Cost of Living Indexes and Exact Index Numbers Revised, August 6, 2009.

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Abstract

The paper reviews and extends the theory of exact and superlative index numbers. Exact index numbers are empirical index number formula that are equal to an underlying theoretical index, provided that the consumer has preferences that can be represented by certain functional forms. These exact indexes can be used to measure changes in a consumer's cost of living or welfare. Two cases are considered: the case of homothetic preferences and the case of nonhomothetic preferences. In the homothetic case, exact index numbers are obtained for square root quadratic preferences, quadratic mean of order r preferences and normalized quadratic preferences. In the nonhomothetic case, exact indexes are obtained for various translog preferences.

Key Words

Exact index numbers, superlative index numbers, flexible functional forms, Fisher ideal index, normalized quadratic preferences, mean of order r indexes, homothetic preferences, nonhomothetic preferences, cost of living indexes, the measurement of welfare change, translog functional form, duality theory, Allen quantity index.

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1. Introduction

The main thesis of this chapter is that for many purposes, it is not necessary to use econometric methods in order to estimate a consumer's preferences. If our purpose is either to measure the change in a consumer's cost of living going from one period to another or to measure the consumer's change in welfare, then instead of using econometric methods, exact index number formulae can be used.[2]

[1] The author is indebted to Bert Balk for helpful comments. This paper will appear as Chapter 8 in *Quantifying Consumer Preferences*, Daniel Slottje (ed.), Emerald Publishing Group, 2008.
[2] However, if our main purpose is to estimate systems of consumer demand functions and the resulting elasticities of demand, then the use of econometric methods is unavoidable.

In section 2, we outline the theory of the cost of living index that was first developed by the Russian economist, Konüs (1939). The approach in this section is completely nonparametric but it sets the stage for later developments.

In section 3, we specialize the general theory developed in section 2 to the case where the consumer's preferences are homothetic; i.e., they can be represented by a linearly homogeneous utility function. At first glance, it may seem that this restriction is not very interesting from an empirical point of view since Engel's Law demonstrates that overall consumer preferences are not homothetic. However, there are too many commodities in the real world; it is necessary to aggregate similar commodities into subaggregates in order to model the economy. In forming subaggregates, it is very useful to assume the existence of a linearly homogeneous subaggregator function so that we obtain a subaggregate price index that is independent of quantities.

In section 4, we establish Shephard's Lemma and Wold's Identity. These results will prove to be very useful in the subsequent sections.

In sections 5-7, we establish various exact index number formulae in the case where the consumer's preferences are homothetic or where the subaggregator function is linearly homogeneous. These formulae can be evaluated using observable price and quantity data pertaining to the two periods under consideration and they are exactly equal to a corresponding theoretical index, provided that the consumer's preferences can be represented by certain functional forms. We restrict our analysis to the case where the underlying functional form for the preference function can provide a second order approximation to an arbitrary preference function of the type under consideration; i.e., we restrict ourselves to *flexible functional forms* for functions that represent preferences.

In section 8, we consider price indexes or cost of living indexes in the case where preferences are general; i.e., we drop the homotheticity assumption in this section and in section 9, where we consider quantity indexes in the nonhomothetic case. The situation is much more complicated in the case of nonhomothetic preferences but the results presented in sections 8 and 9 are reasonably powerful.

Section 10 offers a short conclusion.

2. Konüs True Cost of Living Indexes

In this section, we will outline the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, A. A. Konüs (1939). This theory relies on the assumption of *optimizing behavior* on the part of the consumer. Thus given a vector of commodity or input prices p^t that the consumer faces in a given time period t, it is assumed that the corresponding observed quantity vector q^t is the solution to a cost minimization problem that involves the consumer's preference or utility function f.

We assume that "the" consumer has well defined *preferences* over different combinations of the N consumer commodities or items.[3] Each combination of items can be represented by a nonnegative vector $q \equiv [q_1,\ldots,q_N]$. The consumer's preferences over alternative possible consumption vectors q are assumed to be representable by a nonnegative, continuous, increasing, and quasiconcave utility function f, which is defined over the nonnegative orthant. Thus if $f(q^1) > f(q^0)$, then the consumer prefers the consumption vector q^1 to q^0. We further assume that the consumer minimizes the cost of achieving the period t utility level $u^t \equiv f(q^t)$ for periods t = 0,1. Thus we assume that the observed period t consumption vector q^t solves the following *period t cost minimization problem*:[4]

$$(1)\quad C(u^t,p^t) \equiv \min_q \{p^t\cdot q : f(q) = u^t\} = p^t\cdot q^t ; \qquad t = 0,1.$$

The period t price vector for the n commodities under consideration that the consumer faces is p^t. Note that the solution to the cost or expenditure minimization problem (1) for a general utility level u and general vector of commodity prices p defines the *consumer's cost or expenditure function*, C(u,p). It can be shown[5] that C(u,p) will have the following properties: (i) C(u,p) is jointly continuous in u,p for $p \gg 0_N$ and $u \in U$ where U is the range of f and is a nonnegative function over this domain of definition set; (ii) C(u,p) is increasing in u for each fixed p and (iii) C(u,p) is nondecreasing, linearly homogeneous and concave function of p for each $u \in U$. Conversely, if a cost function is given and satisfies the above properties, then the utility function f that is dual to C can be recovered using duality theory.[6] We shall use the cost function in order to define the consumer's cost of living price index.

The Konüs (1939) family of *true cost of living indexes* pertaining to two periods where the consumer faces the strictly positive price vectors $p^0 \equiv (p_1^0,\ldots,p_N^0)$ and $p^1 \equiv (p_1^1,\ldots,p_N^1)$ in periods 0 and 1 respectively is defined as the ratio of the minimum costs of achieving the same utility level $u \equiv f(q)$ where q is a positive reference quantity vector:

$$(2)\quad P_K(p^0,p^1,q) \equiv C[f(q),p^1]/C[f(q),p^0].$$

We say that definition (2) defines a *family* of price indexes because there is one such index for each reference quantity vector q chosen.

It is natural to choose two specific reference quantity vectors q in definition (2): the observed base period quantity vector q^0 and the current period quantity vector q^1. The first of these two choices leads to the *following Laspeyres-Konüs true cost of living index*:

$$\begin{aligned}(3)\quad P_K(p^0,p^1,q^0) &\equiv C[f(q^0),p^1]/C[f(q^0),p^0] \\ &= C[f(q^0),p^1]/p^0\cdot q^0 \qquad \text{using (1) for } t = 0\end{aligned}$$

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[3] In this section, these preferences are assumed to be invariant over time. In section 8 when we introduce environmental variables, this assumption will be relaxed.

[4] Notation: $p^t\cdot q \equiv \sum_{n=1}^N p_n^t q_n$.

[5] See Diewert (1993b; 124).

[6] See Diewert (1974; 119) (1993b; 129) and Blackorby and Diewert (1979) for the details and for references to various duality theorems.

$= \min_q \{p^1 \cdot q : f(q) = f(q^0)\}/p^0 \cdot q^0$ using the definition of $C[f(q^0),p^1]$
$\leq p^1 \cdot q^0/p^0 \cdot q^0$ since $q^0 \equiv (q_1{}^0,\ldots,q_N{}^0)$ is feasible
$\equiv P_L(p^0,p^1,q^0,q^1)$

where P_L is the observable Laspeyres price index. *Thus the (unobservable) Laspeyres-Konüs true cost of living index is bounded from above by the observable Laspeyres price index.*[7]

The second of the two natural choices for a reference quantity vector q in definition (2) leads to the *following Paasche-Konüs true cost of living index*:

(4) $P_K(p^0,p^1,q^1) \equiv C[f(q^1),p^1]/C[f(q^1),p^0]$
$= p^1 \cdot q^1/C[f(q^1),p^0]$ using (1) for t = 1
$= p^1 \cdot q^1/\min_q \{p^0 \cdot q : f(q) = f(q^1)\}$ using the definition of $C[f(q^1),p^0]$
$\geq p^1 \cdot q^1/p^0 \cdot q^1$ since $q^1 \equiv (q_1{}^1,\ldots,q_N{}^1)$ is feasible and thus $C[f(q^1),p^0] \leq p^0 \cdot q^1$ and $1/C[f(q^1),p^0] \geq 1/p^0 \cdot q^1$
$\equiv P_P(p^0,p^1,q^0,q^1)$

where P_P is the observable Paasche price index. *Thus the (unobservable) Paasche-Konüs true cost of living index is bounded from below by the observable Paasche price index*.[8]

The bound (3) on the Laspeyres-Konüs true cost of living $P_K(p^0,p^1,q^0)$ using the base period level of utility as the living standard is *one sided* as is the bound (4) on the Paasche-Konüs true cost of living $P_K(p^0,p^1,q^1)$ using the *current period* level of utility as the living standard. In a remarkable result, Konüs (1939; 20) showed that there exists an intermediate consumption vector q^* that is on the straight line joining the base period consumption vector q^0 and the current period consumption vector q^1 such that the corresponding (unobservable) true cost of living index $P_K(p^0,p^1,q^*)$ is between the observable Laspeyres and Paasche indexes, P_L and P_P.[9] Thus we have:[10]

Proposition 1: There exists a number λ^* between 0 and 1 such that

(5) $P_L \leq P_K(p^0,p^1,(1-\lambda^*)q^0 + \lambda^*)q^1) \leq P_P$ or $P_P \leq P_K(p^0,p^1,(1-\lambda^*)q^0 + \lambda^*q^1) \leq P_L$.

Proof: Define $g(\lambda)$ for $0 \leq \lambda \leq 1$ by $g(\lambda) \equiv P_K(p^0,p^1,(1-\lambda)q^0 + \lambda q^1)$. Note that $g(0) = P_K(p^0,p^1,q^0)$ and $g(1) = P_K(p^0,p^1,q^1)$. There are 24 = (4)(3)(2)(1) possible a priori inequality relations that are possible between the four numbers g(0), g(1), P_L and P_P. However, the inequalities (3) and (4) above imply that $g(0) \leq P_L$ and $P_P \leq g(1)$. This means that there are only six possible inequalities between the four numbers:

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[7] This inequality was first obtained by Konüs (1939; 17). See also Pollak (1983).
[8] This inequality is also due to Konüs (1939; 19). See also Pollak (1983).
[9] For more recent applications of the Konüs method of proof, see Diewert (1983a;191) (2001; 173) for applications in the consumer context and Diewert (1983b; 1059-1061) for an application in the producer context.
[10] For a generalization of this single consumer result to the case of many consumers, see Diewert (2001; 173).

(6) $g(0) \le P_L \le P_P \le g(1)$;
(7) $g(0) \le P_P \le P_L \le g(1)$;
(8) $g(0) \le P_P \le g(1) \le P_L$;
(9) $P_P \le g(0) \le P_L \le g(1)$;
(10) $P_P \le g(1) \le g(0) \le P_L$;
(11) $P_P \le g(0) \le g(1) \le P_L$.

Using the assumptions that: (a) the consumer's utility function f is continuous over its domain of definition; (b) the utility function is increasing in the components of q and hence is subject to local nonsatiation and (c) the price vectors p^t have strictly positive components, it is possible to use Debreu's (1959; 19) Maximum Theorem (see also Diewert (1993b; 112-113) for a statement of the Theorem) to show that the consumer's cost function $C(f(q),p^t)$ will be continuous in the components of q. Thus using definition (2), it can be seen that $P_K(p^0,p^1,q)$ will also be continuous in the components of the vector q. Hence $g(\lambda)$ is a continuous function of λ and assumes all intermediate values between g(0) and g(1). By inspecting the inequalities (6)-(11) above, it can be seen that we can choose λ between 0 and 1, λ^* say, such that $P_L \le g(\lambda^*) \le P_P$ for case (6) or such that $P_P \le g(\lambda^*) \le P_L$ for cases (7) to (11). Thus at least one of the two inequalities in (5) holds. Q.E.D.

The above inequalities are of some practical importance. If the observable (in principle) Paasche and Laspeyres indexes are not too far apart, then taking a symmetric average of these indexes should provide a good approximation to a true cost of living index where the reference standard of living is somewhere between the base and current period living standards. Note that the theory thus far is completely nonparametric; i.e., we do not have to make any specific assumptions about the functional form of f or C.

If we require a single estimate for the price change between the two periods under consideration, then it is natural to take some sort of evenly weighted average of the two bounding indexes which appear in (5) as our final estimate of price change between periods 0 and 1. This averaging of the Paasche and Laspeyres strategy is due to Bowley:

"If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean … as a first approximation." A. L. Bowley (1901; 227).

"When estimating the factor necessary for the correction of a change found in money wages to obtain the change in real wages, statisticians have not been content to follow Method II only [to calculate a Laspeyres price index], but have worked the problem backwards [to calculate a Paasche price index] as well as forwards. … They have then taken the arithmetic, geometric or harmonic mean of the two numbers so found." A. L. Bowley (1919; 348).[11]

[11] Fisher (1911; 417-418) (1922) also considered the arithmetic, geometric and harmonic averages of the Paasche and Laspeyres indexes.

Examples of such symmetric averages[12] are the arithmetic mean, which leads to the Sidgwick (1883; 68) Bowley (1901; 227)[13] index:

$$(12)\ P_{SB}(p^0,p^1,q^0,q^1) \equiv (1/2)P_L(p^0,p^1,q^0,q^1) + (1/2)P_P(p^0,p^1,q^0,q^1)$$

or the geometric mean, which leads to the Fisher (1922) ideal index:

$$(13)\ P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)\ P_P(p^0,p^1,q^0,q^1)]^{1/2}\ .$$

In order to determine which average of the Laspeyres and Paasche indexes might be "best", we need *criteria* or *tests* or *properties* that we would like our indexes to satisfy. We will conclude this section by suggesting one possible approach to picking the "best" average.

At this point, it is convenient to define exactly what we mean by a symmetric average of two numbers. Thus let a and b be two positive numbers. Diewert (1993c; 361) defined a *symmetric mean* of a and b as a function m(a,b) that has the following properties:

(14) $m(a,a) = a$ for all $a > 0$; (mean property);
(15) $m(a,b) = m(b,a)$ for all $a > 0$, $b > 0$; (symmetry property);
(16) $m(a,b)$ is a continuous function for $a > 0$, $b > 0$; (continuity property);
(17) $m(a,b)$ is a strictly increasing function; (increasingness property).

It can be shown that if m(a,b) satisfies the above properties, then it also satisfies the following property:[14]

(18) $\min\{a,b\} \le m(a,b) \le \max\{a,b\}$; (min-max property);

i.e., the mean of a and b, m(a,b), lies between the maximum and minimum of the numbers a and b. Since we have restricted the domain of definition of a and b to be positive numbers, it can be seen that an implication of (18) is that m also satisfies the following property:

(19) $m(a,b) > 0$ for all $a > 0$, $b > 0$; (positivity property).

If in addition, m satisfies the following property, then we say that m is a *homogeneous symmetric mean*:

(20) $m(\lambda a,\lambda b) = \lambda m(a,b)$ for all $\lambda > 0$, $a > 0$, $b > 0$.

What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical cost of living index? It is very desirable for a price index formula that

[12] For a discussion of the properties of symmetric averages, see Diewert (1993c).
[13] See Diewert (1993a; 36) and Balk (2008; 1-39) for additional references to the early history of index number theory.
[14] To prove this, use the technique of proof used by Eichhorn and Voeller (1976; 10).

depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test*[15]. We say that the index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

$$(21)\ P(p^1,p^0,q^1,q^0) = 1/\ P(p^0,p^1,q^0,q^1)\ ;$$

i.e., if we interchange the period 0 and period 1 price and quantity data and evaluate the index, then this new index $P(p^1,p^0,q^1,q^0)$ is equal to the reciprocal of the original index $P(p^0,p^1,q^0,q^1)$.

Now we are ready to look for a homogeneous symmetric mean of the Laspeyres and Paasche price indexes that satisfies the time reversal test (21).

Proposition 2:[16] The Fisher Ideal price index defined by (13) above is the *only* index that is a homogeneous symmetric average of the Laspeyres and Paasche price indexes, P_L and P_P, and satisfies the time reversal test (21) above.

Proof: In order to prove this proposition, we only require the homogeneous mean function to satisfy the positivity and homogeneity properties, (19) and (20) above.

We define the mean price index P using the function m as follows:

$$(22)\ P(p^0,p^1,q^0,q^1) \equiv m(P_L,P_P) = m(p^1\cdot q^0/p^0\cdot q^0,\ p^1\cdot q^1/p^0\cdot q^1)$$

where we have used the definitions of P_L and P_P which are in (3) and (4) above. Since P is supposed to satisfy the time reversal test, we can substitute definition (22) into (21) in order to obtain the following equation:

$$(23)\ m(p^0\cdot q^1/p^1\cdot q^1,\ p^0\cdot q^0/p^1\cdot q^0) = 1/\ m(p^1\cdot q^0/p^0\cdot q^0,\ p^1\cdot q^1/p^0\cdot q^1).$$

Letting $a \equiv p^1\cdot q^0/p^0\cdot q^0$ and $b \equiv p^1\cdot q^1/p^0\cdot q^1$, we see that equation (23) can be rewritten as:

$$(24)\ m(b^{-1},a^{-1}) = 1/m(a,b).$$

Equation (24) can be rewritten as:

$$\begin{aligned}(25)\ 1 &= m(a,b)\ m(b^{-1},a^{-1}) \\ &= am(1,b/a)\ a^{-1}\ m(a/b,1) && \text{using property (20) of m} \\ &= m(1,x)\ m(x^{-1},1) && \text{letting } x \equiv b/a \\ &= m(1,x)\ x^{-1}\ m(1,x) && \text{using property (20) of m.}\end{aligned}$$

Equation (25) can be rewritten as:

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[15] See Diewert (1992a; 218) for early references to this test.

[16] This result was established by Diewert (1997; 138). See also Balk (2008; 97).

(26) $x = [m(1,x)]^2$.

Thus using (19), we can take the positive square root of both sides of (26) and obtain

(27) $m(1,x) = x^{1/2}$.

Using property (20) of m again, we have

$$\begin{aligned}(28)\ m(a,b) &= a\, m(1,b/a) \\ &= a[b/a]^{1/2} && \text{using (27)} \\ &= a^{1/2}b^{1/2}.\end{aligned}$$

Now substitute (28) into (22) and we obtain the Fisher Index. Q.E.D.

The bounds (3)-(5) are the best bounds that we can obtain on true cost of living indexes without making further assumptions. In the following sections, we will make further assumptions on the class of utility functions that describe the consumer's tastes for the N commodities under consideration. With these extra assumptions, we are able to determine the consumer's true cost of living exactly. However, before we can implement this strategy, we require some preliminary theoretical material, which will be developed in the following two sections.

3. The True Cost of Living Index when Preferences are Homothetic

Up to now, the consumer's preference function f did not have to satisfy any particular homogeneity assumption. In this section, we assume that f is (positively*) linearly homogeneous*[17]; i.e., we assume that the consumer's utility function has the following property:

(29) $f(\lambda q) = \lambda f(q)$ for all $\lambda > 0$ and all $q \geq 0_N$.

Given the continuity of f, it can be seen that property (29) implies that $f(0_N) = 0$ so that the lower bound to the range of f is 0. Furthermore, f also satisfies $f(q) > 0$ if $q > 0_N$.

In the economics literature, assumption (29) is known as the assumption of *homothetic preferences*.[18] Although this assumption is generally not justified when we consider the consumer's overall cost of living index, it can be justified in the context of a *subaggregate* if we assume that the consumer has a separable subaggregator function, f(q), which is linearly homogeneous. In this case, q is no longer interpreted as the entire consumption vector, but refers only to a subaggregate such as "food" or "clothing" or

[17] This assumption is fairly restrictive in the consumer context. It implies that each indifference curve is a radial projection of the unit utility indifference curve. It also implies that all income elasticities of demand are unity, which is contradicted by empirical evidence.

[18] More precisely, Shephard (1953) defined a homothetic function to be a monotonic transformation of a linearly homogeneous function. However, if a consumer's utility function is homothetic, we can always rescale it to be linearly homogeneous without changing consumer behavior. Hence, we simply identify the homothetic preferences assumption with the linear homogeneity assumption.

some more narrowly defined aggregate.[19] Under this assumption, the consumer's subaggregate expenditure or cost function, C(u,p) defined by (1) above (with a new interpretation), decomposes as follows. For a positive subaggregate price vector $p \gg 0_N$ and a positive subaggregate utility level u, we have the following decomposition of C:

$$
\begin{aligned}
(30)\quad C(u,p) &\equiv \min_q \{p\cdot q : f(q) \geq u\} && \\
&= \min_q \{p\cdot q : (1/u)f(q) \geq 1\} && \text{dividing by } u > 0 \\
&= \min_q \{p\cdot q : f(q/u) \geq 1\} && \text{using the linear homogeneity of } f \\
&= u \min_q \{p\cdot q/u : f(q/u) \geq 1\} && \\
&= u \min_z \{p\cdot z : f(z) \geq 1\} && \text{letting } z = q/u \\
&= u\, C(1,p) && \text{using definition (1) with } u = 1 \\
&= u\, c(p)
\end{aligned}
$$

where $c(p) \equiv C(1,p)$ is the *unit cost function* that is corresponds to f.[20] It can be shown that the unit cost function c(p) satisfies the same regularity conditions that f satisfied; i.e., c(p) is positive, concave and (positively) linearly homogeneous for positive price vectors.[21] Substituting (31) into (1) and using $u^t = f(q^t)$ leads to the following equations:

$$(31)\quad p^t\cdot q^t = c(p^t)f(q^t) \qquad \text{for } t = 0,1.$$

Thus under the linear homogeneity assumption on the utility function f, observed period t expenditure on the n commodities (the left hand side of (31) above) is equal to the period t unit cost $c(p^t)$ of achieving one unit of utility times the period t utility level, $f(q^t)$, (the right hand side of (31) above). Obviously, we can identify the period t unit cost, $c(p^t)$, as the period t price level P^t and the period t level of utility, $f(q^t)$, as the period t quantity level Q^t.

The linear homogeneity assumption on the consumer's preference function f leads to a simplification for the family of Konüs true cost of living indices, $P_K(p^0,p^1,q)$, defined by (2) above. Using this definition for an arbitrary reference quantity vector q, we have:

$$
\begin{aligned}
(32)\quad P_K(p^0,p^1,q) &\equiv C[f(q),p^1]/C[f(q),p^0] && \\
&= c(p^1)f(q)/c(p^0)f(q) && \text{using (30) twice} \\
&= c(p^1)/c(p^0).
\end{aligned}
$$

[19] This particular branch of the economic approach to index number theory is due to Shephard (1953) (1970) and Samuelson and Swamy (1974). Shephard in particular realized the importance of the homotheticity assumption in conjunction with separability assumptions in justifying the existence of subindexes of the overall cost of living index.

[20] Economists will recognize the producer theory counterpart to the result C(u,p) = uc(p): if a producer's production function f is subject to constant returns to scale, then the corresponding total cost function C(u,p) is equal to the product of the output level u times the unit cost c(p).

[21] Obviously, the utility function f determines the consumer's cost function C(u,p) as the solution to the cost minimization problem in the first line of (13). Then the unit cost function c(p) is defined as C(1,p). Thus f determines c. But we can also use c to determine f under appropriate regularity conditions. In the economics literature, this is known as *duality theory.* For additional material on duality theory and the properties of f and c, see Samuelson (1953), Shephard (1953) and Diewert (1974) (1993b; 107-123).

Thus under the homothetic preferences assumption, the entire family of Konüs true cost of living indexes collapses to a single index, $c(p^1)/c(p^0)$, the ratio of the minimum costs of achieving unit utility level when the consumer faces period 1 and 0 prices respectively. Put another way, *under the homothetic preferences assumption, $P_K(p^0,p^1,q)$ is independent of the reference quantity vector q.*

If we use the Konüs true cost of living index defined by the right hand side of (32) as our price index concept, then the corresponding implicit quantity index can be defined as the subaggregate value ratio divided by the Konüs price index:

$$\begin{aligned}(33)\ Q(p^0,p^1,q^0,q^1,q) &\equiv p^1\cdot q^1/\{p^0\cdot q^0\ P_K(p^0,p^1,q)\} \\ &= c(p^1)f(q^1)/\{c(p^0)f(q^0)\ P_K(p^0,p^1,q)\} && \text{using (31) twice} \\ &= c(p^1)f(q^1)/\{c(p^0)f(q^0)[c(p^1)/c(p^0)]\} && \text{using (32)} \\ &= f(q^1)/f(q^0).\end{aligned}$$

Thus under the homothetic preferences assumption, the *implicit quantity index* that corresponds to the true cost of living price index $c(p^1)/c(p^0)$ is the *utility ratio* $f(q^1)/f(q^0)$. Since the utility function is assumed to be homogeneous of degree one, this is the natural definition for a quantity index.

4. Wold's Identity and Shephard's Lemma

In subsequent sections, we will need two additional results from economic theory: Wold's Identity and Shephard's Lemma.

Wold's (1944; 69-71) (1953; 145) *Identity* is the following result. Assuming that the consumer satisfies the cost minimization assumptions (1) for periods 0 and 1 and that the utility function f is differentiable at the observed quantity vectors $q^0 >> 0_N$ and $q^1 >> 0_N$ it can be shown[22] that the following equations hold:

$$(34)\ p^t/p^t\cdot q^t = \nabla f(q^t)/q^t\cdot\nabla f(q^t)\ ; \qquad t = 0,1.$$

If we assume that the utility function is linearly homogeneous, then *Wold's Identity* (34) simplifies into the following equations which will prove to be very useful:[23]

$$(35)\ p^t/p^t\cdot q^t = \nabla f(q^t)/f(q^t)\ ; \qquad t = 0,1.$$

[22] To prove this, consider the first order necessary conditions for the strictly positive vector q^t to solve the period t cost minimization problem. The conditions of Lagrange with respect to the vector of q variables are: $p^t = \lambda^t\ \nabla f(q^t)$ where λ^t is the optimal Lagrange multiplier and $\nabla f(q^t)$ is the vector of first order partial derivatives of f evaluated at q^t. Note that this system of equations is the price equals a constant times marginal utility equations that are familiar to economists. Now take the inner product of both sides of this equation with respect to the period t quantity vector q^t and solve the resulting equation for λ^t. Substitute this solution back into the vector equation $p^t = \lambda^t\ \nabla f(q^t)$ and we obtain (34).

[23] Differentiate both sides of the equation $f(\lambda q) = \lambda f(q)$ with respect to λ and then evaluate the resulting equation at $\lambda = 1$. We obtain the equation $\sum_{i=1}^N f_i(q)q_i = f(q)$ where $f_i(q) \equiv \partial f(q)/\partial q_i$.

Shephard's (1953; 11) *Lemma* is the following result. Consider the period t cost minimization problem defined by (1) above. If the cost function $C(u^t,p^t)$ is differentiable with respect to the components of the price vector p, then the period t quantity vector q^t is equal to the vector of first order partial derivatives of the cost function with respect to the components of p; i.e., we have

$$(36)\ q^t = \nabla_p C(u^t,p^t)\ ; \qquad t = 0,1.$$

To explain why (36) holds, consider the following argument. Because we are assuming that the observed period t quantity vector q^t solves the cost minimization problem defined by $C(u^t,p^t)$, then q^t must be feasible for this problem so we must have $f(q^t) = u^t$. Thus q^t is a feasible solution for the following cost minimization problem where the general price vector p has replaced the specific period t price vector p^t:

$$(37)\ C(u^t,p) \equiv \min_q \{p\cdot q : f(q) \geq u^t\} \leq p\cdot q^t \qquad \text{for all } p >> 0_N$$

where the inequality follows from the fact that is a feasible (but usually not optimal) solution for the cost minimization problem in (37). Now define for each strictly positive price vector p the function g(p) as follows:

$$(38)\ g(p) \equiv p\cdot q^t - C(u^t,p).$$

Using (1) and (37), it can be seen that g(p) is minimized (over all strictly positive price vectors p) at $p = p^t$. Thus the first order necessary conditions for minimizing a differentiable function of N variables hold, which simplify to equations (36).

If we assume that the utility function is linearly homogeneous, then using (30), Shephard's Lemma (36) becomes:

$$(39)\ q^t = u^t\ \nabla_p c(p^t)\ ; \qquad t = 0,1.$$

Equations (31) can be rewritten as follows:

$$(40)\ p^t\cdot q^t = c(p^t)f(q^t) = c(p^t)u^t\ ; \qquad t = 0,1.$$

Dividing equations (39) by equation (40), we obtain the following system of equations:

$$(41)\ q^t/p^t\cdot q^t = \nabla c(p^t)/c(p^t)\ ; \qquad t = 0,1.$$

Note the symmetry of equations (35) with equations (41). It is these two sets of equations that we shall use in sections 5-7 below.

5. Superlative Indexes I: The Fisher Ideal Index

Recall that the Fisher price index, $P_F(p^0,p^1,q^0,q^1)$, was defined by (13). The companion *Fisher quantity index*, $Q_F(p^0,p^1,q^0,q^1)$, can be defined as the expenditure ratio for the two periods, $p^1 \cdot q^1/p^0 \cdot q^0$, divided by the price index, $P_F(p^0,p^1,q^0,q^1)$:[24]

$$(42)\ Q_F(p^0,p^1,q^0,q^1) \equiv [p^1 \cdot q^1/p^0 \cdot q^0]/P_F(p^0,p^1,q^0,q^1) = [p^0 \cdot q^1 p^1 \cdot q^1/p^0 \cdot q^0 p^1 \cdot q^0]^{1/2}.$$

Suppose the consumer has the following utility function:

$$(43)\ \ f(q) \equiv [q^TAq]^{1/2}\ ;\ \ A = A^T;\ \ q \in S$$

where $A \equiv [a_{ij}]$ is an N by N symmetric matrix that has one positive eigenvalue (that has a strictly positive eigenvector) and the remaining $N-1$ eigenvalues are zero or negative. The set S is the *region of regularity* where the function f is positive, concave and increasing and hence f can provide a valid representation of preferences over this region. It can be shown[25] that the region of regularity can be defined as follows:

$$(44)\ S \equiv \{q : Aq >> 0_N\ ;\ q >> 0_N\}.$$

Differentiating the f(q) defined by (43) for $q \in S$ leads to the following vector of first order partial derivatives:

$$(45)\ \nabla f(q) = Aq/[q^TAq]^{1/2} = Aq/f(q)$$

where the second equation in (45) follows using (43). We assume that the consumer minimizes the cost of achieving the utility level $u^t = f(q^t)$ for periods t = 0,1 and the observed period t quantity vector q^t belongs to the regularity region S for both periods. Evaluate (45) at $q = q^t$ and divide both sides of the resulting equation by $f(q^t)$. We obtain the following equations:

$$(46)\ \nabla f(q^t)/f(q^t) = Aq^t/f(q^t)^2 = p^t/p^t \cdot q^t\ ; \qquad t = 0,1$$

where the second set of equations in (46) follows using Wold's Identity, (35).

Now use definition (42) for the Fisher ideal quantity index, Q_F:

$$\begin{aligned}(47)\ Q_F(p^0,p^1,q^0,q^1) &= [p^0 \cdot q^1 p^1 \cdot q^1/p^0 \cdot q^0 p^1 \cdot q^0]^{1/2} \\ &= [(p^0/p^0 \cdot q^0) \cdot q^1/(p^1/p^1 \cdot q^1) \cdot q^0]^{1/2} \\ &= [\{q^{0T}A^Tq^1/f(q^0)^2\}/\{q^{1T}A^Tq^0/f(q^1)^2\}]^{1/2} && \text{using (46)} \\ &= [f(q^1)^2/f(q^0)^2]^{1/2} && \text{using } A = A^T \\ &= f(q^1)/f(q^0).\end{aligned}$$

[24] Given either a price index P or a quantity index Q, then a matching index can be defined using the equation $P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1) = p^1 \cdot q^1/p^0 \cdot q^0$. Frisch (1930; 399) called this equation the *product test*. The concept of this test is due to Fisher (1911; 321).

[25] See Diewert and Hill (2009).

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function defined by (43), the Fisher ideal quantity index Q_F is *exactly* equal to the true quantity index, $f(q^1)/f(q^0)$.[26]

Let c(p) be the unit cost function that corresponds to the homogeneous quadratic utility function f defined by (43). Then using (31) and (32), it can be seen that

(48) $P_F(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0)$.

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function defined by (43), the Fisher ideal price index P_F is *exactly* equal to the true price index, $c(p^1)/c(p^0)$.[27]

A twice continuously differentiable function f(q) of N variables q can provide a *second order approximation* to another such function $f^*(q)$ around the point q^* if the level and all of the first and second order partial derivatives of the two functions coincide at q^*. It can be shown[28] that the homogeneous quadratic function f defined by (43) can provide a second order approximation to an arbitrary f^* around any (strictly positive) point q^* in the class of twice continuously differentiable linearly homogeneous functions. Thus the homogeneous quadratic functional form defined by (43) is a *flexible functional form*.[29] Diewert (1976; 117) termed an index number formula $Q_F(p^0,p^1,q^0,q^1)$ that was *exactly* equal to the true quantity index $f(q^1)/f(q^0)$ (where f is a flexible functional form) *a superlative index number formula*.[30] Equation (47) and the fact that the homogeneous quadratic function f defined by (43) is a flexible functional form shows that the Fisher ideal quantity index Q_F is a superlative index number formula. Since the Fisher ideal price index P_F also satisfies (48) where c(p) is the unit cost function that is generated by the homogeneous quadratic utility function, we also call P_F a superlative index number formula.

It is possible to show that the Fisher ideal price index is a superlative index number formula by a different route. Instead of starting with the assumption that the consumer's utility function is the homogeneous quadratic function defined by (43), we can start with the assumption that the consumer's unit cost function is a homogeneous quadratic. Thus we suppose that the consumer has the following unit cost function:

(49) $c(p) \equiv [p^TBp]^{1/2}$; $B = B^T$; $p \in S^*$

[26] This result was first derived by Konüs and Byushgens (1926). For the early history of this result, see Diewert (1976; 116).
[27] We also require the assumption that q^0 and q^1 belong to the regularity region S defined by (44).
[28] See Diewert (1976; 130) and let the parameter r equal 2.
[29] Diewert (1974; 133) introduced this term to the economics literature.
[30] Fisher (1922; 247) used the term superlative to describe the Fisher ideal price index. Thus Diewert adopted Fisher's terminology but attempted to give some precision to Fisher's definition of superlativeness. Fisher defined an index number formula to be superlative if it approximated the corresponding Fisher ideal results using his data set.

where $B \equiv [b_{ij}]$ is an N by N symmetric matrix that has one positive eigenvalue (that has a strictly positive eigenvector) and the remaining $N-1$ eigenvalues are zero or negative. The set S^* is the *price region of regularity* where the function c is positive, concave and increasing and hence c can provide a valid representation of preferences over this region. It can be shown that the region of regularity can be defined as follows:[31]

(50) $S^* \equiv \{p : Bp >> 0_N ; p >> 0_N\}$.

Differentiating the c(p) defined by (49) for $p \in S^*$ leads to the following vector of first order partial derivatives:

(51) $\nabla c(p) = Bq/[p^T Bp]^{1/2} = Bp/c(p)$

where the second equation in (51) follows using (49). We assume that p^0 and p^1 both belong to the regularity region of prices defined by (50). Now evaluate the second equation in (51) at the observed period t price vector p^t and divide both sides of the resulting equation by $c(p^t)$. We obtain the following equations:

(52) $\nabla c(p^t)/c(p^t) = Bp^t/c(p^t)^2 = q^t/p^t \cdot q^t$; $\qquad t = 0,1$

where the second set of equations in (52) follows using Shephard's Lemma, equations (41). Now recall the definition of the Fisher ideal price index, P_F, given by (13) above:

$$
\begin{aligned}
(53)\ P_F(p^0,p^1,q^0,q^1) &= [p^1 \cdot q^0 p^1 \cdot q^1/p^0 \cdot q^0 p^0 \cdot q^1]^{1/2} \\
&= [p^1 \cdot (q^0/p^0 \cdot q^0)/p^0 \cdot (q^1/p^1 \cdot q^1)]^{1/2} \\
&= [p^1 \cdot \{Bp^0/c(p^0)^2\}/p^0 \cdot \{Bp^1/c(p^1)^2\}]^{1/2} && \text{using (52)} \\
&= [c(p^1)^2/c(p^0)^2]^{1/2} && \text{using } B = B^T \\
&= c(p^1)/c(p^0).
\end{aligned}
$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the unit cost function defined by (49), the Fisher ideal price index P_F is *exactly* equal to the true price index, $c(p^1)/c(p^0)$.[32]

Since the homogeneous quadratic unit cost function c(p) defined by (49) is also a flexible functional form, the fact that the Fisher ideal price index P_F exactly equals the true price index $c(p^1)/c(p^0)$ means that P_F is a *superlative index number formula*.[33]

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[31] See Diewert and Hill (2009) for the details and see Blackorby and Diewert (1979) for local duality theorems.

[32] This result was obtained by Diewert (1976; 133-134). We also require the assumption that p^0 and p^1 belong to the regularity region S^*.

[33] Note that we have shown that the Fisher index P_F is exact for the preferences defined by (43) as well as the preferences that are dual to the unit cost function defined by (49). These two classes of preferences do not coincide in general. However, if the N by N symmetric matrix A has an inverse, then it can be shown the corresponding unit cost function is equal to $c(p) \equiv (p^T A^{-1} p)^{1/2} = (p^T Bp)^{1/2}$ where $B \equiv A^{-1}$.

Suppose that the B matrix in (49) is equal to the following matrix of rank 1:

(54) $B \equiv bb^T$; $b >> 0_N$

where b is an N by 1 vector with strictly positive components. In this case, it can be verified that the region of regularity is the entire positive orthant. Note that the cost function defined by (49) simplifies in this case:

(55) $c(p) \equiv [p^TBp]^{1/2} = \equiv [p^Tbb^Tp]^{1/2} = b^Tp = b\cdot p$.

Substituting (55) into Shephard's Lemma (39) yields the following expressions for the period t quantity vectors, q^t:

(56) $q^t = u^t\nabla_p c(p^t) = bu^t$; $t = 0,1.$

Thus if the consumer has the preferences that correspond to the unit cost function defined by (49) where B satisfies the restrictions (54), then the period 0 and 1 quantity vectors are equal to a multiple of the vector b; i.e., $q^0 = bu^0$ and $q^1 = bu^1$. Under these assumptions, the Fisher, Paasche and Laspeyres indices, P_F, P_P and P_L, *all coincide.* However, the (Leontief fixed coefficient) preferences which correspond to the unit cost function defined by (59) and (54) are not consistent with normal consumer behavior since they imply that the consumer will not substitute away from more expensive commodities to cheaper commodities if relative prices change going from period 0 to 1.

6. Superlative Indexes II: Quadratic Mean of Order r Indexes

It turns out that there are many other superlative index number formulae; i.e., there exist many quantity indexes $Q(p^0,p^1,q^0,q^1)$ that are exactly equal to $f(q^1)/f(q^0)$ and many price indexes $P(p^0,p^1,q^0,q^1)$ that are exactly equal to $c(p^1)/c(p^0)$ where the aggregator function f or the unit cost function c is a flexible functional form. We will define two families of superlative indexes below.

Suppose that the consumer has the *following quadratic mean of order r utility function*:[34]

(57) $f^r(q_1,\ldots,q_N) \equiv [\sum_{i=1}^N\sum_{k=1}^N a_{ik}\, q_i^{r/2}\, q_k^{r/2}]^{1/r}$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. The regularity region where f^r is positive, concave and increasing is defined as follows:

(58) $S \equiv \{q : q >> 0_N ; \nabla f^r(q) >> 0_N ; \nabla^2 f^r(q)$ is negative semidefinite$\}$

[34] This terminology is due to Diewert (1976; 129).

where $\nabla^2 f^r(q)$ is the matrix of second order partial derivatives of f^r evaluated at q. Diewert (1976; 130) showed that the utility function f^r defined by (57) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order.[35] Note that when r = 2, f^r equals the homogeneous quadratic function defined by (43) above.

Define the *quadratic mean of order r quantity index* Q^r by:

$$(59)\quad Q^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^N s_i^0\, (q_i^1/q_i^0)^{r/2}\}^{1/r}\, \{\sum_{i=1}^N s_i^1\, (q_i^1/q_i^0)^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^t q_i^t / \sum_{k=1}^N p_k^t q_k^t$ is the period t expenditure share for commodity i. It can be verified that when r = 2, Q^r simplifies into Q_F, the Fisher ideal quantity index.

Using exactly the same techniques as were used in section 5 above, it can be shown that Q^r is exact for the aggregator function f^r defined by (57); i.e., we have

$$(60)\quad Q^r(p^0,p^1,q^0,q^1) = f^r(q^1)/f^r(q^0).$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function defined by (57),[36] the quadratic mean of order r quantity index Q_F is *exactly* equal to the true quantity index, $f^r(q^1)/f^r(q^0)$.[37] Since Q^r is exact for f^r and f^r is a flexible functional form, we see that the quadratic mean of order r quantity index Q^r is a *superlative index* for each $r \neq 0$. Thus there are an infinite number of superlative quantity indexes.

For each quantity index Q^r, we can use the counterpart to (42) (that the product of the price and quantity index must equal the value ratio) in order to define the corresponding *implicit quadratic mean of order r price index* P^{r*}:

$$\begin{aligned}(61)\quad P^{r*}(p^0,p^1,q^0,q^1) &\equiv p^1\cdot q^1/\{p^0\cdot q^0 Q^r(p^0,p^1,q^0,q^1)\}\\ &= c^{r*}(p^1)/c^{r*}(p^0)\end{aligned}$$

where c^{r*} is the unit cost function that corresponds to the aggregator function f^r defined by (57) above. For each $r \neq 0$, the implicit quadratic mean of order r price index P^{r*} is also a superlative index.

When r = 2, Q^r defined by (59) simplifies to Q_F, the Fisher ideal quantity index and P^{r*} defined by (61) simplifies to P_F, the Fisher ideal price index. When r = 1, Q^r defined by (59) simplifies to:

[35] This result holds for any predetermined $r \neq 0$; i.e., we require only the N(N+1)/2 independent a_{ik} parameters in order to establish the flexibility of f^r in the class of linearly homogeneous aggregator functions.

[36] We also require that q^0 and q^1 belong to the regularity region S defined by (58).

[37] See Diewert (1976; 130).

$$
\begin{aligned}
(62)\ Q^1(p^0,p^1,q^0,q^1) &\equiv \{\textstyle\sum_{i=1}^N s_i^0(q_i^1/q_i^0)^{1/2}\}/\{\sum_{i=1}^N s_i^1(q_i^1/q_i^0)^{-1/2}\} \\
&= \{\textstyle\sum_{i=1}^N [p_i^0q_i^0/p^0\cdot q^0](q_i^1/q_i^0)^{1/2}\}/\{\sum_{i=1}^N [p_i^1q_i^1/p^1\cdot q^1](q_i^1/q_i^0)^{-1/2}\} \\
&= \{\textstyle\sum_{i=1}^N p_i^0(q_i^0q_i^1)^{1/2}/p^0\cdot q^0\}/\{\sum_{i=1}^N p_i^1(q_i^0q_i^1)^{1/2}/p^1\cdot q^1\} \\
&= [p^1\cdot q^1/p^0\cdot q^0]/P_W(p^0,p^1,q^0,q^1)
\end{aligned}
$$

where P_W is the *Walsh* (1901; 398) (1921; 97) *price index*. Thus P^{1*} is equal to P_W, the *Walsh price index*, and hence it is also a superlative price index.

Suppose the consumer has the *following quadratic mean of order r unit cost function*:[38]

$$(63)\ \ c^r(p_1,\ldots,p_N) \equiv [\textstyle\sum_{i=1}^N\sum_{k=1}^N b_{ik}\, p_i^{r/2}\, p_k^{r/2}\,]^{1/r}$$

where the parameters b_{ik} satisfy the symmetry conditions $b_{ik} = b_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976; 130) showed that the unit cost function c^r defined by (63) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order. Note that when $r = 2$, c^r equals the homogeneous quadratic unit cost function defined by (49) above. The price regularity region for c^r is defined as follows:

$$(64)\ S^* \equiv \{p : p >> 0_N\ ;\ \nabla c^r(p) >> 0_N\ ;\ \nabla^2 c^r(p) \text{ is negative semidefinite}\}.$$

Define the *quadratic mean of order r price index* P^r by:

$$(65)\ \ P^r(p^0,p^1,q^0,q^1) \equiv \{\textstyle\sum_{i=1}^N s_i^0(p_i^1/p_i^0)^{r/2}\}^{1/r}\ \{\sum_{i=1}^N s_i^1(p_i^1/p_i^0)^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^tq_i^t/\sum_{k=1}^N p_k^tq_k^t$ is the period t expenditure share for commodity i as usual. It can be verified that when $r = 2$, P^r simplifies into P_F, the Fisher ideal quantity index.

Using exactly the same techniques as were used in section 5 above and using the counterparts to (51) and (52), it can be shown that P^r is exact for the unit cost function c^r defined by (63); i.e., we have

$$(66)\ \ P^r(p^0,p^1,q^0,q^1) = c^r(p^1)/c^r(p^0).$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that are dual to the unit cost function defined by (63), the quadratic mean of order r price index P^r is *exactly* equal to the true price index, $c^r(p^1)/c^r(p^0)$.[39] Since P^r is exact for c^r and c^r is a flexible functional form, we see that the quadratic mean of order r price index P^r is a *superlative index* for each $r \neq 0$. Thus there are an infinite number of superlative price indexes.

[38] This terminology is due to Diewert (1976; 130). This unit cost function was first defined by Denny (1974).

[39] See Diewert (1976; 133-134).

For each price index P^r, we can use the product test in order to define the corresponding *implicit quadratic mean of order r quantity index* Q^{r*}:

$$(67)\quad Q^{r*}(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/\{p^0\cdot q^0\, P^r(p^0,p^1,q^0,q^1)\} = f^{r*}(q^1)/f^{r*}(q^0)$$

where f^{r*} is the aggregator function that is dual to the unit cost function c^r defined by (63) above. For each $r \neq 0$, the implicit quadratic mean of order r quantity index Q^{r*} is also a superlative index.

In this section, we have exhibited two families of superlative price and quantity indexes, Q^r and P^{r*} defined by (59) and (61), and P^r and Q^{r*} defined by (65) and (67) for each $r \neq 0$. A natural question to ask at this point is: how different will these indexes be? It is possible to show that all of the price indexes P^r and P^{r*} approximate each other to the second order around any point where the price vectors p^0 and p^1 are equal and where the quantity vectors q^0 and q^1 are equal; i.e., we have the following equalities if the derivatives are evaluated at $p^0 = p^1$ and $q^0 = q^1$ for any r and s not equal to 0:[40]

$$(68)\quad P^r(p^0,p^1,q^0,q^1) = P^s(p^0,p^1,q^0,q^1) = P^{r*}(p^0,p^1,q^0,q^1) = P^{s*}(p^0,p^1,q^0,q^1)\,;$$
$$(69)\quad \nabla P^r(p^0,p^1,q^0,q^1) = \nabla P^s(p^0,p^1,q^0,q^1) = \nabla P^{r*}(p^0,p^1,q^0,q^1) = \nabla P^{s*}(p^0,p^1,q^0,q^1)\,;$$
$$(70)\quad \nabla^2 P^r(p^0,p^1,q^0,q^1) = \nabla^2 P^s(p^0,p^1,q^0,q^1) = \nabla^2 P^{r*}(p^0,p^1,q^0,q^1) = \nabla^2 P^{s*}(p^0,p^1,q^0,q^1).$$

A similar set of equalities holds for the companion quantity indexes, Q^r and Q^{s*} for any r and s not equal to 0. The implication of the above equalities is that if prices and quantities do not change much over the two periods being compared, then all of the mean of order r price indexes will give much the same answer and so will all of the mean of order r quantity indexes.

For an empirical comparisons of some of the above indexes, see Diewert (1978; 894-895) and Hill (2006). Unfortunately, Hill (2006) showed that the second order approximation property of the mean of order r indexes breaks down as r approaches plus or minus infinity. However, in most empirical applications, we generally choose r equal to 2 (the Fisher case) or 1 (the Walsh indexes). For these cases, the resulting indexes generally approximate each other very closely.[41]

7. Superlative Indexes III: Normalized Quadratic Indexes

In addition to the family of quadratic means of order r indexes, there is another family of superlative indexes which we will exhibit in the present section.

[40] The proof is a straightforward differentiation exercise; see Diewert (1978; 889). In fact, these derivative equalities are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$.

[41] The approximations will be close if we are using annual time series data where price and quantity changes are generally smooth. However, if we are making international comparisons or using panel data or using subannual time series data, then the approximations may not be close.

Suppose that a consumer has preferences that are dual to the *normalized quadratic unit cost function* defined as follows:[42]

(71) $c(p) \equiv p^Tb + (1/2)p^TAp/\alpha^Tp$; $\qquad p \gg 0_N$; $\alpha > 0_N$; $A = A^T$;
(72) A is negative semidefinite[43] ;

where p is a positive vector of commodity prices that the consumer faces and the vectors b and α are parameter vectors and the symmetric matrix A is a matrix of parameters.

Let $p^* \gg 0_N$ be a reference commodity price vector. In addition to the restrictions in (71) and (72), we can impose the following restrictions on c:

(73) $Ap^* = 0_N$.

If the restrictions on A given by (73) are satisfied, then it is straightforward to show that we have the following expressions for the first and second order partial derivatives of c evaluated at $p = p^*$:

(74) $\nabla c(p^*) = b$;
(75) $\nabla^2 c(p^*) = A/\alpha^Tp^*$.

Proposition 3: Let α be an arbitrary predetermined vector which satisfies $\alpha > 0_N$. Conditional on this predetermined α, the c(p) defined by (71), (72) and (73) is *flexible* at the point of approximation p^*; i.e., there exists a b vector and an A matrix satisfying (73) such that the following equations are satisfied:

(76) $\quad c(p^*) = c^*(p^*)$;
(77) $\nabla c(p^*) = \nabla c^*(p^*)$;
(78) $\nabla^2 c(p^*) = \nabla^2 c^*(p^*)$

where $c^*(p)$ is an arbitrary twice continuously differentiable, linearly homogeneous, increasing and concave function of p defined for $p \gg 0_N$.

Proof: Substitute (75) into (78) and solve the resulting equation for A:

(79) $A = \alpha^Tp^*\nabla^2 c^*(p^*)$.

Note that $\alpha > 0_N$ and $p^* \gg 0_N$ implies $\alpha^Tp^* > 0$. Since c^* is concave, it must be the case that $\nabla^2 c^*(p^*)$ is a negative semidefinite symmetric matrix. Also, the linear homogeneity

[42] This function was introduced in the producer context by Diewert and Wales (1987; 53) and applied by Diewert and Wales (1992) and Diewert and Lawrence (2002) in this context and by Diewert and Wales (1988a) (1988b) (1993) in the consumer context. The advantages of this flexible functional form are explained in Diewert and Wales (1993).
[43] Diewert and Wales (1987; 66) show that this condition is necessary and sufficient for c(p) to be concave in p.

of c^* implies via Euler's Theorem on homogeneous functions that the following restrictions are satisfied:

(80) $\nabla^2 c^*(p^*)p^* = 0_N$.

Thus the A defined by (79) is negative semidefinite and satisfies the restrictions (73). Now substitute (74) into (77) and we obtain the following equation:

(81) $b = \nabla c^*(p^*)$.

(79) and (81) determine A and b and it can be seen that equations (77) and (78) are satisfied. The final equation that we need to satisfy to prove the flexibility of c(p) is (76) but this equation is implied by (77) and another Euler Theorem on homogeneous functions:

(82) $c(p^*) = p^*\cdot\nabla c(p^*)$ and $c^*(p^*) = p^*\cdot\nabla c^*(p^*)$. Q.E.D.

We note that there are N free b_n parameters in the b vector and N(N−1)/2 free a_{ij} parameters in the A matrix, taking into account the symmetry restrictions on A and the restrictions (73). This is a total of N(N+1)/2 free parameters, which is the minimal number of free parameters that is required for a linearly homogeneous c(p) to be flexible. Thus the normalized quadratic unit cost function defined by (71)-(73) is a *parsimonious flexible functional form*. In what follows, we do not need to impose the restrictions (73).

The region of regularity for the normalized quadratic unit cost function is the following region:

(83) $S^* \equiv \{p : p >> 0_N;\ \nabla c(p) = b + (\alpha^T p)^{-1}Ap - (\alpha^T p)^{-2}App^TA >> 0_N\}$.

Suppose that a consumer has preferences that can be represented by a normalized quadratic expenditure function, C(u,p) equal to uc(p) where c(p) is defined by (71) and (72). Suppose further that the prices that the consumer faces in periods 0 and 1, p^0 and p^1, are in the regularity region defined by (83) and the corresponding quantity vectors, q^t, are equal to $\nabla_p C(u^t,p^t)$ for t = 0,1 (Shephard's Lemma) where $u^0 > 0$ and $u^1 > 0$ are the utility levels that the consumer attains for the two periods. Then Shephard's Lemma gives us the following two equations:

(84) $q^0 = [b + (\alpha^T p^0)^{-1}Ap^0 - (1/2)(\alpha^T p^0)^{-2}p^{0T}Ap^0\alpha]u^0$;
(85) $q^1 = [b + (\alpha^T p^1)^{-1}Ap^1 - (1/2)(\alpha^T p^1)^{-2}p^{1T}Ap^1\alpha]u^1$.

We now derive an exact index number formula that will enable us to calculate the utility ratio u^1/u^0 using just the observable price and quantity data for the two situations, p^0,p^1,q^0,q^1 and the parameter vector α (which is assumed to be known to us).

Premultiply both sides of (84) and (85) by the transpose of the price vector $(\alpha^T p^0)p^1 + (\alpha^T p^1)p^0$. After some simplification, we obtain the following formulae:

(86) $[(\alpha^Tp^0)p^1 + (\alpha^Tp^1)p^0]^Tq^0 = \{[(\alpha^Tp^0)p^1 + (\alpha^Tp^1)p^0]^Tb + p^{1T}Ap^0\}u^0$;
(87) $[(\alpha^Tp^0)p^1 + (\alpha^Tp^1)p^0]^Tq^1 = \{[(\alpha^Tp^0)p^1 + (\alpha^Tp^1)p^0]^Tb + p^{0T}Ap^1\}u^1$.

Since A is symmetric, $p^{1T}Ap^0 = [p^{1T}Ap^0]^T = p^{0T}A^Tp^1 = p^{0T}Ap^1$, and hence, we have:[44]

(88) $u^1/u^0 = [(\alpha^Tp^0)p^1 + (\alpha^Tp^1)p^0]^Tq^1/[(\alpha^Tp^0)p^1 + (\alpha^Tp^1)p^0]^Tq^0 \equiv Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$

where $Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$ is the *normalized quadratic quantity index*.[45] Thus if we know α, $Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$ can be calculated using only observable price and quantity data pertaining to the two situations being considered and (88) tells us that this quantity index is equal to the utility ratio u^1/u^0, which is equal to $f(q^1)/f(q^0)$ where f is the linearly homogeneous utility function that is dual to the expenditure function defined by (71)-(72). Thus $Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$ is a *superlative index number formula* since $Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$ is exactly equal to the utility ratio $f(q^1)/f(q^0)$ where f is dual to a flexible functional form for a unit cost function.

It is possible to rewrite (88) in a more intuitive form. Define the period t real prices or normalized prices ρ^t as the nominal period t prices p^t divided by the period t fixed weight price index (with fixed quantity weights α), $p^t\cdot\alpha$:

(89) $\rho^t \equiv p^t/p^t\cdot\alpha$; $\qquad t = 0,1.$

Now divide the numerator and denominator in (88) by $\alpha^Tp^0\alpha^Tp^1$ and we obtain the following expressions for $Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$:

(90) $Q_{NQ}(p^0,p^1,q^0,q^1;\alpha) = [\rho^0 + \rho^1]\cdot q^1/[\rho^0 + \rho^1]\cdot q^0 = [(½)\rho^0 + (½)\rho^1]\cdot q^1/[(½)\rho^0 + (½)\rho^1]\cdot q^0$.

Thus utility in period t, $f(q^t)$, can be set equal to $[(½)\rho^0 + (½)\rho^1]\cdot q^t$, the inner product of the arithmetic average of the real prices pertaining to the two periods, $(½)\rho^0 + (½)\rho^1$, and the period t quantity vector q^t. Thus we have an additive superlative quantity index![46]

The price index $P_{NQ}(p^0,p^1,q^0,q^1;\alpha)$ that corresponds to the normalized quadratic quantity index defined by (88), $Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$, is defined using the product test as follows:

(91) $P_{NQ}(p^0,p^1,q^0,q^1;\alpha) \equiv p^1\cdot q^1/p^0\cdot q^0Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$.

Since the vector α could be any nonnegative, nonzero vector, there is nothing to prevent us from setting α equal to q^0 or q^1. We will consider these two special cases in turn.

Case 1: $\alpha = q^0$:

[44] This result was obtained by Diewert (1992b; 576).
[45] Diewert (1992b; 576) introduced this index to the economics literature.
[46] The Walsh quantity index, $Q_W(p^0,p^1,q^0,q^1) \equiv [\sum_{n=1}^N (p_n^0p_n^1)^{1/2}q_n^1]/[\sum_{n=1}^N (p_n^0p_n^1)^{1/2}q_n^0] = Q^{1*}(p^0,p^1,q^0,q^1)$, also has this additivity property.

Replacing α by q^0 in (88) leads to the following special case for the normalized quadratic quantity index:

$$\begin{aligned}(92)\ Q_{NQ}(p^0,p^1,q^0,q^1;q^0) &= [(q^0\cdot p^0)p^1 + (q^0\cdot p^1)p^0]\cdot q^1/[(q^0\cdot p^0)p^1 + (q^0\cdot p^1)p^0]\cdot q^0 \\ &= [(q^0\cdot p^0)p^1\cdot q^1 + (q^0\cdot p^1)p^0\cdot q^1]/2(p^0\cdot q^0)(p^1\cdot q^0) \\ &= (1/2)[p^1\cdot q^1/p^1\cdot q^0] + (1/2)[p^0\cdot q^1/p^0\cdot q^0] \\ &= (1/2)Q_P(p^0,p^1,q^0,q^1) + (1/2)Q_L(p^0,p^1,q^0,q^1)\end{aligned}$$

where $Q_L \equiv p^0\cdot q^1/p^0\cdot q^0$ and $Q_P \equiv p^1\cdot q^1/p^1\cdot q^0$ are the Laspeyres and Paasche quantity indexes. Thus when the parameter vector α is equal to q^0, the normalized quadratic quantity index reduces to the arithmetic average of the Paasche and Laspeyres quantity indexes and this index is superlative, which is a new result.

The price index $P_{NQ}(p^0,p^1,q^0,q^1;q^0)$ which corresponds to the normalized quadratic quantity index defined by (92), $Q_{NQ}(p^0,p^1,q^0,q^1;q^0)$, can be defined as follows using (91):

$$\begin{aligned}(93)\ P_{NQ}(p^0,p^1,q^0,q^1;q^0) &\equiv p^1\cdot q^1/p^0\cdot q^0 Q_{NQ}(p^0,p^1,q^0,q^1;q^0) \\ &= p^1\cdot q^1/p^0\cdot q^0\{(1/2)[p^1\cdot q^1/p^1\cdot q^0] + (1/2)[p^0\cdot q^1/p^0\cdot q^0]\} \\ &= \{(1/2)[p^0\cdot q^0/p^1\cdot q^0] + (1/2)[p^0\cdot q^1/p^1\cdot q^1]\}^{-1} \\ &= \{(1/2)[p^1\cdot q^0/p^0\cdot q^0]^{-1} + (1/2)[p^1\cdot q^1/p^0\cdot q^1]^{-1}\}^{-1} \\ &= \{(1/2)[P_L]^{-1} + (1/2)[P_P]^{-1}\}^{-1}.\end{aligned}$$

Thus the superlative price index $P_{NQ}(p^0,p^1,q^0,q^1;q^0)$ which matches up with the normalized quadratic quantity index $Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$ when we choose α equal to q^0 is the *harmonic mean of the Paasche and Laspeyres price indexes*, which were defined in (3) and (4) above.[47]

Case 2: $\alpha = q^1$:

Replacing α by q^1 in (88) leads to the following special case for the normalized quadratic quantity index:

$$\begin{aligned}(94)\ Q_{NQ}(p^0,p^1,q^0,q^1;q^1) &= [(q^1\cdot p^0)p^1 + (q^1\cdot p^1)p^0]\cdot q^1/[(q^1\cdot p^0)p^1 + (q^1\cdot p^1)p^0]\cdot q^0 \\ &= 2p^1\cdot q^1p^0\cdot q^1/\{p^0\cdot q^1p^1\cdot q^0 + p^1\cdot q^1p^0\cdot q^0\} \\ &= \{(1/2)[p^1\cdot q^0/p^1\cdot q^1] + (1/2)[p^0\cdot q^0/p^0\cdot q^1]\}^{-1} \\ &= \{(1/2)[p^1\cdot q^1/p^1\cdot q^0]^{-1} + (1/2)[p^0\cdot q^1/p^0\cdot q^0]^{-1}\}^{-1} \\ &= \{(1/2)Q_P(p^0,p^1,q^0,q^1)^{-1} + (1/2)Q_L(p^0,p^1,q^0,q^1)^{-1}\}^{-1}\end{aligned}$$

where Q_L and Q_P are the Laspeyres and Paasche quantity indexes. Thus when the parameter vector α is equal to q^1, the normalized quadratic quantity index reduces to the

[47] This Harmonic Mean Price Index is mentioned by Fisher (1922; 487) (his formula number 8054) and Balk (2008; 67).

harmonic average of the Paasche and Laspeyres quantity indexes, which is a superlative index.

The price index $P_{NQ}(p^0,p^1,q^0,q^1;q^1)$ which corresponds to the normalized quadratic quantity index defined by (94), $Q_{NQ}(p^0,p^1,q^0,q^1;q^1)$, can be defined as follows using (91):

$$(95)\ P_{NQ}(p^0,p^1,q^0,q^1;q^1) \equiv p^1\cdot q^1/p^0\cdot q^0 Q_{NQ}(p^0,p^1,q^0,q^1;q^1)$$
$$= \{p^1\cdot q^1/p^0\cdot q^0\}\{(1/2)[p^1\cdot q^0/p^1\cdot q^1] + (1/2)[p^0\cdot q^0/p^0\cdot q^1]\}$$
$$= (1/2)[p^1\cdot q^0/p^0\cdot q^0] + (1/2)[p^1\cdot q^1/p^0\cdot q^1]$$
$$= (1/2)P_L + (1/2)P_P$$
$$= P_{SB}(p^0,p^1,q^0,q^1)$$

where $P_{SB}(p^0,p^1,q^0,q^1)$ is the Sidgwick Bowley price index defined by (12). Thus the price index $P_{NQ}(p^0,p^1,x^0,q^1;q^1)$ which matches up with the normalized quadratic quantity index $Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$ when we choose α equal to q^1 is the *arithmetic mean of the Paasche and Laspeyres price indexes*, which is a new result.

As in the previous section, we can ask how different are the various normalized quadratic quantity indexes, $Q_{NQ}(p^0,p^1,q^0,q^1;\alpha)$, as the predetermined vector $\alpha > 0_N$ changes. Again, a straightforward differentiation exercise shows that all of these indexes approximate each other to the second order around an equal price (i.e., $p^0 = p^1$) and equal quantity (i.e., $q^0 = q^1$) point. They also approximate all of the mean of order r quantity indexes, $Q^r(p^0,p^1,q^0,q^1)$ and $Q^{r*}(p^0,p^1,q^0,q^1)$, to the second order around an equal price and equal quantity point.[48] Thus for "normal" data sets that do not fluctuate too violently, all of these superlative indexes will approximate each other reasonably closely.

The theory of superlative indexes presented in sections 5-7 provide reasonable methods for aggregation over commodities when the task at hand is to form subindexes. However, these techniques are not suitable for forming overall cost of living indexes or overall quantity indexes when we deal with broad consumer aggregates, because the assumption of homothetic preferences is not likely to be satisfied. Thus in the following sections, we look for methods of aggregation that do not depend on the homotheticity assumption.

8. Nonhomothetic Preferences and Cost of Living Indexes

Before we derive our main results, we require some preliminary results. Suppose the function of N variables, $f(z_1,\ldots,z_N) \equiv f(z)$, is quadratic; i.e.,

$$(96)\ f(z) \equiv a_0 + a^Tz + (1/2)\, z^TAz\ ;\ A = A^T$$

where a is a vector of parameters and A is a symmetric matrix of parameters. It is well known that the second order Taylor series approximation to a quadratic function is *exact*; i.e., if f is defined by (96) above, then for any two points, z^0 and z^1, we have

[48] Diewert (1992b; 578) noted this result.

(97) $f(z^1) - f(z^0) = \nabla f(z^0)^T(z^1 - z^0) + (1/2)(z^1-z^0)^T\nabla^2 f(z^0)(z^1 - z^0)$.

It is less well known that *an average of two first order Taylor series approximations* to a quadratic function is also *exact*; i.e., if f is defined by (96) above, then for any two points, z^0 and z^1, we have[49]

(98) $f(z^1) - f(z^0) = (1/2)[\nabla f(z^0) + \nabla f(z^1)]^T[z^1-z^0]$.

Diewert (1976; 118) and Lau (1979) showed that equation (98) characterized a quadratic function and called the equation the *quadratic approximation lemma*. We will refer to (98) as the *quadratic identity*.

We now suppose that the consumer's cost function, C(u,p), has the following *translog functional form*:[50]

$$(99)\quad \ln C(u,p) \equiv a_0 + \sum_{i=1}^N a_i \ln p_i + (1/2) \sum_{i=1}^N \sum_{k=1}^N a_{ik} \ln p_i \ln p_k \\ + b_0 \ln u + \sum_{i=1}^N b_i \ln p_i \ln u + (1/2)\, b_{00} [\ln u]^2$$

where ln is the natural logarithm function and the parameters a_i, a_{ik}, and b_i satisfy the following restrictions:

(100) $a_{ik} = a_{ki}$; $\quad i,k = 1,\ldots,N$;
(101) $\sum_{i=1}^N a_i = 1$;
(102) $\sum_{i=1}^N b_i = 0$;
(103) $\sum_{k=1}^N a_{ik} = 0$; $\quad i = 1,\ldots,N$.

The parameter restrictions (100)-(103) ensure that C(u,p) defined by (99) is linearly homogeneous in p. It can be shown that the translog cost function defined by (100)-(103) can provide a second order Taylor series approximation to an arbitrary cost function.[51]

We assume that the consumer engages in cost minimizing behavior during periods 0 and 1 so that equations (1) hold. Applying Shephard's Lemma to the translog cost function leads to the following equations:

(104) $s_i^t = a_i + \sum_{k=1}^N a_{ik} \ln p_k^t + b_i \ln u^t$; $\quad i = 1,\ldots,N$; $t = 0,1$

where as usual, s_i^t is the period t expenditure share on commodity i. Define the geometric average of the period 0 and 1 utility levels as u^*; i.e., define

[49] To prove that (97) and (98) are true, substitute definition (96) and its derivatives into (97) and (98).
[50] Christensen, Jorgenson and Lau (1975) and Diewert (1976) introduced this function into the economics literature.
[51] It can also be shown that if $b_0 = 1$ and all of the $b_i = 0$ for $i = 1,...,N$ and $b_{00} = 0$, then $C(u,p) = uC(1,p) \equiv uc(p)$; i.e., with these additional restrictions on the parameters of the general translog cost function, we have homothetic preferences. Note that we also assume that utility u is scaled so that u is always positive. Finally, we assume that for each of our translog results, the regularity region contains the observed price and quantity data.

(105) $u^* \equiv [u^0u^1]^{1/2}$.

Now observe that the right hand side of the equation that defines the natural logarithm of the translog cost function, equation (99), is a quadratic function of the variables $z_i \equiv \ln p_i$ if we hold utility constant at the level u^*. Hence we can apply the quadratic identity, (98), and get the following equation:

(106) $\ln C(u^*,p^1) - \ln C(u^*,p^0)$

$= (1/2)\sum_{i=1}^N [\partial \ln C(u^*,p^0)/\partial \ln p_i + \partial \ln C(u^*,p^1)/\partial \ln p_i][\ln p_i^1 - \ln p_i^0]$

$= (1/2)\sum_{i=1}^N [a_i+\sum_{k=1}^n a_{ik} \ln p_k^0+b_i \ln u^* + a_i+\sum_{k=1}^N a_{ik} \ln p_k^1+b_i \ln u^*][\ln p_i^1 - \ln p_i^0]$

differentiating (99) at the points (u^*,p^0) and (u^*,p^1)

$= (1/2)\sum_{i=1}^N [a_i+\sum_{k=1}^N a_{ik} \ln p_k^0+b_i\ln[u^0u^1]^{1/2}+a_i+\sum_{k=1}^N a_{ik} \ln p_k^1+b_i\ln[u^0u^1]^{1/2}][\ln p_i^1 - \ln p_i^0]$

using definition (105) for u^*

$= (1/2)\sum_{i=1}^N [a_i+\sum_{k=1}^N a_{ik} \ln p_k^0+b_i \ln u^0 + a_i+\sum_{k=1}^N a_{ik} \ln p_k^1+b_i \ln u^1][\ln p_i^1 - \ln p_i^0]$

$= (1/2)\sum_{i=1}^N [\partial \ln C(u^0,p^0)/\partial \ln p_i + \partial \ln C(u^1,p^1)/\partial \ln p_i][\ln p_i^1 - \ln p_i^0]$

differentiating (99) at the points (u^0,p^0) and (u^1,p^1)

$= (1/2)\sum_{i=1}^N [s_i^0 + s_i^1][\ln p_i^1 - \ln p_i^0]$ using (104)

$\equiv \ln P_T(p^0,p^1,q^0,q^1)$.

The last equation in (106) defines the logarithm of an observable index number formula, $P_T(p^0,p^1,q^0,q^1)$, which is known as the Törnqvist (1936), Törnqvist and Törnqvist (1937) Theil (1967) price index.[52] Hence exponentiating both sides of (106) yields the following equality between the true cost of living between periods 0 and 1, evaluated at the intermediate utility level u^* and the observable price index P_T:[53]

(107) $C(u^*,p^1)/C(u^*,p^0) = P_T(p^0,p^1,q^0,q^1)$.

Since the translog cost function is a flexible functional form, the Törnqvist-Theil price index P_T is also a *superlative index*.[54] The importance of (107) as compared to our earlier exact index number results is that we no longer have to assume that preferences are homothetic. However, we do have to choose a particular utility level on the left hand side of (107) in order to obtain our new exact result, the geometric mean of u^0 and u^1.

It is somewhat mysterious how a ratio of *unobservable* cost functions of the form appearing on the left hand side of the above equation can be *exactly* estimated by an *observable* index number formula but the key to this mystery is the assumption of cost minimizing behavior and the quadratic identity (98) along with the fact that derivatives of cost functions are equal to quantities, as specified by Shephard's Lemma. In fact, all of the exact index number results derived in sections 5 and 6 can be derived using

[52] See Balk (2008; 26) on the history of this index.

[53] This result is due to Diewert (1976; 122).

[54] Diewert (1978; 888) showed that $P_T(p^0,p^1,q^0,q^1)$ approximates the other superlative indexes P^r and P^{r*} to the second order around an equal price and quantity point.

transformations of the quadratic identity along with Shephard's Lemma (or Wold's identity).[55]

It is possible to generalize the above results using some results in Caves, Christensen and Diewert (1982; 1409-1411). We will conclude this section by explaining those results.

We now assume that in period t, the consumer has the utility function $f^t(q,z^t)$ for t = 0,1, where z^t is a period t vector of *environmental or demographic variables* that affect the consumer's choices over market goods and services, q. Note that we are also allowing for taste changes as we move from period 0 to 1. We assume that $f^t(q,z^t)$ is nonnegative, increasing, continuous and quasiconcave in q for $q \geq 0_N$.

For $p >> 0_N$, and u in the range of $f^t(q,z^t)$, we define the consumer's period t cost function C^t as follows:

$$(108)\ C^t(u,p,z^t) \equiv \min_q \{p^t \cdot q : f^t(q,z^t) = u\}\ ; \qquad t = 0,1.$$

Let q^t be the consumer's observed market consumption vector for period t and define the period t utility level as:

$$(109)\ u^t \equiv f^t(q^t,z^t)\ ; \qquad t = 0,1.$$

Suppose the consumer faces the market price vector p^t in period t for t = 0,1. As usual, we assume that the observed period t consumption vector q^t solves the following *period t cost minimization problem*:

$$(110)\ C^t(u^t,p^t,z^t) \equiv \min_q \{p^t \cdot q : f^t(q,z^t) = u^t\} = p^t \cdot q^t\ ; \qquad t = 0,1.$$

Define a *family of generalized Konüs true cost of living indexes* between periods 0 and 1 as follows:

$$(111)\ P_{CCD}(p^0,p^1,u,z,t) \equiv C^t(u,p^1,z)/C^t(u,p^0,z).$$

Note that all variables are exactly the same in the numerator and denominator on the right hand side of (111), except that the period 1 price vector p^1 appears in the numerator and the period 0 price vector p^0 appears in the denominator. Thus the resulting index is a valid measure of pure price change.

Caves, Christensen and Diewert (1982; 1409-1410) singled out the two natural special cases of (111), where the common variables in the numerator and denominator on the right hand side of (109) are chosen to be the period 0 variables or the period 1 variables:

$$(112)\ P_{CCD}(p^0,p^1,u^0,z^0,0) \equiv C^0(u^0,p^1,z^0)/C^0(u^0,p^0,z^0)\ ;$$
$$(113)\ P_{CCD}(p^0,p^1,u^1,z^1,1) \equiv C^1(u^1,p^1,z^1)/C^1(u^1,p^0,z^1).$$

[55] See Diewert (2002).

It turns out that we will not be able to provide empirical approximations to the individual price indexes defined by (112) and (113) but we will be able to provide an exact index number formula for the geometric mean of these two indexes. In order to accomplish this task, we will require the following generalization of the quadratic identity, (98):

Proposition 4: Let x and y be N and M dimensional vectors respectively and let f^1 and f^2 be two general quadratic functions defined as follows:

$$(114)\ f^1(x,y) \equiv a_0^{\ 1} + a^{1T}x + b^{1T}y + (1/2)x^TA^1x + (1/2)\,y^TB^1y + x^TC^1y\ ;\ A^{1T} = A^1;\ B^{1T} = B^1;$$
$$(115)\ f^2(x,y) \equiv a_0^{\ 2} + a^{2T}x + b^{2T}y + (1/2)x^TA^2x + (1/2)\,y^TB^2y + x^TC^2y\ ;\ A^{2T} = A^2\ ;\ B^{2T} = B^2$$

where the $a_0^{\ i}$ are scalar parameters, the a^i and b^i are parameter vectors and the A^i, B^i and C^i are parameter matrices for i = 1,2. Note that the A^i and B^i are symmetric matrices. If $A^1 = A^2$, then the following equation holds for all x^1, x^2, y^1 and y^2:[56]

$$(116)\ f^1(x^2,y^1) - f^1(x^1,y^1) + f^2(x^2,y^2) - f^2(x^1,y^2) = [\nabla_x f^1(x^1,y^1) + \nabla_x f^2(x^2,y^2)]^T[x^2 - x^1]\ .$$

Proof: Straightforward differentiation and substitution establishes (116). Q.E.D.

We now suppose that the consumer's period t cost function, $C^t(u,p,z)$, has the following functional form:[57]

$$\begin{aligned}(117)\ \ln C^t(u,p,z) \equiv{}& a_0^{\ t} + \textstyle\sum_{n=1}^N a_n^{\ t} \ln p_n + b_0^{\ t} \ln u + \sum_{m=1}^M b_{0m}^{\ \ t} z_m \ln u + \sum_{n=1}^N b_n^{\ t} \ln p_n \ln u \\ &+ (1/2)\, b_{00}^{\ \ t}\, [\ln u]^2 + (1/2) \textstyle\sum_{i=1}^N \sum_{n=1}^N a_{in}^{\ \ t} \ln p_i \ln p_n \\ &+ (1/2) \textstyle\sum_{i=1}^M \sum_{m=1}^M b_{im}^{\ \ t}\, z_i\, z_m + \sum_{n=1}^N \sum_{m=1}^M c_{nm}^{\ \ t}\, z_m \ln p_n\end{aligned}$$

where the parameters satisfy the following restrictions, which impose linear homogeneity in prices p on $C^t(u,p,z)$:

$$(118)\ a_{in}^{\ \ t} = a_{ni}^{\ \ t}\ ; \qquad i,n = 1,\ldots,N;$$
$$(119)\ b_{im}^{\ \ t} = b_{mi}^{\ \ t}\ ; \qquad i,m = 1,\ldots,M;$$
$$(120)\ \textstyle\sum_{n=1}^N a_n^{\ t} = 1\ ;$$
$$(121)\ \textstyle\sum_{n=1}^N b_n^{\ t} = 0\ ;$$
$$(122)\ \textstyle\sum_{i=1}^N a_{in}^{\ \ t} = 0\ ; \qquad n = 1,\ldots,N;$$
$$(123)\ \textstyle\sum_{n=1}^N c_{nm}^{\ \ t} = 0\ ; \qquad m = 1,\ldots,M.$$

It can be shown that the $C^t(u,p,z)$ defined by (117) can provide a second order approximation in the variables u,p and z to an arbitrary twice continuously differentiable cost function, C(u,p,z), and hence C^t is a flexible functional form.

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[56] Balk (1998; 225-226) established this result using Diewert's (1976) original quadratic identity. The Translog Lemma in Caves, Christensen and Diewert (1982; 1412) is simply a logarithmic version of (116).
[57] Caves, Christensen and Diewert (1982; 1397) assumed that C^t was a general translog functional form whereas we are assuming a "mixed" translog functional form, which allows the components of the z vector to be 0 if this is required.

If the consumer in period t has preferences that are dual to the C^t defined by (117)-(123), then Shephard's Lemma implies that the period t market expenditure shares, s_n^t, will satisfy the following equations:

$$(124)\ s_n^t = \partial \ln C^t(u^t,p^t,z^t)/\partial \ln p_n = a_n^t + b_n^t \ln u^t + \textstyle\sum_{i=1}^N a_{ni}^t \ln p_i + \sum_{m=1}^M c_{nm}^t z_m \; ; \qquad n = 1,...,N \; ; t = 0,1.$$

With the above preliminaries, we can now prove the following Proposition:

Proposition 5: Suppose the consumer has preferences in period t that are dual to the cost function C^t defined by (117)-(123) for t = 0,1 and the consumer engages in cost minimizing behavior in each period so that equations (110) and (124) are satisfied. Finally, suppose that the quadratic coefficients on prices are the same for the two periods under consideration so that:

$$(125)\ a_{in}^0 = a_{in}^1 \; ; \qquad i,n = 1,\ldots,N.$$

Then the geometric mean of the two CCD true cost of living indexes defined by (112) and (113) is exactly equal to the observable Törnqvist Theil price index $P_T(p^0,p^1,q^0,q^1)$ defined in (106) above; i.e., we have:

$$(126)\ [P_{CCD}(p^0,p^1,u^0,z^0,0)\ P_{CCD}(p^0,p^1,u^1,z^1,1)]^{1/2} = P_T(p^0,p^1,q^0,q^1).$$

Proof: Take twice the logarithm of the left hand side of (126). Using definitions (112) and (113) and using the quadratic nature of $\ln C^t$ in $\ln p$ and z (see (117)), we obtain the following equation:

$$\begin{aligned}(127)\ & \ln C^0(u^0,p^1,z^0) - \ln C^0(u^0,p^0,z^0) + \ln C^1(u^1,p^1,z^1) - C^1(u^1,p^0,z^1) \\ & = \textstyle\sum_{n=1}^N [\partial \ln C^0(u^0,p^0,z^0)/\partial \ln p_n + \partial \ln C^1(u^1,p^1,z^1)/\partial \ln p_n][\ln p_n^1 - \ln p_n^0] \\ & \qquad \text{using assumption (125) and Proposition 4} \\ & = \textstyle\sum_{n=1}^N [s_n^0 + s_n^1][\ln p_n^1 - \ln p_n^0] \qquad \text{using (124)} \\ & = 2 \ln P_T(p^0,p^1,q^0,q^1) \qquad \text{using the definition of } P_T \text{ in (106).}\end{aligned}$$

Equation (127) is equivalent to (126). Q.E.D.

The above result is essentially equivalent to Theorem 5 in Caves, Christensen and Diewert (1982; 1410).[58] The result in Proposition 5 provides a reasonably powerful justification for the use of the Törnqvist Theil price index as a measure of a consumer's change in his or her cost of living index even if preferences are nonhomothetic.[59]

[58] CCD assumed that their translog cost functions were quadratic in the logs of prices and the logs of the demographic variables. Balk (1989) also obtained a special case of Proposition 5 where there were no demographic variables but there was taste change. Balk's case is also a special case of Theorem 5 in CCD.
[59] Note that we have provided two separate interpretations for Törnqvist Theil price index in the context of nonhomothetic preferences.

Up to this point, we have not studied quantity indexes for the case of nonhomothetic preferences. In the case of a linearly homogeneous aggregator function, f(q) say, we have noted that the companion quantity index to the Konüs price index $c(p^1)/c(p^0)$ (the unit cost ratio) was the ratio of the quantity aggregates $f(q^1)/f(q^0)$. In the following section, we will show how to find quantity indexes when preferences are nonhomothetic.

9. Allen Quantity Indexes

Suppose that we make the same assumptions on preferences that we made at the beginning of section 2. Let C(f(q),p) be the consumer's cost function that is dual to the aggregator function f(q). We again assume cost minimizing behavior in periods 0 and 1 so that equations (1) are satisfied.

The *Allen* (1949) *family of true quantity indexes*, $Q_A(q^0,q^1,p)$, is defined for an arbitrary positive reference price vector p as follows:

$$(128)\ Q_A(q^0,q^1,p) \equiv C(f(q^1),p)/C(f(q^0),p)$$

The basic idea of the Allen quantity index dates back to Hicks (1941-42) who observed that if the price vector p were held fixed and the quantity vector q is free to vary, then C(f(q),p) is a perfectly valid cardinal measure of utility.[60]

As was the case with the true cost of living, the Allen definition simplifies considerably if the utility function happens to be linearly homogeneous. In this case, (128) simplifies to:[61]

$$(129)\ Q_A(q^0,q^1,p) = f(q^1)C(1,p)/f(q^0)C(1,p) = f(q^1)/f(q^0).$$

However, in the general case where the consumer has nonhomothetic preferences, we do not obtain the nice simplification given by (129).

It is useful to specialize the general definition of the Allen quantity index and let the reference price vector equal either the period 0 price vector p^0 or the period 1 price vector p^1:

$$(130)\ Q_A(q^0,q^1,p^0) \equiv C(f(q^1),p^0)/C(f(q^0),p^0)\ ;$$
$$(131)\ Q_A(q^0,q^1,p^1) \equiv C(f(q^1),p^1)/C(f(q^0),p^1).$$

Index number formula that are exact for either of the theoretical indexes defined by (130) and (131) do not seem to exist, at least for the case of nonhomothetic preferences that can be represented by a flexible functional form. However, we can find an index number formula that is exactly equal to the geometric mean of the Allen indexes defined by (130) and (131) where the underlying preferences are represented by a flexible functional form. Thus assume that the consumer's preferences can be represented by the general translog

[60] Samuelson (1974) called this a money metric measure of utility.
[61] See Diewert (1981) for references to the literature.

cost function, C(u,p) defined by (99), with the restrictions (100)-(103). This functional form is a special case of the functional form which appears in Proposition 5, with the demographic variables omitted and with no taste changes between periods 0 and 1. Hence we can apply Proposition 5 in the present context, and conclude that the following simplified version of equation (126) is satisfied for our plain vanilla translog consumer (but with general nonhomothetic preferences):

$$(132)\ [\{C(f(q^0),p^1)/C(f(q^0),p^0)\}\{C(f(q^1),p^1)/C(f(q^1),p^0)\}]^{1/2} = P_T(p^0,p^1,q^0,q^1).$$

The implicit quantity index, $Q_{T*}(p^0,p^1,q^0,q^1)$ that corresponds to the Törnqvist Theil price index $P_T(p^0,p^1,q^0,q^1)$ is defined as the value ratio, $p^1 \cdot q^1/p^0 \cdot q^0$, divided by P_T. Thus we have:

$$\begin{aligned}(133)\ Q_{T*}(p^0,p^1,q^0,q^1) &\equiv [p^1 \cdot q^1/p^0 \cdot q^0]/P_T(p^0,p^1,q^0,q^1) \\ &= [C(f(q^1),p^1)/C(f(q^0),p^0)]/P_T(p^0,p^1,q^0,q^1) && \text{using (1)} \\ &= [C(f(q^1),p^1)/C(f(q^0),p^0)]/[\{C(f(q^0),p^1)/C(f(q^0),p^0)\}\{C(f(q^1),p^1)/C(f(q^1),p^0)\}]^{1/2} \\ & && \text{using (132)} \\ &= [\{C(f(q^1),p^0)/C(f(q^0),p^0)\}\{C(f(q^1),p^1)/C(f(q^0),p^1)\}]^{1/2} \\ &= [Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2}\end{aligned}$$

where the last equality follows using definitions (130) and (131). Thus the observable implicit Törnqvist Theil quantity index, $Q_{T*}(p^0,p^1,q^0,q^1)$, is exactly equal to the geometric mean of the two Allen quantity indexes defined by (130) and (131). This is a very powerful new result.

Note that in general, the geometric mean of the two "natural" Allen quantity indexes defined by (130) and (131) matches up with the geometric mean of the two "natural" Konüs price indexes defined by (3) and (4); i.e., using these definitions, we have:

$$\begin{aligned}(134)\ [P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}\ [Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2} &= C(f(q^1),p^1)/C(f(q^0),p^0) \\ &= p^1 \cdot q^1/p^0 \cdot q^0.\end{aligned}$$

Thus in general, these two "natural" geometric mean price and quantity indexes satisfy the product test. Under our translog assumptions, we have a special case of (134) where $Q_{T*}(p^0,p^1,q^0,q^1)$ is equal to $[Q_A(q^0,q^1,p^0)Q_A(q^0,q^1,p^1)]^{1/2}$ and $P_T(p^0,p^1,q^0,q^1)$ is equal to $[P_K(p^0,p^1,q^0)P_K(p^0,p^1,q^1)]^{1/2}$.

There is an alternative concept for a theoretical quantity index in the case of nonhomothetic preferences that appears frequently in the literature and that is the *Malmquist* (1953) *quantity index*. Results that are similar to the results that we have already derived can be obtained for this concept but we will leave these results to the interested reader.[62]

[62] See Diewert (1981) and Caves, Christensen and Diewert (1982) for additional material on this index concept. Diewert (1976; 123-124) provides a nonhomothetic translog result for this index number concept that is an exact analogue to the result in equation (106) for a nonhomothetic cost function.

10. Conclusion

It can be seen that it is not necessary to use econometric methods in order to form estimates for price and quantity aggregates; instead, exact index numbers can be used. In particular, empirical index number formula can be used to closely approximate a consumer's cost of living index or his or her welfare change, even in the case of nonhomothetic preferences.

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