# Measuring Productivity in the Public Sector: Some Conceptual Problems

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**Abstract**

In many sectors of the economy, governments either provide various services at no cost or at highly subsidized prices. Examples are the health, education and general government sectors. The paper analyzes three possible general methods to measure the price and quantity of nonmarket government outputs. If quantity information on nonmarket outputs is available, then the first two methods of price valuation rely on either purchaser based valuations or on cost based valuations. If little or no information on the quantity of nonmarket outputs produced is available, then the method recommended in the *System of National Accounts 1993* must be used, where aggregate output growth is set equal to aggregate input growth. The paper also discusses various methods of adjusting for quality change.

**Keywords**

Measurement of output, input and productivity, nonmarket sector, health, education, general government, cost functions, duality theory, marginal cost prices, quality adjustment, hedonic regressions, index number theory.

**JEL Classification Numbers**

C43, D24, E23, H40, H51, H52, I12, I20.

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## 1. Introduction

To measure the productivity of a government production unit, we need to measure the prices and quantities of the outputs produced and inputs used by that unit or establishment for two periods of time. Then productivity growth can be defined as a quantity index of outputs produced divided by a quantity index of inputs used by the establishment.[2] It is usually possible to measure the price and quantity of inputs in a fairly satisfactory manner[3] but there are problems in measuring the prices and quantities of government nonmarket outputs. Thus in this paper, we will take a systematic look at possible methods for the valuation of nonmarket outputs produced by government production units.

In section 2 below, we will suggest a hierarchy of methods for valuing the nonmarket outputs of government production units. The most desirable methods rely on some form of purchaser valuations for these nonmarket outputs. The next set of methods rely on valuations that are based on costs of producing these outputs. The final method simply sets aggregate output growth equal to aggregate establishment input growth and does not attempt to construct values for the outputs produced by the government establishment. Sections 3 and 4 below explore the first two methods in some detail with some attention paid to the problems associated with valuing changes in the qualities of the government sector nonmarket outputs. Section 5 concludes.

An Appendix develops the cost based methods for the valuation of nonmarket outputs when there is quality change in some detail.

The present paper is rather technical. For general introductions to many of the issues discussed in this paper, see Atkinson (2005), Diewert (2008), Yu (2009) and Schreyer (2009b).

## 2. How Should Government Outputs be Valued?

In many cases, it is difficult to determine exactly what it is that a government production unit or establishment produces. In this case, we may have neither quantities or prices for the outputs of the government service provider. However, in many cases, we can measure at least the quantities of the outputs produced by the government establishment but not the corresponding prices. Finally, in some cases, we can measure both the prices and

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[2] For reviews of various approaches to the measurement of productivity, see Diewert and Morrison (1986), Diewert (1992b), Balk (1998) (2003), Schreyer (2001) and Diewert and Nakamura (2003).

[3] There is still a certain amount of controversy on how exactly to measure capital services; see Diewert (1980; 475-486) and Schreyer (2001) (2009a). A major problem with the System of National Accounts 1993 and 2008 is that capital services in the government sector are to be measured by depreciation only; i.e., there is no allowance for the opportunity cost of capital in government sector user costs whereas market sector user costs of capital include both depreciation and the opportunity cost of capital. This omission will lead to a substantial underestimate of public sector costs (from an opportunity cost point of view).

quantities produced; i.e., the government establishment under consideration produces at least some outputs that are sold at market prices. Thus the "best" practices that can be suggested in order to value the government outputs will depend to a large extent on what information is available.

We suggest the following hierarchy for valuing government outputs in the order of their desirability:

- *First best*: valuation at market prices or purchaser's valuations;
- *Second best*: valuations at producer's unit costs of production;
- *Third best*: government establishment output growth is set equal to real input growth and the aggregate establishment output price growth is set equal to an index of input price growth.

The third best option outlined above is the only option that can be used when there is little or no information on both the prices and quantities produced by a government establishment. This is the option which is recommended in the *System of National Accounts* 1993 and 2008 to value government production when direct information on the prices and quantities of government outputs is not available. The quantity or volume measure for establishment output that results from using this methodology can be interpreted as a *measure of real resources used* by that establishment and as such, it is an acceptable indicator of the output produced by a government unit. Since this third best option is fairly straightforward in principle (and has been extensively discussed in the national income accounting literature), we will not discuss it in more detail. However, the first and second best options listed above merit further discussion. Thus in the following two sections, we will break down the first and second best options for the valuation of government outputs into a finer subdivision of possible treatments, depending on the information that is available to the economic statistician.

## 3. Valuation of Government Outputs at Market Prices or Purchaser's Indirect Valuations

**Case 1**: **Price and quantity information on outputs is available; no quality change.**

We begin our analysis in this section with a very simple case where government outputs and inputs for a government production unit can be observed for say periods 0 and 1 and these outputs are sold at observable market prices in each period. Thus we assume that we can observe the vector of *inputs used* by a government establishment in period t, $x^t \equiv [x_1^t,...,x_N^t]$, and the corresponding vector of *market outputs produced* during period t, $y^t \equiv [y_1^t,...,y_M^t]$ for t = 0,1. We also assume that we can observe the corresponding period t *input and output price vectors*, $w^t \equiv [w_1^t,...,w_N^t]$ and $p^t \equiv [p_1^t,...,p_M^t]$ for t = 0,1. Finally, we assume that there are *no quality changes* in the output quantity vectors in this introductory simple case.

In this simple case, standard index number methodology can be used to measure the output, input and productivity growth of the government establishment. Thus *aggregate* *output and input growth* can be measured by the *Fisher* (1922) *output index* $Q_F$ and the *Fisher input index* $Q_F^*$ defined as follows:[4]

(1) $Q_F \equiv [(p^0 \cdot y^1/p^0 \cdot y^0)(p^1 \cdot y^1/p^1 \cdot y^0)]^{1/2}$ ;
(2) $Q_F^* \equiv [(w^0 \cdot x^1/w^0 \cdot x^0)(w^1 \cdot x^1/w^1 \cdot x^0)]^{1/2}$.

The corresponding measures of *aggregate output and input price growth* are the *Fisher output and input price indexes* defined as follows:

(3) $P_F \equiv [(p^1 \cdot y^0/p^0 \cdot y^0)(p^1 \cdot y^1/p^0 \cdot y^1)]^{1/2}$ ;
(4) $P_F^* \equiv [(w^1 \cdot x^0/w^0 \cdot x^0)(w^1 \cdot x^1/w^0 \cdot x^1)]^{1/2}$.

Finally, the *productivity growth* of the government establishment can be defined as the Fisher index of output growth divided by the Fisher index of input growth, $Q_F/Q_F^*$.

In the following cases, we will assume that price and quantity information on inputs is available and so our focus will be on constructing a suitable output quantity and price index when market prices for outputs are not directly available. In the following cases, direct market prices for the outputs of the government establishment are not available for the two periods under consideration.

**Case 2**: **Quantity but not price information on outputs is available; no quality change; comparable market sector prices are available**

In this case, we assume that information on the outputs produced by the government establishment in each period is available but these outputs are not market outputs and so direct prices are not available. However, for this case, it is assumed that there are private sector producers in the same marketplace that are producing outputs that closely resemble the outputs being produced by the government establishment. In this case, we simply use these comparable prices, say $p^{t*}$ for period t, in place of the missing government establishment output prices, $p^t$.

**Case 3**: **Quantity but not price information on outputs is available; there is quality change over the two periods; somewhat comparable market sector prices are available**

Obviously, Case 3 is similar to Case 2 except that the "comparable" market sector prices for the vector of government establishment outputs is not quite comparable; i.e., there are quality differences between these "comparable" market sector outputs and those produced by the government sector establishment.

A possible solution to this lack of comparability problem is to run a *hedonic regression model*. The basic structure of a hedonic regression model[5] can be explained as follows.

[4] Fisher indexes can be given strong justifications both from the viewpoint of the economic approach to index number theory (see Diewert's (1976) theory of superlative indexes) as well as from the axiomatic point of view; see Diewert (1992a). Notation: $p \cdot y \equiv \sum_{n=1}^{N} p_n y_n$ is the inner product of the vectors p and y.

Consider output n produced by a government establishment. Suppose that in period t, the establishment produces $y_n^t$ units of this commodity but it is supplied to users at nonmarket prices so that we have no immediately available price to value this output. Suppose further that we can associate a vector $z_n^{t*}$ of *quality determining characteristics* with this period t government output. Thus for example, if we are attempting to value the annual output of a publicly funded school, output $y_n^t$ for this school might be the number of hours of faculty instruction that students receive for a particular grade or program of study in year t. Quality determining characteristics contained in the vector $z_n^{t*}$ might be:[6]

- Average class size for the program of study under consideration;
- The number of classroom hours offered per school day for the program;
- The quality of the teachers (measured somehow);
- The quality of the students, measured by their beginning of the year standardized test scores;
- The attendance records of students (and perhaps also of the faculty);
- The number of optional facilities that the school has such as libraries, shop facilities, gyms, etc.

The next assumption that is made in order to implement a hedonic regression model is that *there are market sector alternatives to the public production unit*. Thus we suppose that in period t, there are J market sector production units that offer much the same service as the public sector unit. Thus using the above schooling example, we assume that there are J private schools in the same market area as the public school and that the market price for private school j in period t for a program of study n similar to the public school program is $p_{nj}^t$ for j = 1,...,J. With each private school j in period t, there is an associated vector of quality characteristics, $z_{nj}^t$ and we assume that we can observe both the prices $p_{nj}^t$ and the associated quality vectors $z_{nj}^t$. The next assumption made is that there is a *hedonic function*, $h^{nt}(z)$, that relates the market prices $p_{nj}^t$ to the amounts of the various quality determining characteristics; i.e., we have the following *hedonic regression model*:

$$(5)\quad p_{nj}^t = h^{nt}(z_{nj}^t) + \varepsilon_{nj}^t\,; \qquad j = 1,\ldots,J$$

where $\varepsilon_{nj}^t$ is an error term. Once the unknown parameters that determine the hedonic function $h^{nt}(z)$ have been estimated, we can insert the vector of public production unit

[5] The basic methodology for a hedonic regression model was worked out by Court (1939) and popularized by Griliches (1971).

[6] See Schreyer (2009c; 17-18) for a similar list of quality determining characteristics for the treatment of a medical condition: "For example, if health product $y_1$ is 'treatment of a heart attack', this will encompass strokes of different severity, and patients of different age suffering from a stroke. If old patients require more intense treatment than young patients or if more severe strokes necessitate more intense care than less severe strokes, there are in fact different services involved. To some degree, this can be accommodated by stratification and matching (see above) but only up to a point. Additional heterogeneity is best captured by identifying, through knowledge of the product, those characteristics that make one service distinct from another. The variable $z_{11}$ could thus be 'age of patient', the variable $z_{12}$ 'degree of severity of stroke' and so on."

characteristics $z_n^{t*}$ into this function and obtain an *imputed price* $p_n^{t*}$ for the nth output of the government production unit; i.e., we have

(6) $p_n^{t*} \equiv h^{nt}(z_n^{t*})$.

There are many practical and conceptual issues associated with running hedonic regression models. Some of the conceptual and methodological issues are:

- How should the *functional form* for the hedonic function be determined?
- Should the individual observations in the hedonic regression (5) be weighted by the *economic importance* of each observation j or not and if so, should value or quantity weights be used?
- Should single period hedonic regressions of the type defined by (5) be run or should the data for periods 0 and 1 be combined into a single regression?[7]
- Which *characteristics* should be included in the regression? Should the characteristics be limited to those characteristics which purchasers of the service value or should characteristics which affect costs but are not valued by users also be included?[8]

The above issues are discussed extensively in the important monograph by Triplett (2006). For additional discussions on these technical issues and references to the literature, see Griliches (1971), Triplett and McDonald (1978), Diewert (2003a) (2003b) and Diewert, Heravi and Silver (2009).

In addition to the above technical issues, there are a number of practical issues which will limit the usefulness of the above hedonic regression model method for imputing prices:

- There may be very few or no market sector producers of outputs that are comparable to the outputs produced by the government establishment;
- Even if there is an adequate sample of market sector producers of "similar" outputs, the quality characteristics of the market sector producers may be so different from the quality characteristics of the public producer that it would be extremely hazardous to extrapolate from the market sector part of the characteristics space into the public sector part and thus the imputed prices that would be obtained using equation (6) would be meaningless.

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[7] In the present context, it seems preferable to run separate hedonic regressions for each period unless there is a shortage of observations. This advice is in agreement with the advice given in Pakes (2003) and Diewert, Heravi and Silver (2009).

[8] There is a considerable amount of debate in the literature on this point and the related debate over the "user value" or "resource cost" approach to the interpretation of hedonic regressions; see Triplett (1983). In this section, we take the user valuation point of view and in the following section, we take the resource cost point of view. In the present section, we prefer to choose characteristics from the viewpoint that it is the characteristics that are important to users or purchasers that should appear in the hedonic regression model (5). Thus we follow the example of Griliches (1971; 14), Diewert (2002b) and Schreyer (2009c), who argued that it is the characteristics which determine the user value or utility of a model or service that is the first best set of characteristics to insert into a hedonic regression model where the focus is on obtaining final demander quality adjusted prices.

- Many government services are provided to users or recipients of the services at highly subsidized prices; e.g., medical services, educational services and subsidized housing services that are tied to certain classes of recipients. *It would not be appropriate to use these prices in a hedonic regression model* which purports to represent purchaser valuations because the prices in such a regression model are supposed to represent marginal rates of substitution between the service being purchased and other goods (and thus the relative utility of the service will be revealed to external observers in such a situation). The tied nature of these subsidized purchases prevents these utility tradeoffs from being revealed; i.e., recipients cannot generally resell these highly subsidized services on the open marketplace.

Thus while the purchaser oriented hedonic regression approach to the imputation of nonmarket prices for public sector production units can be useful in some situations, there are many situations where it will not be suitable and so other approaches will have to be used.

**Case 4**: **Quantity but no price information on outputs is available; there are no comparable market sector output prices**

In many situations, public establishments supply measurable units of various goods and services at nonmarket prices and there are no comparable private producers supply these commodities. In these situations, the methodologies for estimating appropriate imputed prices for these nonmarket outputs explained in Cases 1-3 above are not applicable. Thus for each period t, the vector of government establishment nonmarket outputs $y^t$ can be determined but there are no readily available prices $p^t$ that can be used to value these outputs.

At first sight, it would seem that there is little that can be done in this situation, except to give up on obtaining these price valuations and simply move on to our less preferred methods of valuation that rely on cost information. These cost based valuation methods will be covered in the following section. However, if very detailed information on the behavior of the rest of the economy is available, then it turns out that it is possible to come up with a first best method for obtaining user based valuations for the nonmarket government outputs. The basic idea is to imbed the government producer in a general equilibrium model of the economy. Then the vectors of government nonmarket production will be explanatory variables in the production functions for the market sector of the economy as well as in the utility functions for the households which are resident in the economy.[9] Now it is possible to adapt the methodology initially developed by Allais (1977), Boiteux (1951), Debreu (1951) and Diewert (1983b) and work out a measure of the value of the change in government nonmarket production and to determine approximate prices for these government outputs. Such an *indirect pricing methodology*

[99] Equivalently, the government vectors of nonmarket production will be explanatory variables in the profit or cost functions that are dual to the market sector production possibilities sets and in the expenditure functions which are dual to household utility functions. For material on dual representations, see Diewert (1974).

*based on user valuations* was worked out in some detail by Diewert (1986; 56-70). We will not outline this methodology since it would require too many equations and moreover, *the information required to implement this approach is just too great*: expenditure functions for all major classes of households and cost functions for all market sector producers would have to be econometrically estimated in order to implement this indirect approach. Thus this Case 4 methodology is not really practical at the present time even though it is theoretically sound.

We now turn our attention to the class of second best methods for valuing government nonmarket outputs.

## 4. Valuation of Government Outputs using Cost Function Valuations

For this class of second best valuation methods, we assume that information on the outputs produced by a government establishment is available for the two periods under consideration along with basic price and quantity data for inputs. Thus we assume that the two output vectors, $y^0$ and $y^1$ are observable along with the corresponding two input vectors, $x^0$ and $x^1$, along with their input price vectors, $w^0$ and $w^1$. Additional assumptions will be made below as we discuss Cases 5-8 below. However, throughout this section, we assume that no direct user valuations to price the nonmarket government outputs are available. Thus valuations based on cost are used in this section.

### Case 5: Cost functions $C^t(y,w)$ are available for both periods; no quality changes

In this case, we assume that a period t cost function has been econometrically estimated, say $C^t(y,w)$ for $t = 0,1$.[10] In this case, we can define a *family of theoretical output quantity indexes*, $\alpha(y^0,y^1,w,t)$, as follows:

$$(7)\ \alpha(y^0,y^1,w,t) \equiv C^t(y^1,w)/C^t(y^0,w).$$

Note that this output quantity index depends not only on the two quantity vectors for periods 0 and 1, $y^0$ and $y^1$, but it also depends on a reference period t technology and a reference vector of input prices w. Following the example of Konüs (1939), it is natural to single out two special cases of the family of output quantity indexes defined by (7): one choice where we use the period 0 technology and set the reference prices equal to the period 0 input prices $w^0$ and another choice where we use the period 1 technology and set the reference prices equal to the period 1 input prices $w^1$. These special cases are defined as $\alpha_0$ and $\alpha_1$ below:

$$(8)\ \alpha_0 \equiv C^0(y^1,w^0)/C^0(y^0,w^0)\ ;$$
$$(9)\ \alpha_1 \equiv C^1(y^1,w^1)/C^1(y^0,w^1).$$

[10] See definition (A2) in the Appendix where the vector of quality variables z can be suppressed for this Case and the following Case. This is the case considered in Diewert (2008). We assume cost minimizing behavior on the part of the government enterprise, an assumption which may not be entirely justified!

Since the theoretical output quantity indexes, $\alpha_0$ and $\alpha_1$, are both equally representative, a single estimate of cost based output quantity growth should be set equal to a symmetric average of these two estimates. We will choose the geometric mean as our preferred symmetric average and thus our preferred theoretical measure of cost based output quantity growth is the following *theoretical Fisher type output index*, $\alpha_F$:

(10) $\alpha_F \equiv [\alpha_0\alpha_1]^{1/2}$.

We now turn our attention to theoretical measures of input price growth. We use the joint cost function $C^t$ in order to define a *family of input price indexes*, $\beta(w^0,w^1,y,t)$, as follows:

(11) $\beta(w^0,w^1,y,t) \equiv C^t(y,w^1)/C^t(y,w^0)$.

Again following the example of Konüs (1939), it is natural to single out two special cases of the family of input price indexes defined by (11): one choice where we use the period 0 technology and set the reference quantities equal to the period 0 quantities $y^0$ and another choice where we use the period 1 technology and set the reference quantities equal to the period 1 quantities $y^1$. These special cases are defined as $\beta_0$ and $\beta_1$ below:

(12) $\beta_0 \equiv C^0(y^0,w^1)/C^0(y^0,w^0)$ ;
(13) $\beta_1 \equiv C^1(y^1,w^1)/C^1(y^1,w^0)$.

Since both theoretical input price indexes, $\beta_0$ and $\beta_1$, are equally representative, a single estimate of input price change should be set equal to a symmetric average of these two estimates. We again choose the geometric mean as our preferred symmetric average and thus our preferred theoretical measure of input price growth is the following Fisher type theoretical input price index, $\beta_F$:

(14) $\beta_F \equiv [\beta_0\beta_1]^{1/2}$.

We now define our last family of theoretical indexes. We again use the joint cost functions $C^0$ and $C^1$ in order to define a *family of reciprocal indexes of technical progress*, $\gamma(y,w)$, as follows:

(15) $\gamma(y,w) \equiv C^1(y,w)/C^0(y,w)$.

The family of theoretical reciprocal technical progress indexes (or reciprocal productivity indexes) $\gamma(y,w,z)$ defined by (15) is equal to the (hypothetical) total cost $C^1(y,w)$ of producing the reference vector of outputs y when the government establishment faces the reference vector of input prices w using the period 1 technology, divided by the total cost $C^0(y,w)$ of producing the same reference vector of outputs y and facing the same reference vector of input prices w, where we now use the period 0 technology. Thus $\gamma(y,w)$ is a measure of the *proportional reduction in costs* that occurs due to technical progress between periods 0 and 1 and it can be seen that this is an inverse measure of technical progress; i.e., there is positive technical progress between the two periods if $\gamma(y,w)$ is less than one. For each choice of a reference vector of output quantities y and reference vector of input prices w, we obtain a measure of exogenous cost reduction.

Instead of singling out the reference vectors y and w that appear in the definition of $\gamma(y,w)$ to be the period t quantity and price vectors $(y^t,w^t)$ for t = 0,1, we will choose the *mixed reference vectors* $(y^0,w^1)$ and $(y^1,w^0)$ for our usual two special cases. The reason for these somewhat odd looking choices will be explained below.

We want to explain the growth in total costs going from period 0 to 1, $C^1(y^1,w^1)/C^0(y^0,w^0)$, as the product of 3 growth factors:

- *Growth in outputs*; i.e., a factor of the form $\alpha(y^0,y^1,w,t)$ defined above by (7);
- *Growth in input prices*; i.e., a factor of the form $\beta(w^0,w^1,y,t)$ defined by (11) and
- *Exogenous reduction in costs due to technical progress*; i.e., a factor of the form $\gamma(y,w)$ defined by (15).

Simple algebra shows that we have the following decompositions of the cost ratio $C^1(y^1,w^1)/C^0(y^0,w^0)$ into explanatory factors of the above type:

(16) $C^1(y^1,w^1)/C^0(y^0,w^0)$
$= [C^1(y^1,w^1)/C^1(y^0,w^1)][C^0(y^0,w^1)/C^0(y^0,w^0)][C^1(y^0,w^1)/C^0(y^0,w^1)]$
$= \alpha_1\beta_0\gamma(y^0,w^1)$ using definitions (7), (11) and (15);

(17) $C^1(y^1,w^1)/C^0(y^0,w^0)$
$= [C^0(y^1,w^0)/C^0(y^0,w^0)][C^1(y^1,w^1)/C^1(y^1,w^0)][C^1(y^1,w^0)/C^0(y^1,w^0)]$
$= \alpha_0\beta_1\gamma(y^1,w^0)$ using definitions (7), (11) and (15).

The above decompositions show that the following two special cases of $\gamma(y,w)$ defined by (15) are of particular interest:

(18) $\gamma(y^0,w^1) \equiv C^1(y^0,w^1)/C^0(y^0,w^1) \equiv \gamma_0$ ;
(19) $\gamma(y^1,w^0) \equiv C^1(y^1,w^0)/C^0(y^1,w^0) \equiv \gamma_1$ .

We will define $\gamma_F$ as the geometric mean of $\gamma_0$ and $\gamma_1$:

(20) $\gamma_F \equiv [\gamma_0\gamma_1]^{1/2}$.

Using (16) and (17) and definitions (10), (14) and (20), it can be seen that we have the following theoretical decomposition of the ratio of period 1 total costs to period 0 total costs into explanatory factors:

(21) $C^1(y^1,w^1)/C^0(y^0,w^0) = w^1\cdot x^1/w^0\cdot x^0 = \alpha_F\,\beta_F\,\gamma_F$.

The exact decomposition of (one plus) cost growth over the two periods under consideration given by (21) is our preferred decomposition of cost growth into explanatory factors which can be implemented if the economic statistician has estimates for the period 0 and 1 cost functions at hand.

**Case 6: Output quantity data and input price and quantity data available for both periods; estimates of marginal costs or incremental costs are available; no quality changes**

In this case, we assume that information on $y^0$ and $y^1$ (output quantities), $x^0$ and $x^1$ (input quantities), $w^0$ and $w^1$ (input prices) is available along with information on *marginal costs* for each period, $p^0$ and $p^1$.[11] Thus the situation here is that we do not have complete information on the two cost functions but we do have information on the marginal or incremental costs of producing extra units of each output in the two periods under consideration. The model explained in the Appendix[12] can be used in order to derive the following approximations to the cost function based indexes defined in the previous Case 5:

(22) $\alpha_F \cong [(p^0\cdot y^1/p^0\cdot y^0)(p^1\cdot y^1/p^1\cdot y^0)]^{1/2} \equiv Q_F$ :
(23) $\beta_F \cong [(w^1\cdot x^0/w^0\cdot x^0)(w^1\cdot x^1/w^0\cdot x^1)]^{1/2} \equiv P_F^*$ ;
(24) $\gamma_F \cong [Q_F/Q_F^*]^{-1}$

where $Q_F$ is the Fisher ideal index of outputs (using marginal costs as prices in this case), $Q_F^*$ is the Fisher index of inputs and $P_F^*$ is the Fisher input price index. If we substitute the approximations on the right hand sides of (22)-(24) into the exact decomposition given by (21), we get the following exact decomposition of the growth of costs into explanatory factors using the fact that $P_F^*Q_F^* = w^1\cdot x^1/w^0\cdot x^0$:

(25) $w^1\cdot x^1/w^0\cdot x^0 = Q_F\, P_F^*[Q_F/Q_F^*]^{-1}$.

Thus in this case, normal index number theory can be used to provide input, output and productivity indexes provided that we have estimates of marginal (or incremental) costs for each period that we can use to value the nonmarket outputs of the government establishment. Put another way, we can use the algebra associated with Case 1 above where the market price vectors $p^t$ used in Case 1 are replaced by marginal cost valuations in Case 7.

**Case 7: Cost functions $C^t(y,w,z)$ with quality variables z are available for both periods**

In this case, we assume that a period t cost function has been econometrically estimated, say $C^t(y,w,z)$ for t = 0,1 where y is a vector of nonmarket outputs produced by the government establishment, w is a vector of input prices that the unit faces and z is a vector of variables that describes the quality characteristics of the vector of outputs y.

[11] These marginal costs could be approximated by allocated accounting costs or incremental costs.
[12] Simply set $z^0 = z^1$ and use the results in the Appendix.

The analysis of this case is very similar to the above analysis for Case 5 above. The main difference is that we package together changes in the quality characteristics, $z^0$ and $z^1$, along with changes in the (unadjusted) output vectors, $y^0$ and $y^1$, into a composite index of output growth, which of course, includes changes in quality. Thus the quality changes are valued from a cost perspective rather than from a demander or user perspective. The details for Case 7 are explained in the Appendix and will not be repeated here.

**Case 8: Output quantity data and input price and quantity data available for both periods; estimates of marginal costs are available; quality variables are available along with estimates of marginal costs of changes in the quality variables for each period**

In this case, we assume that information on $y^0$ and $y^1$ (output quantities), $x^0$ and $x^1$ (input quantities), $w^0$ and $w^1$ (input prices) is available along with information on marginal costs for each period, $p^0$ and $p^1$. We also assume that information on the period 0 and 1 characteristics vectors $z^0$ and $z^1$, which describe the qualities of the nonmarket output vectors produced in periods 0 and 1, is also available along with estimates of the marginal or incremental costs of changing the quality variables in each period. Thus we assume that estimates of the cost based characteristics prices, $\omega^0$ and $\omega^1$, are available where

(26) $\omega^0 \equiv \nabla_z C^0(y^0,w^0,z^0)$ ; $\omega^1 \equiv \nabla_z C^1(y^1,w^1,z^1)$.

Thus this case is similar to Case 6 above where we do not necessarily have complete information on the period 0 and 1 cost functions but we do have estimates of marginal costs and cost based characteristics prices for these two periods.[13] This model is explained in detail in the Appendix where theoretical counterparts to the indexes $\alpha_F$, $\beta_F$ and $\gamma_F$ defined above by (10), (14) and (20) respectively are generalized to take into account changes in quality in the Appendix. The model explained in the Appendix can be used in order to derive the following approximations to the theoretical cost function based indexes defined in the Appendix by (A14), (A18) and (A29) :

(27) $\alpha_F \cong [\{p^0\cdot y^1 + \omega^0\cdot(z^1-z^0)\}/p^0\cdot y^0][p^1\cdot y^1/\{p^1\cdot y^0 - \omega^1\cdot(z^1-z^0)\}]^{1/2} \equiv Q_{AF}$ :
(28) $\beta_F \cong [(w^1\cdot x^0/w^0\cdot x^0)(w^1\cdot x^1/w^0\cdot x^1)]^{1/2} \equiv P_F^*$ ;
(29) $\gamma_F \cong [Q_{AF}/Q_F^*]^{-1}$

where $Q_{AF}$ is defined in (27) and is the *quality adjusted Fisher ideal index of outputs* (using marginal costs as prices $p^t$ in this case), $Q_F^*$ is the Fisher index of inputs and $P_F^*$ is the Fisher input price index. It should be noted that it is not actually necessary to have a knowledge of the quality vectors pertaining to each period; it is only necessary to know their differences, $z^1-z^0$, in order to evaluate the index on the right hand side of (27). It should also be noted that the approximations on the right hand sides of (27)-(29) are nonparametric ones.

---

[13] Schreyer (2009c) considers a very useful special case of this model where each output is produced by a separate constant returns to scale production function. He also considers useful translog parametric approximations to his general model.

Thus in this case, with a few adjustments, normal index number theory can be used to provide input, output and productivity indexes provided that we have estimates of marginal (or incremental) costs for each period and we have estimates of the incremental changes in cost due to incremental changes in the quality vector z for each period.

## 5. Conclusion

We could also consider *Case 9* where we have little or no information on both the magnitudes of the outputs produced by the government establishment and no information on output values or prices. As was mentioned in section 2, there is little that can be done in this situation except to follow System of National Accounts methodology and assume that aggregate output growth is equal to input growth and aggregate output price growth is equal to input price growth.

We have provided a somewhat systematic overview of possible methods for determining the price and quantity of nonmarket outputs produced by public sector establishments. We considered 9 different scenarios where it was assumed that the government statistician had various bits of information at his or her disposal. Some of the methods assumed that there were no changes in the qualities of the outputs being produced (see Cases 1, 2, 4, 5 and 6) while some methods allowed for the possibility of quality changes (see Cases 3, 7, 8 and 9). The Case 3 methodology relied on traditional hedonic regression analysis (from a user perspective) whereas Cases 7 and 8 relied on a cost based form of hedonic regression analysis. Case 9 is consistent with quality changes but a defect with this methodology is that no productivity improvements in the government establishment will ever be discovered using this methodology.[14] However, in many applications, the Case 9 methodology will be the only one which can be implemented by the economic statistician.

## Appendix: Cost Function Based Output, Input and Productivity Indexes with Quality Change

In this Appendix, we will generalize Diewert's (2008) cost function based derivation of theoretical output, input and productivity indexes to situations where the *quality* of the outputs can change from period to period. We will also define various potentially observable indexes that can approximate these theoretical indexes to the accuracy of at least a first order approximation.

This Appendix takes the *cost based approach* to the measurement of the price and quantity of the outputs produced by a government production unit or establishment that is producing primarily nonmarket outputs. As was mentioned in the main text above, this is only a second best approach to the measurement of the value of government sector outputs but in many situations, it is the only practical approach to value these outputs. In what follows, we will develop various output, input and productivity indexes for

[14] This point was noted by Schreyer (2009b).

government establishments that can only be implemented if econometric or accounting based estimates of the establishment's cost functions are available for the base period and the current period in the comparison. However, we will also develop various approximations to these theoretical indexes which can be implemented with a knowledge of price and quantity data on inputs, quantity data and marginal costs pertaining to outputs and data on various quality characteristics of the outputs for the two periods under consideration; i.e., these approximations can be calculated with only incomplete information on the underlying cost functions.

We assume that we can observe the vector of *inputs used* by a government establishment in period t, $x^t \equiv [x_1^t,...,x_N^t]$, and the corresponding vector of *nonmarket outputs produced* during period t, $y^t \equiv [y_1^t,...,y_M^t]$, for t = 0,1.[15] We also assume that the output vectors are not necessarily expressed in constant quality units in each period; thus we assume that in period t, there is a vector of *quality characteristics*, $z^t \equiv [z_1^t,...,z_K^t]$, associated with the period t output vector $y^t$. In general, if the quality of the outputs improves going from period 0 to 1, greater amounts of input will be required in order to produce the higher quality outputs and hence costs will also increase, holding all else constant.

We assume that the set of feasible inputs and outputs (of varying quality) for period t is the production possibilities set $S^t$ for t = 0,1. Thus if (x,y,z) belongs to $S^t$, then in period t, the government establishment can produce the vector of outputs y with quality characteristics z using the vector of inputs x. We will assume that there are *constant returns to scale in production* if quality is held constant; i.e., we will assume that in each period t, the production possibilities set $S^t$ satisfies the following property:[16]

(A1) $(x,y,z)\in S^t$ implies that $(\lambda x,\lambda y,z)\in S^t$ for all scalars $\lambda > 0$.

The government establishment's *period t joint cost function*, $C^t(y,w,z)$, is defined as follows:

(A2) $C^t(y,w,z) \equiv \min_x \{w\cdot x : (x,y,z)\in S^t\}$ ; $\qquad t = 0,1$

where $w >> 0_N$ is a vector of strictly positive input prices that the establishment faces. The joint cost function will satisfy various regularity conditions, given regularity conditions on the underlying production possibilities sets $S^t$. In particular, with our present assumptions, it can be shown that $C^t(y,w,z)$ will be *linearly homogeneous* in the

[15] If the establishment produces some market outputs, then these market outputs can be treated as negative components of the input vectors, $x^t$.

[16] More formally, we assume that for all z in a nonempty closed convex set of feasible quality vectors, the set of (x,y) such that $(x,y,z)\in S^t$ is a nonempty, closed, convex cone for t = 0,1. The cone assumption is the assumption that implies constant returns to scale for constant quality outputs, which is assumption (A1). With these regularity conditions, it can be shown that $C^t(y,w,z)$ will be a convex, linearly homogeneous function in the components of y and a concave, linearly homogeneous function in w. The assumption of constant returns to scale is somewhat restrictive since in many cases, the government provides a service because the underlying technology that provides the service is subject to increasing returns to scale.

components of y and in the components of w; i.e., $C^t$ will satisfy the following conditions:[17]

(A3) $C^t(\lambda y,w,z) = \lambda C^t(y,w,z)$ for all $\lambda > 0$;
(A4) $C^t(y,\lambda w,z) = \lambda C^t(y,w,z)$ for all $\lambda > 0$.

We suppose that the establishment's period t observed output and input vectors are $y^t$ and $x^t$ respectively and the vector of input prices that is faced in period t is $w^t$ for $t = 0,1$. We assume that a period t vector of output quality characteristics, $z^t$, is observed. We also assume that in each period t, the observed input vector $x^t$ is a solution to the period t cost minimization problem so that we have:

$$\text{(A5)} \quad C^t(y^t,w^t,z^t) = w^t \cdot x^t \; ; \qquad t = 0,1.$$

A final assumption is that the period t cost function $C^t(y,w,z)$ is differentiable with respect to the components of y,w,z at the observed period t data, $y^t,w^t,z^t$, for $t = 0,1$. Given that $C^t(y^t,w^t,z^t)$ is differentiable with respect to the components of the input price vector, by Shephard's (1953; 11) Lemma, the observed period t input vector $x^t$ is equal to the vector of first order partial derivatives of $C^t(y^t,w^t,z^t)$ with respect to the components of w; i.e., we have:[18]

$$\text{(A6)} \quad x^t = \nabla_w C^t(y^t,w^t,z^t) \; ; \qquad t = 0,1.$$

Note that (A5) and (A6) imply that the following equations will hold:

$$\text{(A7)} \quad C^t(y^t,w^t,z^t) = w^t \cdot x^t = w^t \cdot \nabla_w C^t(y^t,w^t,z^t) \; ; \qquad t = 0,1.$$

It is useful to introduce some notation for the vectors of first order partial derivatives of $C^t$ with respect to the components of the output vector y and the vector of quality characteristics z:

$$\text{(A8)} \quad p^t \equiv \nabla_y C^t(y^t,w^t,z^t) \; ; \qquad t = 0,1;$$
$$\text{(A9)} \quad \omega^t \equiv \nabla_z C^t(y^t,w^t,z^t) \; ; \qquad t = 0,1.$$

From (A8), it can be seen that $p^t$ is the period t vector of *marginal cost prices* for the nonmarket outputs produced by the government establishment in period t; i.e., $p_m^t$ is the *incremental cost* of producing an additional unit of output m in period t. These incremental costs could be approximated by *allocated accounting costs*; i.e., $p_m^t$ could be approximated by the period t share of total period t cost that could be attributed to the production of output m, say $C_m^t$, divided by the total period t production of output m, $y_m^t$.

[17] Use the techniques outlined in Diewert (1974; 134-136) to establish these results.
[18] Notation: $w^t \cdot x^t \equiv \sum_{n=1}^N w_n^t x_n^t$ is the inner product of the vectors $w^t$ and $x^t$ and $\nabla_w C^t(y^t,w^t,z^t) \equiv [\partial C^t(y^t,w^t,z^t)/\partial w_1,\ldots,\partial C^t(y^t,w^t,z^t)/\partial w_N]$ is the vector of first order partial derivatives of $C^t(y^t,w^t,z^t)$ with respect to the components of w.

Since $C^t(y,w,z)$ is linearly homogeneous in the components of y, Euler's Theorem on homogeneous functions implies the following relations:

$$\text{(A10)}\quad C^t(y^t,w^t,z^t) = y^t\cdot\nabla_y C^t(y^t,w^t,z^t) \qquad t = 0,1$$
$$= p^t\cdot y^t \qquad \text{using (A8)}$$
$$= w^t\cdot x^t \qquad \text{using (A7).}$$

Thus if we use period t marginal costs $p^t$ as prices for the period t nonmarket outputs $y^t$, then the resulting period t *imputed revenues*, $p^t\cdot y^t$, will be exactly equal to period t *costs*, $w^t\cdot x^t$.

From (A9), it can be seen that $\omega^t$ is the period t vector of *marginal increases in cost due to incremental changes in quality*. Thus if the kth quality variable is measured in such a way that an increase in $z_k$ corresponds to an increase in the quality of one or more outputs, then $\omega_k{}^t \equiv \partial C^t(y^t,w^t,z^t)/\partial z_k$ is the incremental increase in period t costs due to an incremental increase in $z_k$. Following Triplett (1983) (2006), we can interpret $\omega^t$ as a vector of period t cost based *quality adjustment factors* or *characteristics prices*.

Diewert (2008) introduced output, input and productivity indexes using joint cost functions but did not consider problems associated with changes in the quality of the outputs produced by the government establishment in each period.[19] In this Appendix, we will generalize his analysis to cover situations where there are changes in quality of the outputs produced going from period 0 to 1. The main difference in our present analysis as compared to the earlier analysis is that we now group together the effects of changes in unadjusted outputs (the change from $y^0$ to $y^1$) with the effects of changes in the quality of the outputs produced (the change from $z^0$ to $z^1$).

We use the two joint cost functions, $C^0$ and $C^1$, in order to define a *family of cost based quality adjusted output quantity or volume indexes*, $\alpha(y^0,y^1,z^0,z^1,w,t)$, as follows:

$$\text{(A11)}\quad \alpha(y^0,y^1,z^0,z^1,w,t) \equiv C^t(y^1,w,z^1)/C^t(y^0,w,z^0).$$

Note that this output quantity index depends not only on the two quantity vectors for periods 0 and 1, $y^0$,$y^1$, and the two quality vectors, $z^0$,$z^1$, but it also depends on a reference period t technology and a reference vector of input prices w. Thus the theoretical output quantity index $\alpha(y^0,y^1,z^0,z^1,w,t)$ defined by (A11) is equal to the (hypothetical) total cost $C^t(y^1,w,z^1)$ of producing the vector of observed period 1 outputs $y^1$ with observed period 1 qualities $z^1$, divided by the (hypothetical) total cost $C^t(y^0,w,z^0)$ of producing the vector of observed period 0 procedure outputs $y^0$ with observed period 0 qualities $z^0$, where in both cases, we use the technology of period t and assume that the establishment faces the same vector of reference input prices, w. Thus for each choice of

[19] Diewert's (2008) approach to cost function based index number theory was a reasonably straightforward adaptation of the earlier work on theoretical price and quantity indexes by Konüs (1939), Fisher and Shell (1972), Samuelson and Swamy (1974), Archibald (1977), Diewert (1980; 461) (1983a; 1054-1083), Diewert and Morrison (1986) and Kohli (1990).

technology (i.e., t could equal 0 or 1) and for each choice of a reference vector of input prices w, we obtain a (different) cost based output quantity index. Note that this output quantity index combines the effects of changes in the unadjusted output vectors, $y^0$ and $y^1$, and the changes in the quality of the outputs, $z^0$ and $z^1$, holding everything else fixed in the cost comparison.

Following the example of Konüs (1939), it is natural to single out two special cases of the family of output quantity indexes defined by (A11): one choice where we use the period 0 technology and set the reference prices equal to the period 0 input prices $w^0$ and another choice where we use the period 1 technology and set the reference prices equal to the period 1 input prices $w^1$. These special cases are defined as $\alpha_0$ and $\alpha_1$ below:

$$
\begin{aligned}
\text{(A12)}\quad \alpha_0 &\equiv C^0(y^1,w^0,z^1)/C^0(y^0,w^0,z^0) \\
&= C^0(y^1,w^0,z^1)/p^0\cdot y^0 && \text{using (A10)} \\
&\cong [C^0(y^0,w^0,z^0) + \nabla_y C^0(y^0,w^0,z^0)\cdot(y^1-y^0) + \nabla_z C^0(y^0,w^0,z^0)\cdot(z^1-z^0)]/p^0\cdot y^0 \\
& && \text{forming a first order Taylor series approximation to } C^0(y^1,w^0,z^1) \\
&= [p^0\cdot y^1 + \omega^0\cdot(z^1-z^0)]/p^0\cdot y^0 && \text{using (A8)-(A10)} \\
&\equiv Q_{AL}
\end{aligned}
$$

where the *quality adjusted Laspeyres output quantity or volume index* $Q_{AL}$ is defined by $[p^0\cdot y^1 + \omega^0\cdot(z^1-z^0)]/p^0\cdot y^0$. Note that if there is no change in the quality of the outputs produced by the establishment so that $z^1$ equals $z^0$, then $Q_{AL}$ reduces to the ordinary Laspeyres output quantity index, $Q_L \equiv p^0\cdot y^1/p^0\cdot y^0$. In the general case where there is quality change, we need to add to the numerator of the ordinary Laspeyres output quantity index, $p^0\cdot y^1$, the quality adjustment factor, $\omega^0\cdot(z^1-z^0)$, which values the net increase in the quality variables, $z^1-z^0$, at the period 0 characteristics prices $\omega^0$.

We turn now to our second special case of the family of output quantity indexes defined by (A11).

$$
\begin{aligned}
\text{(A13)}\quad \alpha_1 &\equiv C^1(y^1,w^1,z^1)/C^1(y^0,w^1,z^0) \\
&= p^1\cdot y^1/C^1(y^0,w^1,z^0) && \text{using (A10)} \\
&\cong p^1\cdot y^1/[C^1(y^1,w^1,z^1) + \nabla_y C^1(y^1,w^1,z^1)\cdot(y^0-y^1) + \nabla_z C^1(y^1,w^1,z^1)\cdot(z^0-z^1)] \\
& && \text{forming a first order Taylor series approximation to } C^1(y^0,w^1,z^0) \\
&= p^1\cdot y^1/[p^1\cdot y^0 - \omega^1\cdot(z^1-z^0)] && \text{using (A8)-(A10)} \\
&\equiv Q_{AP}
\end{aligned}
$$

where the *quality adjusted Paasche output quantity or volume index* $Q_{AP}$ is defined by $p^1\cdot y^1/[p^1\cdot y^0 - \omega^1\cdot(z^1-z^0)]$. Note that if there is no change in the quality of the outputs produced by the establishment so that $z^1$ equals $z^0$, then $Q_{AP}$ reduces to the ordinary Paasche output quantity index, $Q_P \equiv p^1\cdot y^1/p^1\cdot y^0$. In the general case where there is quality change, we need to subtract from the denominator of the ordinary Paasche output quantity index, $p^1\cdot y^0$, the quality adjustment factor, $\omega^1\cdot(z^1-z^0)$, which values the net increase in the quality variables, $z^1-z^0$, at the period 1 characteristics prices $\omega^1$.

Since the theoretical output quantity indexes, $\alpha_0$ and $\alpha_1$, are both equally representative, a single estimate of cost based output quantity growth should be set equal to a symmetric average of these two estimates. We will choose the geometric mean as our preferred symmetric average[20] and thus our preferred theoretical measure of cost based output quantity growth is the following *theoretical Fisher type output index*, $\alpha_F$:

$$\begin{aligned}\text{(A14)}\ \alpha_F &\equiv [\alpha_0\alpha_1]^{1/2} \\ &\cong [Q_{AL}Q_{AP}]^{1/2} \qquad \text{using (A12) and (A13)} \\ &\equiv Q_{AF}\end{aligned}$$

where the *quality adjusted Fisher output quantity index*, $Q_{AF}$, is defined as the geometric mean of the quality adjusted Laspeyres and Paasche output quantity indexes, $Q_{AL}$ and $Q_{AP}$.

If we have estimated cost functions at our disposal for each period, then our preferred measure of cost based output growth is equal to the index defined by the first line of (A14). If we have only price and quantity information available plus information on quality variables and their prices, then our preferred quality adjusted output index is the quality adjusted Fisher quantity index, $Q_{AF}$, defined as the geometric mean of the corresponding Laspeyres and Paasche quality adjusted indexes defined in (A12) and (A13). It should be noted that while $Q_{AL}$ and $Q_{AP}$ only approximate the corresponding theoretical indexes $\alpha_0$ and $\alpha_1$ to the first order, it is likely that $Q_{AF}$ approximates the theoretical index $\alpha_F$ to the accuracy of a second order approximation.[21]

We now turn our attention to theoretical measures of input price growth. We use the joint cost function $C^t$ in order to define a *family of input price indexes*, $\beta(w^0,w^1,y,z,t)$, as follows:

$$\text{(A15)}\ \beta(w^0,w^1,y,z,t) \equiv C^t(y,w^1,z)/C^t(y,w^0,z).$$

Thus the theoretical input price index $\beta(w^0,w^1,y,z,t)$ defined by (A15) is equal to the (hypothetical) total cost $C^t(y,w^1,z)$ of producing the reference vector of outputs y with qualities z facing the period 1 input prices $w^1$, divided by the (hypothetical) total cost of producing the reference vector of outputs y with qualities z facing the period 0 input prices $w^0$, where in both cases, we use the technology of period t. Thus for each choice of technology (i.e., t could equal 0 or 1) and for each choice of a reference vector of outputs y with qualities z, we obtain a (different) cost based input price index. Note that only the vector of input prices changes in the numerator and denominator of definition (A15).

Again following the example of Konüs (1939), it is natural to single out two special cases of the family of input price indexes defined by (A15): one choice where we use the period 0 technology and set the reference quantities equal to the period 0 quantities $y^0$

[20] Diewert (1997) explained why the geometric mean is a good choice for the symmetric average. In the case where there are no changes in quality over the two periods under consideration, the quality adjusted Fisher output quantity index $Q_{AF}$ reduces to the ordinary Fisher (1922) ideal quantity index.

[21] This is certainly the case if there are no quality changes; i.e., if $z^0 = z^1$; see Diewert (2002a) (2008).

and $z^0$ and another choice where we use the period 1 technology and set the reference quantities equal to the period 1 quantities $y^1$ and $z^1$. These special cases are defined as $\beta_0$ and $\beta_1$ below:

$$
\begin{aligned}
\text{(A16)}\quad \beta_0 &\equiv C^0(y^0,w^1,z^0)/C^0(y^0,w^0,z^0) \\
&= C^0(y^0,w^1,z^0)/w^0\cdot x^0 && \text{using (A10)} \\
&\cong [C^0(y^0,w^0,z^0) + \nabla_w C^0(y^0,w^0,z^0)\cdot(w^1-w^0)]/w^0\cdot x^0 \\
& && \text{forming a first order Taylor series approximation to } C^0(y^0,w^1,z^0) \\
&= w^1\cdot x^0/w^0\cdot x^0 && \text{using (A6) and (A7)} \\
&\equiv P_L^*
\end{aligned}
$$

where $P_L^*$ is the ordinary Laspeyres input price index, $w^1\cdot x^0/w^0\cdot x^0$. Thus the theoretical cost function based input price index $\beta_0$ defined by the first line in (A16) is approximately equal to the Laspeyres input price index $P_L^*$. We turn now to our second special case of the family of input price indexes defined by (A15).

$$
\begin{aligned}
\text{(A17)}\quad \beta_1 &\equiv C^1(y^1,w^1,z^1)/C^1(y^1,w^0,z^1) \\
&= w^1\cdot x^1/C^1(y^1,w^0,z^1) && \text{using (A10)} \\
&\cong w^1\cdot x^1/[C^1(y^1,w^1,z^1) + \nabla_w C^1(y^1,w^1,z^1)\cdot(w^0-w^1)] \\
& && \text{forming a first order Taylor series approximation to } C^1(y^1,w^0,z^1) \\
&= w^1\cdot x^1/w^0\cdot x^1 && \text{using (A6) and (A7)} \\
&\equiv P_P^*
\end{aligned}
$$

where $P_P^*$ is the ordinary Paasche input price index, $w^1\cdot x^1/w^0\cdot x^1$. Thus the theoretical cost function based input price index $\beta_1$ defined by the first line in (A17) is approximately equal to the Paasche input price index $P_P^*$.

Since both theoretical input price indexes, $\beta_0$ and $\beta_1$, are equally representative, a single estimate of input price change should be set equal to a symmetric average of these two estimates. We again choose the geometric mean as our preferred symmetric average and thus our preferred theoretical measure of input price growth is the following Fisher type theoretical input price index, $\beta_F$:

$$
\begin{aligned}
\text{(A18)}\quad \beta_F &\equiv [\beta_0\beta_1]^{1/2} \\
&\cong [P_L^* P_P^*]^{1/2} && \text{using the approximations (A16) and (A17)} \\
&\equiv P_F^*
\end{aligned}
$$

where the Fisher (1922) index of input price change, $P_F^*$, is defined as the geometric mean of the Laspeyres and Paasche input price indexes. Given the fact that $P_L^*$ is a first order approximation to $\beta_0$ and $P_P^*$ is a first order approximation to $\beta_1$, it is obvious that $P_F^*$ is at least a first order approximation to the theoretical input price index $\beta_F$. But in most cases, the approximation of $P_F^*$ to $\beta_F$ will be much better than a first order approximation since the usual upward bias in $P_L^*$ will generally offset the usual downward bias in $P_P^*$.

We now define our last family of theoretical indexes. We again use the joint cost functions $C^0$ and $C^1$ in order to define a *family of reciprocal indexes of technical progress*, $\gamma(y,w,z)$, as follows:

(A19) $\gamma(y,w,z) \equiv C^1(y,w,z)/C^0(y,w,z)$.

The family of theoretical reciprocal technical progress indexes (or reciprocal productivity indexes) $\gamma(y,w,z)$ defined by (A19) is equal to the (hypothetical) total cost $C^1(y,w,z)$ of producing the reference vector of outputs y with quality characteristics z when the government establishment faces the reference vector of input prices w using the period 1 technology, divided by the total cost $C^0(y,w,z)$ of producing the same reference vector of outputs y with the same quality characteristics z and facing the same reference vector of input prices w, where we now use the period 0 technology.[22] Thus $\gamma(y,w,z)$ is a measure of the *proportional reduction in costs* that occurs due to technical progress between periods 0 and 1 and it can be seen that this is an inverse measure of technical progress; i.e., there is positive technical progress between the two periods if $\gamma(y,w,z)$ is less than one. For each choice of a reference vector of output quantities y with qualities vector z and reference vector of input prices w, we obtain a measure of exogenous cost reduction.

Instead of singling out the reference vectors y,z and w that appear in the definition of $\gamma(y,w,z)$ to be the period t quantity and price vectors $(y^t,w^t,z^t)$ for t = 0,1, we will choose the *mixed reference vectors* $(y^0,w^1,z^0)$ and $(y^1,w^0,z^1)$ for our usual two special cases. The reason for these somewhat odd looking choices will be explained below.

We want to explain the growth in total costs going from period 0 to 1, $C^1(y^1,w^1,z^1)/C^0(y^0,w^0,z^0)$, as the product of 3 growth factors:

- Growth in outputs including improvements in the quality of the outputs; i.e., a factor of the form $\alpha(y^0,y^1,z^0,z^1,w,t)$ defined above by (A11);
- Growth in input prices; i.e., a factor of the form $\beta(w^0,w^1,y,z,t)$ defined by (A15) and
- Exogenous reduction in costs due to technical progress; i.e., a factor of the form $\gamma(y,w,z)$ defined by (A19).

Simple algebra shows that we have the following decompositions of the cost ratio $C^1(y^1,w^1,z^1)/C^0(y^0,w^0,z^0)$ into explanatory factors of the above type:[23]

(A20) $C^1(y^1,w^1,z^1)/C^0(y^0,w^0,z^0)$
$= [C^1(y^1,w^1,z^1)/C^1(y^0,w^1,z^0)][C^0(y^0,w^1,z^0)/C^0(y^0,w^0,z^0)][C^1(y^0,w^1,z^0)/C^0(y^0,w^1,z^0)]$
$= \alpha_1\beta_0\gamma(y^0,w^1,z^0)$ using definitions (A13), (A16) and (A19);

[22] This is a cost function analogue to the revenue function definitions of technical progress defined by Diewert (1983a; 1063-1064), Diewert and Morrison (1986) and Kohli (1990).

[23] The decompositions of cost growth given by (A24) and (A26) are nonparametric analogues to the parametric revenue growth decompositions obtained by Diewert and Morrison (1986), Kohli (1990) and Fox and Kohli (1998) into explanatory factors.

$$\text{(A21) } C^1(y^1,w^1,z^1)/C^0(y^0,w^0,z^0)$$
$$= [C^0(y^1,w^0,z^1)/C^0(y^0,w^0,z^0)][C^1(y^1,w^1,z^1)/C^1(y^1,w^0,z^1)][C^1(y^1,w^0,z^1)/C^0(y^1,w^0,z^1)]$$
$$= \alpha_0\beta_1\gamma(y^1,w^0,z^1) \qquad \text{using definitions (A11), (A16) and (A23).}$$

The above decompositions show that the following two special cases of $\gamma(y,w,z)$ defined by (A19) are of particular interest:

$$\text{(A22) } \gamma(y^0,w^1,z^0) \equiv C^1(y^0,w^1,z^0)/C^0(y^0,w^1,z^0) \equiv \gamma_0 \,;$$

$$\text{(A23) } \gamma(y^1,w^0,z^1) \equiv C^1(y^1,w^0,z^1)/C^0(y^1,w^0,z^1) \equiv \gamma_1 \,.$$

We will now work out potentially observable first order approximations to the two specific measures of reciprocal technical progress defined by (A22) and (A23). Using definition (A22) and taking first order approximations to $C^1(y^0,w^1,z^0)$ and $C^0(y^0,w^1,z^0)$, we have the following first order approximation to the *reciprocal productivity index* $\gamma(y^0,w^1,z^0)$:

$$\text{(A24) } \gamma(y^0,w^1,z^0) \cong [C^1(y^1,w^1,z^1) + \nabla_y C^1(y^1,w^1,z^1)\cdot(y^0-y^1) + \nabla_z C^1(y^1,w^1,z^1)\cdot(z^0-z^1)]$$
$$/[C^0(y^0,w^0,z^0) + \nabla_w C^0(y^0,w^0,z^0)\cdot(w^1-w^0)]$$
$$= [p^1\cdot y^1 + p^1\cdot(y^0-y^1) - \omega^1\cdot(z^1-z^0)]/[w^0\cdot x^0 + x^0\cdot(w^1-w^0)] \qquad \text{using (A6)-(A10)}$$
$$= [p^1\cdot y^0 - \omega^1\cdot(z^1-z^0)]/w^1\cdot x^0$$
$$= \{[p^1\cdot y^0 - \omega^1\cdot(z^1-z^0)]/p^1\cdot y^1\}/\{w^1\cdot x^0/w^1\cdot x^1\} \qquad \text{using } p^1\cdot y^1 = w^1\cdot x^1$$
$$= \{Q_{AP}/Q_P^*\}^{-1}$$

where $Q_{AP}$ is the quality adjusted Paasche output index defined in (A13) and the *Paasche input quantity index* $Q_P^*$ is defined as follows:

$$\text{(A25) } Q_P^* \equiv w^1\cdot x^1/w^1\cdot x^0.$$

Thus the cost function based theoretical index of reciprocal technical progress $\gamma(y^0,w^1,z^0)$ defined by (A22) above is approximately equal to the reciprocal of the quality adjusted Paasche output quantity index $Q_{AP}$ divided by the ordinary Paasche input quantity index $Q_P^*$. In the case where quality does not change between the periods, $Q_{AP}/Q_P^*$ collapses to a "traditional" productivity index, the Paasche quantity index of outputs using the marginal cost weights of period 1, $Q_P \equiv p^1\cdot y^1/p^1\cdot y^0$, divided by the Paasche quantity index of inputs, $Q_P^*$.

We now turn our attention to the cost function based measure of technical progress $\gamma(y^1,w^0,z^1)$ defined by (A23). Using definition (A23) and taking first order approximations to $C^1(y^1,w^0,z^1)$ and $C^0(y^1,w^0,z^1)$, we have the following first order approximation to the *reciprocal productivity index* $\gamma(y^1,w^0,z^1)$:

$$\text{(A26) } \gamma(y^1,w^0,z^1) \cong [C^1(y^1,w^1,z^1) + \nabla_w C^1(y^1,w^1,z^1)\cdot(w^0-w^1)]$$
$$/[C^0(y^0,w^0,z^0) + \nabla_y C^0(y^0,w^0,z^0)\cdot(y^1-y^0) + \nabla_z C^0(y^0,w^0,z^0)\cdot(z^1-z^0)]$$
$$= [w^1\cdot x^1 + x^1\cdot(w^0-w^1)]/[p^0\cdot y^0 + p^0\cdot(y^1-y^0) + \omega^0\cdot(z^1-z^0)] \qquad \text{using (A6)-(A10)}$$

$= w^0 \cdot x^1/[p^0 \cdot y^1 + \omega^0 \cdot (z^1 - z^0)]$
$= \{w^0 \cdot x^1/w^0 \cdot x^0\}/\{[p^0 \cdot y^1 + \omega^0 \cdot (z^1 - z^0)]/p^0 \cdot y^0$ using $p^0 \cdot y^0 = w^0 \cdot x^0$
$= \{Q_{AL}/Q_L^*\}^{-1}$

where $Q_{AL}$ is the quality adjusted Laspeyres output index defined in (A12) and the *Laspeyres input quantity index* $Q_L^*$ is defined as follows:

(A27) $Q_L^* \equiv w^0 \cdot x^1/w^0 \cdot x^0$.

Thus the cost function based theoretical index of reciprocal technical progress $\gamma(y^1,w^0,z^1)$ defined by (A23) above is approximately equal to the reciprocal of the quality adjusted Laspeyres output quantity index $Q_{AL}$ divided by the ordinary Laspeyres input quantity index $Q_L^*$. In the case where quality does not change between the periods, $Q_{AL}/Q_L^*$ collapses to a "traditional" productivity index, the Laspeyres quantity index of outputs using the marginal cost weights of period 0, $Q_L \equiv p^0 \cdot y^1/p^0 \cdot y^0$, divided by the Laspeyres quantity index of inputs, $Q_P^*$.

Since the two cost decompositions for the rate of growth of cost, $C^1(y^1,w^1,z^1)/C^0(y^0,w^0,z^0)$, given by (A20) and (A21) are equally valid, we will take the geometric average of these two decompositions to obtain our preferred overall cost decomposition. This leads to the following theoretical decomposition of $w^1 \cdot x^1/w^0 \cdot x^0$ equal to $C^1(y^1,w^1,z^1)/C^0(y^0,w^0,z^0)$ into explanatory factors:

(A28) $C^1(y^1,w^1,z^1)/C^0(y^0,w^0,z^0) = w^1 \cdot x^1/w^0 \cdot x^0 = \alpha_F \beta_F \gamma_F$

where the theoretical Fisher type quality adjusted output quantity growth factor $\alpha_F$ is defined by (A14), the Fisher type theoretical input price growth factor $\beta_F$ is defined by (A18) and the *Fisher type reciprocal measure of technical progress* $\gamma_F$ is defined as the geometric mean of the two reciprocal productivity indexes defined by (A22) and (A23):

(A29) $\gamma_F \equiv [\gamma(y^0,w^1,z^0)\gamma(y^1,w^0,z^1)]^{1/2} = [\gamma_0 \gamma_1]^{1/2}$.

The exact decomposition of (one plus) cost growth over the two periods under consideration given by (A28) is our preferred decomposition of cost growth into explanatory factors which can be implemented if the economic statistician has estimates for the period 0 and 1 cost functions at hand.

On the other hand, if full information on the cost functions for the two periods is not available but information on marginal costs and characteristics prices is available in addition to basic price and quantity information, then the various theoretical indexes on the right hand side of (A28) can be replaced by their first order approximations. Thus using (A14), $\alpha_F$ can be approximated by the geometric mean of the quality adjusted Laspeyres and Paasche output volume indexes $[Q_{AL}Q_{AP}]^{1/2}$, using (A18), $\beta_F$ can be approximated by the Fisher input price index $P_F^*$ and using (A24) and (A26), $\gamma_F$ can be approximated by the geometric mean of the reciprocal of the quality adjusted Paasche productivity index $\{Q_{AP}/Q_P^*\}^{-1}$ and the quality adjusted Laspeyres productivity index $\{Q_{AL}/Q_L^*\}^{-1}$. Substituting these first order approximations into (A28) leads to the following approximate decomposition of cost growth into explanatory factors:

$$\text{(A30)}\quad w^1\cdot x^1/w^0\cdot x^0 \cong [Q_{AL}Q_{AP}]^{1/2}\,[P_L^*P_P^*]^{1/2}\,\{Q_{AP}/Q_P^*\}^{-1/2}\,\{Q_{AL}/Q_L^*\}^{-1/2}$$
$$= \{P_L^*P_P^*Q_L^*Q_P^*\}^{1/2}$$
$$= P_F^*Q_F^*$$

where the Fisher (1922) *input price and quantity indexes* are defined as follows:

$$\text{(A31)}\quad P_F^* \equiv [P_L^*P_P^*]^{1/2} = [(w^1\cdot x^0/w^0\cdot x^0)(w^1\cdot x^1/w^0\cdot x^1)]^{1/2}\,;$$
$$\text{(A32)}\quad Q_F^* \equiv [Q_L^*Q_P^*]^{1/2} = [(w^0\cdot x^1/w^0\cdot x^0)(w^1\cdot x^1/w^1\cdot x^0)]^{1/2}.$$

However, as Fisher (1922) showed long ago, the product of the Fisher input price and quantity indexes, $P_F^*Q_F^*$, is exactly equal to the input value ratio, $w^1\cdot x^1/w^0\cdot x^0$. Thus equation (A30) holds as an *exact equality* rather than as an approximate equality. Thus our first order approximations for the explanatory factors $\alpha_F$ , $\beta_F$ and $\gamma_F$ in the exact decomposition (A28) also have the property that they are exact; i.e., the product of these approximate explanatory factors is exactly equal to the cost ratio $w^1\cdot x^1/w^0\cdot x^0$.

The above arguments are rather complex but can be summarized as follows:

- If cost functions for a government establishment have been estimated econometrically for two periods that take changes in cost due to changes in the quality of the outputs produced into account, then it is possible to decompose cost growth over the two periods under consideration into a product of explanatory factors, $\alpha_F\,\beta_F\,\gamma_F$, where $\alpha_F$ is the cost function based measure of quality adjusted output growth defined by (A14), $\beta_F$ is the cost function based measure of input growth defined by (A18) and $\gamma_F$ is the cost function based measure of reciprocal productivity growth equal to the geometric mean of $\gamma_0$ and $\gamma_1$ defined by (A22) and (A23);
- If full cost function information is not available for the two periods but basic price and quantity data are available in addition to approximate estimates of incremental changes in cost due to incremental changes in characteristics for the two periods, $\omega^0 \equiv \nabla_z C^0(y^0,w^0,z^0)$ and $\omega^1 \equiv \nabla_z C^1(y^1,w^1,z^1)$, then the quality adjusted Laspeyres and Paasche output volume indexes, $Q_{AL}$ and $Q_{AP}$, can be calculated as $[p^0\cdot y^1 + \omega^0\cdot(z^1-z^0)]/p^0\cdot y^0$ and $p^1\cdot y^1/[p^1\cdot y^0 - \omega^1\cdot(z^1-z^0)]$ respectively. These two indexes along with the Laspeyres and Paasche input price indexes, $P_L^*$ and $P_P^*$, and the Laspeyres and Paasche input quantity indexes, $Q_L^*$ and $Q_P^*$, can be used in order to decompose cost growth over the two periods into the product of observable output quantity, input price and productivity growth factors according to the first line in (A30).

In order to implement the "observable" decomposition of cost growth into explanatory factors given by (A30), in addition to price and quantity data on inputs, we need quantity data on outputs, $y^0$ and $y^1$, price information on marginal or incremental costs, $p^0 \equiv \nabla_y C^0(y^0,w^0,z^0)$ and $p^1 \equiv \nabla_y C^1(y^1,w^1,z^1)$, information on the two vectors of characteristics prices, $\omega^0$ and $\omega^1$, and finally, information on the change in characteristics over the two periods under consideration, $z^1 - z^0$. Thus in principle, with the above additional information, "normal" index number theory can be adapted to deal with the problem of quality change using a cost function based approach to the valuation of quality change.

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