Seasonal Commodities, High Inflation and Index Number Theory

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Abstract

This paper studies the problems of measuring economic growth under conditions of high inflation. Traditional bilateral index number theory implicitly assumes that variations in the price of a commodity within a period can be ignored. In order to justify this assumption under conditions of high inflation, the accounting period must be shortened to a quarter, month or possibly a week. However, once the accounting period is less than a year, the problem of seasonal commodities is encountered; i.e., in some subannual periods, many seasonal commodities will be unavailable, and hence the usual bilateral index number theory cannot be applied. The present paper systematically reviews the problems of index number construction when there are seasonal commodities and high inflation. Various index number formulae are justified from the viewpoint of the economic approach to index number theory by making separability assumptions on consumers' intertemporal preferences. We find that accurate economic measurement under conditions of high inflation is very complex. Statistical Agencies should produce at least three different types of index: (i) year over year "monthly" price and quantity indexes; (ii) a short term "month to month" price index of nonseasonal commodities and (iii) annual Mudgett-Stone quantity indexes that use the short term price index in (ii) to deflate the seasonal prices. In section 8, it is shown how the annual Mudgett-Stone quantity indexes can be calculated for moving years as well as for calendar years. These moving year indexes can be centered and the centered indexes can serve as "monthly" seasonally adjusted indexes at annual rates. In section 9, this index number method of seasonal adjustment is compared with traditional time series methods of seasonal adjustment. The paper is also related to the accounting literature on adjusting for changes in the general price level.

Key Words: Aggregation of commodities, consumer theory, index numbers, inflation, seasonal adjustment, separability, time series.

JEL Classification Numbers: B23, C43, D11, D91, E31, M4

Seasonal Commodities, High Inflation and Index Number Theory

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1. Introduction

> "In a time of general price change, a ledger account may contain entries that each deal with £s of different worth. For instance, in a year when prices are rising the transactions of January are expressed in more valuable £s than are those of the following December. Thus, unlike units are added and subtracted; the totals and balances become 'mixed' and perhaps meaningless; it is almost as if we subtracted 2 horses from 9 oranges." *William T. Baxter*[1975; 51]

> "In the present case, the real problem is how to make the index reflect properly the seasonal variation in prices, taking into account the seasonal variation in consumption; it is not how to get the seasonal element out again once it has been adequately measured. In short, the important and difficult issue here is that of proper index measurement, an analytical as well as a practical problem, not the logically subordinate question of technical deseasonalization."
> *Victor Zarnowitz*[1961; 234]

Ever since the German hyperinflation of the twenties, accountants[1] have noted that high inflation causes normal historical cost accounting measures of income and wealth to become virtually useless. One way to restore credibility to business accounts is to deflate current values by appropriate price indexes. However, it is not widely recognized that the construction of price indexes themselves is not at all straightforward under conditions of high inflation–particularly when seasonal commodities are present. Recently, Hill [1995] has addressed some of these index number problems in the context of adapting the existing United Nations [1993] system of national accounts to situations where the economy is experiencing high inflation. This paper can be regarded as an extension of Hill's contributions, taking into account the existence of seasonal commodities.

Before describing the contents of the present paper in detail, we first address some preliminary questions.

What are seasonal commodities? They are commodities which are either (i) not available in the market place during certain seasons of the year or (ii) are available throughout the year but there are regular fluctuations in prices or quantities that are synchronized with the season or time of the year.[2]

What are the sources of seasonal fluctuations in prices or quantities? There are two main sources:[3] (i) climate and (ii) custom. In the first category, fluctuations in temperature, precipitation, wind and hours of daylight cause fluctuations in the demand for ice skates, fuel oil, umbrellas, snow tires, seasonal clothing, outdoor barbecues and electricity, for example. With respect to custom and convention as a cause of seasonal fluctuations, consider the following quotation:

> "Conventional seasons have many origins–ancient religious observances, folk customs, fashions, business practices, statute law ....
>
> Many of the conventional seasons have considerable effects on economic behaviour. We can count on active retail buying before Christmas, on the Thanksgiving demand for turkeys, on the first of July demand for fireworks, on the preparations for June weddings, on heavy dividend and interest payments at the beginning of each quarter, on an increase in bankruptcies in January, and so on. One of these conventional seasons is especially troublesome to statisticians, because it is movable. Easter may come as early as March 22nd or as late as April 25th. Seasonal variations in series affected by Easter buying are decidedly different in the March and the April years." *Wesley C. Mitchell*[1927; 237]

Granger [1978; 33] lists additional examples of customs or conventionally determined timing decisions that affect demands or supplies seasonally such as: the choice of year ends for tax or accounting purposes, the timing of school years and vacation periods and the choice of local and national holiday dates. There are many additional examples of custom as a source of seasonal fluctuations such as: the fall introduction of new car models and new television series, the frequency of salary payments, the length of the conventional work day, the starting time of the work day, the number of days per week a business chooses to stay open and the choice of billing date along with the number of days of trade credit that are normally granted for accounts receivable.

Climate or weather changes play some role in causing fluctuations in: (i) energy demands (fuel oil in winter and electricity for air conditioning in summer); (iii) recreational activities, both active (e.g., gardening, jogging or walking) and "passive" (e.g., reading, attending outdoor concerts or sporting events along with the associated gambling) and (iii) food consumption (consumption of fresh fruits, vegetables and beer increases in the summer).

As can be seen from the above examples, seasonal fluctuations are present in almost all sectors of most economies.

What are the implications of seasonality for index number theory? If we break the year up into M seasons (e.g., $M = 4$ if the season is a quarter, $M = 12$ if the season is a month or $M = 52$ if the season is a week), then the existence of type (i) seasonal commodities in the set of goods that we are aggregating over means that the dimension of the commodity space will not be constant. Under these conditions, it is impossible[4] to apply the usual bilateral index number theory to the problem of decomposing seasonal value aggregates into comparable price and quantity components.

Even if all commodities are available in all seasons, the existence of type (ii) seasonal commodities may mean that bilateral indexes that are exact for an underlying utility function cannot be justified using the usual economic approach to index number theory[5]. The economic approach assumes that the seasonal aggregator function is the same in each of the two seasons being compared. This is not a reasonable assumption if the seasonal aggregator function shifts across seasons due to changes in climate or custom. This suggests that type (ii) seasonal commodities should be further classified into types (ii) (a) and (ii) (b).

A seasonal commodity of type (ii) is defined to be of type (a) if its seasonal quantity fluctuations can be rationalized by utility maximizing behavior over a set of seasons where the prices fluctuate but the utility aggregator function remains unchanged over these seasons, and it is defined to be of type (ii) (b) if its quantity fluctuations cannot be rationalized by maximizing an unchanging utility function over the periods in question.[6] An example may be helpful in distinguishing types (a) and (b). As harvest conditions vary, the price of potatoes in my local supermarket varies and I purchase more potatoes if the price falls and less if it rises. On the other hand, the price of beer remains more or less constant throughout the year but my consumption increases (sometimes dramatically) during the summer. Thus weather shifts my seasonal demand functions for beer but not for potatoes so beer is a type (ii) (b) seasonal commodity and potatoes is a type (ii) (a) seasonal commodity. The usual economic approach to index number theory can be applied to type (ii) (a) seasonal commodities but not to type (ii) (b) where changes in climate or custom cause the seasonal aggregator functions to change.

The problem of index number construction when there are seasonal commodities has a long history in economics and statistics; e.g., see Flux [1921; 184-185], Crump [1924; 185], Bean and Stine [1924], Mudgett [1955], Stone [1956], Rothwell [1958], Zarnowitz [1961], Turvey [1979] and Balk [1980a], [1980b] [1980c] [1981]. However, what is missing from the above papers is a systematic exposition of what assumptions on the consumer's utility function[7] are required in order to justify a particular seasonal index number formula. Thus in the present paper, we will systematically list various separability assumptions[8] on intertemporal preferences that can be used to justify various seasonal index number formulae from the viewpoint of the economic approach to index number theory.[9]

We now set out the general model of consumer behavior that we will specialize in subsequent sections. Suppose that there are M seasons in the year and the Statistical Agency has collected price and quantity data on the consumer's purchases for $1 + T$ years.[10] Suppose further that the dimension of the commodity space in each season remains constant over the $T + 1$ years; i.e., season m has N_m commodities for $m = 1, \ldots, M$. Denote the vector of positive prices facing the consumer in season m of year t by $p^{tm} \equiv [p_1^{tm}, p_2^{tm}, \ldots, p_{N_m}^{tm}]$ and the vector of commodities consumed in season m of year t by $q^{tm} \equiv [q_1^{tm}, q_2^{tm}, \ldots, q_{N_m}^{tm}]$ for $t = 0, 1, \ldots, T$ and $m = 1, \ldots, M$. In many of our models of consumer behavior, it will prove convenient to have notation for the annual price and quantity vectors so define these annual vectors by:

$$p^t \equiv [p^{t1}, p^{t2}, \ldots, p^{tM}]; q^t \equiv [q^{t1}, q^{t2}, \ldots, q^{tM}]; \quad t = 0, 1, \ldots, T. \tag{1}$$

In order to apply the economic approach to index number theory, it is necessary to assume that the observed quantities of q_n^{tm} are a solution to an optimization problem involving the observed prices p_n^{tm}. In the present context, we follow the example of Fisher [1930], Hicks [1946; 121-126] and Pollak [1989; 72] and assume that the intertemporal quantity vector $[q^0, q^1, \ldots, q^T]$ is a solution to the following intertemporal utility maximization problem:

$$max_{x^0,x^1,\ldots,x^T}\{U(x^0, x^1, \ldots, x^T) : \Sigma_{t=0}^T \delta_t p^t \cdot x^t = W\} \tag{2}$$

where $x^t \equiv [x^{t1}, x^{t2}, \ldots, x^{tm}]$ and each seasonal quantity vector x^{tm} has the dimensionality of $q^{tm}, p^t \cdot x^t \equiv \Sigma_{m=1}^M p^{tm} \cdot x^{tm}$ and $p^{tm} \cdot x^{tm} \equiv \Sigma_{n=1}^{N_m} p_n^{tm} x_n^{tm}, U$ is the consumer's intertemporal preference function (assumed to be continuous and increasing in its arguments), $\delta_t > 0$ is an annual discount factor and "wealth" W is the consumer's current and expected future discounted "income" viewed from the perspective of the beginning of year 0. If the consumer can borrow and lend at a constant annual nominal interest rate r, then $\delta_0 \equiv 1$ and

$$\delta_t = 1/(1 + r)^t, \quad t = 1, 2, \ldots, T. \tag{3}$$

Since we are assuming that the observed intertemporal quantity vector $[q^0, q^1, \ldots, q^T]$ is a solution to (2), it must satisfy the intertemporal budget constraint in (2) and hence we can replace W by

$$W \equiv \Sigma_{t=0}^T \delta_t p^t \cdot q^t. \tag{4}$$

Our assumptions are no doubt unrealistic: the consumer is assumed to: (i) know future year t spot prices p^t; (ii) know his or her future income streams at the beginning of year 0; (iii) be able to freely borrow and lend between years at the same sequence of borrowing and lending rates and (iv) have unchanging tastes over the $T+1$ years. Based on these assumptions, the consumer at the beginning of year 0 chooses a sequence of annual consumption plans, $q^t, t = 0, 1, \ldots, T$, and sticks to them with no changes in preferences as the years pass.

However, if we want to make use of the economic approach to index numbers, strong assumptions on consumer (or producer) behavior are required. Some advantages of the economic approach are: (i) it allows for substitution on the part of consumers (or producers) in response to changes in the prices facing consumers (or producers); (ii) it provides a concrete framework which can be used to assess various operational alternatives that occur when a Statistical Agency actually constructs an index number[11] and (iii) the economic approach leads to definite recommendations about the choice of functional forms for index number formulae. These recommended functional forms can then be evaluated from other perspectives, such as the test approach. Thus the economic approach can be useful in narrowing down the choice of functional forms for index numbers

Having made our basic economic assumptions (namely that the observed sequence of annual quantity vectors $[q^0, q^1, \ldots, q^t]$ solves (2) with W defined by (4)), we can now make additional assumptions on the structure of the intertemporal utility function U that will allow us to justify various indexes involving seasonal commodities.

In section 2, we show how (2) can be specialized to yield the annual indexes first proposed by Mudgett [1955; 97] and Stone [1956; 74-75].

In section 3, we note that our Hicksian intertemporal utility maximization problem (2) needs to be modified when inflation is high. The problem is that the annual discount factors δ_t that appear in (2) and (4) do not provide an adequate approximation to the consumer's intertemporal problem when inflation is moderate or high between seasons: we need to introduce between season intra year discount rates as well.

In section 4, we show that when there are seasonal commodities, the use of annual sums of seasonal quantities and the corresponding annual unit values are unsatisfactory as annual price and quantity aggregates.

In section 5, we leave the problems involved in the construction of annual aggregates and turn our attention to the construction of seasonal aggregates. In this section, we consider the construction of year over year seasonal aggregates.

In section 6, we examine the consistency of the year over year seasonal aggregates of section 5 with the annual indexes of sections 2 and 3.

In section 7, we get into the heart of the seasonal aggregation problem and consider methods for obtaining valid season to season measures of price change when there are seasonal commodities.

In section 8, we consider how to extend the scope of the annual calendar year indexes of section 3 to "moving" year comparisons.

In section 9, we indicate how the moving year indexes of section 8 can be centered. These centered indexes provide an index number solution to the econometric or statistical problem of seasonal adjustment.

Section 10 concludes.

2. The Construction of Annual Indexes Under Conditions of Low Inflation

> "The basic index is a yearly index and as a price or quantity index is of the same sort as those about which books and pamphlets have been written in quantity over the years." *Bruce D. Mudgett*[1955; 97]

> "The existence of a regular seasonal pattern in prices which more or less repeats itself year after year suggests very strongly that the varieties of a commodity available at different seasons cannot be transformed into one another without cost and that, accordingly, in all cases where seasonal variations in price are significant, the varieties available at different times of the year should be treated, in principle, as separate commodities." *Richard Stone*[1956; 74-75]

In order to justify the Mudgett-Stone[12] approach to annual index numbers when there are seasonal commodities, we need to restrict the consumer's intertemporal utility function U as follows: there exist F and f such that

$$U(x^0, x^1, \ldots, x^T) = F[f(x^0), f(x^1), \ldots, f(x^T)] \tag{5}$$

where f is a linearly homogeneous, increasing and concave annual utility function[13] and F is an intertemporal utility function that is increasing and continuous in its $T+1$ annual utility arguments. Note that the annual utility function f is assumed to be unchanging over time.

If $q^0, q^1, \ldots, q^T$ solves (2) with W defined by (4) and U defined by (5), then it can be seen that q^t, the observed annual consumption vector for year t, is a solution to the following year t utility maximization problem:

$$max_{x^t}\{f(x^t) : p^t \cdot x^t = p^t \cdot q^t\} = f(q^t); \quad t = 0, 1, \ldots, T. \tag{6}$$

Now we are in a position to apply the theory of exact index numbers.[14] Assume that the bilateral quantity index $Q(p^s, p^t, q^s, q^t)$ is exact for the linearly homogeneous aggregator function f. Then we have

$$f(q^t)/f(q^s) = Q(p^s, p^t, q^s, q^t); \quad 0 \le s, t \le T. \tag{7}$$

As a concrete example of (7), suppose that the annual aggregator function f is $f(x) \equiv (x \cdot Ax)^{1/2}$ where A is a symmetric N^* by N^* matrix of constants satisfying certain regularity conditions. This functional form is flexible; i.e., it can provide a second order approximation to an arbitrary differentiable linearly homogeneous function. The quantity index that is exact for this functional form is the Fisher [1922] ideal quantity index Q_F defined by[15]

$$Q_F(p^s, p^t, q^s, q^t) \equiv [p^t \cdot q^t p^s \cdot q^t / p^t \cdot q^s p^s \cdot q^s]^{1/2}. \tag{8}$$

Since Q_F is exact for a flexible functional form, it is a superlative index.[16]

Given any bilateral quantity index Q, its associated price index P can be defined as follows using Fisher's [1911; 403] weak factor reversal test[17]:

$$P(p^s, p^t, q^s, q^t) \equiv p^t \cdot q^t / p^s \cdot q^s Q(p^s, p^t, q^s, q^t). \tag{9}$$

Given any linearly homogeneous, increasing and concave aggregator function f, its dual unit cost function can be defined for strictly positive prices $p >> 0_{N^*}$ as:

$$c(p) \equiv min_x\{p \cdot x : f(x) = 1\}. \tag{10}$$

When the utility function f is linearly homogeneous, the Konüs [1924] price index between periods s and t reduces to the ratio of the unit cost functions evaluated at the period s and t prices, $c(p^t)/c(p^s)$. If the bilateral quantity index Q is exact for f, then its companion bilateral price index P defined by (9) is exact for the unit cost function c dual to f; i.e., in addition to (7), we also have

$$c(p^t)/c(p^s) = P(p^s, p^t, q^s, q^t); 0 \leq s, t \leq T. \tag{11}$$

As a concrete example of (11), suppose that the annual aggregator function is the homogeneous quadratic aggregator $f(x) \equiv (x \cdot Ax)^{1/2}$ and that c is its unit cost dual function. Then we have (11) holding with $P = P_F$ where the superlative Fisher ideal price index P_F is defined by

$$P_F(p^s, p^t, q^s, q^t) \equiv [p^t \cdot q^t p^t \cdot q^s / p^s \cdot q^t p^s \cdot q^s]^{1/2}. \tag{12}$$

The above analysis seems to indicate that the construction of annual price and quantity indexes when there are seasonal commodities is straightforward: simply regard each "physical" commodity in each season as a separate economic commodity and apply ordinary index number theory to the enlarged annual commodity space. However, in the next section, we show that annual index number construction is not so straightforward when there is high or moderate inflation between seasons within the year.

3. The Construction of Annual Indexes Under Conditions of High Inflation

> "The critics of orthodox accounting point to several faults that appear when prices change. Historical figures of different dates then tend to be expressed in unlike units, i.e., in units of different purchasing power . . . .

To give better information to users, such figures ought to be made comparable. This can be done with an index; the arithmetic is simple and familiar enough. The main problem is what kind of index to choose."

William T. Baxter[1984; 38-39]

"When there is high inflation prices may be several times higher at the end of an accounting year than at the beginning. A price comparison between two time periods for a single product does not therefore simply involve the compilation of a price relative based on two individual price observations but a comparison between two different ranges of prices. The nature and significance of such comparisons needs to be clarified, especially as little attention is paid to them in the literature on index numbers." *Peter Hill*[1995; ch. 5, p. 1]

In the model presented in the previous section, there was a discount rate δ_t that made the level of prices in year t comparable to the level of prices in the base year, year 0. Under conditions of low inflation, this is an acceptable approximation to the consumer's intertemporal choice problem. However, when inflation is high, we can no longer neglect interseasonal interest rates.

Consider the budget constraint in (2). We now interpret δ_t as the discount factor that makes one dollar at the beginning of year t equivalent to one dollar at the beginning of year 0. From the beginning of year t to the middle of season m in year t, another discount factor is required, say ρ_{tm}, which will make a dollar at the beginning of year t equivalent to a dollar in the middle of season m. Thus the budget constraint in (2) must now be replaced by the following intertemporal budget constraint:

$$\Sigma_{t=0}^{T}\Sigma_{m=1}^{M}\delta_t\rho_{tm}p^{tm}\cdot x^{tm} = W \tag{13}$$

where p^{tm} and q^{tm} are the observed (spot) price and quantity vectors for year t and season m and x^{tm} is a year t, season m vector of decision variables. Similarly, definition (4) for "wealth" W is now replaced by the following definition:

$$W \equiv \Sigma_{t=0}^{T}\Sigma_{m=1}^{M}\delta_t\rho_{tm}p^{tm}\cdot q^{tm}. \tag{14}$$

Making assumption (5) again, we can now derive the following counterparts to equations (6):

$$\begin{aligned} &max_{x^{t1},\ldots,x^{tM}}\{f(x^{t1},\ldots,x^{tM}) : \Sigma_{m=1}^{M}\rho_{tm}p^{tm}\cdot x^{tm} = \Sigma_{m=1}^{M}\rho_{tm}p^{tm}\cdot q^{tm}\} \\ &\quad = f(q^{t1},\ldots,q^{tM}) \equiv f(q^t); \quad t = 0,1,\ldots,T \end{aligned} \tag{15}$$

where the annual year t observed quantity vector q^t is equal to $[q^{t1}, \ldots, q^{tM}]$ and q^{tm} is the year t, season m observed quantity vector.

Note that the seasonal discount factors ρ_{tm} appear in the constraints of the annual utility maximization problems (15). Define the vector of year t, season m discounted (to the beginning of year t) prices p^{tm*} as

$$p^{tm*} \equiv \rho_{tm} p^{tm}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M. \tag{16}$$

The constraints in (15) can now be written as $p^{t*} \cdot x^t = p^{t*} \cdot q^t$ where the year t discounted price vector is defined as $p^{t*} \equiv [p^{t1*}, p^{t2*}, \ldots, p^{tM*}]$. Now we can repeat the analysis in the previous section associated with equations (7) - (12): we need only replace the year t spot price vectors p^t by the year t discounted vectors p^{t*}. In particular, assuming that the bilateral index number formula Q is exact for the homogeneous aggregator function f and its dual unit cost function c, we have the following counterparts to (7) and (11):

$$f(q^t)/f(q^s) = Q(p^{s*}, p^{t*}, q^s, q^t); \quad 0 \leq s, t \leq T; \tag{17}$$

$$c(p^{t*})/c(p^{s*}) = P(p^{s*}, p^{t*}, q^s, q^t); \quad 0 \leq s, t \leq T \tag{18}$$

where P is the bilateral price index associated with the quantity index Q defined using the counterpart to (9) which replaces p^s and p^t by p^{s*} and p^{t*}.

Thus our solution to the problem of constructing annual index numbers when there are seasonal commodities and high inflation is to use the Mudgett-Stone annual indexes but the within the year inflation adjusted prices p^{tm*} defined by (16) are used in place of the year t season m spot prices p^{tm}.

To see why we must use inflation adjusted or discounted prices in our annual index number formulae P and Q, consider the situation when there is a hyperinflation in the economy and we are using the Fisher quantity index defined by (8). If the hyperinflation takes place only in season m of year t, then the Paasche part $p^t \cdot q^t / p^t \cdot q^s$ of the Fisher annual quantity index (8) will be approximately equal to $p^{tm} \cdot q^{tm} / p^{tm} \cdot q^{sm}$; i.e., only consumption in season m in year t, q^{tm}, and consumption in season m in year s, q^{sm}, will enter into the Paasche quantity comparison between *years* s and t, if spot prices p^{tm} are used in place of the discounted prices p^{tm*}. This is obviously an undesirable property.

Note that $\rho_{tm+1}/\rho_{tm} \equiv 1 + r_{tm}$ for $m = 1, 2, \ldots, M-1$ where r_{tm} is the average interest rate that consumers face when borrowing or lending money from the middle of season m to the middle of season $m+1$ in year t. If the general level of prices is expected to increase in season $m+1$ compared to season m, then the nominal interest rate r_{tm} can be expected to increase as well.[18] Thus if the discounted prices $\rho_{tm} p_n^{tm}$ are used in place of the nominal prices p_n^{tm} in an annual index number formula, the effects of high inflation in any season will be nullified by the discount rates ρ_{tm}.

The use of the seasonally discounted prices p^{t*} in (17) and (18) in place of the nominal prices p^t poses difficulties for economic statisticians. Not only must the Statistical Agency collect seasonal data on nominal prices and quantities, but data on season to season interest rates r_{tm} must also be collected in order to calculate the seasonal discount factors ρ_{tm}. In principle, the interest rate r_{tm} should be a weighted average of all interest rates that consumers face (both borrowing and lending rates) where the weights are proportional to the amounts of funds loaned out or borrowed by consumers during season m of year t. This is not a trivial task. Moreover, many statisticians will object to using discounted prices in constructing annual price and quantity indexes on the grounds that the Fisher [1930]–Hicks [1946] intertemporal consumer theory that (17) and (18) are based on is too "unrealistic" to be used by an official Statistical Agency. Thus we consider some alternatives to the use of interest rates as discount factors in forming the seasonally deflated prices p^{tm*} defined by (16).

The simplest alternative to the use of interest rates as discount factors is to use the price of a widely traded commodity as a discount factor. Thus if p_G^{tm} is the price of gold in season m of year t, then define the "gold standard" discount factors ρ_{tm}^G by

$$\rho_{tm}^G \equiv p_G^{t1}/p_G^{tm}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M. \tag{19}$$

Thus when $m = 1$, the first seasonal discount factor is $\rho_{t1}^G = 1$; when $m = 2$, the second seasonal discount factor is $\rho_{t2}^G = p_G^{t1}/p_G^{t2}$, the price of gold in season 1 of year t relative to the price of gold in season 2; etc. The gold deflated prices $p_n^{tm*} \equiv \rho_{tm}^G p_n^{tm}$ could be used as the normalized prices in (17) and (18).[19]

Another alternative to the use of interest rates in stabilizing or making comparable seasonal prices during periods of high inflation is to convert nominal prices into prices expressed in terms of a stable currency.[20] In this case, the "foreign currency" discount factors ρ_{tm}^E are defined by

$$\rho_{tm}^E \equiv e_{tm}/e_{t1}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M. \tag{20}$$

where e_{tm} is the average number of units of foreign currency required to buy 1 unit of domestic currency in season m of year t.

Instead of using the price of gold p_G^{tm} as the deflator in (19), we could use any of the commodity prices for year t and season m, provided that the commodity is traded during each season. Instead of deflating by a single commodity price, the general inflation between seasons might be better captured by using the price or cost of a basket of nonseasonal and type (ii) (a) seasonal commodities as the deflator. Thus divide up the season m year t price vector p^{tm} into the vectors $[\tilde{p}^{tm}, \hat{p}^{tm}]$ where $\tilde{p}^{tm} \equiv [\tilde{p}_1^{tm}, \tilde{p}_2^{tm}, \ldots, \tilde{p}_K^{tm}]$ and each of the K commodities represented in $\tilde{p}^{tm}$ is either a nonseasonal commodity or a type (ii) (a) seasonal commodity.[21] Let $b \equiv [b_1, b_2, \ldots, b_K]$ be a vector of "appropriate" commodity quantity weights. Then the year t season m price of this basket of goods is $\tilde{p}^{tm} \cdot b$ and we define the "commodity standard" discount factors by[22]

$$\rho^B_{tm} \equiv \tilde{p}^{tm} \cdot b / \tilde{p}^{tm} \cdot b; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M. \tag{21}$$

As a further refinement to (21), we could replace the fixed basket index $\tilde{p}^{tm} \cdot b$ by a general price index, $\tilde{P}(\tilde{p}^{t1}, \tilde{p}^{tm}, \tilde{q}^{t1}, \tilde{q}^{tm})$, which compares the prices of commodities (excluding type (i) and type (ii) (b) seasonal commodities) in season m of year t, $\tilde{p}^{tm}$, to the prices of the same commodities in the base period, season 1 of year t, $\tilde{p}^{t1}$. In this case, the discount factor for season m of year t becomes

$$\rho^P_{tm} \equiv 1/\tilde{P}(\tilde{p}^{t1}, \tilde{p}^{tm}, \tilde{q}^{t1}, \tilde{q}^{tm}); \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M. \tag{22}$$

We will pursue this final refinement in section 7 below.

Each of the choices for the seasonal discount factors ρ_{tm} represented by (19) - (22) has advantages and disadvantages. The chief disadvantage is that each choice seems somewhat arbitrary. However, each choice does have the advantage of curing the hyperinflation problem: using (19) to (22) in (16) will lead to sensible index number comparisons using (17).

If we make use of Fisher's [1896; 69] observation that nominal rates of interest are approximately equal to real rates plus the rate of inflation, it can be seen that the inflation rate choices that are imbedded in the discount factor choices (19) - (22) will be approximately equal to our interest rate choice for ρ_{tm} that we advocated originally, provided that the season to season real rates of return are small.

The important conclusion that we should draw from the analysis presented in this section is that when constructing annual quantity indexes in high inflation situations, seasonal prices *must* be deflated for general inflation that occurred from season to season throughout the year. If this deflation is not done, the quantities corresponding to high inflation seasons will receive undue weight in the annual quantity index.

We conclude this section by discussing the interpretation of the annual price index $P(p^{s*}, p^{t*}, q^s, q^t)$ in (18), where we are using the vectors of year t normalized price vectors p^{t*}. We assume that the price and quantity indexes P and Q that appear in (17) and (18) satisfy the weak factor reversal test (9) except that normalized prices p^{t*} are used in place of nominal prices p^t. Thus P and Q satisfy

$$\Sigma^M_{m=1} p^{tm*} \cdot q^{tm} / \Sigma^M_{m=1} p^{sm*} \cdot q^{sm} = P(p^{s*}, p^{t*}, q^s, q^t) Q(p^{s*}, p^{t*}, q^s, q^t); \tag{23}$$

i.e., using our original seasonal interest rate discount factors ρ_{tm}, (23) says that the discounted (to the beginning of year t) sum of year t seasonal values $\Sigma^M_{m=1} p^{tm*} \cdot q^{tm}$ divided by the discounted (to the beginning of year s) sum of year s seasonal values $\Sigma^M_{m=1} p^{sm*} \cdot q^{sm}$

is decomposed into the aggregate price change $P(p^{s*}, p^{t*}, q^s, q^t)$ times the aggregate quantity change $Q(p^{s*}, p^{t*}, q^s, q^t)$. Thus the price index $P(p^{s*}, p^{t*}, q^s, q^t)$ captures the change in discounted year t prices relative to discounted year s prices. The interpretation of $P(p^{s*}, p^{t*}, q^s, q^t)$ when the specific commodity discount factors defined by (19) - (22) are used is less clear. If we use the discount factors defined by (19), then the normalized prices in season 1 of each year t, p_n^{t1*}, are equal to the corresponding nominal season 1 year t prices p_n^{t1}, but the normalized prices for later seasons $m > 1$, p_n^{tm*}, are equal to the corresponding nominal prices, p_n^{tm}, divided by the year t, season m to 1, gold price relative, p_G^{tm}/p_G^{t1}. Thus $P(p^{s*}, p^{t*}, q^s, q^t)$ is basically a measure of price level change going from year s to t but the seasonal prices within each year for seasons $2, 3, \ldots, M$ are "stabilized" in terms of season 1 prices using the price of gold as the deflator of post season 1 prices. Hence this price index does not have a clear interpretation as a measure of the average level of nominal prices in year t relative to the average level of nominal prices in year s.

In the following section, we discuss the possible use of annual unit values as prices in the construction of annual price and quantity indexes.

4. Annual Unit Value Indexes Under Conditions of High Inflation

> "But, if, in fact, the automobile industry produces 10 million cars in 1 year and someone adds up the seasonally adjusted figures, and they total only 9 million cars, the public will think you are nutty." *Julius Shiskin*[1978; 101]

> "Similarly, goods or services provided at different times of the day or at different periods of the year must be treated as different qualities even if they are otherwise identical. For example, electricity or transport provided at peak times must be treated as being of higher quality than the same amount of electricity or transport provided at off-peak times .... The different prices or rates charged at peak and off-peak times provide measures of these differences in quality. Similarly, fruit and vegetables supplied out of season must be treated as higher qualities than the same fruit and vegetables in season which are cheaper to produce and of which consumers may be satiated." *Peter Hill*[1993; 397-398]

The reader may well feel that the annual index number model that we developed in the previous section was a bit too complex and that simpler alternatives should be considered. One such alternative is the following: instead of distinguishing commodities by season, simply add up consumption of each physically distinct commodity over the seasons and use these annual total consumptions as our quantities that would be inserted into an index number formula. The price corresponding to each such annual quantity would be the total annual value of expenditures on that physical commodity divided by the annual quantity–an annual unit value.

This is a perfectly reasonable proposal, particularly when we consider that at some stage of disaggregation, unit values must be used in order to aggregate up individual transactions, if we want to apply bilateral index number theory.[23] However, an important characteristic of a unit value is the time period over which it is calculated. As Fisher [1922; 318], Hicks [1946; 122] and Diewert [1995; 22] noted, the time period should be short enough so that individual variations of price within the period can be regarded as unimportant. In periods of rapid inflation or hyperinflation, nominal prices vary substantially between seasons[24] within a year and so the use of annual nominal unit values as annual prices cannot be justified from the viewpoint of the ideal Fisher-Hicks time period: seasonal values that correspond to high inflation seasons will be weighted too heavily in the annual unit value.

However, the above argument does not rule out the use of annual unit values, provided that nominal prices p_n^{tm} are replaced by the within the year inflation adjusted normalized prices p_n^{tm*} defined by (16) in the previous section, *and* provided that these normalized prices p_n^{tm*} are approximately constant across seasons m for each year t and each commodity n. This last proviso will not be satisfied if there are seasonal commodities.

The problem with the use of (normalized) annual unit values when there are seasonal commodities can be illustrated as follows.[25] Imagine two years, where in the second year, after transportation and storage improvements, a constant quantity of a seasonal fruit, say bananas, is consumed at a constant price. In the first year, the same total annual quantity of bananas is consumed mostly in one season at a price slightly lower than the second year constant price. In the other seasons of the first year, one banana is consumed at a very high price. The prices are such that the value of banana consumption is constant over the two years. In these circumstances, the unit value for bananas would be constant over the two years as would the corresponding total annual quantity index. However, most economists would feel under these circumstances, that the utility of banana consumption is much higher in the second year compared to the first year and an index number comparison ought to show this. The use of a Mudgett-Stone Fisher ideal quantity index would lead to a banana quantity index greater than 1 under the above conditions, (assuming that the seasonal real interest rates were all zero or small). Thus there will generally be real biases in using annual (normalized) unit value indexes if there are substantial seasonal fluctuations in quantities and (normalized) prices.

In order to compare more formally the use of annual unit value indexes using normalized prices with the Mudgett-Stone annual indexes in the previous section, we will make the simplifying assumption that there are no type (i) and no type (ii) (b) seasonal commodities. Thus the dimensionality of the commodity space is constant over each season so that $N_m = N$ for $m = 1, \ldots, M$ and we can aggregate commodities over seasons.

Define the year t annual quantity for commodity n as the sum over the year t season m quantities:

$$Q_n^t \equiv \Sigma_{m=1}^M q_n^{tm}; \quad n = 1, \ldots, N; \quad t = 0, 1, \ldots, T. \tag{24}$$

Using the inflation adjusted normalized prices p_n^{tm*}, an annual normalized value for commodity n in year t is defined as

$$V_n^{t*} \equiv \Sigma_{m=1}^M p_n^{tm*} q_n^{tm}; \quad n = 1, \ldots, N; \quad t = 0, 1, \ldots, T. \tag{25}$$

The normalized unit value for good n is defined as

$$P_n^{t*} \equiv V_n^{t*}/Q_n^t; \quad n = 1, \ldots, N; \quad t = 0, 1, \ldots, T. \tag{26}$$

Define the year t vector of normalized unit values as $P^{t*} \equiv [P_1^{t*}, \ldots, P_N^{t*}]$ and the year t vector of total quantities consumed as $Q^t \equiv [Q_1^t, \ldots, Q_N^t]$ for $t = 0, 1, \ldots, T$.

The annual price and quantity vectors P^{t*} and Q^t can be used in an index number formula to calculate annual quantity indexes between years s and t. We now want to justify the use of such a quantity index from the view point of economic theory. Recall the assumptions (5) on the intertemporal utility function, which we now make. An assumption on the annual aggregator function f (which appears in (5)) which appears to be necessary for total annual year t quantities $Q^t = \Sigma_{m=1}^M q^m$ to solve (15) is

$$f(x^1, x^2, \ldots, x^M) = g(\Sigma_{m=1}^M x^m) \tag{27}$$

where g is an increasing, concave and linearly homogeneous function of N variables.[26] However, to ensure that the observed year t seasonal quantity vectors $[q^{t1}, \ldots, q^{tM}]$ are solutions to (15) when f is defined by (27), we also require equality of the year t normalized price vectors; i.e., we require[27]

$$p^{t1*} = p^{t2*} = \ldots = p^{tM*}; \quad t = 0, 1, \ldots, T. \tag{28}$$

To see why this is so, rewrite (15) when f is defined by (27) as follows:

$$\begin{aligned} &max_{x^1,\ldots,x^M}\{g(\Sigma_{m=1}^M x^m) : \Sigma_{m=1}^M p^{tm*} \cdot x^m = \Sigma_{m=1}^M p^{tm*} \cdot q^{tm}\} \\ &= g(\Sigma_{m=1}^M q^{tm}), \quad t = 0, 1, \ldots, T. \end{aligned} \tag{29}$$

Now if (28) were not true for some t, then in the maximization problem (29) for year t, we would find that *all* of the seasonal purchases in year t for any commodity where unequal prices prevailed would have to be concentrated in the seasons where the lowest prices occurred, which would in general contradict the nature of the observed data.

Assuming that (27) and (28) are satisfied, we can apply exact index number theory and derive the following annual index number equalities:

$$g(Q^t)/g(Q^s) = Q^*(P^{s*}, P^{t*}, Q^s, Q^t); \quad 0 \leq s,\, t \leq T \tag{30}$$

for any index number formula Q^* that is exact for the annual aggregator function g. Thus we have provided an economic justification for the use of annual normalized unit values P^{t*} and total annual quantities Q^t in an index number formula.

Suppose that Q^* in (30) and Q in (17) are both Fisher ideal quantity indexes. Under what conditions will the annual unit value approach (which leads to (30) with $Q^* = Q_F^*$) give us the same *numerical* answer as the less restrictive Mudgett-Stone approach (which leads to (17) with $Q = Q_F$)?

Using definitions (24) - (26), it is easy to see that

$$p^{t*} \cdot q^t = \Sigma_{m=1}^{M} p^{tm*} \cdot q^{tm} = P^{t*} \cdot Q^t; \quad t = 0, 1, \ldots, T. \tag{31}$$

Hence it can be verified that a Fisher ideal index used in (17) will equal a Fisher ideal index used in (30); i.e.,

$$Q_F^*(P^{s*}, P^{t*}, Q^s, Q^t) = Q_F(p^{s*}, p^{t*}, q^s, q^t); \quad 0 \leq s,\, t \leq T, \tag{32}$$

if and only if[28]

$$P^{s*} \cdot Q^t = p^{s*} \cdot q^t \quad \text{for} \quad 0 \leq s,\, t \leq T. \tag{33}$$

A simple set of conditions that will ensure the equalities in (33) are the following Leontief [1936] type aggregation conditions:

$$q^{tm} = \alpha_t \beta_m \bar{q}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots M \tag{34}$$

where $\alpha_t > 0$ is a year t growth factor, $\beta_m > 0$ is a seasonal shift factor for season m and $\bar{q} \equiv [\bar{q}_1, \ldots, \bar{q}_N]$ is a fixed quantity vector. If the β_m form an increasing sequence, then they may be interpreted as "monthly" growth factors. If the β_m fluctuate and have mean 1, then the β_m can be interpreted as pure seasonal fluctuation factors; however, note that *all* commodities are subject to the *same* pattern of seasonal fluctuations.

We now verify that assumptions (34) imply the equalities (33). Using the definition of an inner product, we have for $0 \leq s,\, t \leq T$:

$$
\begin{aligned}
P^{s*} \cdot Q^t &= \Sigma_{n=1}^N P_n^{s*} Q_n^t \\
&= \Sigma_{n=1}^N [\Sigma_{m=1}^M p_n^{sm*} q_n^{sm} / \Sigma_{j=1}^M q_n^{sj}][\Sigma_{i=1}^M q_n^{ti}] \\
&\qquad\qquad \text{using definitions (24) - (26)} \\
&= \Sigma_{n=1}^N [\Sigma_{m=1}^M p_n^{sm*} \alpha_s \beta_m \bar{q}_n / \Sigma_{j=1}^M \alpha_s \beta_j \bar{q}_n][\Sigma_{i=1}^M \alpha_t \beta_i \bar{q}_n] \\
&\qquad\qquad \text{using (34)} \\
&= \Sigma_{n=1}^N \Sigma_{m=1}^M p_n^{sm*} \alpha_t \beta_m \bar{q}_n \\
&= \Sigma_{n=1}^N \Sigma_{m=1}^M p_n^{sm*} q_n^{tm} \\
&\qquad\qquad \text{using (34)} \\
&= p^{s*} \cdot q^t
\end{aligned}
$$

where the last equality follows from the definitions of the annual vectors p^{s*} and q^t.

Thus assumptions (34) do indeed imply the equality of the Fisher indexes in (32); unfortunately, assumptions (34) are not consistent with the simultaneous existence of both seasonal and nonseasonal commodities and assumptions (34) are not consistent with the existence of nonconstant "monthly" growth rates.

Another set of conditions that will ensure that the equalities in (33) hold are the following Hicks [1946; 312] aggregation conditions:

$$
p^{tm*} = \gamma_t \bar{p}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M \tag{35}
$$

where $\gamma_t > 0$ is a year t price level factor and $\bar{p} \equiv [\bar{p}_1, \ldots, \bar{p}_N]$ is a constant price vector.

We now verify that assumptions (35) imply the equalities (33). Using definitions (24) - (26) again, we have for $0 \leq s,\, t \leq T$:

$$
\begin{aligned}
P^{s*} \cdot Q^t &= \Sigma_{n=1}^N [\Sigma_{m=1}^M p_n^{sm*} q_n^{sm} / \Sigma_{j=1}^M q_n^{sj}][\Sigma_{i=1}^M q_n^{ti}] \\
&= \Sigma_{n=1}^N [\Sigma_{m=1}^M \gamma_s \bar{p}_n q_n^{sm} / \Sigma_{j=1}^M q_n^{sj}][\Sigma_{i=1}^M q_n^{ti}] \\
&\qquad\qquad \text{using (35)} \\
&= \Sigma_{n=1}^N [\gamma_s \bar{p}_n][\Sigma_{i=1}^M q_n^{ti}] \\
&= \Sigma_{n=1}^N \Sigma_{m=1}^M \gamma_s \bar{p}_n q_n^{tm} \\
&= \Sigma_{n=1}^N \Sigma_{m=1}^M p_n^{sm*} q_n^{tm} \\
&\qquad\qquad \text{using (35)} \\
&= p^{s*} \cdot q^t.
\end{aligned}
$$

Thus conditions (35) imply the equalities in (33) and (32). Note that conditions (35) are just a different way of writing our earlier conditions (28). These conditions are very restrictive: they require absolute equality of all discounted seasonal price vectors p^{tm*} within each year t. In particular, these conditions rule out seasonal fluctuations in prices.

The above analysis indicates that the existence of seasonal commodities will generally cause the annual unit value index numbers to differ (perhaps substantially) from the Mudgett-Stone annual indexes studied in the previous 2 sections. Since the assumptions on the underlying annual aggregator function needed to derive exact indexes are much less restrictive for the Mudgett-Stone indexes, we recommend the use of these indexes over the use of annual unit value indexes.

We turn now to the task of justifying the use of season specific year over year indexes.

5. Year Over Year Seasonal Indexes

> "In the daily market reports, and other statistical publications, we continually find comparisons between numbers referring to the week, month, or other parts of the year, and those for the corresponding parts of a previous year. The comparison is given in this way in order to avoid any variation due to the time of the year. And it is obvious to everyone that this precaution is necessary. Every branch of industry and commerce must be affected more or less by the revolution of the seasons, and we must allow for what is due to this cause before we can learn what is due to other causes." *W. Stanley Jevons*[1884; 3]

> "Each month the average price-change compared with the corresponding month of the preceding year is to be computed. The combination of the monthly variations into a mean-annual figure is a matter the precise detail of which remains open to further consideration, after careful examination of the data yielded by the monthly calculations." *A.W. Flux*[1921; 184-185]

> "There is of course no difficulty in measuring the average price change between the same months of successive years, if a month is our unit 'season', and if a constant seasonal market basket can be used, for traditional methods of price index construction can be applied in such comparisons."
> *Victor Zarnowitz*[1961; 266]

The separability assumptions on the annual aggregator function f which appears in (5) that are required to justify year over year seasonal indexes can be phrased as follows: there exists an increasing continuous function of M variables h and there exist functions f^m in N_m variables, $m = 1, \ldots, M$, such that

$$f(x^1, \ldots, x^M) = h[f^1(x^1), \ldots, f^M(x^M)] \qquad (36)$$

where the seasonal aggregator functions $f^m(x^m)$ are increasing, linearly homogeneous and concave in their arguments.

Assumption (36) says that the annual aggregator function f which appeared in sections 2 and 3 above now has a more restrictive functional form which aggregates the seasonal commodity vectors x^m in two stages. In the first stage, the commodities in season $m, x^m \equiv [x_1^m, x_2^m, \ldots, x_{N_m}^m]$, are aggregated by the season specific utility function $f^m(x^m) \equiv u_m$ and then the seasonal utilities u_m are aggregated together in the second stage by the function h to form annual utility, $u \equiv h(u_1, u_2, \ldots, u_M)$.

Making assumption (5) again and assuming that the consumer's intertemporal budget constraint is defined by (13) and (14), we can again derive equations (15). Substituting (36) into (15) and making use of the assumption that h is increasing in its arguments leads to the following equalities:

$$max_{x^m}\{f^m(x^m) : p^{tm} \cdot x^m = p^{tm} \cdot q^{tm}\} = f^m(q^{tm}); \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M. \tag{37}$$

Let the unit cost dual c^m to the seasonal aggregator function f^m be defined by:

$$c^m(p^m) \equiv min_{x^m}\{p^m \cdot x^m : f^m(x^m) = 1\}; \quad m = 1, \ldots, M. \tag{38}$$

Let P^m and Q^m be price and quantity indexes that are exact for the season m aggregator function f^m. Then under our optimizing assumptions, we have the following equalities, applying the usual theory of exact index numbers: for $0 \leq s,\ t \leq T$ and $m = 1, \ldots, M$, we have:

$$f^m(q^{tm})/f^m(q^{sm}) = Q^m(p^{sm}, p^{tm}, q^{sm}, q^{tm}); \tag{39}$$

$$c^m(p^{tm})/c^m(p^{sm}) = P^m(p^{sm}, p^{tm}, q^{sm}, q^{tm}). \tag{40}$$

Equation (39) says that the ratio of seasonal utility in season m of year t to seasonal utility in the same season m of year s is exactly equal to the quantity index $Q^m(p^{sm}, p^{tm}, q^{sm}, q^{tm})$ which is a function of the nominal price vectors for season m of years s and t, p^{sm} and p^{tm}, and the observed quantity vectors for season m of years s and t, q^{sm} and q^{tm}. If the seasonal aggregator functions are chosen to be the flexible homogeneous quadratic functions $f^m(x^m) \equiv [x^m \cdot A^m x^m]^{1/2}$, where A^m is a square symmetric matrix of constants for $m = 1, \ldots, M$, then the corresponding exact Q^m and P^m will be the superlative Fisher ideal indexes Q_F^m and P_F^m for $m = 1, \ldots, M$. Equation (40) tells us that the theoretical Konüs [1924] price index for season m between years s and t, $c^m(p^{tm})/c^m(p^{sm})$, is exactly equal to the price index $P^m(p^{sm}, p^{tm}, q^{sm}, q^{tm})$ which in turn will equal the Fisher ideal price index $P_F^m(p^{sm}, p^{tm}, q^{sm}, q^{tm})$ if f^m is the homogeneous quadratic aggregator function defined above.

Note that the nominal price vectors for season m in years s and t, p^{sm} and p^{tm}, appear in (40). Thus the index number on the right hand side of (40) is a valid indicator of the amount of nominal price change or inflation that has occurred going from season m of year s to the same season m in year t. Presumably, we have chosen the seasons to be short enough so that prices can be assumed to be approximately constant within each season and hence we have avoided the weighting problems that we encountered in the previous two sections when we were constructing annual indexes.

Summarizing the results of this section, we have shown how the separability assumption (36) justifies the use of the year over year seasonal price and quantity indexes that appeared in (39) and (40). These year over year seasonal indexes have been proposed by Flux [1921; 184], Zarnowitz [1961; 266] and many others[29] but explicit economic justifications for these indexes seem to be lacking.

In the following section, we ask whether the year over year seasonal indexes that were derived in this section, (39) and (40), can be used as building blocks in the construction of annual indexes.

6. The Consistency of the Year Over Year Seasonal Indexes With An Annual Index

> "It is a proposal for replacement of a practice, which can never be anything but incorrect, of using annual weights, say, for February in place of February weights, and of using, say, November prices for February when no price ever existed for February. To introduce these non-appropriate and non-existent data into an index number in order to satisfy a public demand for monthly indexes is to create something out of nothing; and that it introduces a spurious element into index-number construction can scarcely be denied."
>
> *Bruce D. Mudgett*[1955; 97-98]

> "Very often, an index number used in an economic model has been constructed in two or more stages. If the two stage procedure gives the same answer as a single stage procedure, then Vartia calls the index number formula 'consistent in aggregation'. Paasche and Laspeyres indexes have this consistency in aggregation property, but these index number formulae are consistent only with very restrictive functional forms for the underlying aggregator (i.e., utility or production) function. The present paper shows that the class of superlative index number formulae has an approximate consistency in aggregation property, where a superlative index number formula is one which is consistent with a flexible functional form for the underlying aggregator function."
>
> *W. Erwin Diewert*[1978; 883]

Recall the results of section 3 and specialize them so that the year s which appears in (17) and (18) is the base year, year 0. Let f be the linearly homogeneous, concave and increasing annual aggregator function which appears in (15) and (17) and let Q and P be exact for f. Then using (17) with $s = 0$, we have for $t = 0, 1, \ldots, T$:

$$f(q^t)/f(q^0) = f(q^{t1}, \ldots, q^{tM})/f(q^{01}, \ldots, q^{0M}) = Q(p^{0*}, p^{t*}, q^0, q^t). \tag{41}$$

Equations (41), along with a base period normalization for $f(q^0)$ such as $f(q^0) \equiv p^{0*} \cdot q^0$, can be used to compute the annual quantity aggregates $f(q^t)$ by utilizing the index number formula Q that is exact for f.

Now consider the model in the previous section where the annual aggregator function f had the more restrictive separable functional form defined by (36). How can the annual aggregates $f(q^t) = h[f^1(q^1), \ldots, f^M(q^M)]$ be computed exactly in this case?[30]

As in the previous section, assume that the seasonal aggregator functions $f^m(x^m)$ are linearly homogeneous, increasing and concave in their arguments and we now assume that h has the same mathematical properties. Assume that the f^m have exact index number formulae P^m and Q^m. We can again derive the equalities (39) and (40) but we can also derive the following counterparts to (39) and (40) (with $s = 0$)[31] where normalized prices p^{tm*} replace the nominal price vectors p^{tm}, for $t = 0, 1, \ldots, T$ and $m = 1, \ldots, M$:

$$f^m(q^{tm})/f^m(q^{0m}) = Q^m(p^{0m*}, p^{tm*}, q^{0m}, q^{tm}); \tag{42}$$

$$c^m(p^{tm*})/c^m(p^{0m*}) = P^m(p^{0m*}, p^{tm*}, q^{0m}, q^{tm}). \tag{43}$$

Choose units of measurement to measure base period seasonal utilities $f^m(q^{0m})$ as follows:

$$f^m(q^{0m}) \equiv p^{0m*} \cdot q^{0m} \equiv Q^0_m; \quad m = 1, \ldots, M; \tag{44}$$

i.e., we set utility in season m of year 0, $f^m(q^{0m})$ or Q^0_m, equal to base period expenditures in season $m, p^{0m} \cdot q^{0m}$, times the inflation conversion factor ρ_{0m} which converts the dollars of season m in year 0 to dollars at the beginning of year 0; (remember that $p^{0m*} = \rho_{0m} p^{0m}$). The normalizations (44) imply that base year seasonal unit costs, $c^m(p^{0m*})$, are all equal to unity; i.e., we also have

$$c^m(p^{0m*}) = 1 \equiv P^{0*}_m; \quad m = 1, \ldots, M. \tag{45}$$

Note that we have used equations (44) and (45) to define the base year seasonal quantity aggregates Q^0_m and (normalized) seasonal price levels P^{0*}_m for $m = 1, \ldots, M$. Now substitute (44) and (45) into (42) and (43) to obtain the following computable formulae for the year t seasonal price and quantity aggregates, $c^m(p^{tm*})$ and $f^m(q^t)$ for $t = 1, \ldots, T$ and $m = 1, \ldots, M$:

$$f^m(q^{tm}) = Q^m(p^{0m*}, p^{tm*}, q^{0m}, q^{tm}) p^{0m*} \cdot q^{0m} \equiv Q^t_m; \tag{46}$$

$$c^m(p^{tm*}) = P^m(p^{0m*}, p^{tm*}, q^{0m}, q^{tm}) \qquad\qquad \equiv P^{t*}_m. \tag{47}$$

Note that we have used equations (46) and (47) to define year t and season m seasonal price and quantity aggregates, P_m^t and Q_m^t.

Now consider the year t utility maximization problems (15) when f has the separable form (36): for $t = 0, 1, \ldots, T$:

$$\begin{aligned}
&max_{x^1,\ldots,x^M}\{h[f^1(x^1),\ldots,f^M(x^M)] : \Sigma_{m=1}^M p^{tm*}\cdot x^m = \Sigma_{m=1}^M p^{tm*}\cdot q^{tm}\} \\
=&max_{x^1,\ldots,x^M}\{h[f^1(x^1),\ldots,f^M(x^M)] : \Sigma_{m=1}^M c^m(p^{tm*})f^m(x^m) = \Sigma_{m=1}^M c^m(p^{tm*})f^m(q^{tm})\} \\
&\qquad\qquad \text{since maximization of utility implies cost minimization}^{32} \\
=&max_{Q_1,\ldots,Q_M}\{h[Q_1,\ldots,Q_m] : \Sigma_{m=1}^M P_m^{t*}Q_m = \Sigma_{m=1}^M P_m^{t*}Q_m^t\} \\
&\qquad\qquad \text{letting } Q_m \equiv f^m(x^m), Q_m^t \equiv f^m(q^{tm}) \text{ and } P_m^{t*} \equiv c^m(p^{tm*}) \\
=&h[Q_1^t,\ldots,Q_M^t].
\end{aligned} \tag{48}$$

The equalities in (48) follow from the assumption that the observed quantity data for year $t, q^t \equiv [q^{t1}, \ldots, q^{tM}]$, solve the year t utility maximization problem (15) when f has the separable structure (36) and the homogeneous seasonal aggregator functions f^m have the exact index number formulae $Q^m(p^{0m*}, p^{tm*}, q^{0m}, q^{tm})$ and $P^m(p^{0m*}, p^{tm*}, q^{0m}, q^{tm})$ that enabled us to construct the seasonal price and quantity aggregates P_m^{t*} and Q_m^t via (44) - (47).

Let the annual aggregate quantity index Q^a be exact for the linearly homogeneous aggregator function h. Then the equalities in (48) imply the following exact index number relationships for $t = 1, 2, \ldots, T$:

$$\begin{aligned}
&h[f^1(q^{t1}),\ldots,f^M(q^{tM})]/h[f^1(q^{01}),\ldots,f^M(q^{0M})] \\
&\quad = Q^a(P_1^{0*},\ldots,P_M^{0*}; P_1^{t*},\ldots,P_M^{t*}; Q_1^0,\ldots,Q_M^0; Q_1^t,\ldots,Q_M^t).
\end{aligned} \tag{49}$$

The index number formula on the right hand side of (49) is an example of a two stage aggregation formula. In the first stage, we use the year over year "monthly" indexes Q^m and P^m to form the "monthly" aggregate prices and quantities P_m^{t*} and Q_m^t using (44) - (47). In the second stage, the annual aggregate quantity index Q^a aggregates the "monthly" information using the right hand side of (49) to form an estimator for the ratio of real consumption in year t to real consumption in year 0.[33]

The two stage estimator of the annual consumption ratio defined by (49) can be compared with the single stage estimator defined by the right hand side of (41). In general, the two stage estimator (49) will not coincide with the one stage estimator (41). However, there are special cases of interest to Statistical Agencies where the two index number approaches will yield exactly the same answer: if all of the aggregator functions $f, f^1, \ldots, f^m$ and h are of the Leontief [1936] no substitution variety[34], then corresponding exact price and quantity indexes are (i) the Laspeyres price indexes $P_L, P_L^1, \ldots, P_L^M$ and P_L^a and the Paasche quantity indexes $Q_P, Q_P^1, \ldots, Q_P^M$ and Q_P^a and (ii) the Paasche price indexes $P_P, P_P^1, \ldots, P_P^M$ and P_P^a and the Laspeyres quantity indexes $Q_L, Q_L^1, \ldots, Q_L^M$ and Q_L^a. Thus if Paasche or Laspeyres indexes are used throughout, then the year over year seasonal indexes can be used as building blocks in a two stage procedure to construct an annual index, and the two stage procedure will give the same answer as the single stage procedure.

However, from the viewpoint of economic theory, the use of Paasche and Laspeyres indexes cannot be readily justified. The problem is that these indexes are exact only for Leontief aggregator functions which assume zero substitutability between all commodities, and this is simply not credible from the viewpoint of observed economic behavior.[35]

If we make the reasonable assumption that all of the homogeneous aggregator functions $f, f^1, \ldots, f^M$ and h can be closely approximated by homogeneous quadratic utility functions, then the corresponding exact index number formulae for Q, $Q^1, \ldots,$ $Q^M, P^1,$ $\ldots, P^M$ and Q^a are all (superlative) Fisher ideal indexes. In this case, the single stage annual aggregate quantity ratio defined by the right hand side of (41), $Q_F(p^{0*}, p^{t*}, q^0, q^t)$, will not be precisely equal to the corresponding two stage annual aggregate quantity ratio defined by the right hand side of (49) where Q^a is the Fisher ideal quantity index Q_F^a. However, Diewert [1978; 889], drawing on some results due to Vartia [1974] [1976], showed that, numerically, the right hand side of (41) would approximate the right hand side of (49) to the second order[36], provided that superlative index number formulae were used for all of the indexes $Q, Q^1, \ldots, Q^M, P^1, \ldots, P^M$ and Q^a. Some limited empirical evidence on the closeness of single stage superlative indexes to their two stage counterparts can be found in Diewert [1978; 895] where Canadian annual data on 13 categories of consumer expenditures were used and in Diewert [1983c; 1036-1040] where Turvey's [1979] artificial data on 5 seasonal commodities over 48 months were used. In the latter case, Diewert found a very close correspondence between the single stage and two stage procedures.[37]

If the single stage number differs considerably from the two stage number, which number should be used? If superlative indexes are being used in both procedures, then from the viewpoint of economic theory, the single stage number should be preferred, since the assumptions on the annual preference function f are the weakest using this procedure.

We turn now to the difficult problem of making index number comparisons between seasons within the same year when there are seasonal commodities.

7. Short Term Season To Season Indexes

> "No conventional price index formula can handle a situation in which the 'market basket' varies between two consecutive periods. This is the hard core of the seasonality problem. To make real sense economically, the solution of this problem must seek an approximation to constant-utility indexes through

the use of seasonal goods complexes that approach equivalence in the eyes of the representative consumer or producer." *Victor Zarnowitz*[1961; 246]

"La source de la difficulté est claire: c'est la prétention du statisticien de calculer un indice *mensuel* pour des articles dont la consommation obéit une périodicité *annuelle*."

Institut National de la Statistique et des Etudes Economiques[1976; 67]

Under conditions of high or low inflation, it is important to have reliable short term measures of it for many purposes: indexation, wage negotiations, calculation of real rates of return, etc. Thus we need to be able to compare the price level of the current season with the immediately preceding ones. The annual price indexes defined earlier in this paper are obviously not suitable for this purpose nor are the year over year seasonal indexes defined by (40) in section 5. These year over year price indexes $P^m(p^{sm}, p^{tm}, q^{sm}, q^{tm})$ are not comparable over seasons or "months" m since the commodity baskets change over the seasons due to the existence of type (i) seasonal commodities.

To make this lack of comparability problem a bit clearer, make the separability assumption (36) on the annual utility function f. Assume that the season m aggregator function f^m has the unit cost dual c^m, and P^m is an exact bilateral price index for f^m, for $m = 1, \ldots, M$. Then we can again derive (40). Setting $s = 0$, equations (40) become:

$$c^m(p^{tm})/c^m(p^{0m}) = P^m(p^{0m}, p^{tm}, q^{0m}, q^{tm}); \quad t = 1, \ldots, T; \quad m = 1, \ldots, M. \tag{50}$$

We can interpret $c^m(p^{tm})$ as the price or unit cost of one unit of season m subutility in year t, but there is no way of comparing these subutilities across seasons.[38] Thus equations (50) are of no help in obtaining comparable (across seasons) price indexes.

The thrust of the quotations by Mudgett [1955; 97-98], Zarnowitz [1961; 246] and the economic statisticians at INSEE [1976; 67] that appeared at the front of this section and the previous section is that the existence of type (i) seasonal goods makes it impossible to carry out normal bilateral index number comparisons between consecutive seasons.

A solution to this real problem of a lack of comparability is to make a different separability assumption on the intertemporal utility function U that was introduced in section 1 above. Recall the notation introduced in section 3 where we partitioned the season m year t price vector p^{tm} into the vectors $[\tilde{p}^{tm}, \hat{p}^{tm}]$ where the commodities represented in $\tilde{p}^{tm}$ were either nonseasonal commodities or type (ii) (a) seasonal commodities. Partition the year t season m quantity vectors in a similar manner; i.e., $q^{tm} \equiv [\tilde{q}^{tm}, \hat{q}^{tm}]$ and $x^{tm} \equiv [\tilde{x}^{tm}, \hat{x}^{tm}]$ for $t = 0, 1, \ldots, T$ and $m = 1, \ldots, M$. We now assume that the intertemporal utility function U has the following structure: there exists an increasing, continuous function G and an increasing, linearly homogeneous and concave function ϕ such that

$$\begin{aligned} &U(x^{01},\ldots,x^{0M};\ldots;x^{T1},\ldots,x^{TM}) \\ &\qquad = G[\phi(\tilde{x}^{01}),\hat{x}^{01},\ldots,\phi(\tilde{x}^{0M}),\hat{x}^{0M};\ldots;\phi(\tilde{x}^{T1}),\hat{x}^{T1},\ldots,\phi(\tilde{x}^{TM}),\hat{x}^{TM}]. \end{aligned} \tag{51}$$

The assumptions on the structure of intertemporal preferences represented by (51) are similar to the separability assumptions made by Pollak [1989; 77] to justify the usual annual indexes (recall sections 2 and 3 above). The only difference is that we now want to justify comparable "monthly" indexes and thus our "monthly" aggregator function ϕ must not include type (i) and type (ii) (b) seasonal goods, since these goods are simply not comparable across all seasons.[39]

Using our new notation for $p^{tm} \equiv [\tilde{p}^{tm},\hat{p}^{tm}]$, $x^{tm} \equiv [\tilde{x}^{tm},\hat{x}^{tm}]$ and $q^{tm} \equiv [\tilde{q}^{tm},\hat{q}^{tm}]$, we can rewrite the consumer's intertemporal budget constraint defined by (13) and (14) as follows:

$$\Sigma_{t=0}^{T}\Sigma_{m=1}^{M}\delta_t\rho_{tm}[\tilde{p}^{tm}\cdot\tilde{x}^{tm}+\hat{p}^{tm}\cdot\hat{x}^{tm}] = \Sigma_{t=0}^{T}\Sigma_{m=1}^{M}\delta_t\rho_{tm}[\tilde{p}^{tm}\cdot\tilde{q}^{tm}+\hat{p}^{tm}\cdot\hat{q}^{tm}]. \tag{52}$$

As usual, we assume that $[q^0,q^1,\ldots,q^T]$ solves the intertemporal utility maximization problem when U is defined by (51) and the budget constraint is defined by (52), where the year t observed quantity vector is $q^t \equiv [q^{t1},\ldots,q^{tm}]$ and the year t season m quantity vector is $q^{tm} \equiv [\tilde{q}^{tm},\hat{q}^{tm}]$. Using the assumptions that G and ϕ are increasing in their arguments, we can deduce that[40]

$$max_{\tilde{x}^{tm}}\{\phi(\tilde{x}^{tm}):\tilde{p}^{tm}\cdot\tilde{x}^{tm}=\tilde{p}^{tm}\cdot\tilde{q}^{tm}\}=\phi(\tilde{q}^{tm}); \quad t=0,1,\ldots,T; \quad m=1,\ldots,M. \tag{53}$$

Let $\gamma(\tilde{p}^{tm})$ be the unit cost function that is dual to the short run aggregator function ϕ. Assume that the bilateral price and quantity indexes $\tilde{P}$ and $\tilde{Q}$ are exact for the aggregator function ϕ. Then the equalities (53) imply the following equalities for $0 \le s,\, t \le T;\ m = 1,\ldots,M$ and $j=1,2,\ldots,M$:

$$\phi(\tilde{q}^{tm})/\phi(\tilde{q}^{sj}) = \tilde{Q}(\tilde{p}^{sj},\tilde{p}^{tm},\tilde{q}^{sj},\tilde{q}^{tm}); \tag{54}$$

$$\gamma(\tilde{p}^{tm})/\gamma(\tilde{p}^{sj}) = \tilde{P}(\tilde{p}^{sj},\tilde{p}^{tm},\tilde{q}^{sj},\tilde{q}^{tm}). \tag{55}$$

We normalize the theoretical "monthly" price level function $\gamma(\tilde{p}^{tm})$ so that the seasonal price level in season 1 of year 0 is unity; i.e., we place the following restriction on the unit cost function γ:

$$\gamma(\tilde{p}^{01}) = 1. \tag{56}$$

Equations (55) and the normalization (56) allow us to use the exact bilateral index number formula $\tilde{P}$ to provide estimates for the theoretical short term seasonal price levels $\gamma(\tilde{p}^{tm})$. The fixed base sequence of short term inflation estimates is

$$\begin{aligned} &1,\tilde{P}(\tilde{p}^{01},\tilde{p}^{02},\tilde{q}^{01},\tilde{q}^{02}),\ldots,\tilde{P}(\tilde{p}^{01},\tilde{p}^{0M},\tilde{q}^{01},\tilde{q}^{0M});\ldots;\tilde{P}(\tilde{p}^{01},\tilde{p}^{T1},\tilde{q}^{01},\tilde{q}^{T1}),\ldots, \\ &\qquad\qquad \tilde{P}(\tilde{p}^{01},\tilde{p}^{TM},\tilde{q}^{01},\tilde{q}^{TM}). \end{aligned} \tag{57}$$

Using the chain principle, the sequence of short run inflation estimates is

$$1, \tilde{P}(\tilde{p}^{01}, \tilde{p}^{02}, \tilde{q}^{01}, \tilde{q}^{02}), \tilde{P}(\tilde{p}^{01}, \tilde{p}^{02}, \tilde{q}^{01}, \tilde{q}^{02})\tilde{P}(\tilde{p}^{02}, \tilde{p}^{03}, \tilde{q}^{02}, \tilde{q}^{03}), \ldots \; . \tag{58}$$

The first two numbers in the chain sequence (58) coincide with the first two numbers in the fixed base sequence (57) but then the chain estimate for a given year t and month $m+1$ is equal to the chain estimate for the immediately preceding month m times the month to month bilateral link, $\tilde{P}(\tilde{p}^m, \tilde{p}^{m+1}, \tilde{q}^m, \tilde{q}^{m+1})$. There are other ways of utilizing the exact index number bilateral relationship defined by (55) to obtain estimates for the sequence of "month" to "month" theoretical price levels

$$\gamma(\tilde{p}^{01}), \gamma(\tilde{p}^{02}), \ldots, \gamma(\tilde{p}^{0M}); \ldots; \gamma(\tilde{p}^{T1}), \gamma(\tilde{p}^{T2}), \ldots, \gamma(\tilde{p}^{TM}); \tag{59}$$

but the fixed base and chain methods are the most practical ones.[41]

Should the fixed base sequence of price levels (57) or should the chain sequence (58) be used by Statistical Agencies to measure short term price change? The merits of fixed base versus chain index numbers have been debated by economists for a long time[42] but it seems appropriate to review some of the arguments.

One of the main arguments in favour of the chain system is that it is better adapted to solving the problems of disappearing goods and the appearance of new goods.[43] Recently, the problem of the proliferation of new goods has intensified due to the growth of knowledge and the increasing specialization of the world economy. Hence, the Statistical Agency, in making "month" to "month" index number comparisons, will be forced to use what Keynes [1930; 95] called the highest common factor method: the bilateral index number formula $\tilde{P}$ is applied only to the subset of commodities that are transacted in both periods.[44] If the chain system is used, then the subset of commodities transacted will be larger than the subset obtained using the fixed base system. Hence the chain comparisons will be more reliable than the fixed base comparisons.

Another argument that favours the use of the chain system is that most "reasonable" index number formulae will more closely approximate each other if the chain system is used rather than the fixed base system, since period to period price and quantity changes are likely to be smaller over adjacent periods compared to widely separated periods. Thus in particular, Diewert [1978; 895] and Hill [1988; 143] [1993; 387-388] noted that chaining will tend to reduce the spread between the Paasche and Laspeyres indexes and hence the use of either a chained Paasche or Laspeyres index should more closely approximate a superlative index like the Fisher ideal and hence approximate the underlying economic index more closely.[45]

However, Hill [1988; 136-137], drawing on some analysis by Szulc [1983; 548] on the "bouncing" phenomenon, also made an argument that favours the use of the fixed base system: the chain system should *not* be used if prices and quantities have a tendency to oscillate in a regular fashion and hence should not be used to aggregate seasonal data. To see why the chain principle can give poor results with regularly oscillating seasonal data, consider a situation where the price and quantity data of quarter 1 in year 0 coincides with the price and quantity data of quarter 1 in year $t > 1$. Then if the price index $\tilde{P}$ satisfies the identity test, the fixed base price level for quarter 1 of year t will be $\tilde{P}(\tilde{p}^{01}, \tilde{p}^{t1}, \tilde{q}^{01}, \tilde{q}^{t1}) = \tilde{P}(\tilde{p}^{01}, \tilde{p}^{01}, \tilde{q}^{01}, \tilde{q}^{t1}) = 1$, the correct answer, whereas the chain index will not in general give the correct answer.[46] However, in the present context, this criticism of the chain system loses all or most of its force, since we are excluding most seasonal commodities in the index number formula $\tilde{P}$.[47] Furthermore, if we take our index number formula for $\tilde{P}$ to be the Fisher ideal price index $\tilde{P}_F$, then in most cases, we will find that the chain principle will give satisfactory results even if there are type (ii) (a) seasonal goods included in the list of goods that $\tilde{P}_F$ operates on.[48]

Although our focus in this section is on measuring short term price change using the bilateral price index $\tilde{P}$, we can also use the companion quantity index $\tilde{Q}$ to measure short term quantity change for nonseasonal quantities. Furthermore, the exact index number relations (54), along with a base period normalization such as

$$\phi(\tilde{q}^{01}) = \tilde{p}^{01} \cdot \tilde{q}^{01} \tag{60}$$

which sets season 1 utility in the base year 0 equal to expenditure on nonseasonal goods $\tilde{p}^{01} \cdot \tilde{q}^{01}$, can be used to form estimates for annual sums of seasonal utilities. If we define year t aggregate utility by $\Sigma_{m=1}^{M} \phi(\tilde{q}^{tm})$, then using the fixed base principle, this theoretical real quantity aggregate can be estimated in units of season 1 year 0 constant dollars by

$$[\Sigma_{m=1}^{M} \tilde{Q}(\tilde{p}^{01}, \tilde{p}^{tm}, \tilde{q}^{01}, \tilde{q}^{tm})]\tilde{p}^{01} \cdot \tilde{q}^{01}. \tag{61}$$

We leave to the reader the task of working out chain system or multilateral estimates for the year t utility aggregate. However, annual quantity estimates of the form (61) will be of limited interest due to the exclusion of type (i) and type (ii) (b) seasonal goods in the bilateral quantity index $\tilde{Q}$. To obtain comprehensive annual quantity estimates that include all seasonal goods, it will be necessary to use the Mudgett-Stone indexes described in section 2 (if there is low inflation) or section 3 (if there is high or moderate inflation).

Our discussion of the problems involved in constructing measures of short term changes in consumer prices can be summarized as follows: (1) a "month" to "month" Fisher ideal chain index of nonseasonal commodities (and of type (ii) (a) seasonal commodities) is our preferred alternative; see (58) with $\tilde{P} \equiv \tilde{P}_F$; (2) if quantity information is not available in a timely manner, fixed base Laspeyres price indexes will have to be used; i.e., (57) will have to be used with $\tilde{P} \equiv \tilde{P}_L$. However, the base period should be changed as frequently as possible, say within every 5 years at least.

Some seasonal bilateral index number procedures that work over commodity spaces of varying dimensions have been proposed by Diewert [1980; 506-508] and Balk [1980a; 27]

[1981]. We shall now review these proposals and compare them to our preferred proposal (58), which depended on the separability assumptions (51).

Diewert [1980; 507] attempted to deal with the problem of disappearing and then reappearing seasonal goods by utilizing Hicks' [1940; 114][49] treatment of new goods in conventional index number theory: in seasons when a good is unavailable, determine the reservation price that would just ration the consumer's demand for the good down to zero. These reservation prices, along with the associated zero quantities, could then be used as prices and quantities that could be inserted into a bilateral season to season index number formula. There are two problems with this proposal: (1) Statistical Agencies do not have the resources required to estimate econometrically or statistically these reservation prices[50] and (2) even if appropriate reservation prices could be estimated, the assumptions required to justify the economic approach would not generally be satisfied. With respect to this second problem, recall our earlier classification of type (ii) seasonal goods (price and quantity data are available in each season) into types (ii) (a) and (ii) (b). Once we have estimated reservation prices for type (i) seasonal goods, we have essentially converted them into type (ii) seasonal goods; i.e., we have prices and quantities for them in each season. Hence it can be seen that we have the same problem that we had with type (ii) seasonal commodities–some type (i) seasonal commodities can have their prices and quantities rationalized by maximizing an underlying utility aggregator function over the seasons (call these type (i) (a) seasonal commodities) and some cannot, because custom shifts the aggregator function over the seasons (call these type (i) (b) seasonal commodities). Thus to rigorously justify Diewert's [1980; 507] earlier economic approach to the construction of season to season indexes, we have to rule out type (i) (b) and type (ii) (b) seasonal commodities, or simply restrict the index number comparisons to type (i) (a) and type (ii) (a) seasonal commodities. But this latter case is essentially the case that we analyzed in this section, except that we now add type (i) (a) commodities to our list of K type (a) seasonal commodities (and we have to provide reservation prices for the type (i) (a) commodities).

Balk's [1980a; 27] [1981] proposal for dealing with type (i) seasonal commodities makes use of the Vartia II [1974; 70] [1976] price index[51] so it is necessary to define this index. To do this, first define the logarithmic mean[52] of two positive numbers, x and y, by

$$L(x,y) \equiv \begin{cases} [x-y]/[lnx - lny] & \text{if} \quad x \neq y \\ x & \text{if} \quad x = y. \end{cases} \tag{62}$$

Balk [1981; 73] observed that definition (62) could be extended to the case where one of the numbers x or y is zero and the other is positive. In this case, $L(x,y) = 0$. To define the Vartia II price index, let $p^t \equiv [p_1^t, \ldots, p_N^t]$ and $q^t \equiv [q_1^t, \ldots, q_N^t]$ be two generic price and quantity vectors pertaining to periods $t = 0, 1$. Define the period t expenditure share on commodity n by

$$w_n^t \equiv p_n^t q_n^t / p^t \cdot q^t; \quad t = 0, 1; \quad n = 1, \ldots, N. \tag{63}$$

Define the logarithmic mean average share for commodity n between periods 0 and 1 by

$$w_n^{01} \equiv \begin{cases} L(w_n^0, w_n^1) & \text{if at least one of } w_n^0, w_n^1 \text{ is positive} \\ 0 & \text{if both } w_n^0 \text{ and } w_n^1 \text{ are } 0 \end{cases} \tag{64}$$

Finally, define the Sato [1976a; 224]–Vartia [1974; 70] price index P_{SV} by

$$ln P_{SV}(p^0, p^1, q^0, q^1) \equiv \Sigma_{n=1}^N w_n^{01} ln(p_n^1/p_n^0) / \Sigma_{n=1}^N w_n^{01}. \tag{65}$$

We have added Sato's name to the price index P_{SV} defined by (65) because he showed that P_{SV} is exact for a constant elasticity of substitution (CES) aggregator function.[53]

Balk [1995b] and Reinsdorf and Dorfman [1995] have studied the axiomatic properties of the Sato-Vartia price index and their conclusion is that it almost rivals the Fisher ideal price index in satisfying a priori desirable tests.[54] Furthermore, the fact that the Sato-Vartia index is exact for CES functional forms has proved to be very useful in many empirical applications; e.g., see Feenstra [1994]. However, it should be pointed out that the Sato-Vartia price index P_{SV} defined by (65) is *not* superlative; i.e., it is not exact for an aggregator function that can provide a second order approximation to an arbitrary twice differentiable linearly homogeneous function when the number of commodities N exceeds 2.[55]

We now return to Balk's [1980a; 27] [1981] proposal for dealing with seasonal commodities in the context of bilateral index number theory. His proposed method works as follows: when comparing type (i) commodities between two seasons when both are absent from the marketplace, drop the commodity from the index number computation; for all other cases where the commodity is present in one or both periods, use the Sato-Vartia price index. This procedure will set the weight of the commodity equal to zero in the index number formula if it is not present in both periods. Obviously, another way of describing Balk's proposed method is: use what Keynes [1930; 94] called the highest common factor method (use the prices and quantities in the bilateral index number formula only if they are present in *both* periods) and use the Sato-Vartia price index as the index number formula.

Balk's suggested approach[56] to the treatment of type (i) seasonal commodities is perfectly satisfactory from the viewpoint of the test approach to index number theory but there are two problems with his proposal from the viewpoint of the economic approach: (1) since the Sato-Vartia index is not superlative, it would be better to apply the highest common factor method but use a superlative price index in place of the Sato-Vartia index[57] and (2) Balk's procedure ignores the existence of type (i) (b) and (ii) (b) seasonal commodities. The prices and quantities corresponding to these type (b) seasonal commodities cannot be rationalized by utility maximizing behavior where the utility function remains constant over the two periods in question. Put another way, custom shifts the demand for type (b) seasonal commodities as we move from season to season and hence the economic approach fails for these commodities. The above second criticism of Balk's proposed procedure is the same as our earlier second criticism of Diewert's [1980; 507] economic approach to seasonal indexes and the cure to this problem is the same: simply restrict the season to season index number comparisons to nonseasonal commodities and type (i) (a) and (ii) (a) seasonal commodities.

There is one final problem that needs to be addressed in our discussion of the short term inflation index and that is the domain of definition of the index; i.e., over what set of commodities and what set of transactions should the index be defined?

Fortunately, Hill [1995] has recently addressed this question. His extensive discussion of the issues can be summarized by the following quotations:

> "A general index of inflation is needed for a variety of purposes. In the SNA it is used to calculate the following: neutral and real holding gains and losses, internal and external trading gains and losses, real national and disposable income, real interest and constant intra-period price level (CPL) accounts. In business accounting it may be used for similar purposes, such as Current Purchasing Power accounting. A general price index is needed for policy purposes to monitor the general rate of inflation and to set inflation targets. It may also be used to implement indexation agreements under conditions of high or chronic inflation ....
>
> "In general, the most suitable multi-purpose general price indices seem to be those for total final uses or for total domestic final uses. Whatever index is preferred, however, it must be stressed that there remains a need for a range of other price indices to meet more specific analytic and policy purposes. A general index of inflation should not drive out other indices."
>
> *Peter Hill*[1995; ch. 1; 4-5]

In order to use economic theory to help us choose an appropriate domain of definition for the short term index of general inflation, let us partition the economy into two sets of economic agents: (1) households, governments and the rest of the world (ROW); (2) private and public domestic producers. We also distinguish two classes of goods: (1) outputs produced by domestic producers: consumption C, government final demands G, gross investment expenditures I and exports X; (2) primary inputs utilized by domestic producers: imports M,[58] labour inputs L, capital inputs K (which include depreciation) and natural resource and land inputs R. Putting the two classifications together, the economy can be approximately[59] described as follows:

	Households, Governments, ROW	*Domestic Producers*
Output Markets:	(i) C + G + I + X	(ii) Outputs less Intermediates
Input Markets:	(iv) M + L + K + R	(iii) Primary input Utilization

The transactions in (i), $C + G + I + X$, represent total final expenditures. By the usual national income accounting arguments (i.e., demands equal supplies and each economic agent balances his or her budget during the accounting period), the flows in (i) can be estimated by the flows in (ii), (iii), or (iv). Each of the equivalent value changes represented by (i) - (iv) could be decomposed into broad measures of price and quantity change. Before we examine each of these 4 alternatives in detail, it is necessary to explain why we classified imports as a primary input rather than as a negative export. The reason is that we want our measures of price change to be as stable as possible; if we treat imports as negative exports, the resulting price will tend to fluctuate more violently than an index which excludes these negative exports.[60] Put another way, if we were to include imports as a negative output in (i), then why not include L, K and R also as negative outputs? This could be done in a consistent manner, but most experts would agree that the resulting net price index would not represent inflation; (in fact, it would approximate the negative of a productivity index, which is not without interest).

We now consider the deflation of the flows represented by (i) - (iv) above from the viewpoint of the economic theory of index numbers.

Consider the household income flows in (iv). It is possible to develop an income price index that is analogous to the Konüs [1924] cost of living index that is used to deflate consumption expenditures; see the constant utility income deflator concept in Diewert and Bossons [1987]. However, since this concept is totally unfamiliar to economic statisticians and the general public, we can dismiss it as a practical alternative at the present time.

Now consider the primary input expenditure flows in (iii), and restrict attention to the private (for profit) portion of these expenditures (i.e., exclude general government expenditures on primary inputs). The economic theory of the input price index is well developed and could in theory be applied to these expenditures.[61] However, at present, Statistical Agencies are unable to calculate these theoretical input price indexes for a number of reasons: (1) it is difficult to construct meaningful classifications for the thousands of types of labour in a modern economy and in any case, accurate decompositions of labour expenditures into price and quantity components simply do not exist in most countries; (2) national income statisticians have refused to construct rental prices or user costs for capital and land inputs for a variety of reasons[62] and hence appropriate input prices for the capital components of input cost are lacking, and (3) accurate estimates for resource depletion effects are also largely absent from the archives of Statistical Agencies. Thus we can rule out the deflation of expenditures in (iii).

Now consider the expenditure flows in (ii) but again exclude the public provision of goods and services. The economic theory of the output price index is well developed[63] and could in theory be applied to these net revenues. However, again data limitations will make the aggregate output price index rather unreliable: for most sectors of the economy, an accurate knowledge of intermediate input flows is lacking.

This leaves us with total final expenditure flows (i), $C + G + I + X$. However, from the viewpoint of measuring the impact of inflation on domestic final demanders, we should exclude exports which belong to the rest of the world. This leaves us with $C+G+I$, which Hill [1995] calls total gross domestic final expenditures. However, the prices of investment goods are not relevant to the deflation of current period household expenditures on goods and services that they consume in the current period[64] and hence gross investment expenditures can be deleted.[65] This leaves us with $C+G$. We cannot readily justify the deletion of government final expenditures on goods and services since many government goods (i.e., subsidized housing and transportation) are direct substitutes for privately provided consumer goods and other government outputs (i.e., garbage collection, road maintenance and protection services) certainly provide a flow of current period services to consumers. The problem with including government expenditures in the index number formula is that it is usually difficult to obtain meaningful prices for deflating these expenditures.[66] Thus in the end, for a variety of reasons, we end up with the consumer price index as being perhaps the best indicator of short run inflation in the economy. In view of our earlier discussion, this season to season short run consumer price index should exclude type (b) seasonal goods, since these goods are not comparable across seasons using the economic approach to index number theory.

To conclude this section, we note that many consumer goods are durable; i.e., they provide services beyond the initial season of purchase. Hence from the viewpoint of the economic approach to the short term consumer price index, seasonal rental prices or user costs should be used as the prices for durable consumer goods and the quantity weights should reflect not only the purchases made during the season but also the available stocks of consumer durables.[67] Note that as inflation increases, the “season” will generally have to shrink (so that within season price variation can be neglected) and thus the number of consumer durables will increase (and more user costs will have to be constructed).

In the following section, we return to the annual calendar year indexes defined in sections 2 and 3 above, but we no longer restrict ourselves to calendar years.

8. Moving Year Annual Indexes

"The year is either a calendar or a 'moving' year".

Horst Mendershausen[1937; 245]

"To make further progress, we next observe that there is no reason why we should always compare the base year's January-to-December with the current year's January-to-December observations. Why not compare February 1970 through December 1970 plus January 1971 with February 1971 through December 1971 plus January 1971? If we continue on in this manner, the resulting annual index number series will be just as consistent with economic theory as the usual calendar year comparisons that we have just tabled".

W. Erwin Diewert[1983c; 1028]

In sections 2 and 3 above, we assumed that there were M seasons in the year and the index number comparisons always compared the M seasons in one calendar year with the M seasons in another calendar year. However, the fact that the last month in a year is December is a completely arbitrary way of deciding which month should end the year. Thus we could choose any month (or season) as our year ending month and the prices and quantities of this new noncalendar year could be compared to the prices and quantities of a similar earlier noncalendar year. However, note that the separability assumptions required to justify these new noncalendar year comparisons will be analogous to our earlier separability assumptions (5) that justified calendar year comparisons, but the new assumptions will be slightly different: the annual aggregator function f will now be defined over the seasonal commodity vectors that correspond to a noncalendar year. What are restrictions on intertemporal preferences that will simultaneously justify both calendar year and noncalendar year comparisons? These noncalendar year comparisons can be taken a step further: we could think about comparing the prices and quantities of a *noncalendar* year with the prices and quantities of a base *calendar* year. What are the restrictions on intertemporal preferences that would justify this type of comparison, which we will call a variable year end comparison or a moving year[68] comparison? We provide an answer to both of these questions below.

Recall the seasonal aggregator functions $f^1(x^1), \ldots, f^M(x^M)$ that occurred in sections 5 and 6 above. In those sections, we assumed the existence of an aggregator function h that allowed us to define the annual utility function $f(x^1, \ldots, x^M) \equiv h[f^1(x^1), \ldots, f^M(x^M)]$. In the present section, we again assume the existence of the linearly homogeneous, increasing and concave seasonal aggregator functions $f^1, \ldots, f^M$ but we now make the following stronger assumptions on the structure of the intertemporal utility function U:

$$U(x^{01}, \ldots, x^{0M}; x^{11}, \ldots, x^{1M}; \ldots; x^{T1}, \ldots, x^{TM}) \equiv \psi^{-1}\{\Sigma_{t=0}^{T}\Sigma_{m=1}^{M}\beta_m\psi[f^m(x^{tm})]\} \quad (66)$$

where the $\beta_m > 0$ are parameters that allow the consumer to cardinally compare the transformed seasonal utilities $\psi[f^m(x^{tm})]$ and $\psi(z)$ is a monotonic function of one positive variable z defined by

$$\psi(z) \equiv f_\alpha(z) \equiv \begin{cases} z^\alpha & \text{if } \alpha \neq 0 \\ lnz & \text{if } \alpha = 0. \end{cases} \tag{67}$$

Substituting (67) into (66) reveals that the intertemporal utility function U is a CES (constant elasticity of substitution) aggregate of the seasonal utilities $f^m(x^{tm})$. Using the assumptions that the seasonal aggregator functions $f^m(x^{tm})$ are linearly homogeneous in the elements of x^{tm}, it can be verified that the intertemporal utility function U is linearly homogeneous in its arguments.[69]

Assume that the observed quantity data, $q^{01}, \ldots, q^{0M}; \ldots; q^{T1}, \ldots, q^{TM}$ solve the intertemporal utility maximization problem (2) where the intertemporal utility function U is defined by (66) and (67) and the intertemporal budget constraint is defined by (13) and (14). Then since ψ^{-1} is a monotonic function of one variable[70], it can be seen that for any year t, we must have for $t = 0, 1, \ldots, T$:

$$\begin{aligned} \Sigma_{m=1}^M \beta_m \psi[f^m(q^{tm})] = max_{x^1,\ldots,x^M}\{\Sigma_{m=1}^M \beta_m \psi[f^m(x^m)] : \Sigma_{m=1}^M \delta_t \rho_{tm} p^{tm} \cdot x^m \\ = \Sigma_{m=1}^M \delta_t \rho_{tm} p^{tm} \cdot q^{tm}\}. \end{aligned} \tag{68}$$

Recall that $\delta_t > 0$ is the market discount factor that makes one dollar at the beginning of year t equivalent to one dollar at the beginning of year 0. Recall also that $p^{tm} \equiv [p_1^{tm}, p_2^{tm}, \ldots, p_{N_m}^{tm}]$ is the vector of spot or nominal prices for season m of year t and ρ_{tm} is the market discount factor that makes a dollar in the middle of season m of year t equivalent to a dollar at the beginning of year t.

In section 3 above when we were dealing with calendar years, we defined the vector of year t, season m discounted (to the beginning of year t) prices, p^{tm*}, by (16). In the present section when we will be dealing with noncalendar years, it is convenient to redefine p^{tm*} as follows:

$$p^{tm*} \equiv \delta_t \rho_{tm} p^{tm}; \quad t = 0, 1, \ldots, T; \quad m = 1, \ldots, M; \tag{69}$$

i.e., p^{tm*} is now the nominal year t, season m price vector p^{tm} discounted to the beginning of year 0.

Now return to the equalities (68). The annual utility $\Sigma_{m=1}^M \beta_m \psi[f^m(q^{tm})]$ can be rescaled or transformed by the monotonic function ψ^{-1} to make the resulting annual utility function linearly homogeneous. Making this transformation, we obtain the following equalities for $t = 0, 1, \ldots, T$:

$$\begin{aligned} &\psi^{-1}\{\Sigma_{m=1}^M \beta_m \psi[f^m(q^{tm})]\} \\ &\quad = max_{x^1,\ldots,x^M}\{\psi^{-1}(\Sigma_{m=1}^M \beta_m \psi[f^m(x^m)]) : \Sigma_{m=1}^M p^{tm*} \cdot x^m \\ &\qquad = \Sigma_{m=1}^M p^{tm*} \cdot q^{tm}\}. \end{aligned} \tag{70}$$

Recall equations (44) - (47) in section 6 above. These equations remain valid in the present section, with the understanding that the normalized prices p^{tm*} that appear in these equations are now defined by (69) instead of (16). It can now be seen that the calendar year utility maximization problems in (70) are special cases of the year t utility maximization problems in (48), where the p^{tm*} are defined by (69). The general utility function h which appears in (48) is now the following specialized functional form:

$$\begin{aligned} h(Q_1, \ldots, Q_M) &\equiv \psi^{-1}[\Sigma_{m=1}^{M} \beta_m \psi(Q_m)] \\ &= [\Sigma_{m=1}^{M} \beta_m (Q_m)^{\alpha}]^{1/\alpha} \quad \text{if } \alpha \neq 0. \end{aligned} \tag{71}$$

Assuming that we have exact index number formulae for the seasonal aggregator functions $f^1, \ldots, f^M$, we can use equations (44) - (47) to calculate the seasonal (discounted to the beginning of year 0) prices P_m^{t*} and the seasonal aggregate quantities Q_m^t for $t = 0, 1, \ldots, T$ and $m = 1, \ldots, M$. Furthermore, equations (49) can be used to calculate the calendar year aggregates, $h[f^1(q^{t1}), \ldots, f^M(q^{tM})]/h[f^1(q^{01}), \ldots, f^M(q^{0M})]$, provided that we can find the bilateral index number formula Q^a that is exact for the aggregator function h defined by (71). Note that the h defined by (71) has a CES (or mean of order α) functional form. Sato [1976a; 225] showed that the Vartia II [1974; 66-70] [1976] quantity index Q_{SV} is exact for this functional form. Thus we have for $t = 1, 2, \ldots, T$:

$$\begin{aligned} &ln\{h[f^1(q^{t1}), \ldots, f^M(q^{tM})]/h[f^1(q^{01}), \ldots, f^M(q^{0M})]\} \\ &= lnQ_{SV}(P_1^{0*}, \ldots, P_M^{0*}; P_1^{t*}, \ldots, P_M^{t*}; Q_1^0, \ldots, Q_M^0; Q_1^t, \ldots, Q_M^t) \\ &\equiv \Sigma_{m=1}^{M} w_m^{0t} ln(Q_m^t/Q_m^0)/\Sigma_{j=1}^{M} w_j^{0t} \end{aligned} \tag{72}$$

where $w_m^{0t} \equiv L(w_m^0, w_m^t), w_m^t \equiv P_m^{t*}Q_m^t/\Sigma_{j=1}^{M} P_j^{t*}Q_j^t$ for $m = 1, \ldots, M$ and $t = 0, 1, \ldots, T$ and $L(x, y)$ is the logarithmic mean defined by (62).

The reader is entitled to feel somewhat puzzled at this point (and perhaps at many other points), since all we have done is establish the exact index counterpart to (49), assuming that the functional form for the annual aggregator function h has the more restrictive CES functional form defined by (71) and (67), instead of a general flexible functional form as was allowed in section 6. However, with the special structure of intertemporal preferences defined by (66) and (67), it can be shown that the equalities (70) and (72) established for calendar years can be extended to noncalendar years; i.e., to any consecutive run of M seasons. For example, with our present assumptions on the structure of intertemporal preferences and under the assumption of maximizing behavior over the $T + 1$ years in our sample of observations, we can establish the following counterparts to (70) and (48) for

$t = 0, 1, \ldots, T-1$:

$$\begin{aligned}
&\psi^{-1}\{\Sigma_{m=2}^{M}\beta_m\psi[f^m(q^{tm})] + \beta_1\psi[f^1(q^{t+1,1})]\} \\
&\quad = max_{x^1,\ldots,x^M}\{\psi^{-1}(\Sigma_{m=2}^{M}\beta_m\psi[f^m(x^m)] + \beta_1\psi[f^1(x^1)]) : \\
&\qquad\qquad \Sigma_{m=2}^{M}p^{tm*}\cdot x^m + p^{t+1,1*}\cdot x^1 = \Sigma_{m=2}^{M}p^{tm*}\cdot q^{tm} + p^{t+1,1*}\cdot q^{t+1,1}\} \\
&\quad = \psi^{-1}\{\Sigma_{m=2}^{M}\beta_m\psi[Q_m^t] + \beta_1\psi[Q_1^{t+1}]\} \\
&\quad = max_{Q_1,\ldots,Q_M}\{\psi^{-1}[\Sigma_{m=2}^{M}\beta_m\psi(Q_m) + \beta_1\psi(Q_1)] : \\
&\qquad\qquad \Sigma_{m=2}^{M}P_m^{t*}Q_m + P_1^{t+1*}Q_1 = \Sigma_{m=2}^{M}P_m^{t*}Q_m^t + P_1^{t+1*}Q_1^{t+1}\}
\end{aligned} \tag{73}$$

where the P_m^{t*} and Q_m^t are defined by (44) - (47) with the p^{tm*} now defined by (69) instead of (16). The moving year utility maximization problems in (73) have dropped the quantities of season 1 in year t and added the quantities of season 1 in year $t+1$. Equations (70) when $t = 0$ can be combined with equations (73) and the fact that the Sato-Vartia quantity index Q_{SV} is exact for the h defined by (71) and (67) to yield the following exact index number relationships for $t = 0, 1, \ldots, T-1$:

$$\begin{aligned}
&\psi^{-1}\{\beta_1\psi[f^1(q^{t+1,1})] + \Sigma_{m=2}^{M}\beta_m\psi[f^m(q^{tm})]\}/\psi^{-1}\{\Sigma_{m=1}^{M}\beta_m\psi[f^m(q^{0m})]\} \\
&\quad = \psi^{-1}\{\beta_1\psi(Q_1^{t+1}) + \Sigma_{m=2}^{M}\beta_m\psi(Q_m^t)\}/\psi^{-1}\{\Sigma_{m=1}^{M}\beta_m\psi(Q_m^0)\} \\
&\quad = Q_{SV}(P_1^{0*},\ldots,P_M^{0*};P_1^{t+1*},P_2^{t*},\ldots,P_M^{t*};Q_1^0,\ldots,Q_M^0;Q_1^{t+1},Q_2^t,\ldots,Q_M^t).
\end{aligned} \tag{74}$$

Note that when we evaluate the Sato-Vartia quantity index on the right hand side of (74), we use the base year aggregate discounted seasonal prices $P_1^{0*},\ldots,P_M^{0*}$, the base year seasonal aggregates $Q_1^0,\ldots,Q_M^0$, the year $t+1$ aggregate season 1 discounted price P_1^{t+1*} followed by the year t season 2 to M discounted prices $P_2^{t*},\ldots,P_M^{t*}$ and the year $t+1$ season 1 quantity aggregate Q_1^{t+1} followed by the year t season 2 to M quantity aggregates $Q_2^t,\ldots,Q_M^t$.

In a similar fashion, the aggregate seasonal price and quantity data constructed using (44) - (47) for any run of M consecutive seasons, say P_m^{t*}, $P_{m+1}^{t*},\ldots,P_M^{t*}$, P_1^{t+1*}, $P_2^{t+1*},\ldots,P_{m-1}^{t+1*}$ and $Q_m^t, Q_{m+1}^t,\ldots,Q_M^t, Q_1^{t+1}, Q_2^{t+1},\ldots,Q_{m-1}^{t+1}$, can be rearranged and inserted into the Sato-Vartia index number formula, and the resulting number times the (discounted) value of base year consumption, $\Sigma_{j=1}^{M}p^{0j*}\cdot q^{0j} = \Sigma_{j=1}^{M}P_j^{0*}Q_j^0$,

$$\begin{aligned}
Q_{tm} \equiv Q_{SV}(&P_1^{0*},\ldots,P_M^{0*};P_1^{t+1*},\ldots,P_{m-1}^{t+1*},P_m^{t*},\ldots,P_M^{t*}; \\
&Q_1^0,\ldots,Q_M^0;Q_1^{t+1},\ldots,Q_{m-1}^{t+1},Q_m^t,\ldots,Q_M^t)\Sigma_{j=1}^{M}p^{0j*}\cdot q^{0j}
\end{aligned} \tag{75}$$

is an estimate of the consumer's real consumption in the moving year starting in season m of year t expressed in constant dollars pertaining to the beginning of calendar year 0.

We can divide the quantity index Q_{tm} into the discounted value ratio of the moving year starting in season m of year t to the base year to obtain a price index P_{tm}:

$$P_{tm} \equiv [\Sigma_{i=m}^{M}p^{ti*}\cdot q^{ti} + \Sigma_{j=1}^{m-1}p^{t+1,j*}\cdot q^{t+1,j}]/[\Sigma_{i=1}^{M}p^{0i*}\cdot q^{0i}]Q_{tm}. \tag{76}$$

Due to the fact that discounted price vectors p^{tm*} appear in (75) and (76) instead of the nominal price vectors p^{tm}, it is difficult to interpret the moving year price index P_{tm} that is defined by (76), just as it was difficult to interpret our earlier calendar year price indexes defined by (18). However, our focus in this section is on the moving year quantity indexes Q_{tm} defined by (75). The main advantage of these moving year quantity indexes over the single and two stage calendar year quantity indexes defined earlier by (17) and (49) is their *timeliness*: at the end of *each* season of each year, a moving year quantity index can be calculated that will enable economic policy makers to accurately determine the progress of the economy over the current noncalendar year compared to the base calendar year. A second advantage of the moving year quantity indexes is that they are *comprehensive*; i.e., they include *all* of the seasonal commodities whereas the short term season to season quantity indexes defined in the previous section by (54) were also timely but they had to exclude most seasonal commodities. A third advantage of the moving year quantity indexes Q_{tm} is that they *do not have to be seasonally adjusted*, since the quantities pertaining to an entire year starting at season m of year t are compared to the quantities pertaining to a base year. Thus the moving year quantity indexes Q_{tm} defined by (75) can be viewed as seasonally adjusted constant dollar consumption series at annual rates and the analysis in this section provides a rigorous justification for the use of these series from the viewpoint of the economic approach to index number theory.

In section 6, we recommended that the seasonal aggregates Q_m^t and P_m^{t*} defined by (46) and (47) be defined using Fisher ideal indexes for the seasonal bilateral indexes Q^m and P^m that appeared in (46) and (47). We continue to make that recommendation in the present section. Of course, Statistical Agencies may have to approximate these Fisher indexes by Paasche and Laspeyres indexes and it may also be necessary to approximate the Sato-Vartia quantity and price indexes in (75) and (76) by Paasche and Laspeyres indexes as well. Provided that the base year is changed fairly frequently, these first order approximations provided by the Paasche and Laspeyres indexes should provide adequate approximations to our preferred alternatives. In low inflation contexts (i.e., less than 5% per year), it may also be possible to approximate adequately the moving year quantity indexes Q_{tm} defined by (75) by replacing the discounted price vectors p^{tm*} defined by (69) by the nominal price vectors p^{tm}; this replacement will also occur in (44) -(47). Replacing discounted prices by nominal prices in (76) means that the resulting moving year price index P_{tm} can be regarded as a normal (seasonally adjusted) annual price index. Making these Paasche and Laspeyres approximations and using nominal prices p^{tm} in place of the discounted prices p^{tm*} causes (76) to become the "indice sensible" that was used as a seasonally adjusted consumer price index by the French Statistical Agency INSEE [1976; 67-68] for several years. Diewert [1983c; 1040], using Turvey's [1979] artificial data on seasonal consumption, also calculated some approximations to the moving year price indexes defined by (76): Diewert used Turvey's nominal prices instead of discounted prices and compared the results of using Laspeyres, Paasche, Fisher ideal and translog or Törnqvist [1936] price indexes as the index number formula in both stages of the aggregation. The resulting indexes are tabled in Appendix 2 below (see Table 2) along with Turvey's [1979] original data (see Table 1). As the reader will see, the choice of index number formula does not matter very much for this data set.[71]

In the following section, we regard (75) as an index number method of seasonal adjustment and compare this method with more traditional statistical methods of seasonal adjustment.

9. Econometric Versus Index Number Methods of Seasonal Adjustment

> "A preliminary survey of the graphs of such fundamental monthly series as bank clearings, iron production, commodity prices, and new building permits has led to the following working hypothesis, namely that each series is a composite consisting of four types of fluctuations. The four types are:
>
> 1. A long-time tendency or secular trend; in many series, such as bank clearings or production of commodities, this may be termed the growth element;
> 2. A wave-like or cyclical movement superimposed upon the secular trend; these waves appear to reach their crests during periods of industrial prosperity and their troughs during periods of industrial depression, their rise and fall constituting the business cycle;
> 3. A seasonal movement within the year with a characteristic shape for each series;
> 4. Residual variations due to developments which affect individual series, or to momentous occurrences, such as wars or national catastrophes, which affect a number of series simultaneously". *Warren M. Persons*[1919; 8]

> "Clearly a smoother seasonally adjusted series and shorter MCD's may be achieved by ascribing some of the irregular variation to the seasonal component by means of moving seasonals. Unfortunately, a concomitant result of this procedure are seasonal factors that are susceptible to revision. Conversely, rigidly defined seasonal factors leave much more residual variation in the seasonally adjusted series and complicate analysis. A general purpose seasonal adjustment procedure must come to grips with balancing these alternatives and, at present, there are no theoretical guidelines for making the compromise".
>
> *Shirley Kallek*[1978; 11]

In section 6 above, we showed how year over year seasonal indexes could be aggregated (using a superlative index number formula) to form annual indexes, which compared a calendar year with a base calendar year. What we have done in the previous section is to show that we do not have to restrict these annual comparisons to calendar years: if we use the Sato-Vartia quantity index, Q_{SV} defined by (75), to aggregate up the year over year seasonal indexes, then we can make exact index number comparisons for any consecutive string of M seasons with the base year. These moving year indexes have no seasonal components and hence can be regarded as seasonally adjusted "monthly" series at annual rates.

Instead of using the Sato-Vartia index Q_{SV} in (75) to aggregate the seasonal year over year indexes, a superlative quantity index such as the Fisher ideal Q_F could be used to provide a reasonably close approximation to Q_{SV}.[72] In this case, if the moving year is a calendar year, then the resulting Fisher annual index $Q_F(P_1^{0*}, \ldots, P_M^{0*}; P_1^{t*}, \ldots, P_M^{t*}; Q_1^0, \ldots, Q_M^0; Q_1^t, \ldots, Q_M^t)$ reduces to our preferred two stage annual index defined by (49), where $Q^a \equiv Q_F$. Note that in the general case where the Fisher quantity index $Q_F(P_1^{0*}, \ldots, P_M^{0*}; P_1^{t+1*}, \ldots, P_{m-1}^{t+1*}, P_m^{t*}, \ldots, P_M^{t*}; Q_1^0, \ldots, Q_M^0; Q_1^{t+1}, \ldots, Q_{m-1}^{t+1}, Q_m^t, \ldots, Q_M^t)$ is defined for the moving year starting at season m of year t, $Q_F \equiv [Q_P Q_L]^{1/2}$ where the Paasche and Laspeyres quantity indexes, Q_P and Q_L, are evaluated at the same aggregate seasonal prices and quantities and can be regarded as share weighted moving averages of the moving year seasonal quantity aggregates $Q_m^t, Q_{m+1}^t, \ldots, Q_M^t, Q_1^{t+1}, Q_2^{t+1}, \ldots, Q_{m-1}^{t+1}$.

As a further refinement of our suggested index number method for seasonal adjustment, we can "center" the series of moving year quantity indexes $Q_{t,m}$ defined by (75). Suppose that we have monthly data so that the number of seasons M equals 12. Then $Q_{t,m}$ represents the aggregate quantity of a moving year starting at month m of year t relative to the aggregate quantity of a base year. An estimate of the annual quantity *centered* around month m of year t compared to the quantity of the base year is

$$Q_{t,m}^c \equiv \begin{cases} (1/2)Q_{t,m-6} + (1/2)Q_{t,m-5}; & t = 0, 1, \ldots, T-1; \quad m = 7, \ldots, 12 \\ (1/2)Q_{t-1,m+6} + (1/2)Q_{t-1,m+7}; & t = 1, 2, \ldots, T; \quad m = 1, \ldots, 5 \\ (1/2)Q_{t-1,12} + (1/2)Q_{t,1}; & t = 1, \ldots, T; \quad m = 6. \end{cases} \tag{77}$$

Note that we cannot provide centered monthly quantity estimates for the first 6 months and the last 6 months in our sample; i.e., $Q_{t,m}^c$ is not defined for $t = 0$ and $m = 1, 2, \ldots, 6$ and for $t = T$ and $m = 7, 8, \ldots, 12$.

The two stage method of seasonal adjustment defined by (44) - (47) (with the discounted prices p^{tm*} defined by (69) instead of (16)) and (75) and (77) aggregates over commodities within a season in the first stage. If there is only one (seasonal) commodity in each season, then our overall index number method of seasonal adjustment can be viewed as a "pure" seasonal adjustment procedure which can be compared to classical statistical methods for seasonally adjusting a single time series, which could be denoted by $Q_1^0, \ldots, Q_M^0; Q_1^1, \ldots, Q_M^1; \ldots; Q_1^T, \ldots, Q_M^T$ using our notation. In order to compare our suggested index number method for seasonal adjustment with leading statistical methods, it is necessary to indicate the characteristics of these statistical methods.

Statistical or econometric methods for seasonally adjusting economic time series have a long history dating back to the primitive methods of daily, weekly or monthly averages.[73] However, it soon became apparent that in order to distinguish the seasonal fluctuations in an economic time series, one also had to identify other components as well. Thus Persons [1919; 8] refined Cournot's[74] earlier classification of types of fluctuations and postulated that an economic time series could be statistically decomposed into four components: (i) a long term (or secular) trend; (ii) a cyclical component reflecting the ups and downs of business cycles; (iii) seasonal components and (iv) irregular or random fluctuations. Macaulay's [1931] monograph represented a systematic attempt to separate the trend (equal to the secular plus cyclical) from the seasonal components[75]; his central tool was the use of moving averages to represent trends.[76] These early moving average methods were further developed by Shiskin [1942] [1955], Shiskin and Eisenpress [1957] and Shiskin, Young and Musgrave [1967] into the rather complex $X-11$ seasonal adjustment method that is still in use today. Dagum [1975] [1983] developed the $X-11-ARIMA$ method which extended the original time series at the end points using extrapolated values from $ARIMA$ models of the type described by Box and Jenkins [1970] and then the $X-11$ program was run on the extended series.

Many additional statistical and econometric methods have been developed in the past 70 years in order to seasonally adjust economic time series, including: (i) linear regression techniques (e.g., see Hart [1922], Mendershausen [1939], Lovell [1963] and Jorgenson [1964]); (ii) spectral analysis methods (e.g., see Hannan [1960] and Nerlove [1964]); (iii) Whittaker [1923] - Henderson [1924] penalized least squares or smoothing splines (e.g., see Leser [1963], Akaike [1980], Schlicht [1981] and Engle, Granger, Rice and Weiss [1986]) and (iv) other time series methods (e.g., see Zellner [1978], Pierce [1978] [1980], Cleveland [1983], Bell and Hillmer [1984], Findley and Monsell [1986], Hylleberg [1992] and Ghysels [1993]).

The basic problem with all of the above statistical methods of seasonal adjustment is that each method is more or less arbitrary. For example, let us consider the problem of seasonally adjusting a "monthly" quantity series Q_t where we use Person's [1919; 8] classification of unobserved components. Then we might postulate that Q_t can be decomposed into its four components as follows:[77]

$$Q_t = T_t + C_t + S_t + E_t; \quad t = 1, \ldots, N \tag{78}$$

where T_t is the long term trend at period t, C_t is the business cycle component of the series, S_t is the seasonal component and E_t is an "error" or residual component for period

t. However, after further reflection, we might find that the following multiplicative model is more plausible:

$$Q_t = T_t C_t S_t E_t; \quad t = 1, \ldots, N. \tag{79}$$

Finally, we might decide that both of the models (78) and (79) are too restrictive so we postulate the existence of a function F such that

$$Q_t = F(T_t, C_t, S_t, E_t); \quad t = 1, \ldots N. \tag{80}$$

It is obvious that, in order to *identify* the unobserved components T_t, C_t, S_t and E_t from our observations $Q_t,\ t = 1, \ldots, N$, we are going to have to make more or less arbitrary assumptions: (i) to determine the functional form for F and (ii) even when F is completely specified as in (78) or (79), we need to make additional assumptions to identify the four unobserved components.

The above *fundamental identification problem* has been noticed in the literature. The most complete statement of it is due to Anderson [1927] but has been largely forgotten:

> "We must either obtain the missing $(mN - N)$ equations from other sources, which can happen only in very exceptional cases, or introduce some preliminary assumptions, some hypotheses concerning the construction of the aggregates V, which would replace the missing equations. Thus, in most cases with which the social investigator has to deal in practice, *in the decomposition of series into components, neither the definition of the function F nor the finding of the numerical meanings of the effects caused by the aggregates of cases V', V'', V''', V'''' is possible without the introduction of different hypotheses which are more or less arbitrary."* *Oskar Anderson*[1927; 552-553]

To adapt the above quotation to our present situation, assume that our model is (80) with the number m of unobserved "explanatory" variables equaling 4 with $V' \equiv T,\ V'' \equiv C,\ V''' \equiv S$ and $V'''' \equiv E$. Hence we have N equations in (80) with $4N$ unknown T_t, C_t, S_t and E_t variables to be determined–a rather formidable identification problem!

Assuming that we have chosen say, the additive model (78), Anderson went on to explain how the various components in the right hand side of (78) might be identified:

> "Further, the investigator again limits arbitrarily the circle of his possibilities. For example:
>
> (a) assuming that the secular component represents a polynomial function of the argument t (time or ordinal number) . . .
>
> (b) assuming that the cyclical component can be represented as a more or less complex trigonometrical function;

(c) assuming that the residual component e is a random series; ..."

Oskar Anderson[1927; 554]

From the above quotations, it can be seen that Anderson had a very clear conception of the problem of identifying unobserved components in time series analysis.

In the context of identifying unobserved seasonal components, we have at least two specific additional difficulties: (i) should "normal" monthly growth be included in the seasonal components and (ii) how can we distinguish a changing seasonal component from the business cycle component? Wisniewski was aware of the latter identification problem:

> "Another point is the dependence of the cyclical variation on the seasonal one .... I tried various methods but no one gave quite satisfactory results. It seems as if the two ways of interdependence were not capable of being untangled. Very frequently we cannot say whether the cyclical variation is the cause of the seasonal change or vice versa." *Jan Wisniewski*[1934; 180]

Finally, in more recent times, Pierce also recognized the existence of Anderson's fundamental time series identification problem:

> "It is necessary, in these situations, to restrict the class (20) of models so that the seasonal component of a series can be determined, theoretically and empirically. Often, restrictions are provided by the nature of the problem or by specific information .... The problem here, as elsewhere, is that a consensus on this theory is lacking. One person prefers to define trend or cyclic effects in one way, another differently. In multivariate approaches, there are probably as many varieties of variables $y, z, \ldots$ relevant to seasonally adjusting a variable x, and as many varieties of plausible specifications of relationships among and between their components, all essentially compatible with the data, as their are social scientists (economists,statisticians, etc.) to specify these variables and relationships. This situation is evidently a general one in econometric modelling, where a variety of specifications, including a purely autoregressive equation, are all compatible with the data and all have comparable predictive power."
>
> *David A. Pierce*[1978; 245-246]

The above analysis indicates that statistical or econometric models generate a wide range of implied seasonal adjustment factors: how can we reduce this range? One possible solution would be to take an axiomatic approach to the determination of the unobserved components in the general model (80). This axiomatic or desirable properties approach to seasonal adjustment has in fact been pursued by Hart [1922; 342-347], Lovell [1963] [1966], Grether and Nerlove [1970] and Pierce [1978; 246-247] among others but this research program is still in a preliminary phase. Moreover, different axioms will lead to different seasonal factors. Hence until economists and statisticians can agree on a "reasonable" set of axioms for seasonal adjustment (as well as for the determination of trend and cycle components), this test approach to seasonal adjustment will not be of much help to Statistical Agencies.[78]

Now consider our index number method of seasonal adjustment in the light of the above discussion. It is evident that our suggested index number method is not really a seasonal adjustment method in the sense that it determines the unobserved S_t components that appear in (78) - (80). In fact, our index numbers Q_{tm} defined by (75) simply compare a moving year aggregate to a corresponding base year aggregate. *Thus we have changed the question that we are trying to answer.* The centered index number comparisons Q^c_{tm} of the form (77) are simply averages of the more fundamental comparisons made in (75), where the averaging is done so that the resulting centered estimates will more closely resemble a conventional seasonally adjusted series at annual rates.

In Appendix 3 below, we compared official U.S. seasonally adjusted at annual rates data on quarterly GDP over the years 1959-1988 with moving year centered index numbers which aggregated the quarterly unadjusted data published in the Bureau of Economic Analysis [1992].[79] We found that our suggested index number method for seasonal adjustment performed as well as the official X-11 method in that the turning points for the two series were basically the same. The main differences between the two seasonally adjusted series were: (i) the index number adjusted series was smoother and (ii) the X-11 adjusted series grew more slowly.[80] The reason for the second difference is that the X-11 series was constructed by seasonally adjusting the U.S. fixed base quarterly (unadjusted) quantity series whereas we used the unadjusted quarterly chain data as our input into the index number formula.[81] Our results are consistent with the fixed 1987 base year Laspeyres and chained comparisons of U.S. real GDP over the years 1959-1987 made by Young [1992; 36], who found that the average rate of growth of the fixed base GDP index numbers was 3.1% compared to 3.4% per year for the chain indexes.[82] Users of U.S. seasonally adjusted data should be made aware that it is fixed base data that is being seasonally adjusted and hence when the base year is changed, fairly substantial changes in growth rates can occur in the official seasonally adjusted fixed base data.

The fact that our suggested index number method for seasonal adjustment defined by (75) and (77) (or by approximations to these formulae) can perform as well as the X-11 method is significant since the index number method offers a number of advantages over the X-11 method: (i) The index number method can be explained fairly simply whereas it is virtually impossible to explain to the public how the X-11 method works.[83] (ii) There are many significant unannounced choices that must be made by the statistican-operator of the X-11 method (i.e., multiplicative or additive seasonals, treatment of outliers, etc.), whereas the index number method involves only two choices which can be announced to the public rather easily.[84] (iii) Final seasonal adjustment factors using the X-11 method (and most other econometric methods) are not available until 2 or 3 years of additional unadjusted data become available. In contrast, indexes of the form (75) will be available (after normal processing delays) immediately after the last season in the moving year and the centered indexes of the form (77) will be available after an additional 6 months. Since these indexes will not usually be revised, the Statistical Agency will avoid the current embarrassing problem of trying to explain why the seasonally adjusted series are still being revised years after the preliminary series are released. (iv) The index number method of aggregation simultaneously seasonally adjusts (normalized) prices and quantities (recall (75) and (76) above) whereas statistical methods of seasonal adjustment separately adjust prices, quantities and values without respecting the fact that only 2 of these 3 variables are independent. (v) Finally, statistical seasonal adjustment methods that allow for changing seasonals run into a severe identification problem and the resulting seasonal factors that these statistical methods churn out are not well defined from a theoretical point of view; recall our earlier discussion of Anderson's [1927] criticisms of unobserved components models.

It should be noted that econometric methods for seasonal adjustment do have one substantial advantage: they can be applied to situations where there is only information on quantities and no price information, i.e., the X-11 method can seasonally adjust an unemployment series but an index number method cannot.

It should be emphasized that the moving year quantity indexes defined by (75) are sufficient statistics for defining the centered moving year quantity indexes defined by (77). Thus the Statistical Agency should strive to provide moving year quantity and price indexes of the form (75) and (76) on a timely basis: users can easily perform the simple arithmetic operations inherent in forming the centered moving year indexes of the form (77).

Our specific assumptions on intertemporal preferences represented by (66) and (67) led to the specific Sato-Vartia exact index number formula (75) where the "monthly" aggregates P_m^{t*} and Q_m^t were formed using superlative index number formulae in (44) - (47). Since in many situations, the Statistical Agency will not have access to current period quantity information, it may be necessary to approximate both the "monthly" price indexes P^m which appear in (47) and the Sato-Vartia price index P_{SV} which appears in (76) by Laspeyres price indexes. If these Laspeyres approximations for the price indexes are used, then the corresponding quantity indexes in (46) and (75) will be Paasche quantity indexes. However, these Paasche and Laspeyres indexes will approximate their superlative and Sato-Vartia counterparts to the first order around an equal price and quantity point and hence will be acceptable approximations, provided that the base year is changed fairly frequently, say at least once every 5 years.

Another approximation to our recommended theoretically exact indexes defined by (44) - (47) and (75) - (76) occurs if the inflation adjusted prices p^{tm*} defined by (69) and used in (44) - (47) are replaced by the corresponding unadjusted spot price vectors p^{tm}. This replacement of general inflation adjusted prices by unadjusted prices will make little difference to the moving year quantity indexes defined by (75) and (77) provided that: (i) inflation is "low" and (ii) seasonal fluctuations in prices or quantities are not "too" erratic. Some numerical experiments will be required before we can be more precise.

10. Conclusion

> "It is entirely appropriate for research institutes and academic centers to try to distill the trend cycle or to measure cycles and random components of economic time series, but such measures should not be permitted to cloud the regular publication of official statistics". *Lawrence R. Klein*[1978; 30-31]

> "As is well known among the many official statistical agencies in charge of compiling monthly price index numbers, seasonal commodities, i.e. horticultural and agricultural products, present difficulties both in a practical and in a methodological sense .... Now the inconvenient property of seasonal commodities is their absence from the market during a number of months per year and consequently the impossibility of determining their price during these months. Within the Laspeyres-bound methodology, various proposals to deal with this problem have been brought forward. They are all reviewed by Balk [1980a]; cf. also Balk [1980b]. It appears there is no generally accepted method. Moreover, every method hitherto used or proposed suffers from one or more drawbacks".
> *Bert M. Balk*[1981; 72]

In this paper, we have discussed the problem Statistical Agencies face when constructing price and quantity aggregates under conditions of high inflation when there are seasonal commodities. If there were no seasonal commodities, then the index number problem in the context of high inflation is straightforward (but expensive): the Statistical Agency simply has to collect subannual price and quantity (or value) information more frequently in order to make the subannual periods of time short enough so that variations in prices within the periods can be neglected.[85] However, when there are seasonal commodities, this solution to the high inflation index number problem is simply not valid: we cannot make meaningful bilateral index number comparisons (from the viewpoint of the economic approach) between consecutive months or quarters if the dimensionality of the commodity space varies from period to period. No useful purpose is served in trying to compare the incomparable!

The assumptions on consumer preferences that we have made provide justifications for three types of seasonal index number comparisons that Statistical Agencies should provide to the public:

(i) For measuring short term price change, the approach outlined in section 7 above should be used; i.e., a season to season short run price index using only nonseasonal (and type (a) seasonal) commodities should be constructed. These short term indexes would be used as deflators[86] when constructing the annual quantity indexes in (iii) below under conditions of high inflation.

(ii) The year over year seasonal indexes defined by (39) and (40) in section 5 above should also be constructed. The assumptions made on preferences required to justify these indexes are the least restrictive. Moreover, the business community will probably find these indexes the most useful for their purposes.

(iii) Finally, the moving year price and quantity indexes defined by (75) and (76) in section 9 should also be calculated.[87] These indexes will serve as seasonally adjusted price and quantity indexes (at annual rates). If there is low inflation, spot prices p^{tm} can be used in place of the normalized prices p^{tm*} in (45) - (48) and (75) - (76).

For each of the above three indexes, the Statistical Agency will have to decide whether to provide Paasche and Laspeyres versions of each index or to provide superlative versions. From the viewpoint of economic theory, the superlative versions will be more accurate but they will be more costly to produce and there will be a loss of timeliness. In the long run, I believe that it will be feasible to produce timely superlative indexes if Statistical Agencies could make use of electronically recorded data on the sales of commodities.[88] However, in the short run, Statistical Agencies will have to make many difficult choices on how to produce price and quantity indexes when there are seasonal commodities and high inflation.

APPENDIX

Appendix 1: Plutocratic Price Indexes and Aggregate Reservation Prices

In this Appendix, we show how Fisher ideal price indexes can be justified when we are using aggregate consumer data rather than individual data. We also justify the use of Fisher and Griliches' [1995; 236] aggregate reservation prices in the context of type (i) seasonal goods.

Pollak [1981; 328] defined what he called a Scitovsky-Laspeyres group cost of living index as the ratio of the total expenditure required to enable each household to attain its reference indifference surface at period 1 prices to that expenditure required at period 0 prices. This same concept of a plutocratic price index was suggested by Prais [1959] in less precise language. Assume that there are H households with preference functions $F^h(x)$ defined over N goods $x \equiv [x_1, \ldots, x_N]$ and let C^h be the associated dual cost or expenditure function defined for reference utility level u_h and reference prices $p \equiv [p_1, \ldots, p_N]$ by

$$C^h(u_h, p) \equiv min_x\{p \cdot x : F^h(x) \geq u_h\}, \quad h = 1, \ldots, H. \tag{A1}$$

Then assuming that all households face the same prices $p^0 \equiv [p_1^0, \ldots, p_N^0]$ and $p^1 \equiv [p_1^1, \ldots, p_N^1]$ in periods 0 and 1, then the Prais-Pollak plutocratic cost of living index for reference utility levels $u_1, \ldots, u_H$ can be defined as

$$P_{PP}(p^0, p^1, u_1, \ldots, u_H) \equiv \Sigma_{h=1}^H C^h(u_h, p^1)/\Sigma_{h=1}^H C^h(u_h, p^0). \tag{A2}$$

Pollak [1989; 122 and 140] and Diewert [1983a; 190-191] develop various bounds for (A2), which we shall now adapt, following Fisher and Griliches [1995; 236], to deal with the case of seasonally unavailable commodities.

Suppose that commodity N is available in season 1 but not in season 0 (and it is a type (i) (a) seasonal commodity). Let $q^{1h} \equiv [q_1^{1h}, \ldots, q_N^{1h}]$ and $q^{0h} \equiv [q_1^{0h}, \ldots, q_{N-1}^{0h}, 0]$ denote the observed household h consumption vectors for periods 1 and 0 (note that $q_N^{0h} = 0$ for each h due to the unavailability of good N in season 0), let $p^1 \equiv [p_1^1, \ldots, p_N^1]$ be the season 1 vector of prices that each household faces and let $p^{0h} \equiv [p_1^0, \ldots, p_{N-1}^0, p_N^{0h}] \equiv [\tilde{p}^0, p_N^{0h}]$ be the vector of prices that household h faces in period 0. Note that $\tilde{p}^0 \equiv [p_1^0, \ldots, p_{N-1}^0]$ is the same for each household and p_N^{0h} is the household h reservation price for unavailable seasonal commodity in period 0, for $h = 1, \ldots, H$. Under these circumstances, the plutocratic cost of living index is now defined as

$$P_{PP}(\tilde{p}^0; p_N^{01}, \ldots, p_N^{0H}; p^1; u_1, \ldots, u_H) = \Sigma_{h=1}^H C^h(u_h, p^1)/\Sigma_{h=1}^H C^h(u_h, \tilde{p}^0, p_N^{0h}). \tag{A3}$$

Now insert the household base period utility levels, $u_h^0 \equiv F^h(q^{0h}), \quad h = 1, \ldots, H$, into (A3). Under the assumption of cost minimizing behavior in period 0, we have

$$C^h(u_h^0, \tilde{p}^0, p_N^{0h}) = p^{0h} \cdot q^{0h} = \tilde{p}^0 \cdot \tilde{q}^{0h};\ h = 1, \ldots, H \tag{A4}$$

since $q^{0h} \equiv [\tilde{q}^{0h}, 0]$ for $h = 1, \ldots, H$. Also, since q^{0h} will be feasible (but not necessarily optimal) for the cost minimization problem defined by $C^h(u_h^0, p^1)$, we also have

$$C^h(u_h^0, p^1) \leq p^1 \cdot q^{0h} = \tilde{p}^1 \cdot \tilde{q}^{0h}; \quad h = 1, \ldots, H. \tag{A5}$$

Substituting (A4) and (A5) into (A3) when $u_h = u_h^0$ for $h = 1, \ldots, H$ yields

$$\begin{aligned} P_{PP}(\tilde{p}^0; p_N^{01}, \ldots, p_N^{0H}; p^1; u_1^0, \ldots, u_H^0) &\leq \Sigma_{h=1}^H p^1 \cdot q^{0h} / \Sigma_{h=1}^H p^{0h} \cdot q^{0h} \\ &= \Sigma_{h=1}^H \tilde{p}^1 \cdot \tilde{q}^{0h} / \Sigma_{h=1}^H \tilde{p}^0 \cdot \tilde{q}^{0h} \\ &= \tilde{p}^1 \cdot (\Sigma_{h=1}^H \tilde{q}^{0h}) / \tilde{p}^0 \cdot (\Sigma_{h=1}^H \tilde{q}^{0h}) \\ &= \tilde{p}^1 \cdot \tilde{Q}^0 / \tilde{p}^0 \tilde{Q}^0 \end{aligned} \tag{A6}$$

where the aggregate amount of consumption of commodity n in season m is defined as

$$Q_n^m \equiv \Sigma_{h=1}^H q_n^{mh}; \quad n = 1, \ldots, N; \quad m = 0, 1 \tag{A7}$$

and $\tilde{Q}^0 \equiv [Q_1^0, Q_2^0, \ldots, Q_{N-1}^0]$. Since $q_N^{0h} = 0$ for each h, $Q_N^0 = 0$ and this explains why Q_N^0 does not appear in the right hand side of (A6), which is an aggregate Laspeyres price index. Without the complication of reservation prices, the inequality (A6) was obtained by Pollak [1980; 276] and Diewert [1983a; 191] [1993b; 294].

Now insert the household period 1 utility levels, $u_h^1 \equiv F^h(q^{1h}), h = 1, \ldots, H$, into (A3). Under the assumption of cost minimizing behavior in period 1, we have

$$C^h(u_h^1, p^1) = p^1 \cdot q^{1h}, \quad h = 1, \ldots, H. \tag{A8}$$

Since q^{1h} will be feasible (but not necessarily optimal) for the cost minimization problem defined by $C^h(u_h^1, p^0)$, we also have

$$C^h(u_h^1, p^{0h}) \leq p^{0h} \cdot q^{1h} = \tilde{p}^0 \cdot \tilde{q}^{1h} + p_N^{0h} q_N^{1h}; \quad h = 1, \ldots, H. \tag{A9}$$

Substituting (A8) and (A9) into (A3) when $u_h = u_h^1$ for $h = 1, \ldots, H$ yields, using the positivity of expenditures,

$$\begin{aligned} P_{PP}(\tilde{p}^0; p_N^{01}, \ldots, p_N^{0H}; p^1; u_1^1, \ldots, u_H^1) &\geq \Sigma_{h=1}^H p^1 \cdot q^{1h} / \Sigma_{h=1}^H p^{0h} \cdot q^{1h} \\ &= \Sigma_{h=1}^H p^1 \cdot q^{1h} / [\Sigma_{h=1}^H (\tilde{p}^0 \cdot \tilde{q}^{1h} + p_N^{0h} q_N^{1h})] \\ &= p^1 \cdot (\Sigma_{h=1}^H q^{1h}) / [\tilde{p} \cdot (\Sigma_{h=1}^H \tilde{q}^{1h}) + p_N^0 (\Sigma_{h=1}^H q_N^{1h})] \\ &= p^1 \cdot Q^1 / p^0 \cdot Q^1 \end{aligned} \tag{A10}$$

where the average period 0 reservation price for commodity N is defined by

$$p_N^0 \equiv \Sigma_{h=1}^H p_N^{0h} q_N^{1h} / \Sigma_{h=1}^H q_N^{1h} = \Sigma_{h=1}^H p_N^{0h} q_N^{1h} / Q_N^1. \tag{A11}$$

The reservation price defined by (A11) is precisely the Fisher-Griliches [1995; 236] new good reservation price for the base period and the inequality (A10) is their Paasche lower bound inequality; see Fisher and Griliches [1995; 235]. Without the complication of reservation prices, the inequality (A10) was obtained by Diewert [1983a; 191] [1993b; 294].

Once the aggregate reservation price p_N^0 has been defined by (A11), we can define the aggregate base period price vector by $p^0 \equiv [\tilde{p}^0, p_N^0]$. Define $Q^0 \equiv [\tilde{Q}^0, 0]$ and rewrite the inequality (A6) using our new notation as

$$P_{PP}(\tilde{p}^0; p_N^{01}, \ldots, p_N^{0H}; p^1; u_1^0, \ldots, u_H^0) \leq p^1 \cdot Q^0 / p^0 \cdot Q^0. \tag{A12}$$

Note that the right hand side of (A11) is an aggregate Laspeyres price index and the right hand side of (A9) is an aggregate Paasche price index.

Now we are in a position to prove the following result.

Theorem 1: Under appropriate regularity conditions on preferences (see Diewert [1983a; 167]), there exists a reference utility vector $u^* \equiv [u_1^*, \ldots, u_H^*]$, such that component h, u_h^*, lies between u_h^0 and u_h^1 for $h = 1, \ldots, H$, and the plutocratic Prais-Pollak price index defined by (A3) evaluated at this reference utility vector, $P_{PP}(\tilde{p}^0; p_N^{01}, \ldots p_N^{0h}; p^1; u^*)$, lies between the Paasche and Laspeyres aggregate indexes, $p^1 \cdot Q^1 / p^0 \cdot Q^1$ and $p^1 \cdot Q^0 / p^0 \cdot Q^0$ respectively.

The proof of the above result is essentially the same as Diewert's [1983a; 191] proof of Theorem 16, which in turn is essentially due to Konüs [1924]. Note that the above result does *not* require that individual preferences be homothetic.

The connection of Theorem 1 with the use of Fisher ideal price indexes is made as follows: if the Paasche and Laspeyres bounds in (A10) and (A12) are close to each other, then taking a symmetric mean of the Paasche and Laspeyres indexes, such as the geometric mean, should lead to a close approximation to the theoretical index $P_{PP}(\tilde{p}^0; p_N^{01}, \ldots, p_N^{0H}; p^1; u^*)$ defined in Theorem 1. For a discussion of the properties of symmetric means, see Diewert [1993c].

Appendix 2:

Appendix 3: U.S. Seasonally Adjusted and Centered Moving Year Estimates

The raw data for our comparisons were taken from the Bureau of Economic Analysis [1992]: seasonally unadjusted estimates of U.S. GDP from quarter 1 of 1959 to quarter 4 of 1988 (120 quarters in all) were taken from Table 9.1; implicit price deflators for GDP using chain type weights were taken from Table 7.2 and estimates of quarterly GDP seasonally adjusted at annual rates were taken from Table 1.2. The price index was normalized to equal 1 in the third quarter of 1987. Dividing the seasonally unadjusted GDP by the price index gave us the series $Y_t,\ t = 1, \ldots, 120$ (note that we have changed our notation for distinguishing seasons and years). The series Y_t is plotted in Figure 1 below; the observation numbers run from 1 to 120. The units are 100 millions of 1987 third quarter dollars. The series of 4th quarter observations, say Y_4, are indicated by the sharp peaks joined up by dashed lines. It can be seen that the seasonal fluctuations are evolving over time.

Denote the Fisher ideal fixed base moving year index by Q_t, where t indicates the first quarter of the moving year and the base year is taken to be the 4 quarters of 1987. (We made no adjustment for general inflation since it was not high). Note that Q_t is defined for $t = 1, 2, \ldots, 117$. Define the centered moving year Fisher ideal indexes by

$$Q_t^c \equiv (1/2)Q_{t-1} + (1/2)Q_{t-2}; \quad t = 3, 4, \ldots, 118. \tag{A13}$$

These centered moving year Fisher indexes Q_t^c are plotted as the solid line in Figure 2 (and denoted by QF). The units of measurement are again 100 millions of 1987 dollars. The official seasonally adjusted U.S. constant dollar GDP series for the same 116 quarters is also plotted as the dashed line in Figure 2 (an denoted by SAY). Note that SAY grows more slowly and is more erratic than QF but both series have roughly the same turning points and hence both can serve as guides to the current progress of the economy with respect to business cycle movements.

Figure 1

Figure 2

Footnotes

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1. See Middleditch [1918], Paton [1920; 2], Sweeney [1927] [1928] [1964] and Baxter [1975; 17-35] [1984; 38-57]. For additional references to the early French and German accounting literature, see Sweeney [1964; 9-11].

2. This classification corresponds to Balk's [1980a; 7] [1980b; 110] [1980c; 68] narrow and wide sense seasonal commodities.

3. This classification is due to Mitchell [1927; 236]:

> "Two types of seasons produce annually recurring variations in economic activities–those which are due to climates and those which are due to conventions."

4. This is not quite true: both Diewert [1980; 506-508] and Balk [1981] proposed bilateral index number procedures which will work over commodity spaces of varying dimensions. We discuss their proposals in section 7 below.

5. References to the economic approach to index number theory include Konüs [1924], Pollak [1989], Samuelson and Swamy [1974] and Diewert [1976] [1981] [1993a].

6. Using the nonparametric tests for maximizing behavior due to Afriat [1967] and Diewert [1973; 424], we can test whether a given set of price and quantity data are consistent with the maximization of a homothetic or linearly homogeneous utility function; see Diewert [1981; 198-199]. If a combination of seasonal and nonseasonal data pass this test, then the seasonal commodities are of type (ii) (a).

7. Diewert[1980; 506-508] [1983c] made a start on the economic approach to seasonal indexes. The present paper focuses on the theory of the seasonal consumer price index. An analogous theory exists for the theory of the seasonal producer price index; separability assumptions on the intertemporal utility function are replaced with separability assumptions on the producer's intertemporal production function or factor requirements function. See Fisher and Shell [1972] and Diewert [1980] [1983b] for material on the economic approach to producer price indexes.

8. The role of separability assumptions in the economic approach to index number theory is laid out in chapters 2 and 3 of Pollak [1989] and chapter 9 of Blackory, Primont and Russell [1978]. For a general exposition of separability and its role in economic models, see Blackorby, Primont and Russell [1978].

9. The present paper generalizes Diewert's [1983c] earlier economic approach.

10. Our single consumer theory can be extended to groups of consumers using Pollak's [1981; 328], Social Cost of Living Index; see Diewert [1983a; 190-192] [1993b; 294] and Appendix 1 of the present paper. Note that we do not deal with sampling problems in the present paper.

11. For example, should indirect taxes be included in the consumer price index? The structure of the relevant utility maximization problem gives us guidance on this issue. This advantage of the economic approach has been stressed by Jack Triplett over the years.

12. Some precursors of Mudgett and Stone in recognizing that seasonal commodities should be distinguished as separate commodities in the seasons that they are available were Marshall and Bean and Stine as the following quotations indicate.

> "This difficulty has been commonly recognized; but there is another closely connected with it, which seems to have escaped notice. It is that of a thing which is supplied at a time of the year at which it used not be available. The best plan seems to be to regard it as a new commodity when it first appears out of its old season." *Alfred Marshall*[1887; 373]

However, Marshall did not follow up on his insight and recommend that annual indexes distinguish commodities by season. Bean and Stine [1924; 34] have only the following single sentence on the topic, which however does capture the essence of the Mudgett-Stone proposals: "The yearly index for Type D is obtained by summing the twelve monthly aggregates for the year and dividing by the similar sum for the base period."

13. f is defined over the annual commodity space of dimension $\Sigma_{m=1}^{M} N_m \equiv N^*$; i.e., each "physical" commodity in each season is treated as a separate economic commodity from the perspective of the annual utility f. The concavity assumption on f can be replaced by the weaker condition of continuity; see Diewert [1974; 111].

14. See Diewert [1976; 116] [1981; 180-193] [1983a; 184] [1993a; 45-50].

15. See Diewert [1976; 116] for the history of this result.

16. See Diewert [1976; 117] for this terminology. Diewert [1976; 137-138] regarded Q_F as the best superlative index number formula since it is exact for both Leontief (no substitution) and linear (perfect substitutability) aggregator functions and it is the only superlative index that is consistent with revealed preference theory. Diewert [1992; 221] also showed that Q_F had good axiomatic properties.

17. The terminology is due to Samuelson and Swamy [1974; 572].

18. This relationship was first noticed by Fisher [1896; 75]. Fisher [1896; 13 and 69] also defined the expected real interest rate $\bar{r}_{tm}$ in terms of the nominal interest rate r_{tm} and an expected commodity inflation rate i_{tm} going from season m to season $m+1$ of year t as follows: $(1 + r_{tm}) = (1 + \bar{r}_{tm})(1 + i_{tm})$.

19. This type of deflation was used by German accountants to stabilize (or make comparable) accounting values pertaining to different time periods during the German hyperinflation of 1923; see Sweeney [1927][1928].

20. This type of reevaluation to make accounting values comparable is also used by accountants in high inflation countries; e.g., see Wasserman [1931; 10].

21. Since in practice, it is going to be difficult to distinguish type (ii) (a) from type (ii) (b) seasonal commodities, it may be more practical to just restrict ourselves to nonseasonal commodities. Of course, there may be difficulties in distinguishing nonseasonal from seasonal commodities as well.

22. This commodity basket approach to deflating the value of money to make it comparable over time dates back at least to William Fleetwood who wrote in 1707; see Ferger [1946]. The use of the cost of a basket of goods as an index number was extensively developed by Lowe [1823; 331-346] and Scrope [1833; 401-425] who applied this idea to many practical problems of indexation. This commodity standard idea for adjusting the value of money has been independently discovered many times; see for example Marshall [1887; 371], Fisher [1911; ch. 13] and the many references in Fisher [1920; 291-294].

23. This point was first made by Walsh [1901; 96] [1921;88] and Davies [1924; 183][1932;59]. For more recent discussions on unit values, see Dalen [1992; 135], Diewert [1995; 20-24] and Balk [1995a].

24. See the data on the German hyperinflation in Sweeney [1927; 182].

25. Our observation here is substantially due to Marshall [1887; 374]:

"This class of consideration is of much more importance than at first sight appears; for a great part of modern agricultural and transport industries are devoted to increasing the period of time during which different kinds of foods are available. Neglect of this has, in my opinion vitiated the statistics of the purchasing power of money in medieval times with regard to nearly all kinds of food except corn; even the well-to-do would hardly get so simple a thing as fresh meat in winter."

26. It should be noted that assumption (27) is quite restrictive: it says that the consumer is indifferent to the annual consumption of each commodity taking place in a single season or spread out uniformly across the seasons.

27. Note that these conditions are Hicksian [1946; 312] aggregation conditions which guarantee the existence of annual aggregates. In fact, if conditions (28) hold, we do not have to make the restrictive assumption (27) in order to determine that $[q^{t1}, q^{t2}, \ldots, q^{tM}]$ solves the year t maximization problem in (15). To determine the annual aggregator function g^* under conditions (28), let $c(p^{t1*}, \ldots p^{tM*})$ be the unit cost function dual to $f(x^1, \ldots, x^M)$. Define the N variable unit cost function $c^*(p^t) \equiv c(p^t, \ldots, p^t)$. Then $g^*(\Sigma_{m=1}^{M} x^m)$ is dual to c^*.

28. Satisfaction of (33) will also ensure that the two Laspeyres indexes, $Q_L^*(P^{s*}, P^{t*}, Q^s, Q^t) \equiv P^{s*} \cdot Q^t / P^{s*} \cdot Q^s$ and $Q_L(p^{s*}, p^{t*}, q^s, q^t) \equiv p^{s*} \cdot q^t / p^{s*} \cdot q^s$ and the two Paasche indexes, $Q_P^*(P^{s*}, P^{t*}, Q^s, Q^t) \equiv P^{t*} \cdot Q^t / P^{t*} \cdot Q^s$ and $P_P(p^{s*}, p^{t*}, q^s, q^t) \equiv p^{t*} \cdot q^t / p^{t*} \cdot q^s$, are also equal.

29. For example see Bean and Stine's [1924; 31] Type D index number or Rothwell [1958; 70]. Incidentally, Flux [1921; 185] also proposed (and used) Bean and Stine's Type B index and Crump, in his discussion of Flux's [1921; 207] paper, proposed Bean and Stine's Type A index number. Finally, Bean and Stine's [1924; 31] Type C index number has come to be known as the Rothwell [1958; 71] index.

30. Actually, the exact index number approach used in this section in conjunction with the homogeneous separability assumptions (36) can be viewed as an extension of Shephard's [1953; 64-71] aggregation theory; see also Diewert [1974; 151].

31. Multiply both sides of the constraint in (37) by the discount factor ρ_{tm} and the resulting constraint becomes $p^{tm*} \cdot x^m = p^{tm*} \cdot q^{tm}$. This means that the nominal price vectors p^{sm} and p^{tm} in (39) and (40) can be replaced by the normalized price vectors p^{sm*} and p^{tm*}.

32. This requires that h be increasing. This technique was used by Shephard [1953; 64-71] [1970; 114-123] and Diewert [1974; 164-165] under various regularity conditions.

33. If there are no type (i) seasonal commodities, then consider an alternative two stage procedure where for each "physical" commodity, we aggregate over seasons within a year in the first stage and then aggregate over these "annual" commodities in the second stage, using say Laspeyres price indexes and Paasche quantity indexes at each stage. This two stage procedure would also give the same answer as the single stage procedure (41) if the Q in (41) were the Paasche index $Q_P(p^{0*}, p^{t*}, q^0, q^t) \equiv p^{t*} \cdot q^t / p^{t*} \cdot q^0$. These alternative two stage aggregation procedures when there are seasonal commodities were considered by Balk [1980a; 25] and Diewert [1980; 506-508]. Note that different separability assumptions are required to justify each procedure from the viewpoint of the economic approach.

34. In this case, equation (36) will hold even though Leontief aggregator functions are not strictly increasing in all arguments. If h is Leontief, then $h(Q_1, \ldots, Q_M) \equiv min_m\{Q_m/b_m : m = 1, \ldots, M\}$ where the b_m are positive constants. The corresponding unit cost function is $\Sigma_{m=1}^{M} b_m P_m$.

35. Put another way, the Leontief functional form is not flexible; i.e., its dual unit cost function can provide only a first order approximation to an arbitrary differentiable unit cost function.

36. Diewert's [1978; 888] results required that the second order approximations be taken around a point where the period t price vector equals the period 0 price vector and the period t quantity vector equals the period 0 quantity vector. However, the same second order approximation result will hold if these equality restrictions are relaxed to proportionality restrictions, since superlative indexes are homogeneous of degree 0 or 1 in their price and quantity vector arguments.

37. Diewert's [1983c; 1038] conclusions on this topic were summarized as follows: "Note that the two stage Fisher and Translog indexes in Table 8 coincide with the corresponding single stage Fisher and Translog indexes in both Tables 4 and 5 to four significant figures. Hence, the two stage indexes may be used in order to approximate very closely the corresponding single stage indexes".

38. As the quotation at the beginning of this section indicates, Zarnowitz [1961; 246] seems to feel that it is possible to somehow estimate cardinally comparable seasonal subutility functions f^m. Balk [1980a; 21] comments on the possibility of doing this as follows: "Here, a solution could only be provided by a reasoning based on the economic concepts

of ‘indifference curve’ and ‘production possibility curve’. So far, however, no success has been achieved in the practical implementation of such concepts”.

39. We want to include all nonseasonal and type (ii) (a) seasonal commodities in the $\tilde{x}^{tm}$ vectors to make the commodity coverage of the resulting “monthly” price indexes as broad as possible. However, we must exclude type (i) and type (ii) (b) seasonal commodities from the “monthly” aggregator function ϕ, since inclusion of these commodities would cause the resulting ϕ to shift as climate and customs changed across the seasons, thus making “monthly” index number comparisons impossible. In practice, it will be difficult to decide what is a type (ii) (a) seasonal commodity.

40. We also use the positivity of the discount factors δ_t and ρ_{tm} in deriving (53).

41. Rothwell [1958; 71] noted that the problem of making price comparisons between seasons with different market baskets is formally identical to the problem of making international comparisons between countries with different market baskets. This suggests that the symmetric methods used in making international comparisons could be applied to the problem of aggregating up the many bilateral price comparisons in (55) into a consistent sequence of “monthly” price levels. Balk [1981; 74] in fact implemented this idea, calculating a system of EKS (see Gini [1931; 12], Eltetö and Köves [1964] and Szulc [1964]) monthly purchasing power parities for Dutch fruit and vegetables. However, Walsh [1901; 399] and Balk [1981; 77] also noted a practical disadvantage to the use of these symmetric methods: the price levels have to be recalculated each time a new observation is added.

42. See the discussion and references in Diewert [1993a; 52-55].

43. In fact, it was these problems that led Julius Lehr [1885; 45-46] and Alfred Marshall [1887; 373] to introduce the chain system.

44. Obviously, our formal economic model needs to be modified to deal with the problem of new commodities. Note that Mudgett [1951; 46] called the error in an index number comparison that was introduced by the use of the highest common factor method, the homogeneity error.

45. Using annual Canadian data for 13 categories of consumption over the years 1947 to 1971, Diewert [1978; 894] provided some evidence to support these theoretical approximation results. By 1971, the fixed base Paasche and Laspeyres price indexes were 2.2763 and 2.3621 respectively whereas the chained Paasche and Laspeyres indexes were 2.3172 and 2.3285–a much narrower spread.

46. The chain index will give the correct answer if $\tilde{P}$ satisfies Walsh's [1901; 389] [1924; 506] multiperiod identity test; see Diewert [1993a; 40] for a discussion of this test. However, the Paasche, Laspeyres and all known superlative indexes do not satisfy this test and thus are vulnerable to the Szulc-Hill criticism of the chain system. Incidentally, Walsh [1901; 401] [1924; 506] was the first to make this criticism.

47. We are including only type (ii) (a) seasonal commodities in $\tilde{P}$. If it is too difficult to determine these seasonal commodities, then we exclude all seasonal commodities from $\tilde{P}$.

48. The reason for this statement is that Fisher [1922; 280-283] (and others) have found that $\tilde{P}_F$ satisfies Walsh's multiperiod identity test to a high degree of approximation.

49. See also Hofsten [1952; 97] and Fisher and Shell [1972; 101]

50. Diewert [1980; 502-503] suggested an econometric approach to the estimation of reservation prices but did not implement it. Hausman [1995] seems to have been the first to implement such an econometric approach. Fisher and Griliches [1995; 236] note that there is a further complication in the use of reservation prices when we are aggregating over consumers: the correct aggregate reservation price in the base period is a weighted average of individual base period reservation prices, where the weights are proportional to the purchases of the seasonal good in the current period when the commodity is available. We outline and extend their argument in Appendix 1 below.

51. As was mentioned earlier in section 6, the Vartia I [1974; 66-67] [1976] price index was used to establish the approximate consistency in aggregation of superlative indexes. It is interesting to note that Montgomery [1937; 37] defined the Vartia I index much earlier and also established its consistency in aggregation properties; see Montgomery [1937; 40-48].

52. For the properties of the logarithmic mean and references to the mathematics literature, see Carlson [1972].

53. Lau [1979; 75-81] clarified and extended the class of functions that P_{SV} is exact for. The CES aggregator function is equal to a positive constant times a weighted mean of order r. For the properties and axiomatic characterizations of means of order r, see Hardy, Littlewood and Polya [1934; 12-19] and Diewert [1993c; 381].

54. Reinsdorf and Dorfman [1995] show that P_{SV} fails to satisfy the four monotonicity axioms that are listed in Diewert [1992; 220] that are due to Eichhorn and Voeller [1976; 23] and Vogt [1980; 70]. The Fisher index P_F satisfies these axioms. See also Eichhorn [1978] on the test approach to index number theory.

55. P_{SV} is not pseudosuperlative (see Diewert [1978; 888]) either; i.e., when we evaluate the first and second order partial derivatives of $P_{SV}(p^0, p^1, q^0, q^1)$ around an equal price ($p^0 = p^1$) and equal quantity ($q^0 = q^1$) point, we find that the first order derivatives of P_{SV} coincide with the corresponding first order derivatives of a superlative index tabled in Diewert [1978; 893] but the second order derivatives do not. This is what we would expect since P_{SV} is exact for CES and two stage mixtures of CES and Cobb-Douglas functions (see Lau [1979; 75-81] for the precise results) and these functions can provide only first order approximations to arbitrary N commodity aggregator functions.

56. Actually, we are describing only the first stage in Balk's [1981; 73] procedure. In the second stage of his procedure, Balk [1981; 74] uses the multilateral Gini [1931; 12], Eltetö and Köves [1964] (EKS) index and Szulc [1964] (EKS) index to eliminate the influence of a base period on his seasonal price indexes.

57. Since seasonal price and quantity changes can be huge, the choice of index number formula makes a large difference. When Balk [1980a; 41] compared his Sato-Vartia indexes for Dutch fruit and vegetables with an alternative index number formula, he found some differences in the 50% range. Also Reinsdorf and Dorfman [1995; table 1] found substantial differences between the Sato-Vartia price index and the superlative Fisher and Törnqvist [1936] price indexes for some artificial data.

58. We follow Kohli [1978] [1991] in classifying imports as a primary input into the domestic production sector of the economy.

59. We have not dealt with the allocation of direct and indirect taxes and a host of other complications.

60. This seems consistent with Hill's [1995; ch. 4; 8] criticism of value added and GDP price indexes: "The indices are therefore sensitive to errors in both the output and input indices." The Accounting Research Division of the American Institute of Certified Public Accountants recommended that the GNP implicit price deflator be used as the measure of general price level change in business accounting because its universe encompasses the entire economy; see Tierney [1963; 112]. In our view, a price index based on deflating total final expenditures would be just as comprehensive but more appropriate.

61. See the references in Diewert [1980; 455-467] [1992; 230-237] and Caves, Christensen and Diewert [1982; 1395-1399].

62. Basically, it is not easy to construct user costs; many somewhat arbitrary judgements have to be made to construct rental prices. For discussions of the practical difficulties, see

Diewert [1980; 470-486] [1983b; 1100-1103]. The user cost concept dates back to Walras [1954].

63. See Archibald [1977], Fisher and Shell [1972; 53], Samuelson and Swamy [1974; 588], Sato [1976b; 438], Diewert [1980; 460-464] [1983b; 1063-1077] and Caves, Christensen and Diewert [1982; 1399-1401]. The difficulties involved in aggregating over producers are discussed in Domar [1961] and Diewert [1980; 464-470 and 495-498].

64. Investment goods prices are relevant to the construction of household real wealth estimates but not to real consumption estimates for the current period.

65. A case for dropping investment goods from the index number basket can also be made on reliability grounds; many investment goods are uniquely constructed for particular producers and hence the lack of comparability of investment goods over time is a massive problem.

66. See Hill [1993; 402-403] for some suggestions on how to construct these prices.

67. For a discussion of the problems involved in constructing user costs for consumer durables and references to the literature, see Diewert [1983a; 211-216].

68. The term "moving year" is due to Mendershausen [1937; 245]. Diewert [1983c; 1029] earlier used the term "split year" comparison to describe a variable year end index number comparison. Following the terminology used by Crump [1924; 185] in a slightly different context, we could also use the term "rolling year" comparison.

69. Diewert [1983c; 1034] assumed that U was the simple sum of seasonal utilities, $\Sigma_{t=0}^{T}\Sigma_{m=1}^{M} f^m(x^{tm})$. This is a special case of (66) and (67) with the $\beta_m = 1$ and $\alpha = 1$.

70. Recall that $\psi(z) \equiv z^\alpha$ if $\alpha \neq 0$. If $\alpha < 0$ then the *max* in (68) is replaced by a *min*, but equations (70) are still satisfied.

71. Once the Q_{tm} or P_{tm} have been defined by (75) or (76) for $t = 0$ and $m = 1, \ldots, M$, the chain principle can be used to relate the prices and quantities of each moving year with the prices and quantities of the immediately preceding moving year; see Diewert [1983c; 1031-1032] for some comparisons of fixed base and chained moving year price indexes using the Turvey [1979] data.

72. Since superlative indexes are exact for flexible aggregator functions, the flexible aggregator function can approximate the CES aggregator function in (74) to the second order.

73. See Babbage [1856; 30-43], Gilbart [1856; 152], Jevons [1884; 3-11 and 160-193] and Kemmerer [1910; 15]. Kemmerer [1910; 13] seems to have been the first to use the terms "seasonal variations" and "seasonal fluctuations"; Jevons [1884; 6] spoke of "commercial fluctuations" and " monthly and quarterly variations".

74. Cournot [1838; 25] initially distinguished two components as the following quotation indicates: "Here, as in astronomy, it is necessary to recognize *secular* variations, which are independent of *periodic* variations". Later, Cournot [1863; 149] also distinguished "les perturbations passagères, accidentelles (comme celles des comètes)"; i.e., transitory or accidental perturbations. Thus Cournot's periodic variations include both business cycle and seasonal fluctuations. It is interesting to note that Persons' [1917; 619] earlier classification of time series components did not include seasonal fluctuations.

75. The weights that Macaulay [1931; 141] used in his moving average were constructed so as to eliminate constant seasonal factors and exactly reproduce a quadratic trend over the window length of the moving average.

76. Perhaps the first use of a moving average to represent a long term trend in the price of a commodity was made by Poynting [1884; 36], a Professor of Physics!

77. Since we have run out of symbols, the notation used here is different from the rest of the paper.

78. This is in contrast to the situation in index number theory where current opinion seems to regard the set of tests that characterize the Fisher ideal index as satisfactory; see Diewert [1992], Balk [1995b] and Reinsdorf and Dorfman [1995].

79. Instead of the Sato-Vartia quantity index, we used the Fisher ideal quantity index in (75). We did not deflate the quarterly prices by an index of purchasing power since inflation was "small" over this period.

80. The average quarterly rate of growth for the official X-11 adjusted series was .78% compared to .85% per quarter for our centered Fisher ideal moving year series.

81. Until recently, most U.S. long term quantity series were constructed using a fixed base Laspeyres quantity index so that additivity of components could be preserved. With the recent huge increases in the quantity of computers and their equally huge declines in price, the use of fixed base price and quantity indexes has become unworkable: changing the base year leads to dramatic revisions in economic history. This illustrates the point emphasized by Hill [1988] [1993]: the base period in fixed base index numbers must be changed reasonably frequently.

82. This difference in annual growth rates is .3% per year which is approximately equal to 4 times our quarterly difference in growth rates of .07% per quarter.

83.

> "Even though the public appears for the most part to be comfortable with seasonally adjusted data, we doubt that many users understand the methods by which the data are produced. It may be too much to expect the statistically unsophisticated person to understand the procedures underlying seasonal adjustment, but even statistical experts are often mystified by these procedures, including the most widely used method, Census X-11."
>
> *William R. Bell and Steven C. Hillmer*[1984; 291]

84. The two choices are variants of (75): (i) should the inflation adjusted normalized prices p^{tm*} defined by (69) be replaced by the unadjusted spot prices p^{tm} and (ii) should the Sato-Vartia index number formula Q_{SV} which appears in (75) be replaced by some other index number formula?

85. Diewert [1995; 22] advocated this solution to the index number problem under high inflation but he neglected the seasonal commodities problem.

86. Recall equations (22) and (69).

87. Under conditions of high inflation, the price indexes defined by (76) will be difficult to interpret and hence the Statistical Agency would not have to report them. The primary focus should be on the production of the moving year quantity indexes defined by (75).

88. Mr. William Hawkes informs me that the A.C. Nielsen company based in the U.S. distinguished 1.65 million separate product codes as of September, 1995; i.e., this company has detailed price and quantity information by region on all of these commodities.

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