Methods of Aggregation above the Basic Heading Level within Regions

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Abstract

The paper explains part of the methodology that was used in the 2005 International Comparison Program (ICP) that compared the relative price levels and GDP levels across 146 countries and 5 regions. This paper looks at the methodology which was used in order to calculate relative volumes and Purchasing Power Parities (PPPs) within each region. In previous rounds of the ICP, only two multilateral methods were used: the Gini Eltetö Köves Szulc (GEKS) method and the Geary Khamis (GK) method, which is an additive method. The axiomatic and economic properties of these methods and a relatively new additive method, the Iklé Dikhanov Balk (IDB) method used by the African region, are studied. A fourth method, the Minimum Spanning Tree method, due to Robert Hill is also studied.

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1. Introduction

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Chapter 6 discussed how the 155 Basic Heading (BH) price parities for each of the K countries in a region were constructed for ICP 2005. Once these Purchasing Power Parities (PPPs) have been constructed, aggregate measures of country prices and relative volumes between countries can be constructed using a wide variety of multilateral comparison methods that have been suggested over the years. These aggregate comparisons assume that in addition to BH price parities for each country, national statisticians have provided country expenditures (in their home currencies) for each of the 155 BH categories for the reference year 2005. Then the 155 by K matrices of Basic Heading price parities and country expenditures are used to form average price levels across all commodities and relative volume shares for each country.

There are a large number of methods that can be used to construct these aggregate Purchasing Power Parities and relative country volumes. Hill (2007a) (2007b) surveyed the main methods that have been used in previous rounds of the ICP as well as other methods that could be used.[2] Basically, only two multilateral methods have been used in previous rounds:

- The Gini Eltetö Köves Szulc (GEKS) method based on Fisher (1922) bilateral indexes and
- The Geary (1958) Khamis (1972) (GK) method, which is an additive method.

In the 2005 ICP round, aggregate PPPs and relative country volumes for countries within each region were constructed for five of the six regions using the Gini-EKS method. However, the African region wanted to use an additive method and so this region used a relatively new additive method, the Iklé Dikhanov Balk (IDB) method, for constructing PPPs and relative volumes within the region.[3] The purpose of this chapter is to describe the properties of these three methods (GEKS, GK and IDB) for making multilateral comparisons between countries in a region.[4] These methods will be discussed in sections 2, 3 and 4 below. An extensive Appendix will discuss the properties of the IDB method in more detail, since this method is relatively unknown. This Appendix can be omitted by the casual reader.

A brief comment on the relative merits of the GEKS, GK and IDB methods is warranted. The GK and IDB methods are *additive methods*; i.e., the real final demand of each country can be expressed as a *sum* of the country's individual Basic Heading final demand components where each real final demand component is weighted by an

[2] For additional methods, see Rao (1990), Balk (1996) (2009; 232-260), R.J. Hill (1997) (1999a) (1999b) (2001) (2004) (2009) and Diewert (1999).

[3] Iklé (1972; 203) proposed the equations for the method in a rather difficult to interpret manner and provided a proof for the existence of a solution for the case of two countries. Dikhanov (1994; 6-9) used the much more transparent equations (13) and (14) below, explained the advantages of the method over the GK method and illustrated the method with an extensive set of computations. Balk (1996; 207-208) used the Dikhanov equations and provided a proof of the existence of a solution to the sytem for an arbitrary number of countries. Van Ijzeren (1983; 42) also used Iklé's equations and provided an existence proof for the case of two countries.

[4] These methods can also be used to make comparisons between regions as will be seen in Chapter 8.

international price which is constant across countries. This feature of an additive method is tremendously convenient for users since components of final demand can be aggregated consistently across both countries and commodity groups and so for many purposes, it is useful to have available a set of additive international comparisons. However, additive methods are not consistent with the *economic approach* to index number theory (which allows for substitution effects), whereas the GEKS method is consistent. Section 6 will explain the economic approach and explain why additive methods are not fully consistent with the economic approach.

In order to discriminate between the various multilateral index number methods that have been suggested for the ICP, it is useful to look at the *axiomatic properties* of the various methods. Thus in Section 5, we will list various axioms or properties or tests that have been suggested for multilateral indexes and see which tests are satisfied by GEKS, GK and IDB.

The GEKS multilateral method is fully consistent with the economic approach to making multilateral comparisons. The GEKS approach also has the property that each country in the comparison is treated in a fully symmetric manner; i.e., the method is a democratic one. This aspect of GEKS can be considered as an advantage of the method. However, from a technical point of view, there are some disadvantages to the method in that countries that are at very different stages of development and which face very different relative prices are given the same weight in the method as countries which are at a very similar stage of development and face the same structure of relative prices. Bilateral comparisons between similar in structure countries are likely to be much more accurate than comparisons between countries which are very dissimilar. Thus in Section 7, an economic approach is introduced that builds up a complete multilateral set of comparisons that rests on making bilateral comparisons between very *similar* in structure countries. This method is called the *Minimum Spanning Tree* (MST) *method* by Robert Hill (1999a) (1999b) (2001) (2004) (2009), who introduced the method.[5] This method has some advantages over GEKS and thus it could be considered for use in the next ICP round.

Section 8 uses the artificial data example in Diewert (1999) to illustrate how the four methods (GEKS, GK, IDB and MST) differ in a rather extreme numerical example. Two less extreme numerical examples will be presented in Chapter 8.

Section 9 concludes.

2. The GEKS Method

The GEKS method is due to Gini (1924; 110) (1931; 12). It was derived in a different fashion by Eltetö and Köves (1964) and Szulc (1964).

[5] Fisher (1922; 272-274) in his discussion on comparing the price levels of Norway, Egypt and Georgia, came close to introducing this method.

In order to explain the method, it will be useful to introduce some notation at this point. Let N equal 155 and let K be the number of countries in the regional comparison for the reference year. Denote the Basic Heading PPP for final demand commodity category n and for country k in the region by $p_n^k > 0$ and the corresponding expenditure (in local currency units) on commodity class n by country k in the reference year by e_n^k for $n = 1,\ldots,N$ and $k = 1,\ldots,K$.[6] Given this information, we can define *volumes*[7] or *implicit quantity levels* q_n^k for each Basic Heading category n and for each country k as the category expenditure deflated by the corresponding Basic Heading commodity PPP for that country:

$$(1)\ q_n^k \equiv e_n^k/p_n^k\ ; \qquad n = 1,\ldots,N\ ;\ k = 1,\ldots,K.$$

It will be useful to define *country commodity expenditure shares* (in domestic currency), s_n^k, for BH class n and country k as follows:

$$(2)\ s_n^k \equiv e_n^k/\textstyle\sum_{i=1}^N e_i^k\ ; \qquad n = 1,\ldots,N\ ;\ k = 1,\ldots,K.$$

Now define *country vectors of BH PPPs* as $p^k \equiv [p_1^k,\ldots,p_N^k]^T$ [8], *country vectors of BH volumes* as $q^k \equiv [q_1^k,\ldots,q_N^k]^T$, *country expenditure vectors* as $e^k \equiv [e_1^k,\ldots,e_N^k]$ and *country expenditure share vectors* as $s^k \equiv [s_1^k,\ldots,s_N^k]^T$ for $k = 1,\ldots,K$.

In order to define the GEKS parities $P^1,P^2,\ldots,P^K$ between the K countries in the comparison, we first need to define the *Fisher* (1922) *ideal bilateral price index* P_F between country j relative to k:[9]

$$(3)\ P_F(p^k,p^j,q^k,q^j) \equiv [p^j\cdot q^j\ p^j\cdot q^k/p^k\cdot q^j\ p^k\cdot q^k]^{1/2}\ ; \qquad j = 1,\ldots,K\ ;\ k = 1,\ldots,K.$$

It can be seen that the Fisher ideal price index is the geometric mean of the Laspeyres price index between countries j and k, $P_L(p^k,p^j,q^k,q^j)$ and the Paasche price index, $P_P(p^k,p^j,q^k,q^j)$. These indexes are defined as follows:

$$\begin{aligned}(4)\ P_L(p^k,p^j,q^k,q^j) &\equiv p^j\cdot q^k/p^k\cdot q^k \\ &= \textstyle\sum_{n=1}^N (p_n^j/p_n^k)p_n^k q_n^k/p^k\cdot q^k \\ &= \textstyle\sum_{n=1}^N (p_n^j/p_n^k)s_n^k \qquad \text{using definitions (1) and (2);}\end{aligned}$$

$$\begin{aligned}(5)\ P_P(p^k,p^j,q^k,q^j) &\equiv p^j\cdot q^j/p^k\cdot q^j \\ &= 1/[p^k\cdot q^j/p^j\cdot q^j]\end{aligned}$$

[6] Note that the expenditures e_n^k are drawn from the national accounts of country k in the reference year and refer to total expenditures on commodity category n; i.e., these expenditures are *not* in per capita terms.

[7] National income accountants distinguish between a "quantity" and a "volume". A *volume* is an aggregate of a group of actual quantities. Since country expenditures in each of the Basic Heading categories are aggregates over many commodities, it is appropriate to refer to the q_n^k as volumes rather than quantities. The price levels p_n^k that correspond to the q_n^k are called Basic Heading (BH) PPPs.

[8] Notation: if $x = [x_1,\ldots,x_N]$, an N dimensional row vector, then x^T denotes the transpose of x and is an N dimensional column vector with the same components. Thus p^k is an N dimensional column vector.

[9] Notation: $p\cdot q \equiv \sum_{n=1}^N p_n q_n$ denotes the inner product between the vectors p and q.

$= 1/[\sum_{n=1}^{N} (p_n^k/p_n^j)p_n^j q_n^j/p^j \cdot q^j]$
$= 1/[\sum_{n=1}^{N} (p_n^k/p_n^j)s_n^j]$ using definitions (1) and (2)
$= [\sum_{n=1}^{N} (p_n^j/p_n^k)^{-1}s_n^j]^{-1}$.

Equations (4) show that the Laspeyres price index between the Basic Heading PPP parities p^j for country j relative to the Basic Heading PPPs p^k for country k is equal to an expenditure share weighted *arithmetic* average (using the expenditure shares for country k, the s_n^k, as weights) of the relative BH PPPs between countries j and k (the p_n^j/p_n^k). Equations (5) show that the Paasche price index between the Basic Heading PPP parities p^j for country j relative to the Basic Heading PPPs p^k for country k is equal to an expenditure share weighted *harmonic* average (using the expenditure shares for country j, the s_n^j, as weights) of the relative BH PPPs between countries j and k (the p_n^j/p_n^k). Thus the Fisher index defined by (3) can also be written in terms of the expenditure shares (in the local currencies) of the two countries being compared and the Basic Heading relative PPP parities for the two countries j and k, the p_n^j/p_n^k for n = 1,...,N.

Various justifications for the use of the Fisher ideal index in the bilateral context have been made by Diewert (1976) (1992) (2002; 569) and others.[10] The Fisher index can be justified from the point of view of finding the "best" symmetric average of the Laspeyres and Paasche indexes, or from the point of view of the axiomatic or test approach to index number theory, or from the viewpoint of the economic approach to index number theory; see Chapters 15, 16 and 17 in the *Consumer Price Index Manual*, ILO/IMF/OECD/UNECE/Eurostat/World Bank (2004).

The *aggregate PPP for country* j, P^j, is defined as follows:

(6) $P^j \equiv \prod_{k=1}^{K} [P_F(p^k,p^j,q^k,q^j)]^{1/K}$; j = 1,...,K.

The rational for choosing the parities defined by (6) is the following one. Make country k the numeraire country. Then a complete set of PPPs for all countries j using k as a numeraire country can be defined by the bilateral Fisher indexes $P_F(p^k,p^j,q^k,q^j) \equiv P(j/k)$ for j = 1,...,K. This set of PPPs for a fixed k is called the set *Fisher star PPPs with country k* as the star.[11] The final GEKS PPP for country j is simply the geometric mean of all of the country j parities over all possible choices of country k as the star; i.e., $P^j \equiv [\prod_{k=1}^{K} P(j/k)]^{1/K}$.

Once the GEKS P^j's have been defined by (6), the corresponding GEKS *country real expenditures or volumes* Q^j can be defined as the country expenditures $p^j \cdot q^j$ in the reference year divided by the corresponding GEKS purchasing power parity P^j:

(7) $Q^j \equiv p^j \cdot q^j/P^j$; j = 1,...,K.

[10] See Balk (2008; 91-97) for a review of the literature on axiomatic justifications for the Fisher index.
[11] This terminology follows that of Kravis (1984).

If all of the P^j defined by (6) are divided by a positive number, α say, then all of the Q^j defined by (7) can be multiplied by this same α without materially changing the GEKS multilateral method. If country 1 is chosen as the numeraire country in the region, then set α equal to P^1 defined by (6) for j = 1 and the resulting price level P^j is interpreted as the number of units of country j's currency it takes to purchase 1 unit of country 1's currency and get an equivalent amount of utility. The rescaled Q^j is interpreted as the volume of final demand of country j in the currency units of country 1.

It is also possible to normalize the aggregate real expenditure of each country in common units (the Q^k) by dividing each Q^k by the sum $\sum_{j=1}^{K} Q^j$ in order to express each country's real expenditure or real final demand as a fraction or share of total regional real expenditure; i.e., define the country k's *share of regional real expenditures*, S^k, as follows:[12]

$$(8)\ S^k \equiv Q^k/\sum_{j=1}^{K} Q^j\ ; \qquad k = 1,\ldots,K.$$

Of course, the country shares of regional real final demand, the S^k, remain unchanged after rescaling the PPPs by the scalar α.

This completes a brief description of the GEKS method for making multilateral comparisons.[13]

3. The Geary Khamis Method

The method was suggested by Geary (1958) and Khamis (1972) showed that the equations that define the method have a positive solution under certain conditions.

The GK system of equations involves K *country price levels* or PPPs, $P^1,\ldots,P^K$, and N *international Basic Heading commodity reference prices*, $\pi_1,\ldots,\pi_N$. The equations which determine these unknowns (up to a scalar multiple) are the following ones:

$$(9)\ \pi_n = \sum_{k=1}^{K} [q_n^k/\sum_{j=1}^{K} q_n^j][p_n^k/P^k]\ ; \qquad n = 1,\ldots,N\ ;$$
$$(10)\ P^k = p^k\cdot q^k/\pi\cdot q^k\ ; \qquad k = 1,\ldots,K$$

where $\pi \equiv [\pi_1,\ldots,\pi_N]$ is the vector of GK regional average reference prices. It can be seen that if a solution to equations (9) and (10) exists, then if all of the country parities P^k are multiplied by a positive scalar λ say and all of the reference prices π_n are divided by the same λ, then another solution to (9) and (10) is obtained. Hence, the π_n and P^k are only determined up to a scalar multiple and an additional normalization is required such as

[12] There are several additional ways of expressing the GEKS PPP's and relative volumes; see Balk (1996), Diewert (1999; 34-37) and section 6 below.

[13] It should be noted that all of the multilateral methods that are described in this section can be applied to subaggregates of the 155 basic heading categories; i.e., instead of working out aggregate price and volume comparisons across all 155 commodity classifications, one could just choose to include the food categories in the list of N categories and use the multilateral method to compare aggregate food consumption across the countries in the region.

(11) $P^1 = 1$

in order to uniquely determine the parities. It can also be shown that only N + K − 1 of the N equations in (9) and (10) are independent. Once the parities P^k have been determined, the real expenditure or volume for country k, Q^k, can be defined as country k's *nominal value of final demand in domestic currency units*, $p^k \cdot q^k$, divided by its PPP, P^k:

$$(12)\ Q^k = p^k \cdot q^k / P^k\ ; \qquad k = 1,\ldots,K$$
$$= \pi \cdot q^k \qquad \text{using (10).}$$

Finally, if equations (12) are substituted into the regional share equations (8), then country k's share of regional real expenditures is

$$(13)\ S^k = \pi \cdot q^k / \pi \cdot q \qquad k = 1,\ldots,K$$

where the *region's total volume vector* q is defined as the sum of the country volume vectors:

$$(14)\ q \equiv \sum_{j=1}^{K} q^j\ .$$

Equations (12) show how convenient it is to have an additive multilateral comparison method: when country outputs are valued at the international reference prices, values are additive across both countries and commodities. However, additive multilateral methods are not consistent with economic comparisons of utility across countries if the number of countries in the comparison is greater than two; see section 6 below. In addition, looking at equations (9), it can be seen that large countries will have a larger contribution to the determination of the international prices π_n and thus these international prices will be much more representative for the largest countries in the comparison as compared to the smaller ones.[14] This leads to the next method for making multilateral comparisons: an additive method that does not suffer from this problem of big countries having an undue influence in the comparison.

4. The Iklé Dikhanov Balk Method

Iklé (1972; 202-204) suggested this method in a very indirect way, Dikhanov (1994) (1997) suggested the much clearer system (13)-(14) below and Balk (1996; 207-208) provided the first existence proof. Dikhanov's (1994; 9-12) equations that are the counterparts to the GK equations (9) and (10) are the following ones:

$$(15)\ \pi_n = [\sum_{k=1}^{K} s_n^k\, [p_n^k/P^k]^{-1} / \sum_{j=1}^{K} s_n^j]^{-1}\ ; \qquad n = 1,\ldots,N$$
$$(16)\ P^k = [\sum_{n=1}^{N} s_n^k\, [p_n^k/\pi_n]^{-1}]^{-1} \qquad k = 1,\ldots,K$$

[14] Hill (1997) and Dikhanov (1994; 5) made this point.

where the country expenditure shares s_n^k are defined by (2) above.

As in the GK method, equations (15) and (16) involve the K country price levels or PPPs, $P^1,...,P^K$, and N international commodity reference prices, $\pi_1,...,\pi_N$. Equations (15) indicate that the *nth international price*, π_n, is a *share weighted harmonic mean of the country k Basic Heading PPPs for commodity n*, p_n^k, deflated by country k's overall PPP, P^k. The country k share weights for commodity n, s_n^k, do not sum (over countries k) to unity but when s_n^k is divided by $\sum_{j=1}^K s_n^j$, the resulting normalized shares do sum (over countries k) to unity. Thus equations (15) are similar to the GK equations (9), except that now a harmonic mean of the deflated BH commodity n "prices", p_n^k/P^k, is used in place of the old arithmetic mean and in the GK equations, country k's share of commodity group n in the region, $q_n^k/\sum_{j=1}^K q_n^j$, was used as a weighting factor (and hence large countries had a large influence in forming these weights) but now the weights involve country expenditure shares and so each country in the region has a more equal influence in forming the weighted average. Equations (16) indicate that P^k, the *PPP for country* k, P^k, is equal to a *weighted harmonic mean of the country k BH PPPs*, p_n^k, deflated by the international price for commodity group n, π_n, where the summation is over commodities n instead of over countries k as in equations (15). The share weights in the harmonic means defined by (16), the s_n^k, of course sum to one when the summation is over n, so there is no need to normalize these weights as was the case for equations (15).

It can be seen that if a solution to equations (15) and (16) exists, then multiplication of all of the country parities P^k by a positive scalar λ and division all of the reference prices π_n by the same λ will lead to another solution to (15) and (16). Hence, the π_n and P^k are only determined up to a scalar multiple and an additional normalization is required such as (11), $P^1 = 1$.

Although the IDB equations (16) do not appear to be related very closely to the corresponding GK equations (10), it can be shown that these two sets of equation are actually the same system. To see this, note that the country k expenditure share for commodity group n, s_n^k, has the following representation:

$$(17)\ s_n^k = p_n^k q_n^k / p^k \cdot q^k\ ; \qquad n = 1,...,N\ ;\ k = 1,...,K.$$

Now substitute equations (15) into equations (14) to obtain the following equations:

$$\begin{aligned}(18)\ P^k &= 1/\textstyle\sum_{n=1}^N s_n^k\,[p_n^k/\pi_n]^{-1} & k = 1,...,K \\ &= 1/\textstyle\sum_{n=1}^N [p_n^k q_n^k/p^k\cdot q^k][\pi_n/p_n^k] \\ &= p^k\cdot q^k/\textstyle\sum_{n=1}^N \pi_n q_n^k \\ &= p^k\cdot q^k/\pi\cdot q^k.\end{aligned}$$

Thus equations (16) are equivalent to equations (10) and the IDB system is an additive system; i.e., equations (12)-(14) can be applied to the present method just as they were applied to the GK method for making international comparisons.

In the Appendix, several different ways of representing the IDB system of parities will be obtained and fairly weak conditions for the existence and uniqueness of the IDB parities will be obtained. Effective methods for obtaining solutions to the system of equations (15) and (16) (with a normalization) will also be presented.

As was mentioned in the introduction to this chapter, the IDB method was used by the African region in order to construct regional aggregates. Basically, this method appears to be an improvement over the GK method in that large countries no longer have a dominant influence on the determination of the international reference prices π_n and so if an additive method is required with more democratic reference prices, IDB appears to be "better" than GK. In addition, Deaton and Heston (2010) have shown empirically that the IDB method generates aggregate PPPs that are much closer to the GEKS PPPs than are GK PPPs, using ICP 2005 data. However, in section 6 below, it will be shown that if one takes the economic approach to index number comparisons, then any additive multilateral method will be subject to some substitution bias.

For many users, the issue of possible substitution bias in the multilateral method is not an important one: these users want an additive multilateral method so that they can aggregate in a consistent fashion across countries and commodity groups. For these users, it may be useful to look at the axiomatic properties of the GK and IDB multilateral methods in order to determine a preference for one or the other of these additive methods. Thus in the next section, various multilateral axioms or tests are listed and the consistency of GK, IDB and GEKS with these axioms will be determined.

5. The Test or Axiomatic Approach to Making Multilateral Comparisons

Balk (1996) proposed a system of nine axioms for multilateral methods based on the earlier work of Diewert (1988).[15] Diewert (1999; 16-20) further refined his set of axioms and in this section, eleven of his thirteen "reasonable" axioms he proposed for a multilateral system will be listed. Some new notation will be used in the present section: $P \equiv [p^1,...,p^K]$ will signify an N by K matrix which has the domestic Basic Heading parities (or "price" vectors) $p^1,...,p^K$ as its K columns and $Q \equiv [q^1,...,q^K]$ will signify an N by K matrix which has the country Basic Heading volumes (or "quantity" vectors) $q^1,...,q^K$ as its K columns.

Any multilateral method applied to K countries in the comparison determines the country aggregate volumes, $Q^1,...,Q^K$, along with the corresponding country PPPs, $P^1,...,P^K$, The country volumes Q^k can be regarded as functions of the data matrices P and Q, so that the country volumes can be written as functions of the two data matrices, P and Q; i.e., the multilateral method the functions, $Q^k(P,Q)$ for k = 1,...,K. Once these functions $Q^k(P,Q)$ have been determined by the multilateral method, then *country k's share of total regional real expenditures*, $S^k(P,Q)$, can be defined as follows:

[15] Balk's axioms were somewhat different from those proposed by Diewert since Balk also introduced an extra set of country weights into Diewert's axioms. Balk's example will not be followed here since it is difficult to determine precisely what these country weights should be. For the most up to date review of the axiomatic approach to multilateral indexes, see Balk (2008; 232-260).

(19) $S^k(P,Q) \equiv Q^k(P,Q)/[Q^1(P,Q) + ... + Q^K(P,Q)]$; $k = 1,...,K.$

Both Balk (1996) (2008) and Diewert (1988) (1999) used the system of regional share equations $S^k(P,Q)$ as the basis for their axioms.

Eleven of Diewert's (1999; 16-20) 13 tests or axioms for a multilateral share system, $S^1(P,Q),...,S^K(P,Q)$, will now be listed.[16] It will be assumed that the two data matrices, P and Q, satisfy some mild regularity conditions which are listed in the first in section A.2.1 in the Appendix to this Chapter. In keeping with the literature on test approaches to index number theory, the components of the data matrix Q will be referred to as "quantities" (when they are actually BH volumes by commodity group and country) and the components of the data matrix P will be referred to as "prices" (when they are actually BH PPPs by commodity group and by country).

T1: *Share Test*: There exist K continuous, positive functions, $S^k(P,Q)$, $k = 1,...,K$, such that $\sum_{k=1}^K S^k(P,Q) = 1$ for all P,Q in the appropriate domain of definition.

This is a very mild test of consistency for the multilateral system.

T2: *Proportional Quantities Test*: Suppose that $q^k = \beta_k q$ for some $q >> 0_N$ and $\beta_k > 0$ for $k = 1,...,K$ with $\sum_{k=1}^K \beta_k = 1$. Then $S^k(P,Q) = \beta_k$ for $k = 1,...,K$.

This test says that if the quantity vector for country k, q^k, is equal to the positive fraction β_k times the total regional quantity vector q, then that country's share of regional real expenditures, $S^k(P,Q)$, should equal that same fraction β_k. Note that this condition is to hold no matter what P is.

T3: *Proportional Prices Test*: Suppose that $p^k = \alpha_k p$ for $p >> 0_N$ and $\alpha_k > 0$ for $k = 1,...,K$. Then $S^k(P,Q) = p \cdot q^k/[p \cdot \sum_{i=1}^K q^i]$ for $k = 1,...,K$.

This test says the following: suppose that the all of the country price vectors p^k are proportional to a common "price" vector p. Then the country k share of regional real expenditure, $S^k(P,Q)$, is equal to the value of its quantity vector, valued at the common prices p, $p \cdot q^k \equiv \sum_{n=1}^N p_n q_n^k$, divided by the regional value of real expenditures, also valued at the common prices p, $p \cdot \sum_{i=1}^K q^i$. Thus if prices are proportional to a common set of prices p across all countries, then these prices p can act as a set of *reference international prices* and the real expenditure volume of country k, Q^k, should equal $p \cdot q^k$ up to a normalizing factor.

[16] Diewert's (1999; 18) bilateral consistency in aggregation test is omitted, since this test depends on choosing a "best" bilateral quantity index and there may be no consensus on what this "best" functional form is. His final axiom involving the consistency of the multilateral system with the economic approach to index number theory will be discussed in section 6 below.

T4: *Commensurability Test*: Let $\delta_n > 0$ for n = 1,...,N and let Δ denote the N by N diagonal matrix with the δ_n on the main diagonal. Then $S^k(\Delta P,\Delta^{-1}Q) = S^k(P,Q)$ for k = 1,...,K.

This test implies that the country shares $S^k(P,Q)$ are invariant to changes in the units of measurement. This is a standard (but important) test in the axiomatic approach to index number theory that dates back to Fisher (1922; 420).

T5: *Commodity Reversal Test*: Let Π denote an N by N permutation matrix. Then $S^k(\Pi P,\Pi Q) = S^k(P,Q)$ for k = 1,...,K.

This test says that the ordering of the N commodity groups should not affect each country's share of regional real expenditure. This test also dates back to Fisher (1922; 63) in the context of bilateral index number formulae.

T6: *Multilateral Country Reversal Test*: Let S(P,Q) denote a K dimensional column vector that has the country shares $S^1(P,Q),.., S^K(P,Q)$ as components and let Π^* be a K by K permutation matrix. Then $S(P\Pi^*,Q\Pi^*) = S(P,Q)\Pi^*$.

This test implies that countries are treated in a symmetric manner; i.e., the country shares of world output are not affected by a reordering of the countries. The next two tests are homogeneity tests.

T7: *Monetary Units Test*: Let $\alpha_k > 0$ for k = 1,...,K. Then $S^k(\alpha_1 p^1,...,\alpha_K p^K,Q) = S^k(p^1,...,p^K,Q) = S^k(P,Q)$ for k = 1,...,K.

This test implies that the absolute scale of domestic prices in each country does not affect each country's share of world output; i.e., only relative prices within each country affect the multilateral volume parities.

T8: *Homogeneity in Quantities Test*: For i = 1,...,K, let $\lambda_i > 0$ and let j denote another country not equal to country i. Then $S^i(P,q^1,...,\lambda_i q^i,...,q^K)/S^j(P,q^1,...,\lambda_i q^i,...,q^K) = \lambda_i S^i(P, q^1,..., q^i,...,q^K)/S^i(P,q^1,..., q^i,...,q^K) = \lambda_i S^i(P,Q)/S^j(P,Q)$.

This test is equivalent to saying that the volume share of country i relative to country j is linearly homogeneous in the components of the country i quantity vector q^i.

T9: *Monotonicity Test in Quantities Test*: For each k, $S^k(P,q^1,...,q^{k-1},q^k,q^{k+1},...,q^K) = S^k(P,Q)$ is increasing in the components of q^k.

This test says that country k's share of world output increases as any component of the country k quantity vector q^k increases.

T10: *Country Partitioning Test*: Let A be a strict subset of the indexes (1,2,...,K) with at least two members. Suppose that for each $i \in A$, $p^i = \alpha_i p^a$ for $\alpha_i > 0$, $p^a >> 0_N$ and $q^i = \beta_i q^a$ for $\beta_i > 0$, $q^a >> 0_N$ with $\sum_{i\in A} \beta_i = 1$. Denote the subset of {1,2,...,K} that does not belong to A by B and denote the matrices of country price and quantity vectors that belong to B by P^b and Q^b respectively. Then: (i) for $i \in A$, $j \in A$, $S^i(P,Q)/S^j(P,Q) = \beta_i/\beta_j$ and (ii) for $i \in B$, $S^i(P,Q) = S^{i*}(p^a,P^b,q^a,Q^b)$ where $S^{k*}(p^a,P^b,q^a,Q^b)$ is the system of share functions that is obtained by adding the group A aggregate price and quantity vectors, p^a and q^a respectively, to the group B price and quantity data, P^b,Q^b.

Thus if the aggregate quantity vector for the countries in group A were distributed proportionally among its members (using the weights β_i) and each group A country faced prices that were proportional to p^a, then part (i) of T10 requires that the group A share functions reflect this proportional allocation. Part (ii) of T10 requires that the group B share functions are equal to the same values no matter whether we use the original share system or a new share system where all of the group A countries have been aggregated up into the single country which has the price vector p^a and the group A aggregate quantity vector q^a. Conversely, this test can be viewed as a *consistency in aggregation test* if a single group A country is partitioned into a group of smaller countries.

T11: *Additivity Test*: For each set of price and quantity data, P,Q, belonging to the appropriate domain of definition, there exists a set of positive world reference prices $\pi >> 0_N$ such that $S^k(P,Q) = \pi \cdot q^k/[\pi \cdot \sum_{i=1}^K q^i]$ for k = 1,...,K.

Thus if the multilateral system satisfies test T11, then it is an *additive method* since the real expenditure Q^k of each country k is proportional to the inner product of the vector of international prices π with the country k vector of commodity volumes (or "quantities"), q^k.

It is useful to contrast the axiomatic properties of the IDB method with the other additive method that has been used in ICP, the GK system. Using the results in Diewert (1999) on the GK system and the results on the IDB system in the Appendix, it can be seen that both methods satisfy tests T1-T7 and T11 and both methods fail the monotonicity in quantities test T9. Thus the tests that discriminate between the two methods are T8 and T10: the IDB multilateral system passes the homogeneity test T8 and fails the country partitioning test T10 and vice versa for the GK system.[17] There has been more discussion about test T10 than test T8. Proponents of the GK system like the fact that it has good aggregation (across countries) properties and the fact that big countries have more influence on the determination of the world reference price vector π is regarded as a reasonable price to pay for these "good" aggregation properties.[18] On the other hand, proponents of the IDB method like the fact that the world reference prices are more democratically determined (large countries play a smaller role in the determination of the

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[17] Balk (1996; 212) also compares the performance of the two methods (along with other multilateral methods) using his axiomatic system.

[18] Note that the fact that big countries play a more important role in the determination of the international prices when test T10 is satisfied is analogous to a property that national prices have to regional prices when a country's national accounts by product are constructed: the national price for a commodity is taken to be the unit value price for that commodity over the regions within the country. Thus large regions with large final demands will have a more important role in the determination of the national price vector than the smaller regions.

vector of international prices π) and they place less weight on having good aggregation properties. Also, from evidence presented by Deaton and Heston (2010) using the ICP 2005 data base, it appears that the IDB parities are closer to the GEKS parities than the GK parities. Thus the IDB method has the advantage that it is an additive method that does not depart too far from the parities that are generated by the GEKS method.

Diewert (1999; 18) showed that the GEKS system (using the Fisher ideal index as the basic building block) passed Tests 1-9 but failed the country partitioning test T10 and the additivity test T11. Thus all three of the multilateral methods considered thus far fail two out of the eleven tests.

At this point, it is difficult to unambiguously recommend any one of the three multilateral methods over the other two. In the following section, an economic approach to making multilateral comparisons will be considered which may help in evaluating the three methods.

6. Additive Multilateral Methods and the Economic Approach to Making Index Number Comparisons

It is useful to begin this section by reviewing what are the essential assumptions for the *economic approach to index number theory*:

- Purchasers have preferences over alternative bundles of goods and services that they purchase.
- As a result, they buy more of things that have gone down in relative price and less of things which have gone up in relative price.

The above type of substitution behaviour is well documented and hence, it is useful to attempt to take it into account when doing international comparisons.

The economic approach to index number theory does take substitution behavior into account. This approach was developed by Diewert (1976) in the bilateral context[19] and by Diewert (1999) in the multilateral context. Basically this theory works as follows:

- Assume that all purchasers have the same preferences over commodities and that these preferences can be represented by a homogeneous utility function.
- Find a functional form that can approximate preferences to the second order[20] and has an exact index number formula associated with it. The resulting index number formula is called a *superlative index number formula*.[21]
- Use the superlative index number formula in a bilateral context so that the real output of every country in the region can be compared to the real output of a numeraire country using this formula. The resulting relative volumes are dependent on the choice of the numeraire country.

[19] The pioneers in this approach were Konüs and Byushgens (1926).
[20] Diewert (1974;113) termed such functional forms *flexible*.
[21] Diewert (1976; 117) introduced this concept and terminology.

- Take the geometric average of all K sets of relative volumes using each country in the region as the numeraire country. This set of average relative volumes can then be converted into regional shares as in section 2 above. The resulting method is called a *superlative multilateral method*.[22]

It turns out that the GEKS method discussed in section 2 above is a superlative multilateral method; see Diewert (1999; 36). The GEKS method also has quite good axiomatic properties as will be seen in section 5 below.

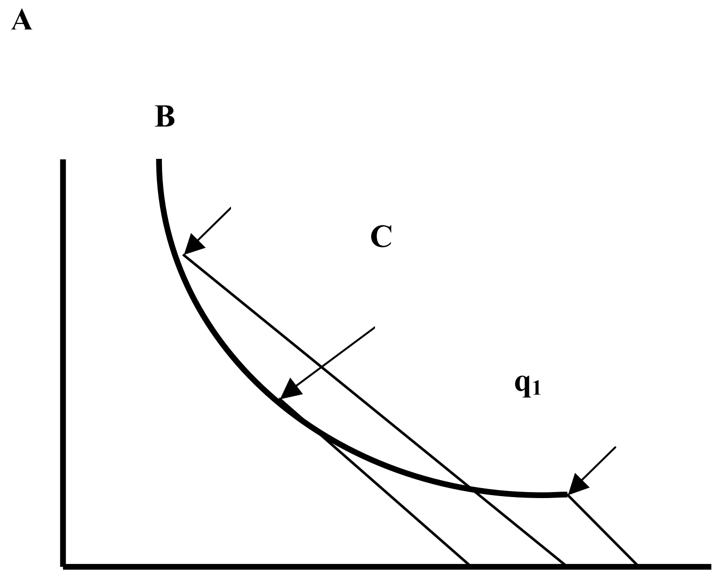
Given the importance of the GEKS multilateral method, it is worth explaining that the GEKS volume parities can be obtained by alternative methods.

The first alternative method is explained by Deaton and Heston (2010). In this method, the GEKS parities can be obtained by using a least squares minimization problem, originally due to Eltetö and Köves (1964) and Szulc (1964), that will essentially make an K by K matrix of bilateral Fisher volume parities that are not transitive into a best fitting set of transitive parities. The second method for deriving the GEKS parities was explained above. Pick any country as the base country and use the Fisher bilateral quantity index to form the volume of every country relative to the chosen base country. This gives estimated volumes for all countries in the comparison relative to the chosen base country.[23] Now repeat this process, choosing each country in turn as the base country, which leads to K sets of relative volume estimates. The final step for obtaining the GEKS relative volumes is to take the geometric mean of all of the K base country dependent sets of parities.

The problem with an additive multilateral method (from the perspective of the economic approach) if the number of countries in the region is greater than two can now be explained with the help of a diagram.[24]

Figure 1

[22] See Diewert (1999; 22).
[23] These are star volumes that are similar to the star PPPs that were explained in section 2 above. It should be noted that the GEKS volume shares can be derived starting with the Fisher star PPPs as in section 2 above or by starting with the Fisher star relative volume parities as in this section.
[24] This diagram is basically due to Marris (1984; 52) and Diewert (1999; 48-50).



The solid curved line in the above Figure represents an indifference curve for purchasers of the two goods under consideration. The consumption vectors of Countries A, B and C are all on the same indifference curve and hence, the multilateral method should show the same volume for the three countries. If we use the relative prices that country B faces as "world" reference prices in an additive method, then country B has the lowest volume or real consumption, followed by country A and finally, C has the highest volume. But they all have equal volumes! It can be seen that we can devise an additive method that will make the volumes of any two countries equal but we cannot devise an additive method that will equalize the volumes for all three countries. On the other hand, the common indifference curve in Figure 1 can be approximated reasonably well by a flexible functional form that has a corresponding exact index number formula (such as the Fisher index) and thus a GEKS method that used the Fisher bilateral index as a basic building block would give the right answer to a reasonable degree of approximation. The bottom line is that an additive multilateral method is not really consistent with economic comparisons of utility across countries if the number of countries in the comparison is greater than two.[25]

Although additive multilateral methods have their problems in that they are not consistent with substitution in the face of changing relative prices, the economic approach as

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[25] "Figure 1.1 also illustrates the Gerschenkron effect: in the consumer theory context, countries whose price vectors are far from the 'international' or world average prices used in an additive method will have quantity shares that are biased upward. ... It can be seen that these biases are simply quantity index counterparts to the usual substitution biases encountered in the theory of the consumer price index. However, the biases will usually be much larger in the multilateral context than in the intertemporal context since relative prices and quantities will be much more variable in the former context. ... The bottom line on the discussion presented above is that the quest for an additive multilateral method with good economic properties (i.e., a lack of substitution bias) is a doomed venture: nonlinear preferences and production functions cannot be adequately approximated by linear functions. Put another way, if technology and preferences were always linear, there would be no index number problem and hundreds of papers and monographs on the subject would be superfluous!" W. Erwin Diewert (1999; 50).

explained above is not without its problems. Two important criticisms of the economic approach are:

- The assumption that all final purchasers have the same preferences over different baskets of final demand purchases is suspect and
- The assumption that preferences are homothetic (i.e., can be represented by a linearly homogeneous utility function) is also suspect.

The second criticism of the economic approach to multilateral comparisons based on superlative bilateral index number formulae has been discussed in the recent literature on international comparisons and some brief comments on this literature are in order here.

An important recent development is Neary's (2004) GAIA multilateral system, which can be described as a consumer theory consistent version of the GK system, which allows for nonhomothetic preferences on the part of final demanders. Deaton and Heston (2010) point out that a weakness of the Neary multilateral system is that it uses a single set of relative prices to value consumption or GDP in all countries, no matter how different are the actual relative prices in each country. This problem was also noticed by Feenstra, Ma and Rao (2009) and these authors generalized Neary's framework to work with two sets of cross sectional data in order to estimate preferences[26] and they also experimented with alternative sets of reference prices. Barnett, Diewert and Zellner (2009) in their discussion of Feenstra, Mao and Rao, noted that a natural generalization of their model would be to use a set of reference prices which would be representative for each country in the comparison. Using representative prices for each country would lead to K sets of relative volumes and in the end, these country specific parities could be averaged just as the GEKS method averages country specific parities. Barnett, Diewert and Zellner conjectured that this geometric average of the country estimates will probably be close to GEKS estimates based on traditional multilateral index number theory, which of course, does not use econometrics. It remains to be seen if econometric approaches to the multilateral index number problem can be reconciled with superlative multilateral methods.[27]

In the following section, another economic approach to constructing multilateral comparisons will be described: a method that is based on linking countries that have similar economic structures.

7. The Minimum Spanning Tree Method for Making Multilateral Comparisons

Recall that the Fisher ideal quantity index can be used to construct real outputs for all K countries in the comparison, using one country as the base country. Thus as each country is used as the base country, K sets of relative outputs can be obtained. The GEKS

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[26] Methods that rely on the econometric estimation of preferences across countries are probably not suitable for the ICP, since it becomes very difficult to estimate flexible preferences for 155 commodity categories.

[27] One limitation of econometric approaches is that it will be impossible to estimate flexible functional forms for preferences when the number of commodity groups is as large as 155 since approximately 12,000 parameters would have to be estimated in this case.

multilateral method treats each country's set of relative outputs as being equally valid and hence an averaging of the parities is appropriate under this hypothesis. Thus the method is "democratic" in that each bilateral index number comparison between any two countries gets the same weight in the overall method. However, it is not the case that all bilateral comparisons of volume between two countries are equally accurate: if the relative prices in countries A and B are very similar, then the Laspeyres and Paasche quantity indexes will be very close to each other and hence it is likely that the "true" volume comparison between these two countries (using the economic approach to index number theory) will be very close to the Fisher volume comparison. On the other hand, if the structure of relative prices in the two countries is very different, then it is likely that the structure of relative quantities in the two countries will also be different and hence the Laspeyres and Paasche quantity indexes will likely differ considerably and it is no longer so certain that the Fisher quantity index will be close to the "true" volume comparison. The above considerations suggest that a more accurate set of world product shares could be constructed if initially a bilateral comparison was made between the two countries which had the most *similar relative price structures*.[28] At the next stage of the comparison, look for a third country which had the most similar relative price structure to the first two countries and link in this third country to the comparisons of volume between the first two countries and so on. At the end of this procedure, a *minimum spanning tree* would be constructed, which is a path between all countries that minimized the sum of the relative price dissimilarity measures. This linking methodology has been developed by Robert Hill (1999a) (1999b) (2004) (2009). The conclusion is that similarity linking[29] using Fisher ideal quantity indexes as the bilateral links is an alternative to GEKS which has some advantages over it.[30] Both methods are consistent with the economic approach to index number theory.

A key aspect of this methodology is the choice of the measure of similarity (or dissimilarity) of the relative price structures of two countries. Various measures of the similarity or dissimilarity of relative price structures have been proposed by Allen and Diewert (1981), Kravis, Heston and Summers (1982; 105), Sergeev (2001) (2009), Hill

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[28] Note that if all countries in the multilateral comparison have proportional "price" vectors, then the GEKS relative volume for any two countries j relative to i, S^j/S^i, is simply the Fisher ideal quantity index between the two countries, which in turn is equal to $p^i \cdot q^j/p^i \cdot q^i$ and to $p^j \cdot q^j/p^j \cdot q^i$, the Laspeyres and Paasche quantity indexes between the two countries. It can be seen that if we choose a vector of international prices π to be any one of the country price vectors, then $S^j/S^i = \pi \cdot q^j/\pi \cdot q^i = Q^j/Q^i$. Thus under the hypothesis of price proportionality across countries, the country real expenditure levels, Q^k, are proportional to $\pi \cdot q^k$ and the GEKS multilateral method can be regarded as an additive method.

[29] Perhaps more descriptive labels for the MST method for making international comparisons is the *similarity linking method* or the *spatial chaining method.*

[30] Deaton (2010; 33-34) noticed the following problem with the GEKS method: suppose we have two countries where the expenditure share on commodity 1 is tiny for country A and very big for country B. Suppose also that the price of commodity 1 in country A is very large relative to the price in country B. Then looking at the Törnqvist price index between A and B, it can be seen that the overall price level for country A will be blown up by the relatively high price for good 1 in A relative to B and by the big expenditure share in B on commodity 1. Since the Törnqvist will generally approximate the corresponding Fisher index closely, it can be seen that we have ended up exaggerating the price level of country A relative to B. This problem can be mitigated by spatial linking of countries that have similar price and quantity structures.

(1997) (2009), Diewert (2009) and Aten and Heston (2009). A few of these suggested measures of dissimilarity will now be discussed.

Suppose that we wish to compare how similar the structure of relative prices is for two countries, 1 and 2, which have the strictly positive Basic Heading PPP vectors p^k and the Basic Heading volume vectors q^k for $k = 1,2$. A *dissimilarity index*, $\Delta(p^1,p^2,q^1,q^2)$, is a function defined over the "price" and "quantity" data pertaining to the two countries, p^1,p^2,q^1,q^2, which indicates how similar or dissimilar the structure of relative prices is in the two countries being considered. If the two price vectors are proportional, so that relative prices in the two countries are equal, then we want the dissimilarity index to equal its minimum possible value, 0; i.e., we want $\Delta(p^1,p^2,q^1,q^2)$ to equal 0 if $p^2 = \lambda p^1$ for any positive scalar λ. If the price vectors are not proportional, then we want the dissimilarity measure to be positive.[31] Thus the larger is $\Delta(p^1,p^2,q^1,q^2)$, the more dissimilar is the structure of relative prices between the two countries.

The first measure of dissimilarity in relative price structures was suggested by Kravis, Heston and Summers (1982; 105) and Robert Hill (1999a) (1999b) (2001) (2004) and it is essentially a normalization of the relative spread between the Paasche and Laspeyres price indexes, so it is known as the *Paasche and Laspeyres Spread relative price dissimilarity measure*, $\Delta_{PLS}(p^1,p^2,q^1,q^2)$:

$$(20)\ \Delta_{PLS}(p^1,p^2,q^1,q^2) \equiv \max\{P_L/P_P, P_P/P_L\} - 1$$

where $P_L \equiv p^2{\cdot}q^1/p^1{\cdot}q^1$, and $\equiv p^2{\cdot}q^2/p^1{\cdot}q^2$. Diewert (2009; 184) pointed out a major problem with this measure of relative price dissimilarity; namely that it is possible for P_L to equal P_P but yet p^2 could be very far from being proportional to p^1. The following two measures of dissimilarity do not suffer from this problem.

Diewert (2009; 207) suggested the following measure of relative price similarity, the *weighted log quadratic measure of relative price dissimilarity*, $\Delta_{WLQ}(p^1,p^2,q^1,q^2)$:

$$(21)\ \Delta_{WLQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} (1/2)(s_n^1 + s_n^2)[\ln(p_n^2/p_n^1 P_F(p^1,p^2,q^1,q^2))]^2$$

where $P_F(p^1,p^2,q^1,q^2) \equiv [p^2{\cdot}q^1\ p^2{\cdot}q^2/p^1{\cdot}q^1\ p^1{\cdot}q^2]^{1/2}$ is the Fisher ideal price index between countries 2 and 1 and $s_n^c \equiv p_n^c q_n^c/p^c{\cdot}q^c$ is the country c expenditure share on commodity n for $c = 1,2$ and $n = 1,\ldots,N$.

There is a problem with the dissimilarity measure defined by (21) if for some commodity group n, either p_n^1 or p_n^2 equal 0 (or both Basic Heading PPPs equal 0), because in these cases, the measure can become infinite.[32] If both PPPs are 0, then commodity group n is irrelevant for both countries and the nth term in the summation in (21) can be dropped. In the case where one of the PPPs, say p_n^1 equals 0, but the other PPP p_n^2 is positive, then it would be useful to have an imputed PPP for commodity group n in country 1 which will

[31] For a more complete discussion of dissimilarity indexes and their properties, see Diewert (2009).

[32] If a PPP p_n^k equals 0, then we will assume that the corresponding volume is also 0.

make final demand for that commodity group equal to 0. This reservation PPP, p_n^{1*} say, could be approximated by simply setting p_n^{1*} equal to $p_n^2/P_F(p^1,p^2,q^1,q^2)$. If p_n^1 equal to 0 in (21) is replaced by this imputed PPP p_n^{1*}, then it can be seen that $p_n^2/p_n^{1*}P_F(p^1,p^2,q^1,q^2)$ is equal to 1 and the nth term on the right hand side of (21) vanishes. Similarly, in the case where p_n^2 equals 0, but the other PPP p_n^1 is positive, then set the reservation PPP for the nth commodity group in country 2, p_n^{2*} say, equal to $p_n^1P_F(p^1,p^2,q^1,q^2)$. If the 0 PPP p_n^2 in (21) is replaced by the imputed PPP p_n^{2*}, then it can be seen that $p_n^{2*}/p_n^1P_F(p^1,p^2,q^1,q^2)$ is equal to 1 and the nth term on the right hand side of (21) also vanishes in this case. Thus if there is a zero "price" or BH PPP for either country for commodity group n, then the above convention for constructing an imputed PPP for the zero PPP leads to the dropping of nth term on the right hand side of (21).[33]

It can be seen that if prices are proportional for the two countries so that $p^2 = \lambda p^1$ for some positive scalar λ, then $P_F(p^1,p^2,q^1,q^2) = \lambda$ and the measure of relative price dissimilarity $\Delta_{WLQ}(p^1,p^2,q^1,q^2)$ defined by (21) will equal its minimum of 0. Thus the smaller is $\Delta_{WLQ}(p^1,p^2,q^1,q^2)$, the more similar is the structure of relative prices in the two countries.

The method of spatial linking using the relative price dissimilarity measure defined by (21) will be illustrated in the next section.[34] Basically, instead of using the GEKS country shares defined by (8) in section 2, the shares generated by the minimum spanning tree are used to link all of the countries in the comparison.

Diewert (2009; 208) also suggested the following measure of relative price similarity, the *weighted asymptotically quadratic measure of relative price dissimilarity*, $\Delta_{WAQ}(p^1,p^2,q^1,q^2)$:

$$(22)\ \Delta_{WAQ}(p^1,p^2,q^1,q^2) \equiv \sum_{n=1}^{N} (1/2)(s_n^1 + s_n^2)\{[(p_n^2/p_n^1P_F(p^1,p^2,q^1,q^2) - 1]^2 + [(P_F(p^1,p^2,q^1,q^2)p_n^1/p_n^2) - 1]^2\}.$$

As was the case with the dissimilarity index defined by (21), the index defined by (22) will equal plus infinity if one of the "prices" or Basic Heading PPPs for commodity group n, p_n^1 or p_n^2, equals zero.[35] Again, it is useful to define an imputed price for the zero price to insert into the formula and a reasonable convention is to use the same imputed prices that were suggested for (21); i.e., if $p_n^1 = 0$, then define $p_n^{1*} \equiv p_n^2/P_F(p^1,p^2,q^1,q^2)$ and if if $p_n^2 = 0$, then define $p_n^{2*} \equiv p_n^1P_F(p^1,p^2,q^1,q^2)$.

Recently, Rao, Shankar and Hajarghasht (2010) have used the MST method for constructing PPPs across OECD countries using data for 1996. They used the PLS and WAQ dissimilarity measures defined by (20) and (21) and compared the resulting spatial

[33] Diewert (2009) did not deal with the zero "price" problem but it is a real problem that needs to be addressed in order to implement his suggested dissimilarity measures for relative price structures using real data. For additional discussion on the difficulties associated with making comparisons across countries where different commodities are being consumed, see Deaton and Heston (2010) and Diewert (2010).
[34] Some additional examples will be presented in Chapter 8.
[35] If both prices are 0, then simply drop the nth term in the summation on the right hand side of (22).

chains with the standard GEKS method. They found some fairly significant differences between the three sets of parities for the 24 countries in the comparison, with differences in the PPP for a single country of up to 10%. Thus the choice of method does matter, even if the methods of comparison are restricted to multilateral methods that allow for substitution effects. An interesting aspect of their study is that they found when WAQ was used as the dissimilarity measure as opposed to PLS, the linking of the countries was much more intuitive:

"As is generally the case with MSTs, there are a number of counter intuitive paths. For example, Spain and Greece are connected through Portugal, Denmark, USA, UK, Germany, Switzerland, Austria, Sweden, Italy. Similarly Australia and New Zealand are connected through the UK, Germany, Switzerland and Austria. Now we turn to Figure 2 where MST based on relative price distance measure is provided. The links in WRPD based MST are a lot more intuitive and are consistent with the notion of price similarity of the countries. For example, Spain, Italy, Portugal, Greece and Turkey are all connected directly, USA-Canada has a direct link so is the pair Ireland-United Kingdom. Countries like Sweden, Finland, Iceland, Norway and Denmark are all connected together. The main conclusion emerging from Figure 2 is that the WRPD [WAQ] is a better measure of price similarity than the PLS used in the standard MST applications." D.S. Prasada Rao, Sriram Shankar and Gholamreza Hajarghasht (2010).

Thus it appears that the pattern of bilateral links that emerges when using the MST method is much more "sensible" when a more discriminating measure of dissimilarity is used in the linking algorithm, as compared to the use of the Paasche and Laspeyres Spread measure defined by (20). Hence in future applications of the MST method, it is recommended that (20) *not* be used as the dissimilarity measure that is a key input into the MST method.

The narrowing of Paasche and Laspeyres spreads by the use of a spatial chaining method is not the only advantage of this method of linking countries. There are advantages at *lower levels of aggregation* in that if similar in structure countries are compared, generally, it will be found that product overlaps are maximized and hence the BH PPPs will be more accurately determined for similar in structure countries:

"Many differences in quality and proportion of high tech items discussed above are likely to be more pronounced between countries with very different economic structures. If criteria can be developed to identify countries with similar economic structure and they are compared only with each other, then it may overcome many of the issues of quality and lowest common denominator item comparisons. Economically similar countries are likely to have outlet types in similar proportions carrying the same types of goods and services. So direct comparisons between such countries will do a better job of holding constant the quality of the items than comparisons across more diverse countries." Bettina Aten and Alan Heston (2009; 251).

"Using the same spanning tree for a number of years would dramatically simplify multilateral international comparisons. Each country would only have to compare itself with its immediate neighbors in the spanning tree, thus reducing the cost and increasing the timeliness of international comparisons. Furthermore, by construction, each country's immediate neighbors in the minimum spanning tree will tend to have similar consumption patterns. This may substantially increase the characteristicity of the comparisons. Geary-Khamis by contrast, compares all countries using a single average price vector. In a comparison over rich and poor countries the average price vector may bear little resemblance to the actual price vectors of many of the countries in the comparison. Conversely, EKS uses all possible combinations of bilateral comparisons. This also requires all countries to provide price and expenditure data on the same set of basic headings, thus reducing the characteristicity of each comparison." Robert Hill (2009; 236-237).

Hill (2009; 237) also pointed out that the basic MST methodology could be adapted to impose a priori restrictions on possible links between certain countries:

"Suppose for example ... we do not want India to be linked directly with Hong Kong. This *exclusion* restriction can be imposed by replacing the PLS between India and Hong Kong, in the K×K PLS matrix, by a large dummy value ... Similarly, suppose we want Korea to be linked directly with Japan. This *inclusion* restriction can be imposed by replacing the PLS measure between Korea and Japan with a small dummy value ... This ensures that the corresponding edge is selected." Robert Hill (2009; 237).

Finally, Hill noted that not all statistical agencies produce data of the same quality and the MST method can be adapted to take this fact into account:

"In particular, some countries have better resourced national statistical offices than others. It would make little sense to put a country with an under resourced national statistical office at the center of a regional star even if so specified by the minimum spanning tree." Robert Hill (2009; 237).

The MST algorithm can be modified to ensure that countries with under resourced statistical offices enter the spanning tree with only one bilateral link to the other countries in the comparison. However, in practice, grading countries on the basis of the quality of their statistics would be a very difficult exercise. Thus given the lack of experience with making multilateral comparisons using the method of similarity linking, it is unlikely that this method will be used in ICP 2011.

To sum up, the *advantages* of the MST method for making multilateral comparisons are as follows:

- The MST method, using a superlative index number formula for forming bilateral links, like GEKS is consistent with the economic approach to making multilateral comparisons; i.e., it takes into account substitution effects.
- The MST method is likely to lead to a more accurate set of parities than those generated by the GEKS method, since the bilateral links between pairs of countries are based on comparisons between countries with the most similar structures of relative prices; i.e., the MST method is the spatial counterpart to chained annual indexes in the time series context.
- The influence of countries with under resourced statistical agencies can be minimized in a simple modification of the basic MST method.

There are also some *disadvantages* to the spatial linking method:

- The method is not as familiar as GEKS and GK and hence, it will be more difficult to build up a constituency for the use of this method.
- There are some arbitrary aspects to the method as compared to GEKS in that: (i) different measures of dissimilarity could be used and there is no universal agreement at this stage as to which measure is the most appropriate one to use; (ii) the treatment of zero "prices" and "quantities" in the measures of dissimilarity is not completely straightforward and (iii) the treatment of countries with under resourced statistical agencies is also not completely straightforward and

moreover, it may prove to be difficult to decide exactly which countries are under resourced.

- The path of bilateral links between countries generated by the method could be unstable; i.e., the Minimum Spanning Tree linking the countries could change when we move from one cross sectional comparison between countries to another cross sectional comparison.[36]

8. An Artificial Data Set Numerical Example

Diewert (1999; 79-84) illustrated the differences between various multilateral methods by constructing country PPPs and shares of "world" volumes for a three country, two commodity example. The GEKS, GK, IDB and MST parities will be calculated in this section using his numerical example.

The price and quantity vectors for the three countries are as follows:

(23) $p^1 \equiv [1,1]$; $p^2 \equiv [10, 1/10]$; $p^3 \equiv [1/10,10]$; $q^1 \equiv [1,2]$; $q^2 \equiv [1,100]$; $q^3 \equiv [1000,10]$.

Note that the geometric average of the prices in each country is 1, so that average price levels are roughly comparable across countries, except that the price of commodity 1 is very high and the price of commodity 2 is very low in country 2 and vice versa for country 3. As a result of these price differences, consumption of commodity 1 is relatively low and consumption of commodity 2 is relatively high in country 2 and vice versa in country 3. Country 1 can be regarded as a tiny country, with total expenditure (in national currency units) equal to 3, country 2 is a medium country with total expenditure equal to 20 and country 3 is a large country with expenditure equal to 200.

The Fisher (1922) quantity index Q_F can be used to calculate the volume Q^k of each country k relative to country 1; i.e., calculate Q^k/Q^1 as $Q_F(p^1,p^k,q^1,q^k) \equiv [p^1\cdot q^k\, p^k\cdot q^k/p^1\cdot q^1 p^k\cdot q^1]^{1/2}$ for k = 2,3. Set $Q^1 = 1$ and then Q^2 and Q^3 are determined and these volumes using country 1 as the base or star country are reported in the Fisher 1 column of Table 1. In a similar manner, use country 2 as the base and use the Fisher formula to calculate Q^1, $Q^2 = 1$ and Q^3. Then divide these numbers by Q^1 and the numbers listed in the Fisher 2 column of Table 1 are obtained. Finally, use country 3 as the base and use the Fisher formula to calculate Q^1, Q^2 and Q^3 =1. Then divide these numbers by Q^1 and obtain the numbers listed in the Fisher 3 column of Table 1. Ideally, these Fisher star parities would all coincide but since they do not, take the geometric mean of them and obtain the GEKS parities which are listed in the fourth column of Table 1. Thus for this example, the GEKS economic approach to forming multilateral quantity indexes leads to

[36] However, this evidence of unstable links comes from the results of the MST method using the Paasche and Laspeyres Spread measure of dissimilarity. Drawing on the recent research of Rao, Shankar and Hajarghasht (2010), it is likely that this instability will be reduced if a better measure of dissimilarity is used in the MST algorithm, like those defined by (21) and (22), as opposed to the use of the PLS measure defined by (20).

the volumes of countries 2 and 3 to be equal to 7.26 and 64.81 times the volume of country 1.[37]

Table 1: Fisher Star, GEKS, GK and IDB Relative Volumes for Three Countries

	Fisher 1	Fisher 2	Fisher 3	GEKS	GK	IDB
Q^1	1.00	1.00	1.00	1.00	1.00	1.00
Q^2	8.12	8.12	5.79	7.26	47.42	33.67
Q^3	57.88	81.25	57.88	64.81	57.35	336.67

Turning to the spatial linking method, it can be seen that country 1 has the most similar structure of prices with both countries 2 and 3; i.e., countries 2 and 3 have the most dissimilar structure of relative prices.[38] Thus in this case, the spatial linking method leads to the Fisher star parities for country 1; i.e., the spatial linking relative outputs are given by the Fisher 1 column in Table 1. Note that these parities are reasonably close to the GEKS parities.

The GK parities for P^k and π_n can be obtained by iterating between equations (9) and (10) until convergence has been achieved.[39] Once these parities have been determined, the Q^k can be determined using equations (12). These country volumes were then normalized so that $Q^1 = 1$. The resulting parities are listed in the GK column in Table 1. It can be seen that the GK parity for Q^3/Q^1, 57.35, is reasonable but the parity for Q^2/Q^1, 47.42, is much too large to be reasonable from an economic perspective. The cause of this unreasonable estimate for Q^2 is the fact that the GK international price vector, $[\pi_1,\pi_2]$, is equal to [1, 9.00] so that these relative prices are closest to the structure of relative prices in country 3, the large country. Thus the relatively large consumption of commodity 2 in country 2 gets an unduly high price weight using the GK vector of international reference prices, leading to an exaggerated estimate for its volume, Q^2. This illustrates a frequent criticism of the GK method: the structure of international prices that it gives rise to is "biased" towards the price structure of the biggest countries.

The IDB parities for the above numerical example are now calculated in order to see if the method can avoid the unreasonable results generated by the GK method. The parities for P^k and π_n can be obtained by iterating between equations (15) and (16) until convergence has been achieved.[40] Once these parities have been determined, the Q^k can be determined using equations (12). These country volumes were then normalized so that $Q^1 = 1$. The resulting parities are listed in the IBD column in Table 1. It can be seen that

[37] Since the Fisher star parities are not all equal, it needs to be recognized that the GEKS parities are only an approximation to the "truth". Thus it could be expected that an economic approach would lead to a Q^2/Q^1 parity in the 5 to 9 range and to a Q^3/Q^1 parity in the 50 to 90 range. Note that the IDB parities are well outside these ranges and the GK parity for Q^2/QY^1 is well outside this suggested range.

[38] This MTS result is obtained for all three measures of dissimilarity (20), (21) and (22) considered in the previous section.

[39] Only 5 iterations were required for convergence.

[40] Since all of the prices and quantities are positive in this example, equations (15) and (16) in the main text can be used instead of the more robust (to zero entries) equations (A3) and (A4). Eighteen iterations were required for convergence.

the GK parity for Q^2/Q^1 is 33.67 which is well outside the suggested reasonable range (from the viewpoint of the economic approach) of 5 to 9 and the GK parity for Q^3/Q^1 is 336.7 which is well outside the suggested reasonable range of 50 to 90. What is the cause of these problematic parities?

The problematic IDB volume estimates are not caused by an unrepresentative vector of international prices since the IBD international price vector, $[\pi_1,\pi_2]$, is equal to [1, 1], which in turn is equal to the vector of (equally weighted) geometric mean commodity prices across countries. The problem is due to the fact that *any* additive method cannot take into account the problem of declining marginal utility as consumption increases if there are 3 or more countries in the comparison. Thus the IBD vector of international prices $\pi = [1,1]$ is exactly equal to the country 1 price vector $p^1 = [1,1]$ and so the use of these international prices leads to an accurate volume measure for country 1. But the structure of the IBD international prices is far different from the prices facing consumers in country 2, where the price vector is $p^2 \equiv [10, 1/10]$. The very low relative price for commodity 2 leads consumers to demand a relatively large amount of this commodity (100 units) and the relatively high price for commodity 1 leads to a relatively low demand for this commodity (1 unit). Thus at international prices, the output of country 2 is $\pi\cdot q^2$ which is equal to 101 as compared to its nominal output $p^2\cdot q^2$ which is equal to 20. Thus the use of international prices overvalues the output of country 2 relative to country 1 because the international price of commodity 2 is equal to 1 which is very much larger than the actual price of commodity 2 in country 2 (which is 1/10). Note that Q^2/Q^1 is equal to $\pi\cdot q^2/\pi\cdot q^1 = 101/3 = 33.67$, an estimate which fails to take into account the declining marginal utility of the relatively large consumption of commodity 2 in country 1. A similar problem occurs when the outputs of countries 1 and 3 are compared using international prices except in this case, the use of international prices tremendously overvalues country 3's consumption of commodity 1. The problem of finding international reference prices that are "fair" for two country comparisons can be solved[41] but the problem cannot be solved in general if there are three or more countries in the comparison as was seen in section 6 above.

The tentative conclusion at this point is that additive methods for making international price and quantity comparisons where there are tremendous differences in the structure of prices and quantities across countries are likely to give rather different answers than methods that are based on economic approaches. This is why it is important for the International Comparison Program to provide two sets of results: one set based on a multilateral method like GEKS or MST that allows for substitution effects and another set that is based on an additive method like GK or IDB. Thus users can decide which set of estimates to use in their empirical work based on whether they need an additive method (with all of its accompanying desirable consistency in aggregation properties) or whether they need a method that allows for substitution effects.

9. Conclusion

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[41] See Diewert (1996; 246) for examples of superlative indexes that are additive if there are only two countries or observations.

This chapter discussed four multilateral methods for constructing PPPs and relative volumes for countries in a region.

Two of the methods were additive methods: the Geary Khamis (GK) method and the Iklé Dikhanov Balk (IDB) method. Additive methods are preferred by many users due to the fact that components of real GDP add up across countries and across commodities when an additive multilateral method is used.

Which additive method is "best"? The axiomatic properties of the IDB and GK systems are very similar and so it is difficult to discriminate between the two methods based on their axiomatic properties. The main *advantages of the IDB method* are as follows:

- The IDB international prices are not as influenced by the structure of relative prices in the biggest countries in the region as compared to the GK international prices ; i.e., the IDB method is more "democratic" than the GK method in its choice of international prices.
- From evidence presented by Deaton and Heston (2010) using the ICP 2005 data base, it appears that the IDB parities are closer to the GEKS parities than the GK parities. Thus the IDB method may have the advantage that it is an additive method that does not depart too far from the parities that are generated by the GEKS method.[42]

The main *advantages of the GK system* are as follows:

- The GK system has been widely used in previous ICP rounds and so users are familiar with the method and may want to continue to use the results of this method.
- The GK system has some similarity with the construction of national accounts data when quantities are aggregated over regions and thus GK estimates may be regarded as a reasonable extension of country wide national accounts to the world.

The other two methods discussed in this chapter were the Gini-Eltetö-Köves-Szulc (GEKS) method and the Minimum Spanning Tree (MST) method of similarity or spatial linking developed by Robert Hill using Fisher ideal indexes as basic bilateral building blocks. These two methods can be regarded as being consistent with an economic approach to a multilateral method; i.e., these methods deal adequately with substitution behavior on the part of purchasers of a country's outputs. The spatial linking method was not used in ICP 2005 but it has some attractive features which were discussed in section 7 above. However, there are some uncertainties associated with the use of this method and so it is unlikely to be used in ICP 2011.

Appendix: The Properties of the Iklé Dikhanov Balk Multilateral System

[42] However, the second example in Chapter 8 below indicates that the IDB parities may not always be closer to the GEKS parities than the GK parities.

A.1 Introduction and Overview

Unfortunately, multilateral index number theory is much more complicated than bilateral index number theory. Thus a rather long appendix is required in order to investigate the axiomatic and economic properties of the IDB multilateral system, particularly when some prices and quantities are allowed to be zero.[43] A brief overview of this appendix follows.

There are many equivalent ways of expressing the equations that define the IDB parities. Section A.2 lists these alternative systems of equations that can be used to define the method. Section A.3 provides proofs of the existence and uniqueness of solutions to the IDB equations. Section A.4 considers various special cases of the IDB equations. When there are only two countries so that $K = 2$, a bilateral index number formula is obtained and this case is considered along with the case where $N = 2$, so that there are only 2 commodities. These special cases cast some light on the structure of the general indexes. Section A.5 explores the axiomatic properties of the IDB method while section A.6 looks at the system's economic properties.

Throughout this appendix, it is assumed that the number of countries K and the number of commodities N is equal to or greater than two.

A.2 Alternative Representations

A.2.1 The P^k, π_n Representation

The basic data for the multilateral system are the prices and quantities for commodity n in country k at the Basic Heading level, p_n^k and q_n^k respectively, for $n = 1,...,N$ and $k = 1,...,K$ where the number of basic heading categories $N \geq 2$ and the number of countries $K \geq 2$. The N by 1 vectors of prices and quantities for country k are denoted by p^k and q^k and their inner product is $p^k \cdot q^k$ for $k = 1,...,K$. The share of country k expenditure on commodity n is denoted by $s_n^k \equiv p_n^k q_n^k / p^k \cdot q^k$ for $k = 1,...,K$ and $n = 1,...,N$.

It is assumed that for each n and k, either p_n^k, q_n^k and s_n^k are all zero or p_n^k, q_n^k and s_n^k are all positive. Thus the possibility that some countries do not consume all of the basic heading commodities is allowed for. This complicates the representations of the equations since division by zero prices, quantities or shares leads to difficulties and complicates proofs of existence.[44] For now, the following assumptions are made:

(A1) For every basic heading commodity n, there exists a country k such that p_n^k, q_n^k and s_n^k are all positive so that each commodity is demanded by some country.

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[43] Balk (1996; 207-208) has the most extensive published discussion of the properties of the IDB system but he considered only the case of positive prices and quantities for all commodities across all countries and he did not discuss the economic properties of the method.

[44] Balk's (1996; 208) existence proof assumed that all prices and quantities were strictly positive.

(A2) For every country k, there exists a commodity n such that $p_n{}^k$, $q_n{}^k$ and $s_n{}^k$ are all positive so that each country demands at least one basic heading commodity.

In section A.2, the above assumptions will be strengthened in order to ensure that the IDB equations have unique, positive solutions.

Recall that the IDB multilateral system was defined by the Dikhanov equations (15) and (16) (plus one normalization such as (11)). Taking into account the division by zero problem, these equations can be rewritten as follows:[45]

$$\text{(A3)}\ \pi_n = [\textstyle\sum_{j=1}^{K} s_n{}^j]/[\textstyle\sum_{k=1}^{K} (q_n{}^k P^k/p^k\cdot q^k)]\ ; \qquad n = 1,\ldots,N$$
$$\text{(A4)}\ P^k = p^k\cdot q^k/\pi\cdot q^k\ ; \qquad k = 1,\ldots,K$$

where π is a vector whose components are $\pi_1,\ldots,\pi_N$.

Using assumptions (A1) and (A2), it can be seen that equations (A3) and (A4) will be well behaved even if some $p_n{}^k$ and $q_n{}^k$ are zero. Equations (A3) and (A4) (plus a normalization on the P^k or π_n such as $P^1 = 1$ or $\pi_1 = 1$) provide the second representation of the IDB multilateral equations.[46]

In order to find a solution to equations (A3) and (A4), one can start by assuming that $\pi = 1_N$, a vector of ones and then use equations (A4) to determine a set of P^k. These P^k can then be inserted into equations (A3) in order to determine a new π vector. Then this new π vector can be inserted into equations (A4) in order to determine a new set of P^k. And so on; the process can be continued until convergence is achieved.

A.2.2 An Alternative P^k, π_n Representation using Biproportional Matrices

It can be seen that equations (A3) and (A4) can be rewritten in the following manner:

$$\text{(A5)}\ \textstyle\sum_{k=1}^{K} q_n{}^k\,[p^k\cdot q^k]^{-1}\pi_n P^k = \textstyle\sum_{j=1}^{K} s_n{}^j\ ; \qquad n = 1,\ldots,N;$$
$$\text{(A6)}\ \textstyle\sum_{n=1}^{N} q_n{}^k\,[p^k\cdot q^k]^{-1}\pi_n P^k = \textstyle\sum_{n=1}^{N} s_n{}^j = 1\ ; \qquad k = 1,\ldots,K.$$

Define the N by K *normalized quantity matrix* A which has element a_{nk} in row n and column k where

$$\text{(A7)}\ a_{nk} \equiv q_n{}^k/p^k\cdot q^k\ ; \qquad n = 1,\ldots,N\ ;\ k = 1,\ldots,K.$$

Define the N by K *expenditure share matrix* S which has the country k expenditure share for commodity n, $s_n{}^k$ in row n and column k. Let 1_N and 1_K be vectors of ones of

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[45] Equations (A3) are equivalent to Balk's (1996; 207) equations (38a) in the case where all price $p_n{}^k$ are positive and equations (A4) are Balk's equations (38b).

[46] Equations (15) and (16) provide a first representation in the case where all prices and quantities are positive.

dimension N and K respectively. Then equations (A5) and (A6) can be written in matrix form as follows:[47]

(A8) $\hat{\pi} AP = S1_K$;
(A9) $\pi^T A \hat{P} = 1_N{}^T S$

where $\pi \equiv [\pi_1,...,\pi_N]$ is the vector of IDB international prices, $P \equiv [P^1,...,P^K]$ is the vector of DI country PPPs, $\hat{\pi}$ denotes an N by N diagonal matrix with the elements of the vector π along the main diagonal and $\hat{P}$ denotes an K by K diagonal matrix with the elements of the vector P along the main diagonal. There are N equations in (A8) and K equations in (A9). However, examining (A8) and (A9), it is evident that if N+K−1 of these equations are satisfied, then the remaining equation is also satisfied. Equations (A8) and (A9) are a special case of the *biproportional matrix fitting model* due to Deming and Stephan (1940) in the statistics context and to Stone (1962) in the economics context (the RAS method). Bacharach (1970; 45) studied this model in great detail and gave rigorous conditions for the existence of a unique positive π, P solution set to (A8), (A9) and a normalization such as $P^1 = 1$ or $\pi_1 = 1$.[48] In section A.2 below, Bacharach's analysis will be used in order to provide simple sufficient conditions for the existence and uniqueness of a solution to equations (A8) and (A9) (plus a normalization).

In order to find a solution to (A8) and (A9), one can use the procedure suggested at the end of section A.2.1, since equations (A3) and (A4) are equivalent to (A5) and (A6).[49] Experience with the RAS method has shown that this procedure tends to converge quite rapidly.

A.2.3 The Q^k, π_n Representation

The above representations of the IDB system are in terms of a system of equations involving the N international reference prices π_n and the K country PPPs, P^k. It is useful to substitute equations (10) in the main text, $Q^k = p^k \cdot q^k/P^k$, which define the country volumes or aggregate quantities Q^k in terms of the country k price and quantity vectors p^k and q^k and the country k aggregate PPP, P^k, into equations (A3) and (A4) in order to obtain the following representation of the IDB multilateral system in terms of the Q^k and the π_n:

(A10) $\pi_n = [\sum_{j=1}^K s_n{}^j]/[\sum_{k=1}^K (q_n{}^k /Q^k)]$; $\qquad n = 1,...,N$

[47] Notation: when examining matrix equations, vectors such as π and P are to be regarded as column vectors and π^T and P^T denote their row vector transposes.
[48] It is obvious that if the positive vectors π and P satisfy (A8) and (A9), then $\lambda\pi$ and $\lambda^{-1}P$ also satisfy these equations where λ is any positive scalar. Dikhanov (1997; 12-13) also derived conditions for the existence and uniqueness of the solution set using a different approach.
[49] Bacharach (1970; 46) calls this method of solution the *biproportional process*. Bacharach (1970; 46-59) establishes conditions for the existence and uniqueness of a solution to the biproportional process; i.e., for the convergence of the process. The normalization (say $P^1 = 1$ or $\pi_1 = 1$) can be imposed at each iteration of the biproportional process or it can be imposed at the end of the process when convergence has been achieved.

(A11) $Q^k = \pi \cdot q^k$; $\quad k = 1,...,K.$

Of course, a normalization such as $Q^1 = 1$ or $\pi_1 = 1$ needs to be added in order to obtain a unique positive solution to (A10) and (A11).[50] Obviously, a biproportional iteration process could be set up to find a solution to equations (A10) and (A11) along the lines suggested at the end of section A.2.1, except that now the Q^k are determined rather than the P^k.

A.2.4 The Q^k Representation

If equations (A10) are substituted into equations (A11), the following K equations are obtained, involving only the country volumes, $Q^1,...,Q^K$:

(A12) $Q^k = \sum_{n=1}^{N} \{[s_n^1 +...+ s_n^K]\, q_n^k / [(q_n^1/Q^1) +...+ (q_n^K/Q^K)]\}$; $\quad k = 1,...,K.$

A normalization on the Q^k is required in order to obtain a unique solution, such as $Q^1 = 1$. It also can be seen that the K equations (A12) are not independent; i.e., if both sides of equation k in (A12) are divided by Q^k for each k and then the resulting equations are summed, the identity K equals K is obtained, using the fact that $\sum_{n=1}^{N} s_n^k = 1$ for each k. Thus once any K−1 of the K equations in (A12) are satisfied, the remaining equation is also satisfied.

Equations (A12) can be used in an iterative fashion in order to obtain a $Q^1,...,Q^K$ solution; i.e., make an initial guess at these volume parities and calculate the right hand side of each equation in (A12). This will generate a new set of volume parities, which can then be normalized to satisfy say $\sum_{k=1}^{K} Q^k$ equals 1. Then these new volume parities can again be inserted into the right hand sides of equations (A12) and so on.[51]

A.2.5 The P^k Representation

If the equations $Q^k = p^k \cdot q^k / P^k$ are substituted into equations (A12), the following K equations involving only the country PPPs, $P^1,...,P^K$, are obtained:

(A13) $(P^k)^{-1} = \sum_{n=1}^{N} \{[s_n^1 +...+ s_n^K][q_n^k/p^k \cdot q^k]/[(P^1 q_n^1/p^1 \cdot q^1) +...+ (P^K q_n^K/p^K \cdot q^K)]\}$;
$k = 1,...,K.$

As usual, a normalization on the P^k is needed in order to obtain a unique solution, such as $P^1 = 1$. It also can be seen that the K equations (A13) are not independent; i.e., if both sides of equation k in (A13) are multiplied by P^k for each k and then the resulting equations are summed, the identity K equals K is obtained, using the fact that $\sum_{n=1}^{N} s_n^k =$

[50] It can be verified that if N+K−1 of the equations (A10) and (A11) are satisfied, then the remaining equation is also satisfied; equations (A12) may be used to establish this result.

[51] When this method was tried on the data for the numerical example in Diewert (1999; 79) (see section 8 above), it was found that convergence was very slow. The iterative methods described in section A.2.1 converged much more quickly.

1 for each k. Thus once any K−1 of the K equations in (A13) are satisfied, the remaining equation is also satisfied.

Equations (A13) can be used iteratively in order to find a solution in a manner similar to the method described at the end of section A.2.4.

Equations (A12) and (A13) are difficult to interpret at this level of generality but when the axiomatic properties of the method are studied, it will be seen that the IDB parities have good axiomatic properties.

A.2.6 The π_n Representation

Finally, substitute equations (A4) into equations (A3) in order to obtain the following system of N equations which characterize the IDB international prices π_n:

$$\text{(A14)}\ \sum_{k=1}^{K} [\pi_n q_n^k/\pi\cdot q^k] = \sum_{k=1}^{K} s_n^k\ ; \qquad n = 1,...,N.$$

It can be seen that equations (A14) are homogeneous of degree 0 in the components of the π vector and so a normalization such as $\pi_1 = 1$ is required in order to obtain a unique positive solution. It also can be seen that if the N equations in (A14) are summed, the identity K equals K is obtained and so if any N−1 of the N equations in (A14) are satisfied, then so is the remaining equation.

Equations (A14) can be rewritten as follows:

$$\text{(A15)}\ \pi_n = [\sum_{k=1}^{K} s_n^k]/[\sum_{k=1}^{K} q_n^k/\pi\cdot q^k]\ ; \qquad n = 1,...,N.$$

Equations (A15) can be used iteratively in the usual manner in order to obtain a solution to equations (A14).

Equations (A14) have an interesting interpretation. Using the international reference prices π_n, define *country k's expenditure share for commodity n using these international prices* as:

$$\text{(A16)}\ \sigma_n^k \equiv \pi_n q_n^k/\pi\cdot q^k\ ; \qquad k = 1,...,K\ ;\ n = 1,...,N.$$

Substituting (A16) into (A14) leads to the following system of equations:

$$\text{(A17)}\ \sum_{k=1}^{K} \sigma_n^k = \sum_{k=1}^{K} s_n^k\ ; \qquad n = 1,...,N.$$

Thus for each basic heading commodity group n, the international prices π_n are chosen by the IDB method to be such that the sum over countries expenditure shares for commodity n using these international reference prices, $\sum_{k=1}^{K} \sigma_n^k$, is equal to the corresponding sum over countries expenditure shares using domestic prices in each country, $\sum_{k=1}^{K} s_n^{\ k}$, and this equality holds for all commodity groups n.[52]

A.3 Conditions for the Existence and Uniqueness of Solutions to the IDB Equations

In order to find conditions for positive solutions to any set of the IDB equations, the biproportional matrix representation that was explained in section A.2.2 above will be used.[53]

Bacharach (1970; 43-59) provided very weak sufficient conditions for the existence of a strictly positive solution $\pi_1,...,\pi_N$, $P^1,...,P^K$ to equations (A5) and (A6), assuming that (A1) and (A2) also hold. Bacharach's conditions involve the concept of *matrix connectedness*. Let A be an N by K matrix. Then Bacharach (1970; 44) defines A to be *disconnected* if after a possible reordering of its rows and columns, it can be written in the following *block rectangular* form:

$$\text{(A18) } A = \begin{bmatrix} A_{nxk} & 0_{nx(K-k)} \\ 0_{(N-n)xk} & A_{(N-n)x(K-k)} \end{bmatrix}$$

where $1 \le n < N$, $1 \le k < K$, $A_{n\times k}$ and $A_{(N-n)\times(K-k)}$ are submatrices of A of dimension n by k and N−n by K−k respectively and $0_{n\times(K-k)}$ and $0_{(N-n)\times(K-k)}$ are n by K−k and N−n by K−k matrices of zeros. As Bacharach (1970; 47) noted, the concept of disconnectedness is a generalization to rectangular matrices of the concept of decomposability which applies to square matrices. Bacharach (1970; 47) defined A to be *connected* if it is not disconnected (and it can be seen that this is a generalization of the concept of indecomposibility which applies to square matrices). Bacharach (1970; 47-55) went on to show that if the matrix A defined by (A7) is connected, assumptions (A1) and (A2) hold, and a normalization like $\pi_1 = 1$ or $P^1 = 1$ is added to equations (A5) and (A6), then these equations have a unique positive solution which can be obtained by using the biproportional procedure suggested at the end of section A.2.1, which will converge.

It is useful to have somewhat simpler conditions on the matrix A defined by (A7) which will imply that it is connected. It can be seen that either of the following two simple conditions will imply that A is connected (and hence, we have sufficient conditions for the existence of unique positive solutions to any representation of the IDB equations):

(A19) There exists a commodity n which is demanded by all countries; i.e., there exists an n such that $y_n^{\ k} > 0$ for $k = 1,...,K$;

[52] Dividing both sides of (A17) by K means that for each commodity group, the average (over countries) expenditure share using the IDB international prices is equal to the corresponding average expenditure share using the domestic prices prevailing in each country.
[53] Once the existence and uniqueness of a positive solution to any one of the representations of the IDB equations has been established, using assumptions (A1) and (A2), it is straightforward to show that a unique positive solution to the other representations is also implied.

(A20) There exists a country k which demands all commodities; i.e., there exists a k such that $y_n^k > 0$ for n = 1,...,N.

Conditions (A19) and (A20) are easy to check. These assumptions will be used in the following section.

A.4 Special Cases

In this section, some of the general N and K representations of the IDB equations will be specialized to cases where the number of commodities N or the number of countries K is equal to two.

A.4.1 The Two Country, Many Commodity Quantity Index Case

Suppose that the number of countries K is equal to 2. Set the country 1 volume equal to 1 so that Q^1 equals one and the first equation in (A12) becomes:

(A21) $\sum_{n=1}^{N} \{[s_n^1 + s_n^2]\, q_n^1 / [q_n^1 + (q_n^2/Q^2)]\} = 1.$

Equation (A21) is one equation in the one unknown Q^2 and it implicitly determines Q^2. It can be seen that Q^2 can be interpreted as a Fisher (1922) type bilateral quantity index, $Q_{IDB}(p^1,p^2,q^1,q^2)$, where p^k and q^k are the price and quantity (or more accurately, volume) vectors for country k. Thus in what follows in the remainder of this section, Q^2 will be replaced by Q.

At this point, assume that the data for country 1 satisfy assumption (A20) (so that q^1, p^1 and s^1 are all strictly positive vectors), which guarantees a unique positive solution to (A21). With this assumption, the quantity relatives r_n are well defined as follows:

(A22) $r_n \equiv q_n^2/q_n^1 \geq 0$; n = 1,...,N.

Assumption (A2) implies that at least one quantity relative r_n is positive. Since each q_n^1 is positive and letting Q equal Q^2, (A21) can be rewritten using definitions (A22) as follows:[54]

(A23) $\sum_{n=1}^{N} \{[s_n^1 + s_n^2] / [1 + (r_n/Q)]\} = 1.$

Define the *vector of quantity relatives* r as $[r_1,...,r_N]$. Then the function on the left hand side of (A23) can be defined as $F(Q,r,s^1,s^2)$, where s^k is the expenditure share vector for country k for k = 1,2. Note that $F(Q,r,s^1,s^2)$ is a continuous, monotonically increasing function of Q for Q positive. It is assumed that the components of q^1 and hence s^1 are all positive. Now compute the limits of $F(Q,r,s^1,s^2)$ as Q tends to plus infinity:

(A24) $\lim_{Q\to+\infty} F(Q,r,s^1,s^2) = \sum_{n=1}^{N} \{[s_n^1 + s_n^2] = 2.$

[54] (A23) shows that Q depends only on the components of two N dimensional vectors, r and $s^1 + s^2$.

In order to compute the limit of $F(Q,r,s^1,s^2)$ as Q tends to 0, two cases need to be considered. For the first case, assume that both countries consume all commodities so that $q^2 >> 0_N$ (this is in addition to the earlier assumption that $q^1 >> 0_N$). In this case, it is easy to verify that:

(A25) $\lim_{Q\to 0} F(Q,r,s^1,s^2) = 0$.

For the second case, assume that one or more components of q^2 are zero and let N^* be the set of indexes n such that q_n^2 equals 0. In this case:

(A26) $\lim_{Q\to 0} F(Q,r,s^1,s^2) = \sum_{n\in N^*} s_n^1 < 1$

where the inequality in (A26) follows from the fact that it is assumed that all s_n^1 are positive and the sum of all of the s_n^1 is 1.

The fact that $F(Q,r,s^1,s^2)$ is a continuous, monotonically increasing function of Q along with (A24)-(A26) implies that a finite positive Q solution to the equation $F(Q,r,s^1,s^2) = 1$ exists and is unique. Denote this solution as

(A27) $Q = G(r,s^1,s^2)$.

Now use the Implicit Function Theorem to show that $G(r,s^1,s^2)$ is a continuously differentiable function which is increasing in the components of r. Thus:

(A28) $\partial G(r,s^1,s^2)/\partial r_n$
$= [s_n^1 + s_n^2]\,[1 + (r_n/Q)]^{-2}Q/\{\sum_{i=1}^N [s_i^1 + s_i^2][1 + (r_i/Q)]^{-2}r_i\} > 0\,; \qquad n = 1,\ldots,N$

where Q satisfies (A27). However, the inequalities in (A28) do not imply that the IDB bilateral index number formula $Q_{IDB}(p^1,p^2,q^1,q^2)$ is increasing in the components of q^2 and decreasing in the components of q^1, since the derivatives in (A28) were calculated under the hypothesis that r_n equal to q_n^2/q_n^1 increased but the share vectors s^1 and s^2 were held *constant* as r_n was increased. In fact, it is *not* the case that $Q_{IDB}(p^1,p^2,q^1,q^2)$ is globally increasing in the components of q^2 and globally decreasing in the components of q^1.[55]

It is clear that $Q_{IDB}(p^1,p^2,q^1,q^2)$ satisfies the *identity test*; i.e., if $q^1 = q^2$ so that all quantity relatives r_n equal 1, then the only Q which satisfies (A23) is Q = 1. It is also clear that if $q^2 = \lambda q^1$ for $\lambda > 0$, then $Q_{IDB}(p^1,p^2,q^1,\lambda q^1)$ equals λ.[56]

[55] This negative monotonicity result also applies to the Törnqvist Theil bilateral index number formula, Q_T; see Diewert (1992; 221). The logarithm of Q_T is defined as $\ln Q_T = \sum_{n=1}^N (1/2)[s_n^1 + s_n^2] \ln r_n$.

[56] It is also clear from (A23) that $Q_{IDB}(p^1,p^2,q^1,q^2)$ satisfies the following four homogeneity tests $Q_{IDB}(p^1,p^2,q^1,\lambda q^2) = \lambda Q_{IDB}(p^1,p^2,q^1,q^2)$, $Q_{IDB}(p^1,p^2,\lambda q^1,q^2) = \lambda^{-1}Q_{IDB}(p^1,p^2,q^1,q^2)$, $Q_{IDB}(\lambda p^1,p^2,q^1,q^2) = Q_{IDB}(p^1,p^2,q^1,q^2)$ and $Q_{IDB}(p^1,\lambda p^2,q^1,q^2) = Q_{IDB}(p^1,p^2,q^1,q^2)$ for all $\lambda > 0$. Equations (A21) or (A23) can be used to show that $Q_{IDB}(p^1,p^2,q^1,q^2)$ satisfies the first eleven of Diewert's (1999; 36) thirteen tests for a

Define $\alpha \geq 0$ as the minimum over n of the quantity relatives, $r_n = q_n^2/q_n^1$ and define $\beta > 0$ as the maximum of these quantity relatives. Then using the monotonicity properties of the function $F(Q,r,s^1,s^2)$ defined by the left hand side of (A23), it can be shown that

$$(A29)\ \alpha \leq Q_{IDB}(p^1,p^2,q^1,q^2) \leq \beta$$

with strict inequalities in (A29) if the r_n are not all equal. Thus the IDB bilateral quantity index satisfies the usual *mean value test* for bilateral quantity indexes.[57]

It is possible to develop various approximations to $Q_{IDB}(p^1,p^2,q^1,q^2)$ that cast some light on the structure of the index. Recall that (A23) defined Q_{IDB} in implicit form. This equation can be rewritten as a *weighted harmonic mean* equal to 2 as follows:

$$(A30)\ \{\textstyle\sum_{n=1}^N w_n [1 + (r_n/Q)]^{-1}\}^{-1} = 2$$

where the *weights* w_n in (A30) are defined as follows:

$$(A31)\ w_n \equiv (1/2)[s_n^1 + s_n^2]\ ; \qquad n = 1,...,N.$$

Now approximate the weighted harmonic mean on the left hand side of (A30) by the corresponding weighted arithmetic mean and we obtain the following approximate version of equation (A30):

$$(A31)\ \textstyle\sum_{n=1}^N w_n [1 + (r_n/Q)] \approx 2.$$

Using the fact that the weights w_n sum up to one, (A31) implies that $Q = Q_{IDB}$ is approximately equal to the following expression:

$$(A32)\ Q_{IDB}(r,w) \approx \textstyle\sum_{n=1}^N w_n r_n = \sum_{n=1}^N (1/2)[(p_n^1 q_n^1/p^1\cdot q^1) + (p_n^2 q_n^2/p^2\cdot q^2)][q_n^2/q_n^1]\ .$$

If the weighted arithmetic mean on the right hand side of (A32) is further approximated by the corresponding weighted geometric mean, then it can be seen that $Q_{IDB}(r,w)$ is approximately equal to the following expression:

$$(A33)\ Q_{IDB}(r,w) \approx \textstyle\prod_{n=1}^N r_n^{w_n} \equiv Q_T(r,w)$$

where Q_T is the logarithm of the Törnqvist Theil quantity index defined as $\ln Q_T = \sum_{n=1}^N w_n \ln r_n$. If all of the quantity relatives r_n are equal to the same positive number, λ say, then the approximations in (A31)-(A33) will be exact and under these conditions where q^2 is equal to λq^1, then the following equalities will hold:

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bilateral quantity index, failing only the monotonicity in the components of q^1 and q^2 tests. Thus the axiomatic properties of the IDB bilateral quantity index are rather good.

[57] See Diewert (1992) for the history of these bilateral tests.

(A34) $Q_{IDB}(\lambda 1_N,w) = Q_T(\lambda 1_N,w) = \lambda$.

In the more general case, where the quantity relatives r_n are approximately equal to the same positive number so that q^2 is approximately proportional to q^1, then the Törnqvist Theil quantity index $Q_T(r,w)$ will provide a good approximation to the implicitly defined IDB quantity index, $Q_{IDB}(r,w)$.[58] However, in the international comparison context, it is frequently the case that quantity vectors are far from being proportional and in this nonproportional case, Q_{IDB} can be rather far from Q_T and other superlative indexes like Q_F as was seen in section 8 of the main text.

A.4.2 The Two Country, Many Commodity Price Index Case

Again, suppose that the number of countries K is equal to 2. Set the country 1 PPP, P^1, equal to 1 and the first equation in (A13) becomes:

$$\text{(A35)}\ \sum_{n=1}^{N} \{[s_n^1 + s_n^2](q_n^1/p^1\cdot q^1)/[(q_n^1/p^1\cdot q^1) + (P^2\, q_n^2/p^2\cdot q^2)]\} = 1.$$

Equation (A35) is one equation in the one unknown P^2 (the country 2 PPP) and it implicitly determines P^2. It can be seen that P^2 can be interpreted as a Fisher (1922) type bilateral price index, $P_{IDB}(p^1,p^2,q^1,q^2)$, where p^k and q^k are the price and quantity vectors for country k. Thus in what follows, P^2 will be replaced by P.

Again, it is assumed that the data for country 1 satisfy assumption (A20) (so that q^1, p^1 and s^1 are all strictly positive vectors), which guarantees a unique positive solution to (A35). It is convenient to define the *country k normalized quantity vector* u^k as the country k quantity vector divided by the value of its output in domestic currency, $p^k\cdot q^k$:

$$\text{(A36)}\ u^k \equiv q^k/p^k\cdot q^k\,; \qquad k = 1,2.$$

Since q^1 is strictly positive, so is q^1. Hence definitions (A36) can be substituted into (A35) in order to obtain the following equation, which implicitly determines $P^2 = P = P_{IDB}$:

$$\text{(A37)}\ \sum_{n=1}^{N} \{[s_n^1 + s_n^2]/[1 + P(q_n^2/q_n^1)(p^1\cdot q^1/p^2\cdot q^2)]\} = \sum_{n=1}^{N} \{[s_n^1 + s_n^2]/[1 + P(u_n^2/u_n^1)]\} = 1.$$

Define $r_n \equiv u_n^2/u_n^1$ for n = 1,...,N and rewrite P as 1/Q. Then it can be seen that equation (A37) becomes equation (A23) in the previous section and so the analysis surrounding equations (A23)-(A29) can be repeated to give the existence of a positive solution $P(r,s^1,s^2)$ to (A37) along with some of the properties of the solution.

Equation (A37) can be used to show that the IDB bilateral price index P, which is the solution to (A37), regarded as a function of the price and quantity data pertaining to the

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[58] If we regard $Q_{IDB}(r)$ and $Q_T(r)$ as functions of the vector of quantity relatives, then it can be shown directly that $Q_{IDB}(r)$ approximates $Q_T(r)$ to the second order around the point $r = 1_N$.

two countries, $P_{IDB}(p^1,p^2,q^1,q^2)$, satisfies the first eleven of the thirteen bilateral tests listed in Diewert (1999; 36)[59], failing only the monotonicity in the components of p^1 and p^2 tests; i.e., it is not necessarily the case that $P_{IDB}(p^1,p^2,q^1,q^2)$ is decreasing in the components of p^1 and increasing in the components of p^2. Thus the axiomatic properties of the IDB bilateral price index are rather good.

The bounds on the IDB bilateral quantity index given by (A29) do not have exactly analogous price counterparts. To develop counterparts to the bounds (A29), it is convenient to assume that all of the price and quantity data pertaining to both countries are positive. Under these conditions, define the following N *implicit partial price indexes* ρ_n:

$$\text{(A38)}\quad \rho_n \equiv [p^2\cdot q^2/q_n^2]/[p^1\cdot q^1/q_n^1] = [p^2\cdot q^2/p^1\cdot q^1]/[q_n^2/q_n^1]\ ; \qquad n = 1,\ldots,N.$$

An implicit bilateral price index is defined as the value ratio, $p^2\cdot q^2/p^1\cdot q^1$, divided by a quantity index, say $Q(p^1,p^2,q^1,q^2)$, where Q is generally some type of weighted average of the individual quantity relatives, q_n^2/q_n^1. Thus each quantity relative, q_n^2/q_n^1, can be regarded as a partial quantity index and hence the corresponding implicit quantity index, which is the value ratio divided by the quantity relative, can be regarded as an implicit partial price index. Substitution of definitions (A38) into (A37) leads to the following equation which implicitly determines P equal to $P_{IDB}(p^1,p^2,q^1,q^2)$:

$$\text{(A39)}\quad \textstyle\sum_{n=1}^N \{[s_n^1 + s_n^2]/[1 + (P/\rho_n)]\} = 1.$$

Define α as the minimum over n of the partial price indexes ρ_n and define β as the maximum of these partial price indexes. Then the monotonicity properties of the function defined by the left hand side of (A39) can be used in order to establish the following inequalities:

$$\text{(A40)}\quad \alpha \leq P_{IDB}(p^1,p^2,q^1,q^2) \leq \beta$$

with strict inequalities in (A40) if the ρ_n are not all equal.

An approximate explicit formula for P_{IDB} can readily be developed. Recall that (A39) defined P_{IDB} in implicit form. This equation can be rewritten as a *weighted harmonic mean* equal to 2 as follows:

$$\text{(A41)}\quad \{\textstyle\sum_{n=1}^N w_n [1 + (P/\rho_n)]^{-1}\}^{-1} = 2$$

where the *weights* w_n in (A41) are the average expenditure shares, $(1/2)[s_n^1 + s_n^2]$ for n = 1,...,N. Now approximate the weighted harmonic mean on the left hand side of (A41) by the corresponding weighted arithmetic mean and the following approximate version of equation (A30) is obtained:

[59] The role of prices and quantities must be interchanged; i.e., Diewert's (1999; 36) tests referred to quantity indexes whereas price indexes are now being considered.

(A42) $\sum_{n=1}^{N} w_n [1 + (P/\rho_n)] \approx 2.$

Using the fact that the weights w_n sum up to one, (A42) implies that $P = P_{IDB}$ is approximately equal to the following expression:

$$\begin{aligned}(A43)\ P_{IDB}(\rho,w) &\approx [\sum_{n=1}^{N} w_n (\rho_n)^{-1}]^{-1} \\ &= \{\sum_{n=1}^{N} (1/2)[(p_n^1 q_n^1/p^1\cdot q^1) + (p_n^2 q_n^2/p^2\cdot q^2)][q_n^2/q_n^1][p^1\cdot q^1/p^2\cdot q^2]\}^{-1}\end{aligned}$$

where $\rho \equiv [\rho_1,...,\rho_N]$ and $w \equiv [w_1,...,w_N]$. Thus the IDB bilateral price index P_{IDB} is *approximately* equal to a weighted harmonic mean of the N partial price indexes ρ_n defined earlier by (A38).[60]

A.4.3 The Many Country, Two Commodity Case

Consider the case where there are K countries but only two commodities so that N = 2. Recall that equations (A4) and (A11) determine the IDB country PPPs, P^k, and the country volumes, Q^k, in terms of the country price and quantity vectors, p^k and q^k, and a vector of international reference prices $\pi \equiv [\pi_1,...,\pi_N]$. Thus once π is determined, the P^k and q^k can readily be determined. In this section, it is assumed that N = 2, so that there are only 2 commodities and K countries. In order to ensure the existence of a solution to the IDB equations, it is assumed that commodity 1 is consumed by all countries:

(A44) $q_1^k > 0$; $\qquad\qquad$ k = 1,...,K.

The first international prices will be set equal to one:

(A45) $\pi_1 = 1.$

The equations which determine the π_n are equations (A14) but since N = 2, the second equation in (A14) can be dropped. Using the normalization (A45), the first equation in (A14) becomes the following equation:

(A46) $\sum_{k=1}^{K} q_1^k/[q_1^k + \pi_2 q_2^k] = \sum_{k=1}^{K} s_1^k$

which determines the international price for commodity 2, π_2.

Using assumptions (A44), the *country k commodity relatives* R^k (the quantities of commodity 2 relative to 1 in country k) are well defined as follows:

(A47) $R^k \equiv q_2^k/q_1^k \geq 0$; $\qquad\qquad$ k = 1,...,K.

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[60] The expressions involving the reciprocals of the ρ_n require that q^2 be strictly positive (in addition to our maintained assumption that y^1 be strictly positive). Equations (A35) and (A37) require only that y^1 be strictly positive.

Assumption (A1) implies that at least one quantity relative R^k is positive. Since each q_1^k is positive, rewrite (A46) using definitions (A47) as follows:[61]

(A48) $F(\pi_2,R,s_1) \equiv \sum_{k=1}^K 1/[1 + \pi_2 R^k] = \sum_{k=1}^K s_1^k \equiv s_1$

where s_1 is defined to be the sum over countries k of the expenditure share of commodity 1 in country k, s_1^k.[62] Define the *vector of country quantity relatives* R as $[R^1,...,R^K]$. Then the function on the left hand side of (A48) can be defined as $F(\pi_2,R,s_1)$.[63] Note that $F(\pi_2,R,s_1)$ is a continuous, monotonically decreasing function of π_2 for π_2 positive, since the R^k are nonnegative with at least one R^k positive. Now compute the limits of $F(\pi_2,R,s_1)$ as π_2 tends to zero:

(A49) $\lim_{\pi_2 \to 0} F(\pi_2,R,s_1) = K > \sum_{k=1}^K s_1^k = s_1$.

In order to compute the limit of $F(\pi_2,R,s_1)$ as π_2 tends to plus infinity, consider two cases. For the first case, assume that all countries consume both commodities so that $R >> 0_K$. Using the definition in (A48), the following inequality is obtained:

(A50) $\lim_{\pi_2 \to +\infty} F(\pi_2,R,s_1) = 0 < \sum_{k=1}^K s_1^k = s_1$.

For the second case, assume that one or more components of R are zero and let K^* be the set of indexes k such that R^k equals 0. In this case, the following limit is obtained:

(A51) $\lim_{\pi_2 \to +\infty} F(\pi_2,R,s_1) = \sum_{k \in K^*} s_n^1 < \sum_{k=1}^K s_1^k = s_1$.

The fact that $F(\pi_2,R,s_1)$ is a continuous, monotonically decreasing function of π_2 along with (A49)-(A51) implies that a finite positive π_2 solution to equation (A48) exists and is unique. Denote this solution as $\pi_2 = G(R,s_1)$. It is straightforward to verify that G is decreasing in the components of R and decreasing in s_1.

Suppose that all country quantity relatives R^k are positive and define α and β to be the minimum and maximum over k respectively of these quantity relatives. Then it is also straightforward to verify that π_2 satisfies the following bounds:[64]

(A52) $[(s_1/K)^{-1} - 1]/\beta \leq \pi_2 \leq [(s_1/K)^{-1} - 1]/\alpha$.

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[61] (A23) shows that Q depends only on the components of two N dimensional vectors, r and $s^1 + s^2$.
[62] Note that s_1 satisfies the inequalities $0 < s_1 < K$.
[63] Thus the π_2 solution to (A48) depends only on the vector of country quantity relatives, R, and the sum across countries k of the expenditure shares on commodity 1, s_1^k. Alternatively, π_2 depends on the K dimensional vector R and the sum across countries commodity share vector, $s^1 +...+ s^K$, which is a two dimensional vector in the present context where N = 2.
[64] It can be verified that $0 < s_1 < K$ so that $(s_1/K)^{-1} > 1$ so that the bounds in (A52) are positive when $R >> 0_K$. In the case where $R > 0_K$, the lower bound is still valid but the upper bound becomes $+\infty$.

Thus if all of country quantity relatives $R^k = q_2^k/q_1^k$ are equal to the same positive number λ, then the bounds in (A52) collapse to the common value $[(s_1/K)^{-1} - 1]/\lambda$.

In the case where prices and quantities are positive across all countries (so that all R^k are positive), then it is possible to rewrite the basic equation (A48) in a more illuminating form as follows:

$$\text{(A53)}\ \sum_{k=1}^{K} s_1^k = \sum_{k=1}^{K} 1/[1 + \pi_2 R^k]$$
$$= \sum_{k=1}^{K} \{s_1^k/[s_1^k + \pi_2 s_1^k(y_2^k/y_1^k)]\}$$
$$= \sum_{k=1}^{K} \{s_1^k/[s_1^k + \pi_2 s_2^k(p_2^k/p_1^k)^{-1}]\}.$$

Equation (A53) shows that the π_2 which solves the equation is a function of the K country share vectors, $s^1,...,s^K$ (each of which is of dimension 2) and the vector of K country price relatives, $[p_2^1/p_1^1,..., p_2^K/p_1^K]$. It can be seen that if all of these country price relatives are equal to a common ratio, say $\lambda > 0$, then the solution to (A53) is $\pi_2 = \lambda$. In the case where all of these country price relatives are positive, let α^* and β^* to be the minimum and maximum over k respectively of these price relatives. Then it is straightforward to verify that π_2 satisfies the following bounds:

$$\text{(A54)}\ \alpha^* \le \pi_2 \le \beta^*.$$

A.4.4 The Two Country, Two Commodity Case

In this section, it is assumed that K = 2 (two countries) and that N = 2 (two commodities). In this case, it is possible to obtain an explicit formula for the country 2 volume Q^2 relative to relative to the country 1 volume Q^1 which is set equal to one; i.e., it is possible to obtain an explicit formula for the IDB bilateral quantity index, $Q^2 = Q = Q_{IDB}(p^1,p^2,q^1,q^2)$. The starting point for this case is equation (A21) which determines Q implicitly. In the case where N equals 2, this equation becomes:

$$\text{(A55)}\ \{[s_1^1 + s_1^2]q_1^1/[q_1^1 + (q_1^2/Q)]\} + \{[(1-s_1^1) + (1-s_1^2)]q_2^1/[q_2^1 + (q_2^2/Q)]\} = 1.$$

As usual, it is assumed that the data for country 1 are positive so that $q_1^1 > 0$ and $q_2^1 > 0$. Thus the two quantity relatives, $r_n \equiv q_n^2/q_n^1$ for n = 1,2, are well defined nonnegative numbers. It is assumed that at least one of the relatives r_1 and r_2 are strictly positive. Substitution of these quantity relatives into (A55) leads to the following equation for Q:

$$\text{(A56)}\ \{[s_1^1 + s_1^2]Q/[Q + r_1]\} + \{[(1-s_1^1) + (1-s_1^2)]Q/[Q + r_2]\} = 1.$$

The above equation simplifies into the following quadratic equation:[65]

$$\text{(A57)}\ Q^2 + [s_1^1 + s_1^2-1][r_2 - r_1]Q - r_1 r_2 = 0.$$

[65] This equation can be utilized to show that $Q_{IDB}(p^1,p^2,q^1,q^2)$ is not necessarily monotonically increasing in the components of q^1 or monotonically decreasing in the components of q^1.

In the case where both r_1 and r_2 are positive, there is a negative and a positive root for (A57). The positive root is the desired bilateral quantity index and it is equal to the following expression:

(A58) $Q_{IDB}(p^1,p^2,q^1,q^2) = -(1/2)(s_1^1+s_1^2-1)(r_2-r_1) + (1/2)[(s_1^1+s_1^2-1)^2 (r_2-r_1)^2 + 4r_1r_2]^{1/2}$.

Now suppose that $r_1 = q_1^2/q_1^1 = 0$ so that $q_1^1 > 0$ and $q_1^2 = 0$. Then $s_1^2 = 0$ as well and using (A57):

(A59) $Q = [1 - s_1^1]r_2 = [1 - s_1^1][q_2^2/q_2^1]$.

Formula (A59) makes sense in the present context. Recall that Q is supposed to reflect the country 2 volume or average quantity relative to country 1. If, as a preliminary estimate of this relative volume, Q is set equal to the single nonzero quantity relative, r_2, then this would overestimate the average volume of country 2 relative to 1 since country 2 has a zero amount of commodity 1 while country 1 has the positive amount q_1^1. Thus r_2 is scaled down by multiplying it by one minus country 1's share of commodity 1, s_1^1. The bigger is this share, the more the preliminary volume ratio r_2 is downsized.

Now suppose that $r_2 = y_2^2/y_2^1 = 0$ so that $q_2^1 > 0$ and $q_2^2 = 0$. Then $s_1^2 = 1$ and using (A57):

(A60) $Q = s_1^1r_1 = [1-s_2^1][q_1^2/q_1^1]$.

Again, formula (A60) makes sense in the present context. If Q is set equal to the single nonzero quantity relative, r_1, then this would overestimate the average volume of country 2 relative to 1 since country 2 has a zero amount of commodity 2 while country 1 has the positive amount q_2^1. Thus scale down r_1 by multiplying it by one minus country 1's share of commodity 2, s_2^1. The bigger is this share, the more the preliminary volume ratio r_1 is downsized.

Two other special cases of (A57) are of interest. Consider the cases where the following conditions hold:

(A61) $r_1 = r_2$;
(A62) $s_1^1+s_1^2 = 1$.

If either of the above two special cases hold, then Q equals $(r_1r_2)^{1/2}$, the geometric mean of the two quantity relatives. This first result is not surprising since this result is implied by the earlier N commodity results for two countries; i.e., see (A29). The second result is more interesting. Note that if (A62) holds, so that the sum of the two country expenditure shares on commodity 1 is equal to 1, then the sum of the two country expenditure shares on commodity 2 is also equal to 1; i.e., it is also the case that $s_2^1+s_2^2 = 1$ and the IDB quantity index is equal to the geometric mean of the two quantity relatives, $(r_1r_2)^{1/2}$.[66]

The following section provides a discussion of the axiomatic or test properties of the IDB multilateral system.

A.5 The Axiomatic Properties of the Iklé Dikhanov Balk Multilateral System

Recall section 5 in the main text of this chapter where 11 tests or axioms for multilateral systems were listed. The axiomatic properties of the IDB system are summarized in the following result.

Proposition 1: Assume that the country price and quantity data P,Q satisfy assumptions (A1), (A2) and at least one of the assumptions (A19) and (A20). Then the IDB multilateral system fails only Tests 9 and 10 for the 11 tests listed in section 5 of the main text.

Proof: The existence and uniqueness of a solution to any one of the representations of the IDB equations have been discussed in section A.3 above. The continuity (and once continuous differentiability) of the IDB share functions $S^k(P,Q)$ in the data follow using the Implicit Function Theorem on the system of equations (A8) and (A9) (plus a normalization) by adapting the arguments in Bacharach (1970; 67-68). This establishes T1.

The proofs of tests T2 and T4-T8 follow by straightforward substitution into equations (A12).

The proof of T3 follows by setting $\pi = p$ and then showing that this choice of π satisfies equations (A14). Once π has been determined as p, then the Q^k are determined as $\pi\cdot q^k$ for $k = 1,...,K$ and finally the share functions are determined using (17).

The results in section A.4.4 can be used to show that the monotonicity test T9 fails.

Finally, it can be seen that the "democratic" nature of the IDB system (each country's shares are treated equally in forming the reference prices π) leads to a failure of test T10.[67]

The main text showed that the IDB method satisfied the additivity test T11. Q.E.D.

A.6 The Economic Properties of the Iklé Dikhanov Balk Multilateral System

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[66] Under these conditions, it is also the case that all prices and quantities are positive in the two countries since it was assumed that y^1 is strictly positive and y^2 is nonnegative and nonzero; i.e., $q^1 \gg 0_2$ and $q^2 > 0_2$.
[67] Diewert (1999; 27) showed that the GK system satisfied all of the 11 tests except the homogeneity test T8 and the monotonicity test T9. The GK system is a "plutocratic" method where the bigger countries have a greater influence in the determination of the international price vector π.

An economic approach to bilateral index number theory was initiated by Diewert (1976) and generalized to multilateral indexes in Diewert (1999; 20-23). The properties of the IDB system in this economic framework will be examined in the present section.

The basic assumption in the economic approach to multilateral indexes is that the country k quantity vector q^k is a solution to the following country k utility maximization problem:

(A63) $\max_q \{f(q) : p^k \cdot q = p^k \cdot q^k\} = u_k = Q^k$

where $u^k \equiv f(q^k)$ is the utility level for country k which can also be interpreted as the country's volume Q^k, $p^k >> 0_N$ is the vector of positive prices for outputs that prevail in country k, for $k = 1,...,K$[68] and f is a linearly homogeneous, increasing and concave aggregator function that is assumed to be the same across countries. This aggregator function has a *dual unit cost or expenditure function* c(p) which is defined as the minimum cost or expenditure required to achieve unit volume level if purchasers face the positive commodity price vector p.[69] Since purchasers in country k are assumed to face the prices $p^k >> 0_N$, the following equalities hold:

(A64) $c(p^k) \equiv \min_q \{p^k \cdot q : f(q) \geq 1\} \equiv P^k$; $\quad k = 1,...,K$

where P^k is the (unobserved) minimum expenditure that is required for country k purchasers to achieve unit utility or volume level when the purchasers face prices p^k. P^k can also be interpreted as country k's aggregate PPP. Under assumptions (A63), it can be shown[70] that the country k price and quantity vectors, p^k and q^k, satisfy the following equation:

(A65) $p^k \cdot q = c(p^k)f(q^k) = P^k u_k = P^k Q^k \quad k = 1,...,K.$

In order to make further progress, it is assumed that either the utility function f(q) is once continuously differentiable with respect to the components of q *or* the unit cost function c(p) is once continuously differentiable with respect to the components of p (or both).

In the case where f is assumed to be differentiable, the first order necessary conditions for the utility maximization problems in (A63), along with the linear homogeneity of f, imply the following relationships between the country k price and quantity vectors, p^k and q^k respectively, and the country unit expenditures, e_k defined in (A64):[71]

(A66) $p^k = \nabla f(q^k) P^k$; $\quad k = 1,...,K$

where $\nabla f(q^k)$ denotes the vector of first order partial derivatives of f with respect to the components of q evaluated at the country k quantity vector, q^k.

[68] In this section, it will be assumed that all country prices and quantities are positive so that $p^k >> 0_N$ and $q^k >> 0_N$ for $k = 1,...,K$.
[69] The unit cost function c(p) is an increasing, linearly homogeneous and concave function in p for $p >> 0_N$.
[70] See Diewert (1974) for material on duality theory and unit cost functions.
[71] See Diewert (1999; 21) for more details on the derivation of these equations.

In the case where c(p) is assumed to be differentiable, then Shephard's Lemma implies the following equations:

(A67) $q^k = \nabla c(p^k)u_k = \nabla c(p^k)Q^k$ $\qquad k = 1,...,K$

where $u_k = f(q^k) = Q^k$ denotes the utility level for country k and $\nabla c(p^k)$ denotes the vector of first order partial derivatives of the unit cost function c with respect to the components of p evaluated at the country k price vector p^k.

If f(q) or c(p) are differentiable, then since both of these functions are assumed to be linearly homogeneous, Euler's Theorem on homogeneous functions implies the following relationships:

(A68) $f(q^k) = \nabla f(q^k)\cdot q^k = \sum_{n=1}^{N} [\partial f(q^k)/\partial q_n]q_n^k$; $\qquad k = 1,...,K;$
(A69) $c(p^k) = \nabla c(p^k)\cdot p^k = \sum_{n=1}^{N} [\partial c(p^k)/\partial p_n]p_n^k$; $\qquad k = 1,...,K.$

Recall that the *expenditure share* on commodity n for country k was defined as $s_n^k \equiv p_n^k q_n^k/p^k\cdot q^k$. In the case where f(q) is differentiable, substitution of (A66) and (A68) into these shares leads to the following expressions:

(A70) $s_n^k = q_n^k f_n(q^k)/f(q^k)$; $\qquad n = 1,...,N ; k = 1,...,K$

where $f_n(q^k) \equiv \partial f(q^k)/\partial q_n$. In the case where c(p) is differentiable, substitution of (A67) and (A69) into the expenditure shares s_n^k leads to the following expressions:

(A71) $s_n^k = p_n^k c_n(p^k)/c(p^k)$; $\qquad n = 1,...,N ; k = 1,...,K$

where $c_n(p^k) \equiv \partial c(p^k)/\partial p_n$. With the above preliminaries laid out, we are now ready to attempt to determine what classes of preferences (i.e., differentiable functional forms for f or c) are consistent with the IDB system of equations (A12).

Start out by considering the case of a differentiable utility function, f(q), which is positive, increasing, linearly homogeneous and concave for $q >> 0_N$.[72] Let $q^k >> 0_N$, $Q^k = f(q^k)$ for k = 1,...,K and substitute these equations and (A70) into equations (A12). Then f must satisfy the following system of K functional equations:

(A72) $\sum_{n=1}^{N}\{[q_n^1 f_n(q^1)/f(q^1)]+...+[q_n^K f_n(q^K)/f(q^K)]\}[q_n^k/f(q^k)]/\{[q_n^1/f(q^1)]+...+[q_n^K/f(q^K)]\}$
$= 1$; $\qquad k = 1,...,K.$

Note that all of the terms in the above system of K equations are the same in each equation except the terms $q_n^k/f(q^k)$ in the middle of equation k. Suppose that f(q) is a *linear function* of q so that:

[72] The functions f or c are defined to be *regular* if they satisfies these regularity conditions.

(A73) $f(q) = f(q_1,...,q_N) = a_1q_1 + ... + a_Nq_N$; $a_1 > 0$, ... , $a_N > 0$.

It is straightforward to verify that the linear function f(q) defined by (A73) satisfies the maintained hypotheses on f and it also satisfies the system of functional equations (A75). Thus the IDB multilateral system is consistent with linear preferences.

Now consider the case of a differentiable unit cost function c(p), which is positive, increasing, linearly homogeneous and concave for $p >> 0_N$. Let $p^k >> 0_N$, $P^k = c(p^k)$ for k = 1,...,K and substitute these equations and (A67) into equations (A12). Then c must satisfy the following system of K functional equations:

$$\text{(A74) } \sum_{n=1}^{N} \{[p_n^1c_n(p^1)/c(p^1)] +...+ [p_n^Kc_n(p^K)/c(p^K)]\}c_n(p^k)/\{c_n(p^1) +...+ c_n(p^K)\} = 1 \text{ ; } k = 1,...,K.$$

Note that all of the terms in the above system of K equations are the same in each equation except the partial derivative terms $c_n(p^k)$ in the middle of equation k. Now suppose that c(p) is a *linear function* of p so that:

(A75) $c(p) = c(p_1,...,p_N) = b_1y_1 + ... + b_Ny_N$; $b_1 > 0$, ... , $b_N > 0$.

It is straightforward to verify that the linear function c(p) defined by (A75) satisfies the maintained hypotheses on c and it also satisfies the system of functional equations (A74). Thus the IDB multilateral system is consistent with Leontief (no substitution) preferences.

The above computations show that the IDB multilateral system is consistent with preferences that exhibit perfect substitutability between commodities (the linear utility function case) and with preferences that exhibit no substitution behavior as prices change (the case of Leontief preferences where the unit cost function is linear). It turns out that if the number of countries is three or more, then these are the only (differentiable) preferences that are consistent with the IDB system as is shown by the following result:

Proposition 2: If the number of countries is greater than two, then the linear utility function defined by (A73) is the only regular differentiable utility function that is consistent with the IDB equations (A72) and the preferences that are dual to the linear unit cost function defined by (A75) are the only differentiable dual preferences that are consistent with the IDB equations (A74).

Proof: Let $K \geq 3$ and let $q^k >> 0_N$ for k = 1,...,K. Then the first two equations in (A72) can be rearranged into the following equation:

$$\text{(A76) } f(q^2) - f(q^1) = \sum_{n=1}^{N}\{[q_n^1f_n(q^1)/f(q^1)]+...+[q_n^Kf_n(q^K)/f(q^K)]\}[q_n^2 - q_n^1]/\{[q_n^1/f(q^1)]+...+[q_n^K/f(q^K)]\}.$$

Fix n and let the components of q^1 and q^2 satisfy the following assumptions:

(A77) $q_n^2 \neq q_n^1$; $q_i^2 = q_i^1$ for $i \neq n$.

Now look at the equation (A76) when assumptions (A77) hold. The left hand side is independent of the components of q^3 and hence the right hand side of (A76) must also be independent of q^3. Using the linear homogeneity of f, this is sufficient to show that $f_n(q^3)$ must be a constant for any $q^3 >> 0_N$; i.e., for all $q >> 0_N$, $f_n(q)$ is equal to a constant a_n, which must be positive under our regularity conditions on f. This proof works for n = 1,...,N, which completes the proof of the first part of the proposition.

Let $K \geq 3$ and let $p^k >> 0_N$ for k = 1,...,K. Then equations (A74) can be rewritten as follows:

(A78) $\sum_{n=1}^{N} \rho_n(p^1,...,p^K)c_n(p^k) = 1$; $\qquad$ k = 1,...,K

where the coefficients $\rho_n(p^1,...,p^K)$ in (A78) are defined for n = 1,...,N as follows:

(A79) $\rho_n(p^1,...,p^K) \equiv \{[p_n^1c_n(y^1)/c(p^1)] +...+ [p_n^Kc_n(p^K)/c(p^K)]\}/\{c_n(p^1) +...+ c_n(p^K)\}$.

The first two equations in (A78) can be subtracted from each other to give the following equation:

(A80) $\sum_{n=1}^{N} \rho_n(p^1,...,p^K)[c_n(p^2) - c_n(p^1)] = 0$.

Define the vector $\rho(p^1,...,p^K) \equiv [\rho_1(p^1,...,p^K), ..., \rho_N(p^1,...,p^K)]$. Since $K \geq 3$, looking at definitions (A79), it can be seen that the components of p^3 can be varied (holding the remaining price vectors constant) so that we can find N linearly independent $\rho(p^1,...,p^K)$ vectors. Substitution of these linearly independent vectors into equation (A80) implies that

(A81) $\nabla c(p^2) = \nabla c(p^1)$.

Since equations (A81) hold for all positive p^1 and p^2, the partial derivatives of c(p) are constant, which completes the proof of the proposition. Q.E.D.

Thus the IDB multilateral system suffers from the same defect as the GK system[73]; both of these additive systems are not consistent with an economic approach that allows consumer preferences to be represented by flexible functional forms, whereas the GEKS system is consistent with preferences that are representable by flexible functional forms.[74]

[73] Diewert (1999; 27) showed that when $K \geq 3$, the GK system is only consistent with a linear or Leontief aggregator function.

[74] See Diewert (1999; 46) for descriptions of multilateral methods that have good economic properties; i.e., methods that are consistent with maximizing behavior on the part of consumers with preferences represented by flexible functional forms. See Diewert (1976) for the concept of a flexible functional form and the economic approach to index number theory. In addition to the GEKS system, the Own Share, MTS and van Ijzeren's (1983) weighted and unweighted balanced methods have good economic properties.

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