I^B(p^0, p^1, q^0, q^1)$ and the superlative indicators of price change to the second order at any point where $p^0 = p^1$ and $q^0 = q^1$.

Proof: Straightforward computations.

Instead of using a mean of order $r, M_r(q_n^0, q_n^1)$, to replace the general homogeneous symmetric mean M in the definition of the family of Generalized Bennet indicators of price change, (103) above, we could use the logarithmic mean, $L(q_n^0, q_n^1)$. Thus we define the *Logarithmic Mean Generalized Bennet indicator* of price change I^{GBL} as follows:

$$(117) \quad I^{GBL}(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^N I_n^{GBL}(p_n^0, p_n^1, q_n^0, q_n^1)$$

where the indicator of price change for commodity n is defined as:

$$(118) \quad I_n^{GBL}(p_n^0, p_n^1, q_n^0, q_n^1) \equiv L(q_n^0, q_n^1)[p_n^1 - p_n^0].$$

Perhaps somewhat surprisingly, the Logarithmic Mean Generalized Bennet indicator of price change, I^{GBL}, has almost the same axiomatic properties as the Mean of Order r Generalized Bennet Indicators of price change, I^{GBr}, for $r \neq 1$.

Proposition 13: $I_n^{GBL}(p_n^0, p_n^1, q_n^0, q_n^1)$ defined by (118) satisfies all of the tests listed in section 4 above except the price weights symmetry test A18 and the factor reversal test A19.

Proof: Except for A7 and A8, the proof is analogous to the corresponding proofs of Proposition 11.

Proof of A7: We need to show that the following counterpart to (110) is true:

$$(119) \quad p_n^1[1 - \partial L(q_n^0, q_n^1)/\partial q_n^1] > 0,$$

except possibly at isolated points. Performing the differentiation in (119) and using $p_n^1 > 0$, we see that (119) is equivalent to:

$$(120) \quad [-ln(q_n^0/q_n^1) + (q_n^0/q_n^1) - 1]/[ln(q_n^0/q_n^1)]^2 < 1$$

provided that $q_n^0 \neq q_n^1$. Letting $x \equiv q_n^0/q_n^1$, (120) is equivalent to

$$(130) \quad f(x) \equiv [lnx]^2 + lnx - x + 1 > 0.$$

Using calculus, we find that $f(x)$ attains a strict global minimum over $x > 0$ at $x = 1$ where $f(1) = 0$. Thus (130) is true except at the single point $x = 1$ and hence (120) is true if $q_n^0 > 0, q_n^1 > 0$ and $q_n^0 \neq q_n^1$.

Proof of A8: The counterpart to (119) that we need to show is true is

$$(131) \quad -p_n^0[1 - \partial L(q_n^0, q_n^1)/\partial q_n^0] < 0.$$

Assuming that $q_n^0 \neq q_n^1$, (131) is equivalent to

$$(132) \quad \partial L(q_n^0, q_n^1)/\partial q_n^0 = [ln(q_n^0/q_n^1) - 1 + (q_n^1/q_n^0)]/(ln(q_n^0/q_n^1))^2 < 1.$$

Letting $x = q_n^0/q_n^1$, (132) is equivalent to

$$(133) \quad g(x) \equiv [lnx]^2 - lnx + 1 - x^{-1} > 0.$$

Again using calculus, we find that $g(x)$ attains a strict global minimum over $x > 0$ at $x = 1$ where $g(1) = 0$. Thus (131) is true if $q_n^0 \neq q_n^1$ and A8 follows.

Note that the proof of the above Proposition shows that the derivatives of the logarithmic mean $L(a, b)$ with respect to a or b are between 0 and 1. It is this boundedness property that allows I^{GBL} to satisfy A7 and A8 whereas the unbounded nature of the derivatives of $M_r(a, b)$ causes I^{GBr} to fail A7 and A8 for $r \neq 1$.

Since the tests A18 and A19 are not very important, Proposition 13 tells us that the Logarithmic Mean Generalized Bennet indicator of price change I^{GBL} has almost the same excellent axiomatic properties as the Bennet indicator, I^B. The following Proposition shows that I^{GBL} is also a pseudosuperlative indicator of price change.

Proposition 14: The Logarithmic Mean Generalized Bennet indicator of price change $I^{GBL}(p^0, p^1, q^0, q^1)$ approximates the Bennet indicator $I^B(p^0, p^1, q^0, q^1)$ and the superlative indicators of price change to the second order at any point where $p^0 = p^1$ and $q^0 = q^1$.

Proof: Straightforward computations.

Recall that the Montgomery and Bennet indicators of price change for commodity n could be obtained by integrating under the 'supply' curve $s_n(p_n)$ using definition (23) where s_n was defined by (22) and (32) respectively. It turns out that the commodity n Logarithmic Mean Generalized Bennet indicator defined by (118) can be given a similar interpretation. Thus we define the following exponential 'supply' curve as follows:

$$(134) \quad s_n(p_n) \equiv \alpha_n e^{\beta_n p_n}; \quad n = 1, 2, \ldots, N$$

where the α_n and β_n are constants. If s_n is defined by (134), then it can be verified that

$$(135) \quad \int_{p_n^0}^{p_n^1} s_n(p)dp = I_n^{GBL}(p_n^0, p_n^1, q_n^0, q_n^1); \quad n = 1, \ldots, N.$$

where q_n^0 and q_n^1 are defined by

$$(136) \quad q_n^0 \equiv \alpha_n e^{\beta_n p_n^0}; q_n^1 \equiv \alpha_n e^{\beta_n p_n^1}.$$

Note that $q_n^1/q_n^0 = exp[\beta_n(p_n^1 - p_n^0)]$ and thus if $p_n^0 \neq p_n^1$, then β_n can be determined from the data as

$$(137) \quad \beta_n \equiv [lnq_n^1 - lnq_n^0]/[p_n^1 - p_n^0].$$

Once β_n has been determined, α_n can be determined using (136).

We conclude this Appendix by tabling a few of the Mean of Order r Generalized Bennet indicators (for $r = 0, 1/2, 1$ and 2) and the Logarithmic Mean Generalized Bennet indicator I^{GBL} using the Fisher [1922] data that we used earlier.

Table 2: Comparison of Mean of Order r and Logarithmic GB Indicator

Note that the average I^{GBr} increases as r increases. This result *must* hold if all of the period to period price increases are positive since the mean of order r quantity weights $M_r(q_n^0, q_n^1)$ increase as r increases (by the Schlömilch-Dunkel inequalities). Note also that the Logarithmic Mean Generalized Bennet indicators of price change are between the corresponding Mean of Order r indicators for $r = 0$ and $r = 1/2$. Again, if all price changes are positive, this result must hold since Diewert [1978; 898] showed that the logarithmic mean $L(a, b)$ was bounded from below by $M_0(a, b)$ and from above by $M_{1/3}(a, b)$.

We turn now to generalizations of the Montgomery indicator of price change.

Appendix 3: Generalized Montgomery Indicators of Price change.

Generalized Montgomery indicators of price change have the form

$$(138) \quad I^{GM}(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^N M(p_n^0 q_n^0, p_n^1 q_n^1) ln(p_n^1/p_n^0)$$

where M is a general homogeneous symmetric mean. Of course, Montgomery chose the general mean M to be the logarithmic mean L. In the present appendix, we will replace the general mean M which appears in (138) by the mean of order r defined by (104) and (105). Thus we define the *Mean of Order r Generalized Montgomery Indicator* of price change as

$$(139) \quad I^{GMr}(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^N I_n^{GMr}(p_n^0, p_n^1, q_n^0, q_n^1)$$

where the indicator of price change for commodity n, I_n^{GMr}, is defined as

$$(140) \quad I_n^{GMr}(p_n^0, p_n^1, q_n^0, q_n^1) \equiv M_r(p_n^0 q_n^0, p_n^1 q_n^1) ln(p_n^1/p_n^0).$$

What are the axiomatic properties of the indicator of price change I_n^{GMr}? The following Proposition answers this questions.

Proposition 15: $I_n^{GMr}(p_n^0, p_n^1, q_n^0, q_n^1)$ defined by (140) fails tests A3-A8, A17-A19 and at least one of tests A11 and A12. I_n^{GMR} passes the remaining tests listed in section 4 above.

Proof of A3: Note that if $p_n^1 \neq p_n^0$, then

$$(141) \quad I_n^{GMr}(p_n^0, p_n^1, q_n, q_n) = M_r(p_n^0, p_n^1) q_n ln(p_n^1/p_n^0) \neq (p_n^1 - p_n^0) q_n.$$

In order for equality to hold in (141), we would require that $M_r(p_n^0, p_n^1) = (p_n^1 - p_n^0)/ln(p_n^1/p_n^0)$; i.e., we would require that $M_r(p_n^0, p_n^1)$ equals the logarithmic mean, $L(p_n^0, p_n^1)$, which is not possible for any r.

Proof of A4: Note the A4 reduces to A3 when $q_n^0 - q_n^1$. Since A3 fails, so does A4.

Proof of A5-A8: Similar to the proof of A5-A8 in Proposition 3.

Proof of A11 and A12: From (141), when $q_n^0 = q_n^1$, we have

$$(142) \quad p_n^1 q_n^1 - p_n^0 q_n^0 - M_r(p_n^0, q_n^0, p_n^1 q_n^1) ln(p_n^1/p_n^0) \neq 0.$$

Hence by the continuity of M_r, for $q_n^0 \neq q_n^1$ but q_n^0 close to q_n^1, (142) will still hold. Hence one of the tests A11 and A12 will fail to hold.

Q.E.D.

Proposition 15 shows that the axiomatic properties of the Mean of Order r Generalized Montgomery indicators, $I^{GMr}(p^0, p^1, q^0, q^1)$, are not good. In particular, the failure of the quantity identity test A3 is devastating to the usefulness of these indexes in practice.

Nevertheless, from a numerical point of view, the indicator I^{GMr} are pseudosuperlative as the following Proposition shows.

Proposition 16: The Mean of Order r Generalized Montgomery indicators $I^{GMr}(p^0, p^1, q^0, q^1)$ approximate the Bennet indicator $I^B(p^0, p^1, q^0, q^1)$ and the superlative indicators of price change to the second order at any point where $p^0 = p^1$ and $q^0 = q^1$.

Proof: Routine Computations.

The results in Appendices 2 and 3 solidify the position of the Bennet indicator of price change I^B as being the best (against the competitors we have considered) from the axiomatic point of view. The Logarithmic Mean Generalized Bennet indicator I^{GBL} considered in Appendix 2 above is a close runner up in the axiomatic sweepstakes, failing only the rather unimportant tests A18 and A19. However, the geometric interpretation for the Bennet indicator I^B (recall figure 2 above), which rests on the assumption of partial equilibrium linear 'supply' functions (34), is more attractive than the nonlinear exponential 'supply' functions (134), which in turn provide a geometric interpretation for the logarithmic indicator I^{GBL}. Thus the Bennet indicator I^B appears to be a worthy 'ideal' indicator counterpart to the Fisher 'ideal' index P_F.

Footnotes

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1. Diewert [1993a; 43] [1993c, 24] mentions Bennet's approach and Balk [1996; 358] explains one of Montgomery's [1937] main results. However, there are some more recent papers that are related to our topic, including those of Vartia [1976a] [1976b], Törnqvist, Vartia and Vartia [1985] and Balk [1996].

2. See, for example, Diewert [1995; 20-23] and Hill [1996; 43]. Balk [1998] considered the axiomatic properties of a unit value price index.

3. Index number theorists, from the very beginning of the subject, have repeatedly emphasized that the construction of index numbers cannot be divorced from their ultimate purpose. Consider the following quotations:

> "The answer to the question what is the *Mean* of a given set of magnitudes cannot in general be found, unless there is given also the object for the sake of which a mean value is required." *F.Y. Edgeworth*[1988; 347]
>
> "What index numbers are 'best'? Naturally much depends on the purpose in view." *Irving Fisher*[1921;533]
>
> "In constructing an index number by which to measure the change of prices between two dates there are, in general, four chief problems:
> (1) To decide on the exact *field* of items to be covered, implying specifically:
> (a) A group of *prices* (p's) to be indexed and
> (b) A group of *quantities* (q's) to be used in forming weights expressing the relative importance of the respective items in the included field;
> (2) To select, from this whole field of items, the best *sample* to represent it;
> (3) To select the best *formula* to be used;
> (4) To select the best *base*, or system of bases . . .
>
> The first problem, that of delimiting the field, is one the solution of which depends eminently on the purpose in view." *Irving Fisher*[1927;419-420]

"Once the question has been refined, the development of a measure that would answer the question is a technical task. To be sure, the formulation of indexing objectives is by far the most difficult task, and it is not entirely an economic one." *Jack E. Triplett*[1983;476]

4. Miller [1984;146] describes the net income and productivity analysis system used by AT & T as an example of a firm that uses a profit change decomposition as a performance evaluation tool. The price change term is called "price recovery" and the quantity change term is called "productivity". Compensation committees of boards of directors should also find profit change decompositions useful when determining bonuses for management.

5. "The [cost] system should provide for showing daily, weekly or monthly, as desired, increases or decreases in cost:
Resulting from variations in the purchase price of material,
Resulting from variations in the efficiency of the use of material ...."
G. Chester Harrison[1918;275]

6. More recent contributors include Harberger [1971] and Diewert [1976a] [1992a].

7. Bennet was a civil servant in the Ministry of Finance in Cairo in 1920.

8. See Mensah [1982], Marcinko and Petri [1984] and Darrough [1986]. Barlev and Callen [1986] contrast the ratio approach to the measurement of efficiency change (the economist's traditional productivity measurement approach) with the difference approach (the accountant's variance analysis approach).

9. Bert Balk found this paper in the Statistics Netherlands library. It was privately printed and circulated by Montgomery who was at that time the Chief of Section in the Bureau of Economic Social Studies of the International Institute of Agriculture in Rome. Later, he expanded this paper into the book, Montgomery [1937], where he notes that he was a member of the Econometric Society and had an M.A. and M.Sc. Unfortunately, only Diewert [1993c; 24] seems to have referred to Montgomery [1929] and only Diewert [1993c; 24] and Balk [1995; 73] [1996;358] have referred to Montgomery [1937].

10. However, situations where a commodity changes from being an input in one period to an output in the other period cannot be modeled by Montgomery type indicators.

11. Cisbani [1938;24] refers to a family of generalized means for two variables introduced by L. Galvani in 1927 which includes the logarithmic mean. Dodd [1941;422] refers to the work of Cisbani and uses the term "logarithmic mean". For a generalization of the

logarithmic mean to N variables, see Pittenger [1985]. For more accessible discussions of the logarithmic mean, see Carlson [1972] or Lorenzen [1990].

12. $M(x_1, \ldots, x_N)$ is a homogeneous symmetric mean if M is a continuous, increasing, symmetric and (positively) homogeneous of degree one function over its domain of definition; see Diewert [1993b; 361–364].

13. Bennet [1920;456] used a diagram similar to Figure 2 except he had a negatively sloped demand curve join the points A and B.

14. Montgomery [1929;3] described this test as follows: "The formula which represents Vq_1 must be the same as the formula which represents Vp_1 with the factors p and q reversed." Montgomery's [1929;4] last major test was the following one: If $q_n^1/q_n^0 = (p_n^1/p_n^0)^a$, then $V_n(p_n^0, p_n^1, q_n^0, q_n^1)/I_n(p_n^0, p_n^1, q_n^0, q_n^1) = a$.

15. Actually this technique is due to Walsh [1921b;541-542] who discovered it when discussing Fisher [1921].

16. See Diewert [1992b].

17. Many index number theorists objected to Fisher's [1922] factor reversal test. Samuelson and Swamy [1974;575] commented on this test as follows: "A man and his wife should be properly matched; but that does not mean I should marry my identical twin!"

18. The consumer theory application that we are outlining in this section is isomorphic to the producer's problem of minimizing cost subject to a production function constraint. In this case, interpret $c(p)$ as the producer's unit cost function, q as a vector of inputs and $f(q)$ as output. A similar reinterpretation of the results of this section is possible for the case of a producer maximizing revenue subject to an aggregate input constraint. In this case, $c(p)$ is a unit input revenue function, q is a vector of outputs, min is replaced by max and f is an input requirements function. Finally, our results can also be reinterpreted in a profit maximization model. In this case, we assume that the producer's period t profit function has the form $\alpha^t \pi(p^t), t = 0, 1$ where α^t is interpreted as a period t efficiency factor. In this case, $c(p^t)$ is $\pi(p^t)$ and $f(q^t)$ is replaced by the efficiency factor α^t.

19. See Diewert [1976b;130-134].

20. Törnqvist [1936] does not have (67) explicitly, but Törnqvist and Törnqvist [1937;18] does have (67), where it is termed the "geometric ideal". This index first appeared as formula 123 in Fisher [1922;473]. Fisher [1922;265] listed it as one of his best 29 formulae. Walsh [1921a;97] used the geometric average of the weights s_n^0 and s_n^1 in place of the

arithmetic average in (67). Finally, Persons [1928;21-22] recommended (67) and eight other indexes as being "best" from the viewpoint of his test approach.

21. See Diewert [1976b;121].

22. Fisher [1922;247] was the first to use the term superlative to describe price indexes P which were numerically close to his ideal index P_F defined by (66). Diewert [1976b] [1992a; 578] exhibits several entire families of superlative indexes.

23. See Diewert [1978; 887-888] [1992a; 578].

24. Montgomery [1937;40-48] also established a version of Vartia's [1976a;134][1976b;124-125] results on the consistency in aggregation of the Vartia I price index.

25. We shall call an indicator pseudosuperlative if it has the same first and second partial derivatives as a superlative indicator around an equal price and quantity point. Thus the Bennet and Montgomery indicators, I^B and I^M, are pseudosuperlative. We note that the Paasche and Laspeyres indicators, I^P and I^L, only approximate superlative (and pseudosuperlative) indexes to the first order around an equal price and quantity point.

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