Irving Fisher and Index Number Theory

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Abstract

There are four main approaches to bilateral index number theory: the fixed basket, stochastic, test and economic approaches. The paper reviews the contributions of Irving Fisher to these approaches to index number theory which are still in use today. The paper also reviews Fisher's contributions to multilateral index number theory. The main themes of the paper are developed in the context of a review of the early history of index number theory: a history that conveys a wealth of information and insight into the making and use of index numbers today.

Key words: Price indexes, quantity indexes, bilateral price indexes, multilateral price indexes, the Fisher ideal index, the test approach to index number theory, spatial comparisons, fixed base versus chained index numbers, Walsh, Lowe, Edgeworth.

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1. Introduction

Irving Fisher's accomplishments as one of America's most accomplished economists are generally well known.[2] Our focus in the present paper will be on his contributions to index number theory. But before we look at his contributions to this subject, it will be useful to list the areas of research that Fisher himself thought were of some importance near the end of his life.

In 1946, Fisher gave an address on the founding of the Irving Fisher Foundation, a Foundation which he set up to continue research in areas that he pursued over his productive life. This address was later published as Fisher (1997) and we will review the areas of research that Fisher flagged as important areas where he had contributed and he hoped the Foundation would continue to explore.

Fisher (1997; 23-36) listed the following seven research areas:

- The basic principles of economic science;
- Monetary stabilization;
- The consumption tax;
- General economics;
- World peace;
- Health habits[3] and
- Eugenics.

The last three topics can best be left to others to discuss in more detail but a few sentences about the first four topics follow below.

Fisher summarized his work on the first topic as follows:

"One of my chief objects has been to help make economics into a genuine science through careful and sound analysis, usually carried out with the help of mathematical methods and statistical verification. My chief books written with this object in view are: The thesis just mentioned, 1892; *The Nature of Capital and Income*, 1906; *The Purchasing Power of Money*, 1911; *The Theory of Interest*, 1930; *Booms and Depressions*, 1932." Irving Fisher (1997; 23).

Fisher's doctoral thesis at Yale had the title *Mathematical Investigations in the Theory of Value and Prices* and it was an early study in the mathematics of general equilibrium

[2] His scientific works have been organized in a comprehensive fourteen volume collection of his papers and books edited by William J. Barber and assisted by Robert W. Dimand and Kevin Foster. For a biography of Fisher, see the first chapter in Volume 1 of this collection, Barber (1997). Fisher helped set up the Econometric Society (with Ragnar Frisch and C.F. Roos) and was its first President; see Barber (1997; 17).
[3] In 1898, Fisher contracted tuberculosis and this helps to explain his life long interest in health; see Barber (1997; 8).

theory.[4] There is no doubt that Fisher's contributions to the development of a modern theory of the interest rate in a general equilibrium context were seminal and are still used in introductory economic courses today. His work on operationalizing the quantity theory of money is still relevant today and it was this work that led him to attempt to measure general inflation and this in turn led him into index number theory.

Fisher's second listed area of interest was *monetary stabilization*. He had a keen sense of the injustices that resulted from unanticipated inflation in the context of debt contracts:

"In all mathematical sciences, the first essential is a unit for measurement. The lack of such a unit for measuring the value of money in the sense of its purchasing power soon impressed me. Several times within my life the dollar has more than doubled or been more than halved in value. Yet throughout all these vicissitudes, only a handful of people had the slightest idea of what was happening." Irving Fisher (1997; 23).

Fisher's solution to the injustices generated by unanticipated rates of inflation was to index interest rates to the rate of inflation and so the measurement of general inflation became an important practical problem.[5] Fisher felt strongly about this problem and wrote six books and hundreds of articles on the problem of monetary stabilization and indexation.[6]

Fisher's third listed area was the *consumption tax*, which Fisher (1997; 28) called the *spending tax*. He felt that savings should not be taxed, partly because he felt this led to double taxation (if there were consumption taxes in addition to savings taxes) and partly because of the adverse efficiency and growth effects of taxing interest:

"In several books (especially *The Nature of Capital and Income*, 1906, and *Constructive Income Taxation*, 1942) as well as in many articles, I have endeavored to show that, whenever part of an income is *saved* and thus becomes capital, a tax on that part, followed by a like tax on the subsequent income from that capital, amounts to double taxation. ... I have shown that, had our present income tax been in existence at the beginning of the century, industrial progress could not have been as rapid as it was ... for the resulting tax on savings (from which we now suffer) would have substantially reduced corporate savings—and a reduction of corporate savings is a reduction of *expansion*." Irving Fisher (1997; 28).

Thus Fisher was a strong advocate of a consumption tax as opposed to an income tax.

Fisher's fourth listed area was *general economics*. What Fisher had in mind here was the study of economic efficiency and its tradeoff with possible inequalities in income and wealth:

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[4] Fisher's initial graduate program at Yale was in mathematics and he later switched into economics on the advice of a Yale economist, William Sumner; see Barber (1997; 4). Thus Fisher was mathematically well prepared to write in the new field of mathematical economics.

[5] See Fisher (1911; 340) where he refers to the "tabular standard". This was a terminology introduced by Scope (1833) who followed Lowe (1823) in suggesting the same indexation remedy for interest rates that was suggested by Fisher. As we shall see later, Lowe should be considered the father of the Consumer Price Index.

[6] See Fisher (1997; 27) for the book references.

"In our efforts to improve distribution we must not impair production—must, in fact, improve it. As my first teacher in economics, Professor William Graham Sumner, often said, we can accomplish more by levelling up than by levelling down, meaning that most efforts 'to soak the rich' do not help but rather harm the poor by reducing production." Irving Fisher (1997; 30).

But Fisher was concerned that the non taxation of savings might lead to large disparities in wealth over time and suggested some solutions to this problem:

"In line with this thought I have stressed two measures: (1) exempt all savings from income taxes, for savings in general mean added capital equipment and this means greater production. ... (2) tax the savings only when they pass out of the hands of the accumulator, that is (chiefly) on his death; ... (3) during life, tax all progressively on excessive spending, that is on their disaccumulations." Irving Fisher (1997; 30).

Thus to compensate for the lack of taxation on savings, Fisher advocated a progressive consumption tax along with an inheritance tax.[7]

The above material gives us some indication of why Fisher became interested in index number theory: it was a consequence of his interest in using economics (and statistics) to improve society.

In order to evaluate Fisher's lasting contributions to index number theory, it will be useful to give the reader a review of the current state of index number theory. There are four main approaches to index number theory that are widely used today:[8]

- The fixed basket approach;
- The stochastic approach;
- The test approach and
- The economic approach.

Thus in sections 3-6 below, we review each of these approaches (and note Fisher's contributions to these approaches). However, before discussing these approaches, in section 2 we present a preliminary discussion on the *levels approach* to index number theory versus the *value ratio decomposition approach* (with the latter approach being the one that Fisher followed).

In section 7, we discuss why Fisher was opposed to the stochastic and economic approaches to index number theory. In section 8, we will discuss Fisher's contributions to

[7] With only a consumption tax and no inheritance tax, the rich (who refused to consume their income) would tend to become richer over time and this can lead to counterproductive social unrest.

[8] There is a fifth approach that is widely used in theoretical work and that is the approach due to Divisia (1926). This is a continuous time approach to index number theory but in practice, continuous time derivatives or integrals have to be approximated by discrete time prices and quantities. The problem with this approach was recognized long ago by Frisch (1936; 8): "As the elementary formula of the chaining, we may get Laspeyre's or Paasche's or Edgeworth's or nearly any other formula, according as we choose the approximation principle for the steps of the numerical integration." For more on discrete time approximation to Divisia indexes, see Diewert (1980; 44-446) and for comprehensive reviews of the Divisia approach, see Vogt(1978) and Balk (2005) (2008).

multilateral index number theory. The issue of whether to use fixed base or chained index numbers arises in this context, so in section 8, we also discuss Fisher's thoughts on this issue. Section 9 concludes.

2. The Levels and Value Ratio Approaches to Index Number Theory

It will be useful to follow the example of Fisher (1922; 8) and set the stage for the subsequent discussion of alternative approaches by defining more precisely what the index number problem is.

We specify two accounting periods,[9] $t = 0,1$ for which we have micro price and quantity data for N commodities pertaining to transactions by a consumer or producer (or a well defined group of consumers or producers). Denote the price and quantity of commodity n in period t by p_n^t and q_n^t respectively for $n = 1,2,\ldots,N$ and $t = 0,1$. Before proceeding further, we need to discuss the exact meaning of the microeconomic prices and quantities if there are *multiple* transactions for say commodity n within period t. In this case, it is natural to interpret q_n^t as the *total* amount of commodity n transacted within period t. In order to conserve the value of transactions, it is necessary that p_n^t be defined as a *unit value*[10]; i.e., p_n^t must be equal to the value of transactions for commodity n during period t divided by the total quantity transacted, q_n^t. For $t = 0,1$, define *the value of transactions in period t* as:

$$(1)\ V^t \equiv \sum_{n=1}^{N} p_n^t q_n^t \equiv p^t \cdot q^t$$

where $p^t \equiv (p_1^t,\ldots, p_N^t)$ is the period t price vector, $q^t \equiv (q_1^t,\ldots, q_N^t)$ is the period t quantity vector and $p^t \cdot q^t$ denotes the inner product of these two vectors.

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[9] How long should the accounting period be? Fisher anticipated Hicks (1946; 122) in suggesting that the accounting period should be short enough so that variation in prices within the period could be ignored: "Throughout this book, 'the price' of any commodity or 'the quantity' of it for any one year was assumed given. But what is such a price or quantity? Sometimes it is a single quotation for January 1 or July 1, but usually it is an average of several quotations scattered through the year. The question arises: On what principle should this average be constructed? The *practical* answer is *any* kind of average since, ordinarily, the variations during a year, so far, at least, as prices are concerned, are too little to make any perceptible difference in the result, whatever kind of average is used. Otherwise, there would be ground for subdividing the year into quarters or months until we reach a small enough period to be considered practically a point." Irving Fisher (1922; 318).

[10] The early index number theorists Walsh (1901; 96), Fisher (1922; 318) and Davies (1924; 96) all suggested unit values as the prices that should be inserted into an index number formula. Walsh nicely sums up the case for unit values as follows: "Some nice questions arise as to whether only what is consumed in the country, or only what is produced in it, or both together are to be counted; and also there are difficulties as to the single price quotation that is to be given at each period to each commodity, since this, too, must be an average. Throughout the country during the period a commodity is not sold at one price, nor even at one wholesale price in its principle market. Various quantities of it are sold at different prices, and the full value is obtained by adding all the sums spent (at the same stage in its advance towards the consumer), and the average price is found by dividing the total sum (or the full value) by the total quantities." Correa Moylan Walsh (1921a; 88).

Using the above notation, we can now state the following *levels version of the index number problem using the test or axiomatic approach*: for t = 0,1, find scalar numbers P^t and Q^t such that

(2) $V^t = P^tQ^t$.

The number P^t is interpreted as an aggregate period t price level while the number Q^t is interpreted as an aggregate period t quantity level. The aggregate price level P^t is allowed to be a function of the period t price vector, p^t while the aggregate period t quantity level Q^t is allowed to be a function of the period t quantity vector, q^t; i.e., we have

(3) $P^t = c(p^t)$ and $Q^t = f(q^t)$; $t = 0,1$.

However, from the viewpoint of the *test approach* to index number theory, the levels approach to finding aggregate quantities and prices comes to an abrupt halt: Eichhorn (1978; 144) showed that if the number of commodities N in the aggregate is equal to or greater than 2 and we restrict $c(p^t)$ and $f(q^t)$ to be positive if the micro prices and quantities $p_n{}^t$ and $q_n{}^t$ are positive, then there do not exist any functions c and f such that $c(p^t)f(q^t) = p^t{\cdot}q^t$ for all $p^t >> 0_N$ and $q^t >> 0_N$.[11]

This negative result can be reversed if we take the *economic approach* to index number theory. In this approach, we assume that the economic agent has a linearly homogeneous utility function, f(q), and when facing the prices p^t chooses q^t to solve the following cost minimization problem:

(4) $\min_q \{p^t{\cdot}q : p^t{\cdot}q = Y^t ; q \geq 0_N\}; t = 0,1$

where period t "income" Y^t is defined as $p^t{\cdot}q^t$. In this setup, it turns out that c(p) is the unit cost function that is dual[12] to the linearly homogeneous utility function f(q) and we can define P^t and Q^t as in (3) with $P^tQ^t = c(p^t)f(q^t) = p^t{\cdot}q^t$ for t = 0,1. Why does the economic approach work in the levels version of the index number problem whereas the test approach does not? In the test approach, both p^t and q^t are regarded as completely independent variables, whereas in the economic approach, p^t can vary independently but q^t cannot vary independently; it is a solution to the period t cost minimization problem (4).

Even though the economic approach to the index number problem as formulated above "works", it is not a *practical* solution that statistical agencies can implement and provide suitable aggregates to the public. In order to implement this solution, the statistical agency would have to hire hundreds of econometricians in order to estimate cost functions for all relevant macroeconomic aggregates and it is simply not feasible to do this. Thus we turn to our second formulation of the index number problem and it is this

[11] Notation: $p >> 0_N$ means all components of p are positive; $p \geq 0_N$ means all components of p are nonnegative and $p > 0_N$ means $p \geq 0_N$ but $p \neq 0_N$.
[12] See Diewert (1974) for materials and references to the literature on duality theory.

formulation that was initiated by Fisher (1911) (1922) in his two books on index number theory.

In the second approach to index number theory, instead of trying to decompose the value of the aggregate into price and quantity components for a single period, we instead attempt to decompose a *value ratio* for the two periods under consideration into a *price change component* P times a *quantity change component* Q.[13] Thus we now look for two functions of 4N variables, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ such that:[14]

$$(5)\ p^1 \cdot q^1/p^0 \cdot q^0 = P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1).$$

If we take the test approach, then we want equation (5) to hold for all positive price and quantity vectors pertaining to the two periods under consideration, p^0,p^1,q^0,q^1. If we take the economic approach, then only the price vectors p^0 and p^1 are regarded as independent variables while the quantity vectors, q^0 and q^1, are regarded as dependent variables.

In this second approach to index number theory, the *price index* $P(p^0,p^1,q^0,q^1)$ and the *quantity index* $Q(p^0,p^1,q^0,q^1)$ cannot be determined independently; i.e., if either one of these two functions is determined, then the remaining function is implicitly determined using equation (5). Historically, the focus has been on the determination of the price index but Fisher (1911; 388) was the first to realize that once the price index was determined, then equation (5) could be used to determine the companion quantity index.[15] Fisher (1911; 401) also realized that one could assume that Q had certain properties or satisfied certain tests and that these quantity tests would imply that the companion price index defined via (5) would then satisfy some interesting properties or tests that were not immediately evident. Thus it can already be seen that Fisher made some fundamental contributions to index number theory.[16]

This value ratio decomposition approach to index number is called *bilateral index number theory* and its focus is the determination of "reasonable" functional forms for P and Q. Fisher's 1911 and 1922 books address this functional form issue using the test approach.

We turn now to a discussion of the various approaches that have been used to determine the functional form for the bilateral price index, $P(p^0,p^1,q^0,q^1)$.

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[13] Looking ahead to the economic approach, P will be interpreted to be the ratio of unit cost functions, $c(p^1)/c(p^0)$, and Q will be interpreted to be the utility ratio, $f(q^1)/f(q^0)$. Note that the linear homogeneity assumption on the utility function f effectively cardinalizes utility.

[14] If N = 1, then we define $P(p_1^0,p_1^1,q_1^0,q_1^1) \equiv p_1^1/p_1^0$ and $Q(p_1^0,p_1^1,q_1^0,q_1^1) \equiv q_1^1/q_1^0$, the single price ratio and the single quantity ratio respectively. In the case of a general N, we think of $P(p_1^0,p_1^1,q_1^0,q_1^1)$ as being a weighted average of the price ratios p_1^1/p_1^0, p_2^1/p_2^0, ..., p_N^1/p_N^0. Thus we interpret $P(p_1^0,p_1^1,q_1^0,q_1^1)$ as an aggregate price ratio, P^1/P^0, where P^t is the aggregate price level for period t for t = 0,1.

[15] This approach to index number theory is due to Fisher (1911; 418) who called the implicitly determined Q, the *correlative formula*. Frisch (1930; 399) later called (5) the *product test*.

[16] Fisher (1922; 451) and Frisch (1930; 398) were the first to recognize these two approaches to index number theory: Frisch called levels type index numbers *absolute indexes* and bilateral indexes *ratio type indexes*.

3. Fixed Basket Approaches to Index Number Theory

A very simple approach to the determination of a price index over a group of commodities is the *fixed basket approach*. In this approach, we are given a basket of commodities that is represented by the positive quantity vector q. Given the price vectors for periods 0 and 1, p^0 and p^1 respectively, we can calculate the cost of purchasing this same basket in the two periods, $p^0 \cdot q$ and $p^1 \cdot q$. Then the ratio of these costs is a very reasonable indicator of pure price change over the two periods under consideration, provided that the basket vector q is "representative". Thus define the *Lowe price index*, P_{Lo}, as follows:

(6) $P_{Lo}(p^0,p^1,q) \equiv p^1 \cdot q/p^0 \cdot q$.

Although Lowe (1823) was not the first to suggest this form of price index,[17] he considered the problems associated with its construction and the uses that it could be put to in much more detail than his precursors:

"Of this, some idea may be formed from a table in the Appendix comprising a list of articles of general consumption, corn, butcher-meat, manufactures, tropical products, &c. and containing the probable amount of money expended on each by the public. This table is followed by explanatory remarks, of which the object is to show that contracts for a series of years ought to be made with a reference to the power of money in purchasing the necessaries and comforts of life; that after fixing a given sum, say 100*l*. as the amount of an annual salary, the payment in subsequent years should be not necessarily 100*l*., but either 95*l*., 100*l*., or 105*l*., according to the varying power of money in making purchases…. For the details of the table, and the calculations connected with it, we refer to the Appendix: at present we shall, for the sake of illustration, suppose it in operation, and bestow a few paragraphs on the effects that the adoption of such a measure would have on the interests of the country. In what, it may be asked, would the benefits of it consist? In ascertaining on grounds that would admit of no doubt or dispute, the power in purchase of any given sum in one year, compared to its power of purchase in another. And what would be the practical application of this knowledge? The correction of a long list of anomalies in regard to rents, salaries, wages, &c., arising out of unforeseen fluctuations in our currency." Joseph Lowe (1823; 333-335).

It can be seen that Lowe laid out a fairly modern theory for the construction of a Consumer Price Index.[18] In order that the basket vector be "representative", Lowe (1823; Appendix page 95) suggested that the commodity basket vector q should be updated every five years. Lowe was also concerned with the problem of the unfairness of unforeseen changes in the price level has on fixed interest contracts that concerned Fisher and in fact, Fisher (1911; 248) refers to Lowe's path breaking book.[19]

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[17] The price index (6) was earlier suggested by Bishop William Fleetwood in 1707; see Ferger (1946) for a detailed account of Fleetwood's contributions. It was also used by the Legislature of Massachusetts in 1780 to index the pay of soldiers fighting in the American Revolutionary War; see Fisher (1913; 437) for an account.

[18] Lowe (1823; 336) also advocated different indexes for different demographic groups of households. The Lowe index is widely used even today by statistical agencies; see Chapter 15 of the ILO (2004).

[19] Scrope (1833) followed up Lowe's path breaking work with his own book where he presented his own version of Lowe's material and coined the term, "the tabular standard" to denote the Lowe index. This term caught on; it was used by Walsh (1901; 555), Fisher (1911; 208) and many others. But Walsh (1901; 539) mistakenly attributed formula (6) to Scrope and since Walsh was widely read 100 years ago, Lowe was

As time passed, economists and price statisticians demanded a bit more precision with respect to the specification of the basket vector q. There are two natural choices for the reference basket: the period 0 commodity vector q^0 or the period 1 commodity vector q^1. These two choices lead to the Laspeyres (1871) price index P_L defined by (7) and the Paasche (1874) price index P_P defined by (8):[20]

$$(7)\ P_L(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^0/p^0\cdot q^0 = \sum_{n=1}^N s_n^0(p_n^1/p_n^0)\ ;$$

$$(8)\ P_P(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/p^0\cdot q^1 = [\sum_{n=1}^N s_n^1(p_n^1/p_n^0)^{-1}]^{-1}$$

where the period t expenditure share on commodity n, s_n^t, is defined as $p_n^tq_n^t/p^t\cdot q^t$ for n = 1,…,N and t = 0,1. Thus the Laspeyres price index P_L can be written as a base period expenditure share weighted average of the N price ratios (or price relatives), p_n^1/p_n^0.[21] The last equation in (8) shows that the Paasche price index P_P can be written as a period 1 (or current period) expenditure share weighted *harmonic* average of the N price ratios.[22]

The problem with these index number formulae is that they are equally plausible but in general, they will give different answers. This suggests that if we require a single estimate for the price change between the two periods, then we should take some sort of evenly weighted average of the two indexes as our final estimate of price change between periods 0 and 1. Examples of such symmetric averages are the arithmetic mean, which leads to the Drobisch (1871) Sidgwick (1883; 68) Bowley (1901; 227)[23] index, $(1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the *Fisher*[24] (1922) *ideal index*, P_F, defined as

$$(9)\ P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)\ P_P(p^0,p^1,q^0,q^1)]^{1/2}\ .$$

At this point, the fixed basket approach to index number theory has to draw on the *test approach* to index number theory; i.e., in order to determine which of these fixed basket indexes or which averages of them might be "best", we need *criteria* or *tests* or *properties* that we would like our indexes to satisfy.

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generally forgotten until recently. Walsh later apologized for this mistake: "This is the method used by Lowe within a year or two of one hundred years ago. In my book, I called it Scope's index number; but it should be called Lowe's." Correa Moylan Walsh (1921b; 543).

[20] Note that $P_L(p^0,p^1,q^0,q^1)$ does not actually depend on q^1 and $P_P(p^0,p^1,q^0,q^1)$ does not actually depend on q^0. However, it does no harm to include these vectors and the notation indicates that we are in the realm of bilateral index number theory.

[21] This result is due to Walsh (1901; 428 and 539).

[22] This expenditure share and price ratio representation of the Paasche index is described by Walsh (1901; 428) and derived explicitly by Fisher (1911; 365).

[23] See Diewert (1992) (1993a) and Balk (2008) for additional references to the early history of index number theory.

[24] Bowley (1899; 641) appears to have been the first to suggest the use of this index. Walsh (1901; 429) mentions it and notes that it gave "no better" results than his preferred index defined by (13) below. However, later Walsh (1921a; 102) strongly endorsed this index: "This method still fails to satisfy the circular test; but perhaps it satisfies it best of all."

Let a and b be two positive numbers. A *mean* or an *average* of the two numbers a and b is a function m(a,b) that has certain properties, the most important of which is the property m(a,a) = a; i.e., if the two numbers are the same, then the mean function is equal to this common number. Diewert (1993b; 361) defined a *symmetric homogeneous mean* of a and b as a function m(a,b) that has the following properties: m(a,a) = a for all a > 0; m(a,b) is a continuous and increasing function of a and b; m(a,b) = m(b,a) for all a and b and $m(\lambda a,\lambda b) = \lambda m(a,b)$ for all positive a, b and λ.

What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical cost of living index? It is very desirable for a price index formula that depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test*.[25] We say that the index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

$$(10)\quad P(p^1,p^0,q^1,q^0) = 1/P(p^0,p^1,q^0,q^1)\ ;$$

i.e., if we interchange the period 0 and period 1 price and quantity data and evaluate the index, then this new index $P(p^1,p^0,q^1,q^0)$ is equal to the reciprocal of the original index $P(p^0,p^1,q^0,q^1)$.

Diewert (1997; 138) showed that the Fisher ideal price index defined by (9) above is the *only* index that is a homogeneous symmetric mean of the Laspeyres and Paasche price indexes, P_L and P_P, and satisfies the time reversal test (10) above. Thus our first *symmetric basket approach* to bilateral index number theory leads to the Fisher index (9) as being "best" from the perspective of this approach.[26]

Instead of looking for a "best" average of the two fixed basket indexes that correspond to the baskets chosen in either of the two periods being compared, we could instead look for a "best" average basket of the two baskets represented by the vectors q^0 and q^1 and then use this average basket to compare the price levels of periods 0 and 1.[27] Thus we ask that the nth quantity weight, q_n, be an average or *mean* of the base period quantity q_n^0 and the period 1 quantity for commodity n q_n^1, say $m(q_n^0,q_n^1)$, for n = 1,2,…,N.[28] Price statisticians refer to this type of index as a *pure price index* and it corresponds to Knibbs' (1924; 43) *unequivocal price index*. Under these assumptions, the pure price index can be defined as a member of the following class of index numbers:

[25] The concept of this test is due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test (and the commensurability test to be discussed later) that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 324) and Fisher (1922; 64).

[26] Bowley was an early advocate of taking a symmetric average of the Paasche and Laspeyres indexes: "If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean … as a first approximation." Arthur L. Bowley (1901; 227). Fisher (1911; 418-419) (1922) considered taking the arithmetic, geometric and harmonic averages of the Paasche and Laspeyres indexes.

[27] Walsh (1901) (1921a) and Fisher (1922) considered both averaging strategies in their classic studies on index numbers.

[28] Note that we have chosen the mean function $m(q_n^0,q_n^1)$ to be the same for each commodity n.

(11) $P_K(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} p_n^1 m(q_n^0,q_n^1) / \sum_{j=1}^{N} p_j^0 m(q_j^0,q_j^1)$.

In order to determine the functional form for the mean function m, it is necessary to impose some *tests* or *axioms* on the pure price index defined by (11). Again we ask that P_K satisfy the *time reversal test*, (10) above. Under this hypothesis, it can be shown that m must be a *symmetric mean*; i.e., m must satisfy the following property: $m(a,b) = m(b,a)$ for all $a > 0$ and $b > 0$.

The assumption that m must be a symmetric mean still does not pin down the functional form for the pure price index defined by (13) above. For example, the function m(a,b) could be the *arithmetic mean*, $(1/2)a + (1/2)b$, in which case (11) reduces to the *Marshall* (1887) *Edgeworth* (1925) *price index* P_{ME}, which was the pure price index preferred by Knibbs (1924; 56):

(12) $P_{ME}(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} p_n^1(1/2)(q_n^0 + q_n^1)/\sum_{j=1}^{N} p_j^0(1/2)(q_j^0 + q_j^1)$.

On the other hand, the function m(a,b) could be the *geometric mean*, $(ab)^{1/2}$, in which case (11) becomes the *Walsh* (1901; 398) (1921a; 97) *price index*, P_W[29]:

(13) $P_W(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} p_n^1(q_n^0 q_n^1)^{1/2}/\sum_{j=1}^{N} p_j^0(q_j^0 q_j^1)^{1/2}$.

However, there are many other possibilities for the mean function m, including the mean of order r, $[(1/2)a^r + (1/2)b^r]^{1/r}$ for $r \neq 0$. Obviously, in order to completely determine the functional form for the pure price index P_K, we need to impose at least one additional test or axiom on $P_K(p^0,p^1,q^0,q^1)$.

In order to obtain an additional axiom, we note that there is a problem with the use of the Marshall Edgeworth price index (12) in the context of using the formula to make international comparisons of prices. If the prices of a very large country are compared to the prices of a small country using formula (12), then the quantity vector of the large country may totally overwhelm the influence of the quantity vector corresponding to the small country. In technical terms, the Marshall Edgeworth formula is not homogeneous of degree 0 in the components of both q^0 and q^1. One way of preventing this problem from occurring, we could ask that P_K defined by (11) satisfy the following *invariance to proportional changes in current quantities test*:

(14) $P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

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[29] Walsh endorsed P_W as being the best index number formula: "We have seen reason to believe formula 6 better than formula 7. Perhaps formula 9 is the best of the rest, but between it and Nos. 6 and 8 it would be difficult to decide with assurance." C.M. Walsh (1921a; 103). His formula 6 is P_W defined by (13) and his 9 is the Fisher ideal defined by (9) above. His formula 8 is the formula $p^1 \cdot q^1/p^0 \cdot q^0 Q_W(p^0,p^1,q^0,q^1)$, which is known as the implicit Walsh price index where $Q_W(p^0,p^1,q^0,q^1)$ is the Walsh quantity index defined by (13) except the role of prices and quantities is interchanged. Thus although Walsh thought that his Walsh price index was the best functional form, his implicit Walsh price index and the "Fisher" formula were not far behind.

It can be shown that these two tests, the time reversal test (10) and the invariance test (14), enable us to determine the precise functional form for the pure price index P_K defined by (11) above where m is a symmetric homogeneous mean: the pure price index P_K must be the Walsh index P_W defined by (13).[30]

Thus the fixed basket approach to bilateral index number theory starts out with the Laspeyres and Paasche price indexes. Some form of averaging of these two indexes is called for since both indexes are equally plausible. Averaging these two indexes directly leads to the Fisher ideal index P_F defined by (9) as being "best" while a direct averaging of the two quantity baskets q^0 and q^1 leads to the Walsh price index P_W defined by (13) as being "best".

We turn now to another early approach to the index number problem.

4. Stochastic and Descriptive Statistics Approaches to Index Number Theory

The (unweighted) stochastic approach to the determination of the price index can be traced back to the work of Jevons (1865) (1884) and Edgeworth (1888) (1896) (1901) over a hundred years ago[31].

The basic idea behind the stochastic approach is that each price relative, p_n^1/p_n^0 for n = 1,2,…,N, can be regarded as an estimate of a common inflation rate α between periods 0 and 1; i.e., Jevons and Edgeworth essentially assumed that

$$(15)\ p_n^1/p_n^0 = \alpha + \varepsilon_n \ ; \ n = 1,2,\ldots,N$$

where α is the common inflation rate and the ε_n are random variables with mean 0 and variance σ^2. The least squares estimator for α is the *Carli* (1804) *price index* P_C defined as

$$(16)\ P_C(p^0,p^1) \equiv \sum_{n=1}^N (1/N)(p_n^1/p_n^0).$$

Unfortunately, P_C does not satisfy the time reversal test, i.e., $P_C(p^1,p^0) \neq 1/P_C(p^0,p^1)$[32].

[30] See section 7 of Diewert (2001).

[31] For additional references to the early literature, see Aldrich (1992), Diewert (1993a; 37-38) (1995) and Balk (2008; 32-36).

[32] In fact Fisher (1922; 66) noted that $P_C(p^0,p^1)P_C(p^1,p^0) \geq 1$ unless the period 1 price vector p^1 is proportional to the period 0 price vector p^0; i.e., Fisher showed that the Carli index has a definite upward bias. Walsh (1901; 327) established this inequality for the case N = 2 but Fisher did not acknowledge this contribution by Walsh to the study of bias in index number formulae. Fisher urged users to abandon the use of the Carli index but his advice was generally ignored by statistical agencies until recently: "In fields other than index numbers it is often the best form of average to use. But we shall see that the simple arithmetic average produces one of the very worst of index numbers. And if this book has no other effect than to lead to the total abandonment of the simple arithmetic type of index number, it will have served a useful purpose." Irving Fisher (1922; 29-30).

Now assume that the logarithm of each price relative, $\ln(p_n^1/p_n^0)$, is an independent unbiased estimate of the logarithm of the inflation rate between periods 0 and 1, β say. Thus we have:

$$(17)\ \ln(p_n^1/p_n^0) = \beta + \varepsilon_n \ ; \ n = 1,2,\ldots,N$$

where $\beta \equiv \ln\alpha$ and the ε_n are independently distributed random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for β is the logarithm of the geometric mean of the price relatives. Hence the corresponding estimate for the common inflation rate α is the *Jevons* (1865) *price index* P_J defined as:

$$(18)\ P_J(p^0,p^1) \equiv \prod_{n=1}^N (p_n^1/p_n^0)^{1/N}.$$

The Jevons price index P_J does satisfy the time reversal test and hence is much more satisfactory than the Carli index P_C. However, both the Jevons and Carli price indexes suffer from a fatal flaw: each price relative p_n^1/p_n^0 is regarded as being equally important and is given an equal weight in the index number formulae (16) and (18).[33] Keynes (1930; 76-81) also criticized the unweighted stochastic approach to index number theory on two other grounds: (i) price relatives are not distributed independently and (ii) there is no single inflation rate that can be applied to all parts of an economy; e.g., Keynes demonstrated empirically that wage rates, wholesale prices and final consumption prices all had different rates of inflation. In order to overcome the Keynesian criticisms of the unweighted stochastic approach to index numbers, it is necessary to:

- have a definite domain of definition for the index number and
- weight the price relatives by their economic importance.

Theil (1967; 136-137) proposed a solution to the lack of weighting in (17). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the nth price relative is equal to $s_n^0 \equiv p_n^0q_n^0/p^0\cdot q^0$, the period 0 expenditure share for commodity n. Then the overall mean (period 0 weighted) logarithmic price change is $\sum_{n=1}^N s_n^0\ln(p_n^1/p_n^0)$. Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) logarithmic price change of $\sum_{n=1}^N s_n^1\ln(p_n^1/p_n^0)$. Each of these measures of overall logarithmic price change seems equally valid so we could argue for taking a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change. Theil (1967; 137) argued that a nice symmetric index number formula can be obtained if we make the probability of selection for the nth price relative equal to the arithmetic average of the period 0 and 1 expenditure shares for commodity n. Using these probabilities of selection, Theil's final measure of overall logarithmic price change is

[33] Walsh (1901) (1921a; 82-83), Fisher (1922; 43) and Keynes (1930; 76-77) all objected to the lack of weighting in the unweighted stochastic approach to index number theory.

(19) $\ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N (1/2)(s_n^0+s_n^1)\ln(p_n^1/p_n^0)$.

It is possible to give a *descriptive statistics* interpretation of the right hand side of (19). Define the nth logarithmic price ratio r_n by:

(20) $r_n \equiv \ln(p_n^1/p_n^0)$ for $n = 1,\ldots,N$.

Now define the discrete random variable, R say, as the random variable which can take on the values r_n with probabilities $\rho_n \equiv (1/2)(s_n^0+s_n^1)$ for $n = 1,\ldots,N$. Note that since each set of expenditure shares, s_n^0 and s_n^1, sums to one, the probabilities ρ_n will also sum to one. It can be seen that the expected value of the discrete random variable R is $\ln P_T(p^0,p^1,q^0,q^1)$ as defined by the right hand side of (19). Thus the logarithm of the index P_T can be interpreted as *the expected value of the distribution of the logarithmic price ratios* in the domain of definition under consideration, where the N discrete price ratios in this domain of definition are weighted according to Theil's probability weights, ρ_n.

Taking antilogs of both sides of (19), we obtain the Theil price index, P_T.[34] This index number formula has a number of good properties. In particular, P_T satisfies the time reversal test (10) and the linear homogeneity test (14).[35]

It is possible to consider a descriptive statistics approach to index number theory where we look at the distribution of price ratios, p_n^1/p_n^0, rather than the distribution of the logarithmic price ratios, $\ln(p_n^1/p_n^0)$. Thus, following in the footsteps of Theil leads to price indexes of the form $\sum_{n=1}^N s_n^0(p_n^1/p_n^0)$, $\sum_{n=1}^N s_n^1(p_n^1/p_n^0)$ and more generally of the form:

(21) $P_m(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N m(s_n^0,s_n^1)(p_n^1/p_n^0)$

where $m(s_n^0,s_n^1)$ is a homogeneous symmetric mean of the period 0 and 1 expenditure shares, s_n^0 and s_n^1. In order to interpret the right hand side of (21) as an expected value of the price ratios p_n^1/p_n^0, it is necessary that

(22) $\sum_{n=1}^N m(s_n^0,s_n^1) = 1$.

However, in order to satisfy (22), m cannot be a symmetric geometric or harmonic mean or any of the commonly used homogeneous symmetric means. In fact, the only simple homogeneous symmetric mean that satisfies (22) is the arithmetic mean. With this choice of m, (21) becomes the following (unnamed) index number formula, P_u:

[34] This index first appeared explicitly as formula 123 in Fisher (1922; 473). P_T is generally attributed to Törnqvist (1936) but this article did not have an explicit definition for P_T; it was defined explicitly in Törnqvist and Törnqvist (1937); see Balk (2008; 26). Persons (1928; 21-22) looked for index number formulae that satisfied the time reversal test and his new test, the *absence of weight correlation bias test*. He found nine admissible index number formulae, including the Törnqvist and Fisher ideal indexes.

[35] For a listing of some of the tests that P_T, P_F, and P_W satisfy, see Diewert (1992; 223). In Fisher's 1922 book, these indexes were listed as numbers 123, 353 and 1153 respectively.

$$(23)\ P_u(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} (1/2)(s_n^{\,0} + s_n^{\,1})(p_n^{\,1}/p_n^{\,0}).$$

Unfortunately, the unnamed index P_u does not satisfy the time reversal test. Thus it can be seen that Theil's stochastic approach to index numbers leads to an index number formula, (19) above, that has more satisfactory properties than competing stochastic approaches.

Additional material on stochastic approaches to index number theory and references to the literature can be found in Selvanathan and Rao (1994), Diewert (1995), Wynne (1997), Clements, Izan and Selvanathan (2006) and Balk (2008; 32-36)

5. Test Approaches to Index Number Theory

Fisher wrote two books that contributed substantially to the test approach to index number theory. In the first book, Fisher (1911), his interest in index number theory was more or less incidental to the main purpose of the book, which was to lay out the quantity theory of money in some detail. It was in this book that Fisher laid out the *value ratio decomposition approach to index number theory*; i.e., he looked for price and quantity index functions, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$, that had good axiomatic properties and which satisfied equation (5), namely that the value ratio $p^1\cdot q^1/p^0\cdot q^0$ be equal to $P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1)$. In his second book, Fisher (1922), he was much more systematic and comprehensive. In this section, we will list Fisher's tests that he proposed in these two books.[36]

We begin by listing Fisher's (1911; 410-429) eight tests. His first two tests were the *proportionality of prices test* (24)[37] and the *proportionality of quantities test* (25) below:

$$(24)\ P(p^0,p^1,q^0,q^1) = \lambda \text{ if } p^1 = \lambda p^0 \text{ for all } p^0 > 0_N,\ q^0 > 0_N,\ q^1 > 0_N \text{ and } \lambda > 0;$$
$$(25)\ Q(p^0,p^1,q^0,q^1) = \lambda \text{ if } q^1 = \lambda q^0 \text{ for all} q^0 > 0_N,\ p^0 > 0_N,\ p^1 > 0_N \text{ and } \lambda > 0$$

where $Q(p^0,p^1,q^0,q^1)$ is always defined as the quantity index that matches up with the price index P using equation (5); i.e., $Q(p^0,p^1,q^0,q^1) \equiv [p^1\cdot q^1/p^0\cdot q^0]/P(p^0,p^1,q^0,q^1)$. These tests are very reasonable: if all price ratios $p_n^{\,1}/p_n^{\,0}$ are equal to a common ratio λ, then the overall price index should also equal λ (and an analogous property holds for the companion quantity index).

Fisher's next two tests were new and he called them *determinateness tests*: if an individual price or quantity in the vectors p^0,p^1,q^0,q^1 became zero, then $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ should not become zero or infinite or become indeterminate.[38]

Fisher's (1911) Tests 5 and 6 have fallen out of favour and in fact, he dropped these tests in his 1922 book on index number theory. His Test 5 was described as follows:[39] a price

[36] All price and quantity vectors are nonnegative and nonzero.
[37] This test is due to Walsh (1901; 385). However, Walsh did not formulate the test (25).
[38] Note that P_C, P_J and P_T do not satisfy this test whereas P_L, P_P, P_F and P_W do satisfy it.

index should be unaffected by the withdrawal or entry of a price ratio agreeing with the index. His Test 6 applied the same test to the companion or "correlative" quantity index: the correlative trade (or quantity) index should be unaffected by the withdrawal or entry of a quantity ratio agreeing with the index.

Fisher's (1911; 411) seventh test was the *base period invariance test*: the ratio of two price indexes should remain unchanged if the base period is changed. Let period 0 and 1 be two possible base periods and let s and t be two other arbitrary periods. Then this test requires that

$$(26)\ P(p^0,p^t,q^0,q^t)/P(p^0,p^s,q^0,q^s) = P(p^1,p^t,q^1,q^t)/P(p^1,p^s,q^1,q^s).$$

This is a useful test but it was not new: Jevons (1884; 152) and Edgeworth (1896; 137) gave a clear statement of this test many years before Fisher.

Fisher's final eighth test in his 1911 book was the *invariance to changes in units of measurement test*:

$$(27)\ P(p_1^0,...,p_N^0;p_1^1,...,p_N^1;q_1^0,...,q_N^0;q_1^1,...,q_N^1)$$
$$= P(\alpha_1p_1^0,...,\alpha_Np_N^0;\alpha_1p_1^1,...,\alpha_Np_N^1;q_1^0/\alpha_1,...,q_N^0/\alpha_N;q_1^1/\alpha_1,...,q_N^1/\alpha_N)$$

where $\alpha_1,...,\alpha_N$ are positive numbers. Note that the change in units of measurement of the prices must be offset by a corresponding opposite change in the units of measurement of the quantities. The concept of this test was due to Jevons (1884; 231) and the Dutch economist Pierson (1896; 131). Fisher (1922; 420) later called it the *commensurability test*.

It can be seen that most of Fisher's tests in the 1911 book did not originate with him. But he was more systematic than his predecessors and he did introduce the very useful idea of the correlative or companion quantity index that matches up with the price index using equation (5).

We turn now to Fisher's tests that he used in his 1922 book. Fisher (1922; 62-63) proposed three tests of "fairness" or symmetry[40] for bilateral index number formulae that he regarded as fundamental: the *commodity reversal test* (28), the *time reversal test* (10) above and the *factor reversal test* (29) below.

The *commodity reversal test* may be stated as follows. Denote p^0,p^1,q^0 and q^1 as the price and quantity vectors in their original ordering and denote p^{0*},p^{1*},q^{0*} and q^{1*} as new price and quantity vectors, where the ordering of the commodities has been permuted (the same permutation is applied to each of the original vectors). Then the test is:

[39] See Fisher (1911; 411). This test seems to be taken from Walsh (1901; 311).

[40] "Index numbers to be fair ought to work both ways—both ways as regards any two commodities to be averaged, or as regards the two times to be compared, or as regards the two sets the two sets of associated elements for which index numbers may be calculated—that is, prices and quantities." Irving Fisher (1922; 62).

(28) $P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p^0,p^1,q^0,q^1)$;

i.e., the price index remains invariant if the ordering of the commodities is changed.[41]

Fisher's *factor reversal test* can be stated as follows: let P be given and let Q be defined implicitly by (5). Then Q is also equal to

(29) $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$;

i.e., the functional form for the price index can be used as a quantity index if the role of prices and quantities is interchanged and the resulting quantity index will also satisfy the product test (5). A justification for this test is the following one: if $P(p^0,p^1,q^0,q^1)$ is a good functional form for the price index, then if we reverse the roles of prices and quantities, $P(q^0,q^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which is a valid argument) and thus the product of the price index $P(p^0,p^1,q^0,q^1)$ and the quantity index $Q(p^0,p^1,q^0,q^1)$ defined as $P(q^0,q^1,p^0,p^1)$ ought to equal the value ratio, $p^1{\cdot}q^1/p^0{\cdot}q^0$. The second part of this argument is not compelling and thus many researchers over the years have objected to the factor reversal test.[42]

As noted earlier, Fisher (1922; 82) was not the originator of the time reversal test but he was the originator of the commodity[43] and factor reversal tests.

Fisher (1921; 536) provided a preview of his 1922 book and he placed great importance on index number formulae that satisfied the above three tests. He noted in this paper that only a few formulae satisfied these tests and that the simplest of these formulae and "the best by other tests" was the *Fisher ideal price index* defined earlier by (9).[44] However, Walsh who was a discussant of Fisher's (1921) paper begged to differ on the uniqueness of the Fisher index being the only formula which satisfied the three reversal tests when he made the following comments on the importance of the factor reversal test:

"Professor Fisher seems to think it a very important test. In his abstract, he seemed to use it almost as the crucial test. He there wrote that as far as he knew the 'ideal' index number which he recommends is the only one which fulfills this test. That sounded like an argument for this index number; and I wrote a criticism demolishing it. But yesterday, Professor Fisher told me he was aware that there are other index numbers that fulfill this test. What I wrote therefore, can no longer serve as a criticism. Still, Professor Fisher may not be aware of the great number of index numbers that fulfill this test—a fact which considerably diminishes its importance; and therefore I will read what I wrote, which presents him with at least twenty such index numbers. Not that I intend to describe all these index numbers to you. I will show you only the method of obtaining them." Correa Moylan Walsh (1921b; 541).

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[41] The test (28) and (5) will imply that the companion quantity index also satisfies $Q(p^{0*},p^{1*},q^{0*},q^{1*}) = Q(p^0,p^1,q^0,q^1)$.
[42] See for example Samuelson and Swamy (1974).
[43] Fisher (1922; 63) commented on the commodity reversal test as follows; "This is so simple so as never to have been formulated ." Nevertheless, it is a very useful and important test.
[44] It can readily be verified that the Fisher ideal index defined by (9) satisfies Fisher's three reversal tests.

Walsh (1921b; 542) and Fisher (1922; 125) defined the *factor antithesis* P^* to a given bilateral price index P as follows:

(30) $P^*(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/[p^0\cdot q^0 P(q^0,q^1,p^0,p^1)]$.

Then Walsh (1921b; 542) and later Fisher (1922; 396-397) showed that the geometric mean of P and P^* will always satisfy the factor reversal test. Thus Walsh (1921b) in his discussion of Fisher (1921) was quite right: the Fisher ideal index was not the *only* index number formula that satisfied the factor reversal test and moreover, Walsh showed Fisher how starting from an arbitrary bilateral price index function $P(p^0,p^1,q^0,q^1)$, the "rectified"[45] formula $P^{**}(p^0,p^1,q^0,q^1) \equiv [P(p^0,p^1,q^0,q^1)P^*(p^0,p^1,q^0,q^1)]^{1/2}$ would always satisfy the factor reversal test.[46]

Fisher went on to apply a similar rectification procedure to obtain index number formulae which would automatically satisfy the time reversal test. Thus given a bilateral price index P, Fisher (1922; 119) defined the *time antithesis* $P°$ for P as follows:

(31) $P^{\circ}(p^0,p^1,q^0,q^1) \equiv 1/P(p^1,p^0,q^1,q^0)$.

Thus $P°$ is equal to the reciprocal of the price index which has reversed the role of time, $P(p^1,p^0,q^1,q^0)$. Fisher (1922; 140) then showed that the geometric mean of P and $P°$ satisfies the time reversal test, (10).

In addition to the three reversal tests mentioned above, Fisher mentioned in passing several other desirable properties (or tests), which are discussed below. He used these additional properties to reject certain index number formulae that might not be rejected by a narrow application of his three main reversal tests.

Fisher (1922; 42) asked that P not be *haphazard*; i.e., it should satisfy the *commensurability test* (27) which was noted earlier in his 1911 book.[47]

Fisher (1922; 43) also stressed the importance of the *fairness* property; i.e., that price relatives be *weighted* according to their expenditure importance. This property enabled Fisher to reject all equally weighted index number formulae.[48]

Fisher also asked that an index number not be *freakish*:

"When an index number is highly erratic we have called it *freakish*." Irving Fisher (1922; 116).

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[45] This terminology is due to Fisher (1922; 137).
[46] Fisher (1922; 458-460), who was generally quite meticulous in noting the contributions of earlier index number researchers, failed to mention that the "Fisher" rectification procedure was actually suggested by Walsh (1921b).
[47] Fisher (1922; 451) used this test to reject the Dutot (1738) index defined as the arithmetic average of the period 1 prices divided by the arithmetic average of the period 0 prices.
[48] The importance of weighting was earlier stressed by Walsh (1901; 87-88) (1921a; 80-90).

This property might be translated into more modern terminology as a requirement that $P(p^0,p^1,q^0,q^1)$ be *continuous* in its independent variables, p^0,p^1,q^0,q^1.

Fisher (1922; 210) also condemned index number formulae (like the median or mode of the N price ratios, p_n^1/p_n^0) that were *insensitive* or *unresponsive*; i.e., Fisher thought that if a single price changed, then the overall index should also change. This property could be interpreted as a *monotonicity* property; i.e., $P(p^0,p^1,q^0,q^1)$ should be an increasing function in the components of the period 1 price vector p^1 and a decreasing function in the components of the period 0 price vector p^0.[49]

All of the above tests are reasonable and the Fisher ideal index P_F defined by (9) does satisfy the tests discussed in Fisher (1922). The next test that Fisher (1922; 270) discussed was the following *circularity test*:[50]

(32) $P(p^0,p^2,q^0,q^2) = P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)$.

The price index on the left hand side of (32) directly compares the prices and quantities of period 2 with the prices in period 0 whereas the price change on the right hand side of (32) is decomposed into the product of the index comparing period 0 to 1, $P(p^0,p^1,q^0,q^1)$, times the index comparing period 1 to 2, $P(p^1,p^2,q^1,q^2)$. This is also a very desirable property for an index formula to satisfy but for once, the Fisher ideal index does not satisfy this test.

Note that the index on the left hand side of (32) compares the prices of period 2 directly with the prices of period 0 whereas on the right hand side of (32), the overall change in prices going from period 0 to 2 is calculated as the product of the price change from 0 to 1 times the price change from 1 to 2. Indexes that compare the prices of period 2 (or any subsequent period) *directly* with the prices of a base period 0 are called *fixed base index numbers* and indexes that calculate the overall price change as a product of period to period changes are called *chained index numbers*.[51] In his 1911 book, Fisher favoured the chain system for making intertemporal comparisons:

"It may be said that the cardinal virtue of the successive base or chain system is the facility it affords for the introduction of new commodities, the dropping out of obsolete commodities, and the continued readjustment of the system of weighting to new commodities. A fixed base system soon gets behind the times in every sense of the word." Irving Fisher (1911; 204).

However, in his 1922 book, Fisher condemned the circular test and the use of chained comparisons:

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[49] Somewhat surprisingly, these two monotonicity tests for P were not proposed until much later by Eichhorn and Voeller (1976; 23). Vogt (1980; 70) proposed that the companion quantity index $Q(p^0,p^1,q^0,q^1)$ to P defined using (5) be increasing in the components of q^1 and decreasing in the components of q^0.

[50] This test is due to Westergaard (1890; 218). The terminology "circular test" is due to Walsh (1921a; 98) and Fisher (1922; 413).

[51] Fisher (1911; 203) introduced this terminology. The concept of chaining is due to Lehr (1885) and Marshall (1887; 373).

"But the analogy of the circular test with the time reversal test, while plausible, is misleading. I aim to show that the circular test is theoretically a mistaken one, that a necessary irreducible minimum of divergence from such fulfillment is entirely right and proper, and, therefore, that a *perfect* fulfillment of this so-called circular test should really be taken as proof that the formula which fulfills it is erroneous." Irving Fisher (1922; 271).

Fisher (1922; 274) justified his condemnation of the circularity test by asserting that the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*.[52] Fisher (1922; 413-416) gave two examples of these constant weight indexes that satisfied the circularity test: Lowe type indexes that had the form $p^1 \cdot q/p^0 \cdot q$ where q is a vector of positive constants and weighted Jevons indexes of the following form:

$$(33)\ P_{KB} \equiv \prod_{n=1}^{N} (p_n^{1}/p_n^{1})^{a(n)}$$

where the a(n) are positive constants summing to unity. We have labelled this index as P_{KB} since this particular functional form for a price index is due to Konüs and Byushgens.[53]

Fisher objected to constant weights on the following grounds:

"But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915 we need, theoretically at least, another set of weights. ... We cannot justify using the same weights for comparing the price level of 1913, not only with 1914 and 1915, but with 1860, 1776, 1492 and the times of Diocletian, Rameses II and the Stone Age!" Irving Fisher (1922; 275).

Fisher did not provide a formal proof of his assertion that circularity implies constant weights but his intuition was more or less correct. Frisch (1930) using differential equation techniques was the first to provide some formal characterizations for the functional form for a bilateral index number when it satisfied circularity (along with some other properties). Additional results on the implications of circularity using functional equation techniques are due to Eichhorn and Voeller (1976; 29-58), Eichhorn (1978; 164-172) and Balk (1995). We will state some results on this topic due to Diewert (2011) that draw heavily on the results of Eichhorn. However, it will first be necessary to state a few more tests.

Let $P(p^0,p^1,q^0,q^1)$ be a function of 4N variables that is defined for all strictly positive price and quantity vectors p^0,p^1,q^0,q^1 pertaining to the two periods under consideration. The *positivity* [54] and *continuity*[55] *tests* simply require that $P(p^0,p^1,q^0,q^1)$ be a positive and

[52] Fisher (1922; 413-416) gave two examples of these constant weight indexes that satisfied the circularity test: Lowe type indexes that had the form $p^1 \cdot q/p^0 \cdot q$ and weighted Jevons indexes of the form $\prod_{n=1}^{N} (p_n^{1}/p_n^{1})^{a(n)}$ where q is a constant vector and the a(n) are positive constants summing to unity.

[53] Konüs and Byushgens (1926; 163-166) showed that the index defined by (33) was exact for Cobb-Douglas (1928) preferences. The concept of an exact index number formula will be explained in the following section where we study the economic approach to index number theory.

[54] Eichhorn and Voeller (1976; 23) seem to have been the first to formally suggest this obvious test.

continuous function over its domain of definition. Define the *identity*,[56] *proportionality* [57] and *monotonicity tests in period 1 prices* [58] for P as (34), (35) and (36) respectively:

(34) $P(p,p,q^0,q^1) = 1$ for all strictly positive vectors p,q^0,q^1;
(35) $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$ and all strictly positive p^0,p^1,q^0,q^1;
(36) $P(p^0,p^1,q^0,q^1) < P(p^0,p^2,q^0,q^1)$ where p^2 is equal to or greater than p^1 but $p^1 \neq p^2$.

Diewert (2011) showed that if $P(p^0,p^1,q^0,q^1)$ satisfied the positivity, continuity, identity, proportionality and monotonicity in period 1 prices, commensurability and circularity tests, then P must equal P_{KB} defined by (33), which is a partial vindication of Fisher's assertion. It is also possible to show that if we drop the commensurability test from the above list of tests that P must satisfy, then P must have the following functional form:

(37) $P(p^0,p^1,q^0,q^1) = g(p^1)/g(p^0)$

where g(p) is a positive, continuous, nondecreasing and linearly homogeneous function of p. It can be seen that the class of index number formulae defined by (37) includes the Lowe index as a special case, which provides another partial vindication of Fisher's conjecture on the implications of the circularity test.

Fisher (1922; 83-84) also had some interesting material on the *bias* of index number formulae. His method for evaluating the bias in an index number formula was based on the time reversal test which can be written in the form:

(38) $P(p^0,p^1,q^0,q^1)P(p^1,p^0,q^1,q^0) = 1$.

Fisher (1922; 84) took 28 of the most popular price index number formulae and calculated the left hand side of (38) using each formula for consecutive periods in his data set.[59] He then subtracted 1 from the left hand side of (38) and tabled the resulting "biases" for each pair of consecutive years for each of the 28 index number formulae. Obviously, if the formula satisfied the time reversal test, the bias would always be zero. This is a very useful method for evaluating the bias in an index and is still in use today. Unfortunately, Fisher failed to acknowledge that his method for calculating the bias in a bilateral formula was actually a special case of a method suggested by Walsh (1901; 389), (1921b; 540) (1924; 506). His *multiperiod identity test* is the following test:[60]

(39) $P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)P(p^2,p^0,q^2,q^0) = 1$.

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[55] As we have seen Fisher (1922) informally suggested this test.
[56] Laspeyres (1871; 308), Walsh (1901; 308) and Eichhorn and Voeller (1976; 24) all suggested this test.
[57] This test was proposed by Walsh (1901; 385) and Eichhorn and Voeller (1976; 24).
[58] This test was proposed by Eichhorn and Voeller (1976; 23).
[59] Fisher's (1922; 489-490) data set consisted of annual production of 36 basic commodities in the U.S. over the six years 1913-1918.
[60] This is Diewert's (1992; 40) term for the test. Walsh did not limit himself to just three periods as in (39); he considered an indefinite number of periods. Walsh used his test in a manner similar to that used by Fisher; i.e., as a way of evaluating the "bias" in various index number formulae.

Thus we calculate price change over consecutive periods but we introduce an artificial final period where the prices and quantities revert back to the prices and quantities in the very first period. The Walsh test asks that the product of all of these price changes should equal unity.

Fisher (1922; 277) found that for his data set, the Fisher ideal index satisfied circularity to a reasonably high degree of approximation. We return to this somewhat surprising result in section 8 below.

Fisher's second book on index number theory was certainly a landmark in the history of index number theory. Fisher (1922; 466-487) listed explicitly some 180 bilateral price index formulae.[61] Fisher (1922; 489-518) also numerically evaluated some 134 of these formulae using his data set for 36 commodities over six years. Fisher (1922; 244-247) then graded these 134 formulae according to how far away numerically they were from his ideal index. He classified the formulae into seven groups: *worthless*, *poor*, *fair*, *good*, *very good*, *excellent* and *superlative*.[62]

Fisher did not develop a full axiomatic characterization for his ideal index or the consistency of the various tests that he proposed. The consistency task was left to later researchers, led by the work of Frisch (1930), Eichhorn and Voeller (1976) and Eichhorn (1978). Diewert (1992) showed that the Fisher ideal index satisfies some 21 "reasonable" tests and that these tests imply that the functional form for the price index must be P_F. Other axiomatic characterizations of the Fisher ideal price index have been obtained by Funke and Voeller (1978; 180) and Balk (1985) (1995) (2008; 91-97).

Although there is not complete consensus among index number theorists, the Fisher ideal index is probably the "best" index from the perspective of the test approach to index number theory.

6. The Economic Approach to Index Number Theory

In this section, we will outline the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, A. A. Konüs (1924). This theory relies on the assumption of *optimizing behavior* on the part of the consumer. Thus given a vector of commodity or input prices p^t that the consumer faces in a given time period t, it is assumed that the corresponding observed quantity vector q^t is the solution to a cost minimization problem that involves the consumer's preference or utility function f.

The economic approach assumes that "the" consumer has well defined *preferences* over different combinations of the N consumer commodities or items. The consumer's

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[61] Some of the formulae turned out to be duplicates.

[62] There were eleven formulae in his superlative group including of course, the Fisher ideal index. Other members of his superlative group included the Marshall-Edgeworth and Walsh indexes defined by (12) and (13) as well as the arithmetic mean of the Laspeyres and Paasche indexes defined by (7) and (8). The index P_T defined by (19) was only rated as very good.

preferences over alternative possible consumption vectors q are assumed to be representable by a nonnegative, continuous, increasing, and quasiconcave utility function f, which is defined over the nonnegative orthant. It is further assumed that the consumer minimizes the cost of achieving the period t utility level $u^t \equiv f(q^t)$ for periods t = 0,1. Thus the observed period t consumption vector q^t solves the following *period t cost minimization problem*:[63]

$$(40)\quad C(u^t,p^t) \equiv \min_q \{p^t \cdot q : f(q) = u^t\} = p^t \cdot q^t ; \qquad t = 0,1.$$

The period t price vector for the N commodities under consideration that the consumer faces is p^t. The Konüs (1939) family of *true cost of living indexes* $P_K(p^0,p^1,q)$ between periods 0 and 1 is defined as the ratio of the minimum costs of achieving the same utility level $u \equiv f(q)$ where q is a positive reference quantity vector:

$$(41)\quad P_K(p^0,p^1,q) \equiv C[f(q),p^1]/C[f(q),p^0].$$

We say that definition (41) defines a *family* of price indexes because there is one such index for each reference quantity vector q chosen. However, if we place an additional restriction on the utility function f, then it turns out that the Konüs price index, $P_K(p^0,p^1,q)$, will no longer depend on the reference q.

The extra assumption on f is that f be (positively*) linearly homogeneous* so that $f(\lambda q) = \lambda f(q)$ for all $\lambda > 0$ and all $q \geq 0_N$. In the economics literature, this extra assumption is known as the assumption of *homothetic preferences*.[64] Under this assumption, the consumer's cost function, C(u,p) decomposes into uc(p) where c(p) is the consumer's *unit cost function*, $c(p) \equiv C(1,p)$, which corresponds to f. Under the assumption of cost minimizing behavior in both periods, it can be shown that the homotheticity assumption implies that equations (40) simplify to the following equations:

$$(42)\quad p^t \cdot q^t = c(p^t)f(q^t) \qquad \text{for } t = 0,1.$$

Thus under the linear homogeneity assumption on the utility function f, observed period t expenditure on the n commodities is equal to the period t unit cost $c(p^t)$ of achieving one unit of utility times the period t utility level, $f(q^t)$. Obviously, we can identify the period t unit cost, $c(p^t)$, as the period t price level P^t and the period t level of utility, $f(q^t)$, as the period t quantity level Q^t (as in section 2 above).

The linear homogeneity assumption on the consumer's preference function f leads to a simplification for the family of Konüs true cost of living indexes, $P_K(p^0,p^1,q)$, defined by (41) above. Using this definition for an arbitrary reference quantity vector q and the decomposition $C(f(q),p^t) = c(p^t)f(q)$ for t = 0,1, we have:

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[63] Notation: $p^t \cdot q \equiv \sum_{n=1}^N p_n^t q_n$.

[64] More precisely, Shephard (1953) defined a homothetic function to be a monotonic transformation of a linearly homogeneous function. However, if a consumer's utility function is homothetic, we can always rescale it to be linearly homogeneous without changing consumer behavior. Hence, we simply identify the homothetic preferences assumption with the linear homogeneity assumption.

(43) $P_K(p^0,p^1,q) \equiv C[f(q),p^1]/C[f(q),p^0] = c(p^1)f(q)/c(p^0)f(q) = c(p^1)/c(p^0)$.

Thus under the homothetic preferences assumption, the entire family of Konüs true cost of living indexes collapses to a single index, $c(p^1)/c(p^0)$, which is the ratio of the minimum costs of achieving a unit utility level when the consumer faces period 1 and 0 prices respectively.

If we use the Konüs true cost of living index defined by the right hand side of (43) as our price index concept, then the corresponding implicit quantity index can be defined as the value ratio divided by the Konüs price index:

(44) $Q(p^0,p^1,q^0,q^1,q) \equiv p^1 \cdot q^1/[p^0 \cdot q^0\ P_K(p^0,p^1,q)] = f(q^1)/f(q^0)$.

Thus under the homothetic preferences assumption, the *implicit quantity index* that corresponds to the true cost of living price index $c(p^1)/c(p^0)$ is the *utility ratio* $f(q^1)/f(q^0)$. Since the utility function is assumed to be homogeneous of degree one, this is the natural definition for a quantity index.[65]

Recall that the Fisher price index, $P_F(p^0,p^1,q^0,q^1)$, was defined by (13). The companion *Fisher quantity index*, $Q_F(p^0,p^1,q^0,q^1)$, can be defined using (5). Now suppose that the consumer's preferences can be represented by the homothetic utility function f defined as

(45) $f(q) \equiv [q^TAq]^{1/2}$

where $A \equiv [a_{ij}]$ is an N by N symmetric matrix that has one positive eigenvalue (that has a strictly positive eigenvector) and the remaining N−1 eigenvalues are zero or negative. Under these conditions, there will be a *region of regularity* where the function f is positive, concave and increasing and hence f can provide a valid representation of preferences over this region. Using these preferences and the assumption of cost minimizing behavior in periods 0 and 1, it can be shown that

(46) $Q_F(p^0,p^1,q^0,q^1) = f(q^1)/f(q^0)$.

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function f defined by (45), the Fisher ideal quantity index Q_F is *exactly* equal to the true quantity index, $f(q^1)/f(q^0)$.[66]

Let c(p) be the unit cost function that corresponds to the homogeneous quadratic utility function f defined by (45). Then using (5), (42) and (46), it can be shown that

(47) $P_F(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0)$.

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[65] Samuelson and Swamy (1974) used this homothetic approach to index number theory.

[66] This result was first derived by Konüs and Byushgens (1926). For an alternative derivation and the early history of this result, see Diewert (1976; 116).

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function $f(q) = (q^TAq)^{1/2}$, the Fisher ideal price index P_F is *exactly* equal to the true price index, $c(p^1)/c(p^0)$. The significance of (46) and (47) is that we can calculate the consumer's true rate of utility growth and his or her true rate of price inflation *without having to undertake any econometric estimation*; i.e., the left hand sides of (46) and (47) can be calculated exactly using observable price and quantity data for the consumer for the two periods under consideration. Thus the present economic approach to index number theory using a *ratio approach* leads to practical solutions to the index number problem whereas the earlier *levels approach* explained in section 2 did not lead to practical solutions.

A twice continuously differentiable function f(q) of N variables q can provide a *second order approximation* to another such function $f^*(q)$ around the point q^* if the level and all of the first and second order partial derivatives of the two functions coincide at q^*. It can be shown[67] that the homogeneous quadratic function f can provide a second order approximation to an arbitrary f^* around any point q^* in the class of twice continuously differentiable linearly homogeneous functions. Thus the homogeneous quadratic functional form defined by (45) is a *flexible functional form*.[68] Diewert (1976; 117) termed an index number formula $Q_F(p^0,p^1,q^0,q^1)$ that was *exactly* equal to the true quantity index $f(q^1)/f(q^0)$ (where f is a flexible functional form) *a superlative index number formula*.[69] Equation (46) and the fact that the homogeneous quadratic function f defined by (45) is a flexible functional form shows that the Fisher ideal quantity index Q_F is a superlative index number formula. Since the Fisher ideal price index P_F also satisfies (47) where c(p) is the dual unit cost function that is generated by the homogeneous quadratic utility function, P_F is also a superlative index number formula.

It is possible to show that the Fisher ideal price index is a superlative index number formula by a different route. Instead of starting with the assumption that the consumer's utility function is the homogeneous quadratic function defined by (45), we could start with the assumption that the consumer's unit cost function is a homogeneous quadratic. Thus suppose that the consumer has the following unit cost function:

(48) $c(p) \equiv [p^TBp]^{1/2}$

where $B \equiv [b_{ij}]$ is an N by N symmetric matrix that has one positive eigenvalue (that has a strictly positive eigenvector) and the remaining N−1 eigenvalues are zero or negative. It can be shown that again (47) holds. Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the

[67] See Diewert (1976; 130) and let the parameter r equal 2.
[68] Diewert (1974; 133) introduced this term to the economics literature.
[69] As we have seen earlier, Fisher (1922; 247) used the term superlative to describe the Fisher ideal price index. Thus Diewert adopted Fisher's terminology but attempted to give more precision to Fisher's definition of superlativeness.

N commodities that correspond to the unit cost function defined by (48), the Fisher ideal price index P_F is *exactly* equal to the true price index, $c(p^1)/c(p^0)$.[70]

Since the homogeneous quadratic unit cost function $c(p)$ defined by (48) is also a flexible functional form, the fact that the Fisher ideal price index P_F exactly equals the true price index $c(p^1)/c(p^0)$ means that P_F is a *superlative index number formula*.

It turns out that there are many other superlative index number formulae; i.e., there exist many quantity indexes $Q(p^0,p^1,q^0,q^1)$ that are exactly equal to $f(q^1)/f(q^0)$ and many price indexes $P(p^0,p^1,q^0,q^1)$ that are exactly equal to $c(p^1)/c(p^0)$ where the aggregator function f or the unit cost function c is a flexible functional form. We will define a family of superlative indexes below.

Suppose that the consumer has the *following quadratic mean of order r utility function*:[71]

$$(49)\quad f^r(q_1,\ldots,q_N) \equiv [\textstyle\sum_{i=1}^N \sum_{k=1}^N a_{ik}\, q_i^{r/2}\, q_k^{r/2}\,]^{1/r}$$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976; 130) showed that the utility function f^r defined by (49) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order.[72] Note that when $r = 2$, f^r equals the homogeneous quadratic function defined by (45) above.

Define the *quadratic mean of order r quantity index* Q^r by:

$$(50)\quad Q^r(p^0,p^1,q^0,q^1) \equiv \{\textstyle\sum_{i=1}^N s_i^0\, (q_i^1/q_i^0)^{r/2}\}^{1/r}\ \{\textstyle\sum_{i=1}^N s_i^1\, (q_i^1/q_i^0)^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^t q_i^t/\sum_{k=1}^N p_k^t q_k^t$ is the period t expenditure share for commodity i. It can be verified that when $r = 2$, Q^r simplifies to Q_F, the Fisher ideal quantity index. It can be shown that Q^r is exact for the aggregator function f^r defined by (49); i.e., we have

$$(51)\quad Q^r(p^0,p^1,q^0,q^1) = f^r(q^1)/f^r(q^0).$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function defined by (49), the quadratic mean of order r quantity index Q_F is *exactly* equal to the true quantity index, $f^r(q^1)/f^r(q^0)$.[73] Since Q^r is exact for f^r and f^r is a flexible functional form, we see that the quadratic mean of order r quantity index Q^r is a

[70] This result was first obtained by Diewert (1976; 133-134).
[71] This terminology is due to Diewert (1976; 129).
[72] This result holds for any predetermined $r \neq 0$; i.e., we require only the $N(N+1)/2$ independent a_{ik} parameters in order to establish the flexibility of f^r in the class of linearly homogeneous aggregator functions.
[73] See Diewert (1976; 130).

superlative index for each $r \neq 0$. Thus there are an infinite number of superlative quantity indexes.

For each quantity index Q^r, we can use (5) in order to define the corresponding *implicit quadratic mean of order r price index* P^r:

(52) $P^r(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1/[p^0 \cdot q^0 Q^r(p^0,p^1,q^0,q^1)] = c^r(p^1)/c^r(p^0)$

where c^r is the unit cost function that is dual to the aggregator function f^r defined by (49) above. For each $r \neq 0$, the implicit quadratic mean of order r price index P^r is also a superlative index.

When $r = 2$, Q^r defined by (50) simplifies to Q_F, the Fisher ideal quantity index and P^r defined by (52) simplifies to P_F, the Fisher ideal price index. When $r = 1$, Q^r defined by (50) simplifies to

(53) $Q^1(p^0,p^1,q^0,q^1) = [p^1 \cdot q^1/p^0 \cdot q^0]/P_W(p^0,p^1,q^0,q^1)$

where P_W is the *Walsh* (1901; 398) (1921a; 97) *price index* defined earlier by (13). Thus the Walsh price index is also a superlative price index.

The above results provide reasonably strong justifications for the Fisher and Walsh price indexes from the viewpoint of the economic approach. An even stronger justification[74] can be provided for the Törnqvist Theil index P_T defined by (19) as we will show below.

Suppose that the consumer's cost function, $C(u,p)$, has the following *translog functional form*:[75]

(54) $$\ln C(u,p) \equiv a_0 + \sum_{i=1}^N a_i \ln p_i + (1/2) \sum_{i=1}^N \sum_{k=1}^N a_{ik} \ln p_i \ln p_k + b_0 \ln u + \sum_{i=1}^N b_i \ln p_i \ln u + (1/2)\, b_{00} [\ln u]^2$$

where ln is the natural logarithm function and the parameters a_i, a_{ik}, and b_i satisfy the following restrictions: (i) $a_{ik} = a_{ki}$ for $i,k = 1,\ldots,N$; (ii) $\sum_{i=1}^N a_i = 1$; (iii) $\sum_{i=1}^N b_i = 0$; (iv) $\sum_{k=1}^N a_{ik} = 0$ for $i = 1,\ldots,N$. These restrictions ensure that $C(u,p)$ defined by (54) is linearly homogeneous in p. It can be shown that this translog cost function can provide a second order Taylor series approximation to an arbitrary cost function.[76]

We assume that the consumer engages in cost minimizing behavior during periods 0 and 1 and has the preferences that are dual to the translog cost function defined by (54).

[74] The exact index number formula (55) is stronger than the above results because we no longer have to assume homothetic preferences.

[75] Christensen, Jorgenson and Lau (1975) and Diewert (1976) introduced this function into the economics and index number literature.

[76] It can also be shown that if $b_0 = 1$ and all of the $b_i = 0$ for $i = 1,\ldots,N$ and $b_{00} = 0$, then $C(u,p) = uC(1,p) \equiv uc(p)$; i.e., with these additional restrictions on the parameters of the general translog cost function, we have homothetic preferences.

Define the geometric average of the period 0 and 1 utility levels as $u^* \equiv [u^0u^1]^{1/2}$. Then it can be shown that the log of P_T defined by (19) is exactly equal to the log of the Konüs true cost of living index that corresponds to the reference indifference surface that is indexed by the intermediate utility level u^*; i.e., we have the following exact identity:[77]

(55) $C(u^*,p^1)/C(u^*,p^0) = P_T(p^0,p^1,q^0,q^1)$.

Since the translog cost function is a flexible functional form, the Törnqvist-Theil price index P_T is also a *superlative index*.[78] The importance of (55) as compared to the earlier exact index number results is that it is no longer necessary to assume that preferences are homothetic. However, it is necessary to choose a particular utility level on the left hand side of (55) to be the geometric mean of u^0 and u^1 in order to obtain the new exact index number result.[79]

It is somewhat mysterious how a ratio of *unobservable* cost functions of the form appearing on the left hand side of the above equation can be *exactly* estimated by an *observable* index number formula but the key to this mystery is the assumption of cost minimizing behavior and the quadratic nature of the underlying preferences. In fact, all of the exact index number results derived in this section can be derived using transformations of a quadratic identity.[80]

The important message to take home from this section is that the Fisher, Walsh and Theil indexes, P_F, P_W and P_T, can all be given strong justifications from the viewpoint of the economic approach to index number theory. Note that these same formulae also emerged as being "best" from the viewpoints of the basket, stochastic and test approaches to index number theory. *Thus the four major approaches to bilateral index number theory lead to the same three formulae as being best.* Which formula should then be used by a statistical agency as their target index? It turns out that for "typical" time series data, it will not matter much, since the three indexes numerically approximate each other very closely.[81]

The fact that four rather different approaches to index number theory lead to the same small number of index number formulae as being "best" and the fact that these formulae closely approximate each other for annual time series data has been a positive development. Twenty years ago, measurement economists and price statisticians from North America tended to favor the economic approach to index number theory whereas

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[77] This result is due to Diewert (1976; 122).

[78] Diewert (1978; 888) showed that $P_T(p^0,p^1,q^0,q^1)$ approximates the other superlative indexes P^r and P^{r*} to the second order around an equal price and quantity point.

[79] For exact index number results in the context of quantity indexes and nonhomothetic preferences that are analogous to (55), see Diewert (1976; 123-124) and Diewert (2009a; 241) where the first paper uses Malmquist (1953) quantity indexes and the second one uses Allen (1949) quantity indexes.

[80] See Diewert (2002).

[81] Diewert (1978; 888) showed that all known (at that time) superlative indexes numerically approximated each other to the second order around a point where $p^0 = p^1$ and $q^0 = q^1$. Thus if prices and quantities do not change "too much" between the two periods being compared, P_F, P_W and P_T will generate very similar indexes. It is interesting to note that Edgeworth (1901; 411-412) used the same methodology to show that the Marshall Edgeworth index P_{ME} approximated the Walsh index P_W to the second order around an equal price and quantity point.

their counterparts in Europe tended to favor the test[82] or stochastic approaches. This difference in views led to a great deal of counterproductive discussion on the relative merits of the various approaches to index number theory at international meetings on price measurement. Since for all practical purposes, the various approaches lead to the same small number of index number formulae as being "best", recent international meetings have been far more productive, with participants focused on how to improve price measurement rather than fighting methodological wars.

7. Fisher on the Stochastic and Economic Approaches to Index Number Theory

As we have seen above, Fisher made significant contributions to the test approach to index number theory but he did not contribute to the stochastic and economic approaches to index number theory because he was opposed to them.

Fisher (1922; 43) followed Walsh (1901) (1921a; 82-83) in objecting to Edgeworth's (1888) (1896) (1901) unweighted stochastic approach explained in section 4 above due to the *lack of weighting* in the Carli and Jevons indexes defined earlier by (16) and (18). Edgeworth also advocated taking the median of the price ratios (p_n^1/p_n^0) as an easy to calculate index that would fit in well with his approach.[83] Fisher (1922; 249-250) classified the median formula and the Jevons index as *poor* and the Carli index as *worthless*; i.e., for his data set, they gave numerical results which were rather far from the results generated by his ideal index P_F.

Fisher was well aware of consumer theory but he rejected the economic approach to index number theory on the grounds that it was not practical:[84]

"In this connection it may be well to call attention to another standard of purchasing power of money which has sometimes been suggested for adjusting contracts. This is the utility standard. According to this, each person would be expected to receive or pay back marginal utility equivalent to what he had lent or borrowed. But the marginal utility of the same goods is different for different persons and different for the same person at different periods of his life. Hence, no such standard could be practically applied." Irving Fisher (1911; 220).

The development of the economic approach to price and quantity indexes had to wait until the path breaking 1924 article of Konüs (1924) was translated into English in *Econometrica* in 1939[85]

[82] The work of the Europeans Eichhorn and Voeller (1976), Eichhorn (1978) and Balk (1995) (2008) have been very influential in popularizing the test approach to index number theory.
[83] In fairness, Edgeworth (1888; 363) also considered the use of weighted medians, a robust estimator of inflation that is in use today to measure *core inflation*. Fisher (1922) did not calculate weighted median indexes for his data set.
[84] "Since we cannot measure utility statistically, we cannot measure the corrections in utility required to redistribute the 'benefits of progress'. In the absence of statistical measurement, any practicable correction is out of the question." Irving Fisher (1911; 222).
[85] Konüs(1924) introduced economic price indexes in the nonhomothetic case. Economic quantity indexes when preference are nonhomothetic were introduced later by Allen (1949) and Malmquist (1953).

Thus Fisher rejected the economic and stochastic approaches to index number theory (but he indirectly contributed to the stochastic approach by pointing out the upward bias in the Carli formula).

8. Fisher on Multilateral Indexes and Fixed Base versus Chained Index Numbers

As was noted in section 6 above, Fisher (1911; 204) favoured the use of chained index numbers but in Fisher (1922), he favoured direct comparisons between any two periods or more generally, between any two spatial comparisons:

"Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another." Irving Fisher (1922; 275).

Thus Fisher broadened the scope of index number theory to include *spatial comparisons* between countries or regions as well as the usual time series comparisons that were considered up to that time. The second point to note about Fisher's 1922 position on index number comparisons is that when index number comparisons are made between N price and quantity situations, instead of computing N − 1 indexes between the N points, we would have to compute N(N−1)/2 indexes; i.e., a matrix of bilateral comparisons would be required:

"It follows that, to secure the theoretically most perfect result, for the sake of finding the very best for each pair of years, we should, for a given series of years and with a given formula, work out every possible index number connecting every possible pair of years among all the years considered." Irving Fisher (1922; 297).

Fisher (1922; 298) went on to note that we could compute a sequence of indexes where each time period or country would be the base so for N price quantity situations, we could calculate N separate sequences of index number comparisons. In Fisher's case, he had six observations and so six separate series could be generated using each year as the base. But there will be a demand for a single series of index number comparisons rather than six separate comparisons so what should be done? Fisher had a very modern answer:

"Doubtless the very best as to accuracy, were it practicable, is the blend or average of all six. This blend constitutes Formula 7053." Irving Fisher (1922; 305).

Thus each of Fisher's six series using each year as the base in turn were normalized to equal 1 in the first period and then his best blend was the simple arithmetic average of these six normalized series. Fisher's *blended method* is almost equal to Gini's (1931) method except that Gini took the geometric mean of the six series rather than the arithmetic mean. Gini's method is still widely used in making international comparisons.[86] Thus Fisher was a pioneer in this area.

[86] Gini's method is also known as the EKS method. For surveys of multilateral methods, see Diewert (1999) and Balk (2008; 232-260).

On the narrower issue of using fixed base versus chained indexes in the context of making time series comparisons, the modern literature has suggested that countries (or time periods) should be linked which have the most *similar* structure of prices and quantities, because with similar structures, the functional form for the bilateral index number formula will not matter too much: all "reasonable" indexes will generate much the same answer. A practical problem with this similarity linking approach is: exactly how should the measure of price or quantity similarity be measured?[87] For annual time series data, it turns out that for various "reasonable" similarity measures, chained indexes are generally consistent with the similarity approach to linking observations.

Fisher (1922; 309) found that for his data set, the six sets of index numbers that used each year as the base in turn were so close to the chained index numbers that the seven sets of price index comparisons could not be graphically distinguished. Thus for his data set, the Fisher ideal index satisfied the circularity test to a high degree of approximation. Moreover, the time series generated by the Theil and Walsh indexes, P_T and P_W, were also very close to the fixed base and chained Fisher indexes. It is possible to give a theoretical explanation for these empirical results. Alterman, Diewert and Feenstra (1999; 61) showed that if the logarithmic price ratios $\ln(p_n^t/p_n^{t-1})$ trend linearly with time t and the expenditure shares s_n^t also trend linearly with time, then the Törnqvist Theil index P_T will satisfy the circularity test exactly.[88] Since many economic time series on prices and quantities satisfy these assumptions approximately, then the Törnqvist Theil index P_T will satisfy the circularity test approximately. But Diewert (1978; 888) showed that P_T, P_F and P_W numerically approximate each other to the second order around an equal price and quantity point and so these indexes will generally be very close to each other using annual time series data. Hence since P_T will generally satisfy the circularity test to some degree of approximation, P_F will also satisfy circularity approximately in the time series context. Thus for many economic time series, P_F, P_T and P_W will all satisfy the circularity test to a high enough degree of approximation so that it will not matter whether we use the fixed base or chain principle.

However, the situation changes if we are comparing countries or using sub annual time series data. In these cases, price and quantity vectors may be very different across observations and so the smooth trend assumptions that justified the exact satisfaction of the circularity test for P_T will not be satisfied. Under these conditions, chained indexes can give very unsatisfactory results; i.e., Walsh's multiperiod identity test will be far from being satisfied. Under these conditions, fixed base indexes or multilateral methods will have to be used.[89]

9. Conclusion

[87] This similarity approach to linking bilateral comparisons into a complete set of comparisons across all observations has been pioneered by Robert Hill (1999a) (1999b) (2001) (2004) (2009). For an axiomatic approach to similarity measures, see Diewert (2009b).

[88] This exactness result can be extended to cover the case when there are monthly proportional variations in prices and the expenditure shares have constant seasonal effects in addition to linear trends; see Alterman, Diewert and Feenstra (1999; 65).

[89] See Szulc (1983), Hill (1988) and Ivancic, Diewert and Fox (2011).

It can be seen from the above material that Fisher contributed substantially to all of the main approaches to bilateral index number theory that are in use today except that he did not contribute to the economic approach. Fisher and Walsh are the early fathers of the test approach to index number theory. In addition, Fisher made substantial contributions to the modern theory of multilateral indexes. Fisher's ideal index turns out to be a "best" formula from the viewpoint of the fixed basket, test and economic approaches to index number theory. Finally, the Fisher ideal quantity index is being used as a target index when constructing estimates of real GDP by national statistical agencies such as the U.S. Bureau of Economic Analysis. Fisher has left a lasting legacy to index number theory.

References

Aldrich, J. (1992), "Probability and Depreciation: A History of the Stochastic Approach to Index Numbers", *History of Political Ec*onomy 24, 657-687.

Allen, R.G.D. (1949), "The Economic Theory of Index Numbers", *Economica* 16, 197-203.

Alterman, W.F., W.E. Diewert and R.C. Feenstra, (1999)*, International Trade Price Indexes and Seasonal Commodities*, Bureau of Labor Statistics, Washington D.C.

Balk, B.M. (1985), "A Simple Characterization of Fisher's Price Index," *Statistische Hefte* 26, 59-63.

Balk, B.M. (1995), "Axiomatic Price Index Theory: A Survey", *International Statistical Review* 63, 69-93.

Balk, B.M. (2005), "Divisia Price and Quantity Indices: 80 Years After", *Statistica Neerlandica* 59, 119-158.

Balk, B.M. (2008), *Price and Quantity Index Numbers*, New York: Cambridge University Press.

Barber, W. J. (1997), "Fisher: Career Highlights and Formative Influences, 1867-1947", pp. 3-21 in *The Works of Irving Fisher, Volume 1: The Early Professional Works*, W.J. Barber assisted by R.W. Dimand and K. Foster (eds.), London: Pickering & Chatto.

Bowley, A.L. (1899), "Wages, Nominal and Real", pp. 640-651 in *Dictionary of Political Economy*, Volume 3, London: MacMillan.

Bowley, A.L. (1901), *Elements of Statistics*, Westminster: P.S. King and Son.

Carli, Gian-Rinaldo, (1804), "Del valore e della proporzione de' metalli monetati", pp. 297-366 in *Scrittori classici italiani di economia politica*, Volume 13, Milano: G.G. Destefanis (originally published in 1764).

Christensen, L.R., D.W. Jorgenson and L.J. Lau (1975), " Transcendental Logarithmic Utility Functions", *American Economic Review* 65, 367-383.

Clements, K.W., H.Y. Izan and E.A. Selvanathan (2006), "Stochastic Index Numbers: A Review", *International Statistical Review* 74, 235-270.

Cobb, C. and P.H. Douglas (1928), "A Theory of Production", *American Economic Review* 18, 139-165.

Davies, G.R. (1924), "The Problem of a Standard Index Number Formula", *Journal of the American Statistical Association* 19, 180-188.

Diewert, W.E., 1974. "Applications of Duality Theory," pp. 106-171 in M.D. Intriligator and D.A. Kendrick (ed.), *Frontiers of Quantitative Economics*, Vol. II, Amsterdam: North-Holland.

Diewert, W.E. (1976), "Exact and Superlative Index Numbers", *Journal of Econometrics* 4, 114-145.

Diewert, W.E. (1978), "Superlative Index Numbers and Consistency in Aggregation", *Econometrica* 46, 883-900.

Diewert, W.E. (1980), "Aggregation Problems in the Measurement of Capital", pp. 433-528 in *The Measurement of Capital*, Dan Usher (ed.), Studies in Income and Wealth, Vol. 45, National Bureau of Economics Research, Chicago: University of Chicago Press.

Diewert, W.E. (1992), "Fisher Ideal Output, Input and Productivity Indexes Revisited", *Journal of Productivity Analysis* 3, 211-248.

Diewert, W.E. (1993a), "The Early History of Price Index Research", pp. 33-65 in *Essays in Index Number Theory*, Volume 1, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland.

Diewert, W.E. (1993b), "Symmetric Means and Choice under Uncertainty"', pp. 355-433 in *Essays in Index Number Theory*, Volume 1, W.E. Diewert and A.O. Nakamura (eds.), Amsterdam: North-Holland.

Diewert, W.E. (1995), On the Stochastic Approach to Index Numbers", Discussion Paper 95-31, Department of Economics, University of British Columbia, Vancouver, Canada.

Diewert, W.E. (1997), "Commentary on Mathew D. Shapiro and David W. Wilcox: Alternative Strategies for Aggregating Prices in the CPI", *The Federal Reserve Bank of St. Louis Review,* Vol. 79:3, 127-137.

Diewert, W.E. (1999), "Axiomatic and Economic Approaches to International Comparisons", pp. 13-87 in *International and Interarea Comparisons of Income, Output and Prices*, A. Heston and R.E. Lipsey (eds.), Studies in Income and Wealth, Volume 61, Chicago: The University of Chicago Press.

Diewert, W. E. (2001), "The Consumer Price Index and Index Number Purpose", *Journal of Economic and Social Measurement* 27, 167-248.

Diewert, W. E. (2002), "The Quadratic Approximation Lemma and Decompositions of Superlative Indexes", *Journal of Economic and Social Measurement* 28, 63-88.

Diewert, W.E. (2009a),"Cost of Living Indexes and Exact Index Numbers", pp. 207-246 in *Quantifying Consumer Preferences*, D. Slottje (ed.), Contributions to Economic Analysis Series, United Kingdom: Emerald Group Publishing.

Diewert, W.E. (2009b), "Similarity Indexes and Criteria for Spatial Linking", pp. 183-216 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Diewert, W.E. (2011), "The Axiomatic Approach to Bilateral Index Number Theory", Chapter 3, Lecture Notes for *Index Number Theory and Measurement Economics*, Department of Economics, University of British Columbia, Canada. http://faculty.arts.ubc.ca/ediewert/580ch3.pdf

Divisia, F. (1926), *L'indice monetaire et la theorie de la monnaie*, Paris: Societe anonyme du Recueil Sirey.

Drobisch, M. W. (1871), "Űber die Berechnung der Veränderungen der Waarenpreise und des Geldwerths", *Jahrbücher für Nationalökonomie und Statistik* 16, 143-156.

Dutot, Charles, (1738), *Réflexions politiques sur les finances et le commerce*, Volume 1, La Haye: Les frères Vaillant et N. Prevost.

Edgeworth, F.Y. (1888), "Some New Methods of Measuring Variation in General Prices", *Journal of the Royal Statistical Society* 51, 346-368.

Edgeworth, F.Y. (1896), "A Defense of Index Numbers", *Economic Journal* 6, 132-142.

Edgeworth, F.Y. (1901), "Mr. Walsh on the Measurement of General Exchange Value", *Economic Journal* 11, 404-416.

Edgeworth, F.Y. (1925), *Papers Relating to Political Economy*, Volume 1, New York: Burt Franklin.

Eichhorn, W. (1978), *Functional Equations in Economics*, London: Addison-Wesley.

Eichhorn, W. and J. Voeller (1976), *Theory of the Price Index*, Lecture Notes in Economics and Mathematical Systems, Vol. 140, Berlin: Springer-Verlag.

Ferger, W.F. (1946), "Historical Note on the Purchasing Power Concept and Index Numbers", *Journal of the American Statistical Association* 41, 53-57.

Fisher, I. (1911), *The Purchasing Power of Money*, London: Macmillan.

Fisher, I. (1921), "The Best Form of Index Number", *Quarterly Publication of the American Statistical Association* 17, 533-537.

Fisher, I. (1922), *The Making of Index Numbers*, Boston: Houghton-Mifflin.

Fisher, I. (1997), "An Address on the Irving Fisher Foundation, 11 September 1946", pp. 22-37 in *The Works of Irving Fisher, Volume 1: The Early Professional Works*, W.J. Barber assisted by R.W. Dimand and K. Foster (eds.), London: Pickering & Chatto.

Fisher, W.C. (1913), "The Tabular Standard in Massachusetts History", *Quarterly Journal of Economics* 27, 417-451.

Frisch, R. (1930), "Necessary and Sufficient Conditions Regarding the Form of an Index Number which Shall Meet Certain of Fisher's Tests", *American Statistical Association Journal* 25, 397-406.

Frisch, R. (1936), "Annual Survey of General Economic Theory: The Problem of Index Numbers", *Econometrica* 4, 1-39.

Funke, H. and J. Voeller (1978), "A Note on the Characterization of Fisher's Ideal Index," pp. 177-181 in *Theory and Applications of Economic Indices*, W. Eichhorn, R. Henn, O. Opitz and R.W. Shephard (eds.), Würzburg: Physica-Verlag.

Gini, C. (1931), "On the Circular Test of Index Numbers", *Metron* 9:9, 3-24.

Hicks, J.R. (1946), *Value and Capital*, Second Edition, Oxford: Claredon Press.

Hill, R.J. (1999a), "Comparing Price Levels across Countries Using Minimum Spanning Trees", *The Review of Economics and Statistics* 81, 135-142.

Hill, R.J. (1999b), "International Comparisons using Spanning Trees", pp. 109-120 in *International and Interarea Comparisons of Income, Output and Prices*, A. Heston and R.E. Lipsey (eds.), Studies in Income and Wealth Volume 61, NBER, Chicago: The University of Chicago Press.

Hill, R.J. (2001), "Measuring Inflation and Growth Using Spanning Trees", International Economic Review 42, 167-185.

Hill, R.J. (2004), "Constructing Price Indexes Across Space and Time: The Case of the European Union", *American Economic Review* 94, 1379-1410.

Hill, R.J. (2009), "Comparing Per Capita Income Levels Across Countries Using Spanning Trees: Robustness, Prior Restrictions, Hybrids and Hierarchies", pp. 217-244 in *Purchasing Power Parities of Currencies: Recent Advances in Methods and Applications*, D.S. Prasada Rao (ed.), Cheltenham UK: Edward Elgar.

Hill, T.P. (1988), "Recent Developments in Index Number Theory and Practice", *OECD Economic Studies* 10, 123-148.

ILO/IMF/OECD/UNECE/Eurostat/The World Bank (2004), *Consumer Price Index Manual: Theory and Practice*, Peter Hill (ed.), Geneva: International Labour Office.

Ivancic, L., W.E. Diewert and K.J. Fox (2011), "Scanner Data, Time Aggregation and the Construction of Prices Indexes", *Journal of Econometrics* 161, 24-35.

Jevons, W.S., (1865), "The Variation of Prices and the Value of the Currency since 1782", *Journal of the Statistical Society of London* 28, 294-320; reprinted in *Investigations in Currency and Finance* (1884), London: Macmillan and Co., 119-150.

Jevons, W.S., (1884), "A Serious Fall in the Value of Gold Ascertained and its Social Effects Set Forth (1863)", pp. 13-118 in *Investigations in Currency and Finance*, London: Macmillan and Co.

Keynes, J.M. (1930), *Treatise on Money*, Vol. 1, London: Macmillan.

Knibbs, Sir G.H. (1924), "The Nature of an Unequivocal Price Index and Quantity Index", *Journal of the American Statistical Association* 19, 42-60 and 196-205.

Konüs, A.A. (1924), "The Problem of the True Index of the Cost of Living", translated in *Econometrica* 7, (1939), 10-29.

Konüs, A.A. and S.S. Byushgens (1926), "K probleme pokupatelnoi cili deneg", *Voprosi Konyunkturi* 2, 151-172.

Laspeyres, E. (1871), "Die Berechnung einer mittleren Waarenpreissteigerung", *Jahrbücher für Nationalökonomie und Statistik* 16, 296-314.

Lehr, J. (1885), *Beitrage zur Statistik der Preise*, Frankfurt: J.D. Sauerlander.

Lowe, J. (1823), *The Present State of England in Regard to Agriculture, Trade and Finance*, second edition, London: Longman, Hurst, Rees, Orme and Brown.

Malmquist, S. (1953) "Index Numbers and Indifference Surfaces", *Trabajos de Estatistica* 4, 209-242.

Paasche, H. (1874), "Über die Preisentwicklung der letzten Jahre nach den Hamburger Borsennotirungen", *Jahrbücher für Nationalökonomie und Statistik* 12, 168-178.

Persons, W.M. (1928), *The Construction of Index Numbers*, Cambridge, Massachusetts: The Riverside Press.

Pierson, N.G. (1896), "Further Considerations on Index-Numbers," *Economic Journal* 6, 127-131.

Samuelson, P.A. and S. Swamy (1974), "Invariant Economic Index Numbers and Canonical Duality: Survey and Synthesis", *American Economic Review* 64, 566-593.

Scrope, G.P. (1833), *Principles of Political Economy*, London: Longman, Rees, Orme, Brown, Green and Longman.

Selvanathan, E.A. and D.S. Prasada Rao (1994), *Index Numbers: A Stochastic Approach*, Ann Arbor: The University of Michigan Press.

Shephard, R.W. (1953), *Cost and Production Functions*, Princeton: Princeton University Press.

Sidgwick, H. (1883), *The Principles of Political Economy*, London: Macmillan.

Szulc, B.J. (1983), "Linking Price Index Numbers," pp. 537-566 in *Price Level Measurement*, W.E. Diewert and C. Montmarquette (eds.), Ottawa: Statistics Canada.

Theil, H. (1967), *Economics and Information Theory*, Amsterdam: North-Holland Publishing.

Törnqvist, L. (1936), "The Bank of Finland's Consumption Price Index", *Bank of Finland Monthly Bulletin* 10, 1-8.

Törnqvist, L. and E. Törnqvist (1937), Vilket är förhållandet mellan finska markens och svenska kronans köpkraft?", *Ekonomiska Samfundets Tidskrift* 39, 1-39 reprinted as pp. 121-160 in *Collected Scientific Papers of Leo Törnqvist*, Helsinki: The Research Institute of the Finnish Economy, 1981.

Vogt, A. (1978), “Divisia Indices on Different Paths”, pp. 297-305 in *Theory and Applications of Economic Indices*, W. Eichhorn, R. Henn, O. Opitz and R.W. Shephard (eds.), Würzburg: Physica-Verlag.

Vogt, A. (1980), “Der Zeit und der Faktorumkehrtest als ‘Finders of Tests’”, *Statistische Hefte* 21, 66-71.

Walsh, C.M. (1901), *The Measurement of General Exchange Value*, New York: Macmillan and Co.

Walsh, C.M. (1921a), *The Problem of Estimation*, London: P.S. King & Son.

Walsh, C.M. (1921b), “The Best Form of Index Number: Discussion”, *Quarterly Publication of the American Statistical Association* 17, 537-544.

Walsh, C.M. (1924), “Professor Edgeworth’s Views on Index Numbers”, *Quarterly Journal of Economics* 38, 500-519.

Wynne, M.A. (1997), “Commentary on Measuring short Run Inflation for Central Bankers”, *Federal Reserve Bank of St. Louis Review* 79:3, 161-167.