INDEX NUMBERS

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Abstract

Index numbers are used to aggregate detailed information on prices and quantities into scalar measures of price and quantity levels or their growth. The paper reviews four main approaches to bilateral index number theory where two price and quantity vectors are to be aggregated: fixed basket and average of fixed baskets, stochastic, test or axiomatic and economic approaches. The paper also considers multilateral index number theory where it is necessary to construct price and quantity aggregates for more than two value aggregates. A final section notes some of the recent literature on related aspects of index number theory the construction of indexes when there is seasonality in the underlying data, sources of bias in consumer price indexes, the use of index numbers in measuring productivity, the problem of quality change and index number theory that is based on taking differences rather than ratios.

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1. Introduction

Each individual consumes the services of thousands of commodities over a year and most producers utilize and produce thousands of individual products and services. Index numbers are used to reduce and summarize this overwhelming abundance of microeconomic information. Hence index numbers intrude themselves on virtually every empirical investigation in economics.

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The index number problem may be stated as follows. Suppose we have price data $p^t \equiv (p_1^t, \ldots, p_N^t)$ and quantity data $q^t \equiv (q_1^t, \ldots, q_N^t)$ on N commodities that pertain to the same economic unit at time period t (or to comparable economic units) for t = 0, 1, 2,…,T. The *index number problem* is to find T+1 numbers P^t and T+1 numbers Q^t such that

(1) $P^tQ^t = p^t \cdot q^t \equiv \sum_{n=1}^N p_n^t q_n^t$ for t = 0,1,…,T.

P^t is the *price index* for period t (or unit t) and Q^t is the corresponding *quantity index*. P^t is supposed to be representative of all of the prices p_n^t, n = 1,..., N in some sense, while Q^t is to be similarly representative of the quantities q_n^t, n = 1,..., N. In what precise sense P^t and Q^t represent the individual prices and quantities is not immediately evident and it is this ambiguity which leads to different approaches to index number theory. Note that we require that the product of the price and quantity indexes, P^tQ^t, equals the actual period (or unit) i expenditures on the N commodities, $p^t \cdot q^t$. Thus if the P^t are determined, then the Q^t may be implicitly determined using equations (1), or vice versa.

The number P^t is interpreted as an aggregate period t price level while the number Q^t is interpreted as an aggregate period t quantity level. The *levels* approach to index number theory works as follows. The aggregate price level P^t is assumed to be a function of the components in the period t price vector, p^t while the aggregate period t quantity level Q^t is assumed to be a function of the period t quantity vector components, q^t; i.e., it is assumed that

(2) $P^t = c(p^t)$ and $Q^t = f(q^t)$; t = 0,1,…,T.

The functions c and f are to be determined somehow. Note that we are requiring that the functional forms for the price aggregation function c and for the quantity aggregation function f be independent of time. This is a reasonable requirement since there is no reason to change the method of aggregation as time changes.

Substituting (2) into (1) and dropping the superscripts t means that c and f must satisfy the following functional equation for all strictly positive price and quantity vectors:

(3) $c(p)f(q) = p \cdot q \equiv \sum_{n=1}^N p_nq_n$ for all $p >> 0_N$ and for all $q >> 0_N$.

Note that $p >> 0_N$ means that each component of p is positive, $p \geq 0_N$ means each component is nonnegative and $p > 0_N$ means each component is nonnegative and at least one component is positive. We now could ask what properties should the price aggregation function c and the quantity aggregation function f have? We could assume that c and f satisfied various "reasonable" properties and hope that these properties would determine the functional form for c and f. However, it turns out that we only have to make the following very weak *positivity* assumptions on f and c in order to obtain an impossibility result:

(4) $c(p) > 0$ for all $p >> 0_N$; $f(q) > 0$ for all $q >> 0_N$.

Eichhorn (1978; 144) proved the following result: if the number of commodities N is greater than 1, then there do not exist any functions c and f that satisfy (3) and (4). Thus this *levels approach* to index number theory comes to an abrupt halt. As we shall see later when the economic approach to index number theory is studied, this is not quite the end of the story: in (3) and (4), we allowed p and q to vary independently from each other and this is what leads to the impossibility result. If instead we allow p to vary independently but assume that q is determined as the result of an optimizing model, then equation (3) can be satisfied.

If we change the question that we are trying to answer slightly, then there are practical solutions to the index number problem. The change is that instead of trying to decompose the value of the aggregate into price and quantity components for a single period, we instead attempt to decompose a *value ratio* pertaining to two periods, say periods 0 and 1, into a *price change component* P times a *quantity change component* Q. Thus we now look for two functions of 4N variables, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ such that:

$$(5)\ p^1\cdot q^1/p^0\cdot q^0 = P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1).$$

Note that if some approach to index number theory determines the "best" functional form for the price index $P(p^0,p^1,q^0,q^1)$, then the *product test* (5) can be used to determine the functional form for the corresponding quantity index, $Q(p^0,p^1,q^0,q^1)$.

If we take the *test or axiomatic approach* to index number theory, then we want equation (5) to hold for all positive price and quantity vectors pertaining to the two periods under consideration, p^0,p^1,q^0,q^1. If we take the *economic approach*, then only the price vectors p^0 and p^1 are regarded as independent variables while the quantity vectors, q^0 and q^1, are regarded as dependent variables. In section 4 below, we will pursue the test approach and in sections 5 to 7, we will take the economic approach. In sections 2-7, we take a *bilateral approach to index number theory*; i.e., in making price and quantity comparisons between any two time periods, the relevant indexes use *only* price and quantity information that pertains to the two periods under consideration. It is also possible to take a *multilateral approach*; i.e., we look for functions, P^t and Q^t, that are functions of *all* of the price and quantity vectors, $p^0,p^1,\ldots,p^T,q^0,q^1,\ldots,q^T$. Thus we look for 2(T+1) functions, $P^t(p^0,p^1,\ldots,p^T,q^0,q^1,\ldots,q^T)$ and $Q^t(p^0,p^1,\ldots,p^T,q^0,q^1,\ldots,q^T)$, t = 0,1,…,T, such that

$$(6)\ p^t\cdot q^t = P^t(p^0,p^1,\ldots,p^T,q^0,q^1,\ldots,q^T)\ Q^t(p^0,p^1,\ldots,p^T,q^0,q^1,\ldots,q^T) \quad \text{for } t = 0,1,\ldots,T.$$

We briefly pursue the multilateral approach to index number theory in section 9 below.

The four main approaches to bilateral index number theory will be covered in this review: (i) the *fixed basket approach* (section 2), (ii) the *stochastic approach* (section 3), (iii) the *test approach* (section 4) and (iv) the *economic approach*, which relies on the assumption of maximizing or minimizing behavior (sections 5-7).

Section 8 discusses fixed base versus chained index numbers and section 10 concludes by mentioning some recent areas of active research in the index number literature.

2. Fixed Basket Approaches

The English economist Joseph Lowe (1823) developed the theory of the consumer price index in some detail. His approach to measuring the price change between periods 0 and 1 was to specify an approximate representative commodity basket quantity vector, $q \equiv (q_1,\ldots,q_N)$, which was to be updated every five years, and then calculate the level of prices in period 1 relative to period 0 as

$$(7)\ P_{Lo}(p^0,p^1,q) \equiv p^1 \cdot q/p^0 \cdot q$$

where p^0 and p^1 are the commodity price vectors that the consumer (or group of consumers) face in periods 0 and 1 respectively. The fixed basket approach to measuring price change is intuitively very simple: we simply specify the commodity "list" q and calculate the price index as the ratio of the costs of buying this same list of goods in periods 1 and 0.

As time passed, economists and price statisticians demanded a bit more precision with respect to the specification of the basket vector q. There are two natural choices for the reference basket: the period 0 commodity vector q^0 or the period 1 commodity vector q^1. These two choices lead to the Laspeyres (1871) price index P_L defined by (8) and the Paasche (1874) price index P_P defined by (9):

$$(8)\ P_L(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^0/p^0 \cdot q^0 ;$$
$$(9)\ P_P(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1/p^0 \cdot q^1 .$$

The above formulae can be rewritten in an alternative manner that is very useful for statistical agencies. Define the period t expenditure share on commodity n as follows:

$$(10)\ \ s_n^t \equiv p_n^t q_n^t/p^t \cdot q^t \qquad \text{for } n = 1,\ldots,N \text{ and } t = 0,1.$$

Following Fisher (1911), the Laspeyres index (8) can be rewritten as follows:

$$\begin{aligned}(11)\ \ P_L(p^0,p^1,q^0,q^1) &= \textstyle\sum_{n=1}^N p_n^1 q_n^0/p^0 \cdot q^0 \\ &= \textstyle\sum_{n=1}^N (p_n^1/p_n^0)\, p_n^0 q_n^0/p^0 \cdot q^0 \\ &= \textstyle\sum_{n=1}^N (p_n^1/p_n^0)\, s_n^0 \qquad\qquad \text{using definitions (10).}\end{aligned}$$

Thus the Laspeyres price index P_L can be written as a base period expenditure share weighted average of the N price ratios (or price relatives using index number terminology), p_n^1/p_n^0 . The Laspeyres formula (until the very recent past) has been widely used as the intellectual basis for country Consumer Price Indexes (CPIs) around the world. To implement it, the country statistical agency collects information on expenditure shares s_n^0 for the index domain of definition for the base period 0 and then collects information on *prices* alone on an ongoing basis. Thus a Laspeyres type CPI can be produced on a timely basis without having to know current period quantity information. In fact, the situation is more complicated than this: in actual CPI programs, prices are collected on a monthly or quarterly frequency and with base month 0 say but the quantity vector q^0 is typically *not* the quantity vector that pertains to the price base month 0; rather it is actually equal to a *base year quantity vector* q^b say, which is typically prior to the base month 0. Thus the typical CPI, although loosely based on the Laspeyres index, is actually a form of Lowe index; see (7) above. Instead of using the Lowe formula for their CPI, some statistical agencies use the following Young (1812) index:

$$(12)\ P_Y(p^0,p^1,s^b) \equiv \sum_{n=1}^{N} (p_n^{1}/p_n^{0})\, s_n^{b}$$

where the s_n^{b} are base year expenditure shares on the N commodities in the index. For additional material on Lowe and Young indexes and their use in CPI and PPI (Producer Price Index) programs, see the ILO (2004) and the IMF (2004).

The Paasche index can also be written in expenditure share and price ratio form as follows:

$$\begin{aligned}(13)\ P_P(p^0,p^1,q^0,q^1) &= 1/\,[\textstyle\sum_{n=1}^{N} p_n^{0}q_n^{1}/p^1\cdot q^1] \\ &= 1/\,[\textstyle\sum_{n=1}^{N} (p_n^{0}/p_n^{1})\, p_n^{1}q_n^{1}/p^1\cdot q^1] \\ &= 1/\,[\textstyle\sum_{n=1}^{N} (p_n^{1}/p_n^{0})^{-1}\, s_n^{1}] && \text{using definitions (10)} \\ &= [\textstyle\sum_{n=1}^{N} (p_n^{1}/p_n^{0})^{-1}\, s_n^{1}]^{-1}.\end{aligned}$$

Thus the Paasche price index P_P can be written as a period 1 (or current period) expenditure share weighted *harmonic* average of the N price ratios.

The problem with the Paasche and Laspeyres index number formulae is that they are equally plausible but in general, they will give different answers. This suggests that if we require a single estimate for the price change between the two periods, then we need to take some sort of evenly weighted average of the two indexes as our final estimate of price change between periods 0 and 1. Examples of such symmetric averages are the arithmetic mean, which leads to the Sidgwick (1883; 68) Bowley (1901; 227) index, $(1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the Fisher (1922) ideal index, P_F, which was actually first suggested by Bowley (1899; 641), defined as

$$(14)\ P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)\ P_P(p^0,p^1,q^0,q^1)]^{1/2}\ .$$

At this point, the fixed basket approach to index number theory is transformed into the *test approach* to index number theory; i.e., in order to determine which of these fixed basket indexes or which averages of them might be "best", we need *criteria* or *tests* or *properties* that we would like our indexes to satisfy. We will pursue this topic in more detail in section 4 below but we will give the reader an introduction to this topic in the present section because some of these tests or properties are useful to evaluate other approaches to index number theory.

Let a and b be two positive numbers. Diewert (1993b; 361) defined a *symmetric mean* of a and b as a function m(a,b) that has the following properties: (i) $m(a,a) = a$ for all $a > 0$ (mean property); (ii) $m(a,b) = m(b,a)$ for all $a > 0$, $b > 0$ (symmetry property); (iii) m(a,b) is a continuous function for $a > 0$, $b > 0$ (continuity property) and (iv) m(a,b) is a strictly increasing function in each of its variables (increasingness property). Eichhorn and Voeller (1976; 10) showed that if m(a,b) satisfies the above properties, then it also satisfies the following property: (v) $\min\{a,b\} \leq m(a,b) \leq \max\{a,b\}$ (min-max property); i.e., the mean of a and b, m(a,b), lies between the maximum and minimum of the numbers a and b. Since we have restricted the domain of definition of a and b to be positive numbers, it can be seen that an implication of the last property is that m also satisfies the following property: (vi) $m(a,b) > 0$ for all $a > 0$, $b > 0$ (positivity property). If in addition, m satisfies the following property, then we say that m is a *homogeneous symmetric mean*: (vii) $m(\lambda a,\lambda b) = \lambda m(a,b)$ for all $\lambda > 0$, $a > 0$, $b > 0$.

What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical cost of living index? It is very desirable for a price index formula that depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test*. We say that the index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

$$(15) \quad P(p^1,p^0,q^1,q^0) = 1/ P(p^0,p^1,q^0,q^1) \ ;$$

i.e., if we interchange the period 0 and period 1 price and quantity data and evaluate the index, then this new index $P(p^1,p^0,q^1,q^0)$ is equal to the reciprocal of the original index $P(p^0,p^1,q^0,q^1)$. For the history of this test (and other tests), see Diewert (1992a; 218) (1993a).

Diewert (1997; 138) proved the following result: the Fisher Ideal price index defined by (14) above is the *only* index that is a homogeneous symmetric average of the Laspeyres and Paasche price indexes, P_L and P_P, that also satisfies the time reversal test (15) above.

Thus the symmetric basket approach to index number theory leads to the Fisher ideal index as the "best" formula. It is interesting to note that this symmetric basket approach to index number theory dates back to one of the early pioneers of index number theory, Bowley, as the following quotations indicate:

"If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean … as a first approximation." A. L. Bowley (1901; 227).

"When estimating the factor necessary for the correction of a change found in money wages to obtain the change in real wages, statisticians have not been content to follow Method II only [to calculate a Laspeyres price index], but have worked the problem backwards [to calculate a Paasche price index] as well as forwards. … They have then taken the arithmetic, geometric or harmonic mean of the two numbers so found." A. L. Bowley (1919; 348).

Instead of taking a symmetric average of the Paasche and Laspeyres indexes, an alternative average basket approach takes a symmetric average of the baskets that prevail in the two periods under consideration. For example, the average basket could be the arithmetic or geometric mean of the two baskets, leading the Marshall (1887) Edgeworth (1925) index P_{ME} or the Walsh (1901; 398) (1921a; 97-101) index P_W:

$$(16)\quad P_{ME}(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N p_n^1(1/2)(q_n^0 + q_n^1) / \sum_{m=1}^N p_j^0(1/2)(q_m^0 + q_m^1);$$
$$(17)\quad P_W(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N p_n^1(q_n^0q_n^1)^{1/2} / \sum_{m=1}^N p_m^0(q_m^0q_m^1)^{1/2} .$$

Diewert (2002b; 569-571) showed that the Walsh index P_W emerged as being "best" in this average basket framework; see also Chapters 15 and 16 in ILO (2004).

We turn now to the second major approach to bilateral index number theory.

3. The Stochastic Approach to Index Number Theory

"In drawing our averages the independent fluctuations will more or less destroy each other; the one required variation of gold will remain undiminished." W. Stanley Jevons (1884; 26).

The stochastic approach to the determination of the price index can be traced back to the work of Jevons (1865) (1884) and Edgeworth (1888) (1923) (1925) over a hundred years ago. For additional discussion on the early history of this approach, see Diewert (1993a; 37-38) (1995b).

The basic idea behind the stochastic approach is that each price relative, p_n^1/p_n^0 for $n = 1,2,\ldots,N$ can be regarded as an estimate of a common inflation rate α between periods 0 and 1; i.e., it is assumed that

$$(18)\quad p_n^1/p_n^0 = \alpha + \varepsilon_n \;;\; n = 1,2,\ldots,N$$

where α is the common inflation rate and the ε_n are random variables with mean 0 and variance σ^2. The least squares estimator for α is the Carli (1764) price index P_C defined as

$$(19)\quad P_C(p^0,p^1) \equiv \sum_{n=1}^N (1/N)\,(p_n^1/p_n^0).$$

Unfortunately, P_C does not satisfy the time reversal test, i.e., $P_C(p^1,p^0) \neq 1/ P_C(p^0,p^1)$. In fact, Fisher (1922; 66) noted that $P_C(p^0,p^1)P_C(p^1,p^0) \geq 1$ unless the period 1 price vector p^1 is proportional to the period 0 price vector p^0; i.e., Fisher showed that the Carli (and the Young) index has a definite upward bias. He urged statistical agencies not to use these formulae.

Now assume that the logarithm of each price relative, $\ln(p_n^1/p_n^0)$, is an unbiased estimate of the logarithm of the inflation rate between periods 0 and 1, β say. Thus we have:

$$(20)\quad \ln(p_n^1/p_n^0) = \beta + \varepsilon_n \;;\; n = 1,2,\ldots,N$$

where $\beta \equiv \ln\alpha$ and the ε_n are independently distributed random variables with mean 0 and variance σ^2. The least squares estimator for β is the logarithm of the geometric mean of the price relatives. Hence the corresponding estimate for the common inflation rate α is the Jevons (1865) price index P_J defined as:

(21) $P_J(p^0,p^1) \equiv \prod_{n=1}^N (p_n^1/p_n^0)^{1/N}$.

The Jevons price index P_J does satisfy the time reversal test and hence is much more satisfactory than the Carli index P_C.

Bowley (1928) attacked the use of both (19) and (21) on two grounds. First, from an empirical point of view, he showed that price ratios were not symmetrically distributed about a common mean and their logarithms also failed to be symmetrically distributed. Secondly, from a theoretical point of view, he argued that it was unlikely that prices or price ratios were independently distributed. Keynes (1930) developed Bowley's second objection in more detail; he argued that changes in the money supply would not affect all prices at the same time. Moreover, real disturbances in the economy could cause one set of prices to differ in a systematic way from other prices, depending on various elasticities of substitution and complementarity. In other words, prices are not randomly distributed, but are systematically related to each other through the general equilibrium of the economy. Keynes (1930; 76-77) had other criticisms of this *unweighted stochastic approach* to index number theory, including the point that that there is no such thing as *the* inflation rate; there are only price changes that pertain to well specified sets of commodities or transactions; i.e., the domain of definition of the price index must be carefully specified. Keynes also followed Walsh in insisting that price movements must be weighted by their economic importance; i.e., by quantities or expenditures:

"It might seem at first sight as if simply every price quotation were a single item, and since every commodity (any kind of commodity) has one price-quotation attached to it, it would seem as if price-variations of every kind of commodity were the single item in question. This is the way the question struck the first inquirers into price-variations, wherefore they used simple averaging with even weighting. But a price-quotation is the quotation of the price of a generic name for many articles; and one such generic name covers a few articles, and another covers many. … A single price-quotation, therefore, may be the quotation of the price of a hundred, a thousand, or a million dollar's worths, of the articles that make up the commodity named. Its weight in the averaging, therefore, ought to be according to these money-unit's worth." Correa Moylan Walsh (1921a; 82-83).

Theil (1967; 136-137) proposed a solution to the lack of weighting in (21). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the nth price relative is equal to $s_n^0 \equiv p_n^0 q_n^0/p^0\cdot q^0$, the period 0 expenditure share for commodity n. Then the overall mean (period 0 weighted) logarithmic price change is $\sum_{n=1}^N s_n^0 \ln(p_n^1/p_n^0)$. Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) logarithmic price change of $\sum_{n=1}^N s_n^1 \ln(p_n^1/p_n^0)$. Each of these measures of overall logarithmic price change seems equally valid so we could argue for taking a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change. Theil (1967; 138) argued that a nice symmetric index number formula can be obtained if we make the probability of selection for the nth price relative equal to the arithmetic average of the period 0 and 1 expenditure shares for commodity n. Using these probabilities of selection, Theil's final measure of overall logarithmic price change was

$$(22)\quad \ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N} (1/2)(s_n^0 + s_n^1) \ln(p_n^1/p_n^0).$$

We can give the following *descriptive statistics* interpretation of the right hand side of (22). Define the nth logarithmic price ratio r_n by:

$$(23)\quad r_n \equiv \ln(p_n^1/p_n^0) \quad \text{for } n = 1,\ldots,N.$$

Now define the discrete random variable, R say, as the random variable which can take on the values r_n with probabilities $\rho_n \equiv (1/2)[\, s_n^0 + s_n^1]$ for $n = 1,\ldots,N$. Note that since each set of expenditure shares, s_n^0 and s_n^1, sums to one, the probabilities ρ_n will also sum to one. It can be seen that the expected value of the discrete random variable R is

$$(24)\quad E[R] \equiv \sum_{n=1}^{N} \rho_n\, r_n = \sum_{n=1}^{N} (1/2)(s_n^0 + s_n^1) \ln(p_n^1/p_n^0) = \ln P_T(p^0,p^1,q^0,q^1)$$

using (22) and (23). Thus the logarithm of the index P_T can be interpreted as *the expected value of the distribution of the logarithmic price ratios* in the domain of definition under consideration, where the N discrete price ratios in this domain of definition are weighted according to Theil's probability weights, $\rho_n \equiv (1/2)[\, s_n^0 + s_n^1]$ for $n = 1,\ldots,N$.

Taking antilogs of both sides of (24), we obtain the Törnqvist (1936), Törnqvist and Törnqvist (1937) Theil price index, P_T. This index number formula has a number of good properties. Thus the second major approach to bilateral index number theory has led to the Törnqvist Theil price index P_T as being "best" from this perspective.

Additional material on stochastic approaches to index number theory and references to the literature can be found in Selvanathan and Rao (1994), Diewert (1995b), Wynne (1997), ILO (2004), IMF (2004) and Clements, Izan and Selvanathan (2006).

It turns out that formulae (8), (9), (14) and (22) (the Laspeyres, Paasche, Fisher and Törnqvist Theil formulae) are the most widely used formulae for a bilateral price index. But Walsh (1901) and Fisher (1922) presented hundreds of functional forms for bilateral price indexes—on what basis are we to choose one as being better than the other? Perhaps the next approach to index number theory will narrow the choices.

4. The Test Approach to Index Number Theory

In this section, we will take the perspective outlined in section 1 above; i.e., along with the price index $P(p^0,p^1,q^0,q^1)$, there is a companion quantity index $Q(p^0,p^1,q^0,q^1)$ such that the product of these two indices equals the value ratio between the two periods. Thus, throughout this section, we assume that P and Q satisfy the product test (5) above.

Assuming that the product test holds means that as soon as the functional form for the price index P is determined, then (5) can be used to determine the functional form for the quantity index Q. However, as Fisher (1911; 400-406) and Vogt (1980) observed, a further advantage of assuming that the product test holds is that we can assume that the quantity index Q satisfies a "reasonable" property and then use (5) to translate this test on the quantity index into a corresponding test on the price index P.

If N = 1, so that there is only one price and quantity to be aggregated, then a natural candidate for P is p_1^1/p_1^0 , the single price ratio, and a natural candidate for Q is q_1^1/q_1^0 , the single quantity ratio. When the number of commodities or items to be aggregated is greater than 1, then what index number theorists have done over the years is propose properties or tests that the price index P should satisfy. These properties are generally multi-dimensional analogues to the one good price index formula, p_1^1/p_1^0. Below, following Diewert (1992a), we list twenty tests that turn out to characterize the Fisher ideal price index.

We shall assume that every component of each price and quantity vector is positive; i.e., $p^t >> 0_N$ and $q^t >> 0_N$ for t = 0,1. If we want to set $q^0 = q^1$, we call the common quantity vector q; if we want to set $p^0 = p^1$, we call the common price vector p.

Our first two tests, due to Eichhorn and Voeller (1976; 23) and Fisher (1922; 207-215), are not very controversial and so we will not discuss them.

T1: *Positivity*: $P(p^0,p^1,q^0,q^1) > 0$.

T2: *Continuity*: $P(p^0,p^1,q^0,q^1)$ is a continuous function of its arguments.

Our next two tests, due to Laspeyres (1871; 308), Walsh (1901; 308) and Eichhorn and Voeller (1976; 24), are somewhat more controversial.

T3: *Identity or Constant Prices Test*: $P(p,p,q^0,q^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. The controversial part of this test is that the two quantity vectors are allowed to be different in the above test.

T4: *Fixed Basket or Constant Quantities Test*: $P(p^0,p^1,q,q) = \sum_{i=1}^N p_i^1 q_i / \sum_{i=1}^N p_i^0 q_i$.

That is, if quantities are constant during the two periods so that $q^0 = q^1 \equiv q$, then the price index should equal the expenditure on the constant basket in period 1, $\sum_{i=1}^N p_i^1 q_i$, divided by the expenditure on the basket in period 0, $\sum_{i=1}^N p_i^0 q_i$. The origins of this test go back at least two hundred years to the Massachusetts legislature which used a constant basket of goods to index the pay of Massachusetts soldiers fighting in the American Revolution;

see Willard Fisher (1913). Other researchers who have suggested the test over the years include: Lowe (1823, Appendix, 95), Scrope (1833, 406), Jevons (1865), Sidgwick (1883, 67-68), Edgeworth (1925, 215) originally published in 1887, Marshall (1887, 363), Pierson (1895, 332), Walsh (1901, 540) (1921b; 544), and Bowley (1901, 227). Vogt and Barta (1997; 49) also observed that this test is a special case of Fisher's (1911; 411) proportionality test for quantity indexes which Fisher (1911; 405) translated into a test for the price index using the product test (5).

The following four tests restrict the behavior of the price index P as the scale of any one of the four vectors p^0,p^1,q^0,q^1 changes. The following test was proposed by Walsh (1901, 385), Eichhorn and Voeller (1976, 24) and Vogt (1980, 68).

T5: *Proportionality in Current Prices*: $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. Most index number theorists regard this property as a very fundamental one that the index number formula should satisfy.

Walsh (1901) and Fisher (1911; 418) (1922; 420) proposed the related proportionality test $P(p,\lambda p,q^0,q^1) = \lambda$. This last test is a combination of T3 and T5; in fact Walsh (1901, 385) noted that this last test implies the identity test, T3.

In the next test, due to Eichhorn and Voeller (1976; 28), instead of multiplying all period 1 prices by the same number, we multiply all period 0 prices by the number λ.

T6: *Inverse Proportionality in Base Period Prices*: $P(\lambda p^0,p^1,q^0,q^1) = \lambda^{-1}P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0.

The following two homogeneity tests can also be regarded as invariance tests.

T7: *Invariance to Proportional Changes in Current Quantities*: $P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if current period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 1 quantity vector q^1. Vogt (1980, 70) was the first to propose this test and his derivation of the test is of some interest. Suppose the quantity index Q satisfies the quantity analogue to the price test T5; i.e., suppose Q satisfies $Q(p^0,p^1,q^0,\lambda q^1) = \lambda Q(p^0,p^1,q^0,q^1)$ for $\lambda > 0$. Then using the product test (5), we see that P must satisfy T7.

T8: *Invariance to Proportional Changes in Base Quantities*: $P(p^0,p^1,\lambda q^0,q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if base period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 0 quantity vector q^0. If the quantity index Q satisfies the following counterpart to T8: $Q(p^0,p^1,\lambda q^0,q^1) = \lambda^{-1}Q(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$, then using (5), the corresponding price index P must satisfy T8. This argument provides some additional justification for assuming the validity of T8 for the price index function P. This test was proposed by Diewert (1992a; 216).

T7 and T8 together impose the property that the price index P does not depend on the *absolute* magnitudes of the quantity vectors q^0 and q^1.

The next five tests are invariance or symmetry tests. Fisher (1922; 62-63, 458-460) and Walsh (1921b; 542) seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, 62-63) spoke of fairness but it is clear that he had symmetry properties in mind. It is perhaps unfortunate that he did not realize that there were more symmetry and invariance properties than the ones he proposed; if he had realized this, it is likely that he would have been able to provide an axiomatic characterization for his ideal price index, as will be done shortly below. Our first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed:

T9: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p^0,p^1,q^0,q^1)$$

where p^{t*} denotes a permutation of the components of the vector p^t and q^{t*} denotes the same permutation of the components of q^t for $t = 0,1$. This test is due to Irving Fisher (1922), and it is one of his three famous reversal tests. The other two are the time reversal test and the factor reversal test which will be considered below.

T10: *Invariance to Changes in the Units of Measurement* (commensurability test):

$$P(\alpha_1 p_1^0,\ldots,\alpha_N p_N^0;\ \alpha_1 p_1^1,\ldots,\alpha_N p_N^1;\ \alpha_1^{-1}q_1^0,\ldots,\alpha_N^{-1}q_N^0;\ \alpha_1^{-1}q_1^1,\ldots,\alpha_N^{-1}q_N^1) =$$
$$P(p_1^0,\ldots,p_N^0;\ p_1^1,\ldots,p_N^1;\ q_1^0,\ldots,q_N^0;\ q_1^1,\ldots,q_N^1) \text{ for all } \alpha_1 > 0, \ldots, \alpha_N > 0.$$

That is, the price index does not change if the units of measurement for each commodity are changed. The concept of this test was due to Jevons (1884; 23) and the Dutch economist Pierson (1896; 131), who criticized several index number formula for not satisfying this fundamental test. Fisher (1911; 411) first called this test *the change of units test* and later, Fisher (1922; 420) called it the *commensurability test*.

T11: *Time Reversal Test*: $P(p^0,p^1,q^0,q^1) = 1/P(p^1,p^0,q^1,q^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. We have already encountered this test; recall (15) above. Obviously, in the one good case when the price index is simply the single price ratio; this test is satisfied (as are all of the other tests listed in this section). When the number of goods is greater than one, many commonly used price indices fail this test; e.g., the Laspeyres and Paasche price indexes, P_L and P_P defined earlier by (8) and (9) above, both *fail* this fundamental test. The concept of the test was due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test, that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 368) (1921b; 541) and Fisher (1911; 534) (1922; 64).

Our next two tests are more controversial, since they are not necessarily consistent with the economic approach to index number theory. However, these tests are quite consistent with the weighted stochastic approach to index number theory discussed in section 3 above.

T12: *Quantity Reversal Test* (quantity weights symmetry test): $P(p^0,p^1,q^0,q^1) = P(p^0,p^1,q^1,q^0)$.

That is, if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities are used to weight the prices in the index number formula, then the period 0 quantities q^0 and the period 1 quantities q^1 must enter the formula in a symmetric or even handed manner. Funke and Voeller (1978; 3) introduced this test; they called it the *weight property*.

The next test proposed by Diewert (1992a; 218) is the analogue to T12 applied to quantity indices:

T13: *Price Reversal Test* (price weights symmetry test):
$\{\sum_{i=1}^N p_i^1 q_i^1/ \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) = \{\sum_{i=1}^N p_i^0 q_i^1/ \sum_{i=1}^N p_i^1 q_i^0\}/P(p^1,p^0,q^0,q^1)$.

Thus if we use (5) to define the quantity index Q in terms of the price index P, then it can be seen that T13 is equivalent to the following property for the associated quantity index Q:

(25) $Q(p^0,p^1,q^0,q^1) = Q(p^1,p^0,q^0,q^1)$.

That is, if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property T13 implies that these prices enter the quantity index in a symmetric manner.

The next three tests are mean value tests. The following test was proposed by Eichhorn and Voeller (1976; 10):

T14: *Mean Value Test for Prices*:
$\min_i (p_i^1/p_i^0 : i=1,...,N) \leq P(p^0,p^1,q^0,q^1) \leq \max_i (p_i^1/p_i^0 : i = 1,...,N)$.

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is supposed to be some sort of an average of the N price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfy this test.

The next test proposed by Diewert (1992a; 219) is the analogue to T14 applied to quantity indexes:

T15: *Mean Value Test for Quantities*:
$\min_i (q_i^1/q_i^0 : i=1,...,n) \leq \{V^1/V^0\}/ P(p^0,p^1,q^0,q^1) \leq \max_i (q_i^1/q_i^0 : i = 1,...,n)$

where V^t is the period t value aggregate $V^t \equiv \sum_{n=1}^N p_n^t q_n^t$ for t = 0,1. Using (5) to define the quantity index Q in terms of the price index P, we see that T15 is equivalent to the following property for the associated quantity index Q:

(26) $\min_i (q_i^1/q_i^0 : i = 1,...,N) \leq Q(p^0,p^1,q^0,q^1) \leq \max_i (q_i^1/q_i^0 : i = 1,...,N)$.

That is, the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth q_i^1/q_i^0 of the individual quantities.

In section 2, it was argued that it was very reasonable to take an average of the Laspeyres and Paasche price indices as a single "best" measure of overall price change. This point of view can be turned into a test:

T16: *Paasche and Laspeyres Bounding Test*: The price index P lies between the Laspeyres and Paasche indices, P_L and P_P, defined by (8) and (9) above.

Bowley (1901; 227) and Fisher (1922; 403) both endorsed this property for a price index.

Our final four tests are monotonicity tests; i.e., how should the price index $P(p^0,p^1,q^0,q^1)$ change as any component of the two price vectors p^0 and p^1 increases or as any component of the two quantity vectors q^0 and q^1 increases.

T17: *Monotonicity in Current Prices*: $P(p^0,p^1,q^0,q^1) < P(p^0,p^2,q^0,q^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase, so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1. This property was proposed by Eichhorn and Voeller (1976; 23) and it is a very reasonable property for a price index to satisfy.

T18: *Monotonicity in Base Prices*: $P(p^0,p^1,q^0,q^1) > P(p^2,p^1,q^0,q^1)$ if $p^0 < p^2$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0 . This very reasonable property was also proposed by Eichhorn and Voeller (1976; 23).

T19: *Monotonicity in Current Quantities*: if $q^1 < q^2$, then
$\{\sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) < \{\sum_{i=1}^N p_i^1 q_i^2 / \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^2)$.

T20: *Monotonicity in Base Quantities*: if $q^0 < q^2$, then
$\{\sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) > \{\sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^2\}/P(p^0,p^1,q^2,q^1)$.

If we define the implicit quantity index Q that corresponds to P using (1), we find that T19 translates into the following inequality involving Q:

(27) $Q(p^0,p^1,q^0,q^1) < Q(p^0,p^1,q^0,q^2)$ if $q^1 < q^2$.

That is, if any period 1 quantity increases, then the implicit quantity index Q that corresponds to the price index P must increase. Similarly, we find that T20 translates into:

(28) $Q(p^0,p^1,q^0,q^1) > Q(p^0,p^1,q^2,q^1)$ if $q^0 < q^2$.

That is, if any period 0 quantity increases, then the implicit quantity index Q must decrease. Tests T19 and T20 are due to Vogt (1980, 70).

Diewert (1992a; 221) showed that the only index number formula $P(p^0,p^1,q^0,q^1)$ which satisfies tests T1-T20 is the Fisher ideal price index P_F defined earlier by (14); i.e., as the geometric mean of the Laspeyres and Paasche price indexes.

It turns out that P_F satisfies yet another test, T21, which was Irving Fisher's (1921; 534) (1922; 72-81) third reversal test (the other two being T9 and T11):

T21: *Factor Reversal Test* (functional form symmetry test):
$$P(p^0,p^1,q^0,q^1)\, P(q^0,q^1,p^0,p^1) = \sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^0 .$$

A justification for this test is the following one: if $P(p^0,p^1,q^0,q^1)$ is a good functional form for the price index, then if we reverse the roles of prices and quantities, $P(q^0,q^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument) and thus the product of the price index $P(p^0,p^1,q^0,q^1)$ and the quantity index $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ ought to equal the value ratio, V^1/V^0 . The second part of this argument does not seem to be valid and thus many researchers over the years have objected to the factor reversal test. However, if one is willing to embrace T21 as a basic test, Funke and Voeller (1978; 180) showed that the only index number function $P(p^0,p^1,q^0,q^1)$ which satisfies T1 (positivity), T11 (time reversal test), T12 (quantity reversal test) and T21 (factor reversal test) is the Fisher ideal index P_F defined by (14).

Other characterizations of the Fisher price index can be found in Funke and Voeller (1978) and Balk (1985) (1995).

The Fisher price index P_F satisfies all 20 of the tests listed above. Which tests do other commonly used price indexes satisfy? Recall the Laspeyres index P_L defined by (8), the Paasche index P_P defined by (9) and the Törnqvist Theil index P_T defined by (22). Straightforward computations show that the Paasche and Laspeyres price indexes fail only the three reversal tests, T11, T12 and T13. Since the quantity and price reversal tests, T12 and T13, are somewhat controversial and hence can be discounted, the test performance of P_L and P_P seems at first sight to be quite good. However, the failure of the time reversal test, T11, is a severe limitation associated with the use of these indexes.

The Törnqvist Theil price index P_T fails nine tests: T4 (the fixed basket test), the quantity and price reversal tests T12 and T13, T15 (the mean value test for quantities), T16 (the Paasche and Laspeyres bounding test) and the 4 monotonicity tests T17 to T20. Thus the Törnqvist Theil index is subject to a rather high failure rate from the perspective of this particular axiomatic approach to index number theory.

However, it could be argued that the list of tests or axioms that was used to establish the superiority of the Fisher ideal index might have been chosen to favor this index. Thus Diewert (2004), following the example of Walsh (1901; 104-105) and Vartia (1976), developed a set of axioms for price indexes of the form $P(p^0,p^1,v^0,v^1)$ where v^0 and v^1 are vectors of expenditures on the N commodities in the index and these vectors replace the quantity vectors q^0 and q^1 as weighting vectors for the prices. In this new axiomatic framework, the Törnqvist Theil index P_T emerged as being "best".

The consistency and independence of various bilateral index number tests was studied in some detail by Eichhorn and Voeller (1976). Our conclusion at this point echoes that of Frisch (1936): the test approach to index number theory, while extremely useful, does not lead to a single unique index number formula. However, two test approaches that take alternative approaches to the methods for weighting prices do lead to the Fisher and Törnqvist Theil indexes as being "best" in their respective axiomatic frameworks.

For additional material on the test approach to bilateral index number theory, see Balk (1995), Reinsdorf and Dorfman (1999), Balk and Diewert (2001), Vogt and Barta (1997) and Reinsdorf (2007).

In the following 3 sections, we consider various economic approaches to index number theory. In the economic approach to price index theory, quantity vectors are no longer regarded as being exogenous variables; rather they are regarded as solutions to various economic optimization problems.

5. The Economic Approach to Price Indexes

Before a definition of a microeconomic price index is presented, it is necessary to make a few preliminary definitions.

Let $F(q)$ be a function of N variables, $q \equiv (q_1,...,q_N)$. In the consumer context, F represents a consumer's preferences; i.e. if $F(q^2) > F(q^1)$, then the consumer prefers the commodity vector q^2 over q^1. In this context, F is called a *utility function*. In the producer context, $F(q)$ might represent the output that could be produced using the input vector q. In this context, F is called a *production function*. In order to cover both contexts, we follow the example of Diewert (1976a) and call F an *aggregator function*.

Suppose the consumer or producer faces prices $p \equiv (p_1,\ldots, p_N)$ for the N commodities. Then the economic agent will generally find it is useful to minimize the cost of achieving at least a given utility or output level u; we define the *cost function* or *expenditure function* C as the solution to this minimization problem:

$$(29)\ C(u,p) \equiv \min_q \{p\cdot q : F(q) \geq u\}$$

where $p\cdot q \equiv \sum_{n-1}^{N} p_n q_n$ is the inner product of the price vector p and quantity vector q.

Note that the cost function depends on $1 + N$ variables; the utility or output level u and the N commodity prices in the vector p. Moreover, the functional form for the aggregator function F completely determines the functional form for C.

We say that an aggregator function is *neoclassical* if F is: (i) continuous, (ii) positive; i.e. $F(q) > 0$ if $q >> 0_N$ and (iii) linearly homogeneous; i.e. $F(\lambda q) = \lambda F(q)$ if $\lambda > 0$. If F is neoclassical, then the corresponding cost function $C(u,p)$ equals u times the unit cost function, $c(p) \equiv C(1,p)$, where $c(p)$ is the minimum cost of producing one unit of utility or output; i.e.,

$$(30)\ C(u,p) = uC(1,p) = uc(p).$$

Shephard (1953) formally defined an aggregator function F to be *homothetic* if there exists an increasing continuous function of one variable g such that $g[F(q)]$ is neoclassical. However, the concept of homotheticity was well known to Frisch (1936) who termed it expenditure proportionality. If F is homothetic, then its cost function C has the following decomposition:

$$\begin{aligned}(31)\ C(u,p) &\equiv \min_q \{p\cdot q : F(q) \geq u\} \\ &= \min_q \{p\cdot q : g[F(q)] \geq g(u)\} \\ &= g(u)c(p)\end{aligned}$$

where $c(p)$ is the unit cost function that corresponds to $g[F(q)]$.

Let $p^0 >> 0_N$ and $p^1 >> 0_N$ be positive price vectors pertaining to periods or observations 0 and 1. Let $q > 0_N$ be a nonnegative, nonzero reference quantity vector. Then the Konüs (1924) *price index* or *cost of living index* is defined as:

(32) $P_K(p^0,p^1,q) \equiv C[F(q),p^1]/C[F(q),p^0]$.

In the consumer (producer) context, P_K may be interpreted as follows. Pick a reference utility (output) level $u \equiv F(q)$. Then $P_K(p^0,p^1,q)$ is the minimum cost of achieving the utility (output) level u when the economic agent faces prices p^1 relative to the minimum cost of achieving the same u when the agent faces prices p^0. If N = 1 so that there is only one consumer good (or input), then it is easy to show that $P_K(p_1^0,p_1^1,q_1) = p_1^1q_1/p_1^0q_1 = p_1^1/p_1^0$.

Using the fact that a cost function is linearly homogeneous in its price arguments, it can be shown that P_K has the following homogeneity property: $P_K(p^0,\lambda p^1,q) = \lambda P_K(p^0,p^1,q)$ for $\lambda > 0$ which is analogous to the proportionality test T5 in the previous section. P_K also satisfies $P_K(p^1,p^0,q) = 1/P_K(p^0,p^1,q)$ which is analogous to the time reversal test, T11.

Note that the functional form for P_K is completely determined by the functional form for the aggregator function F which determines the functional form for the cost function C.

In general, P_K depends not only on the two price vectors p^0 and p^1, but also on the reference vector q. Malmquist (1953), Pollak (1983) and Samuelson and Swamy (1974) have shown that P_K is independent of q and is equal to a ratio of unit cost functions, $c(p^1)/c(p^0)$, if and only if the aggregator function F is homothetic.

If we knew the consumer's preferences or the producer’s technology, then we would know F and we could construct the cost function C and the Konüs price index P_K. However, we generally do not know F or C and thus it is useful to develop *bounds* that depend on observable price and quantity data but do not depend on the specific functional form for F or C.

Samuelson (1947) and Pollak (1983) established the following bounds on P_K. Let $p^0 >> 0_N$, and $p^1 >> 0_N$. Then for every reference quantity vector $q > 0_N$, we have

(33) $\min_n\{p_n^1/p_n^0\} \leq P_K(p^0,p^1,q) \leq \max_n\{p_n^1/p_n^0\}$;

i.e., P_K lies between the smallest and largest price ratios. Unfortunately, these bounds are usually too wide to be of much practical use.

To obtain closer bounds, we now assume that the observed quantity vectors for the two periods, $q^i \equiv (q_1^i,\ldots,q_N^i)$, i = 0,1, are solutions to the producer’s or consumer’s cost minimization problems; i.e., we assume:

(34) $p^i \cdot q^i = C[F(q^i),p^i]$, $\quad p^i >> 0_N$, $\quad q^i > 0_N$, $\quad i = 0,1$.

Given the above assumptions, we now have two natural choices for the reference quantity vector q that occurs in the definition of $P_K(p^0,p^1,q)$: q^0 or q^1. The *Laspeyres-Konüs price index* is defined as $P_K(p^0,p^1,q^0)$ and the *Paasche-Konüs price index* is defined as $P_K(p^0,p^1,q^1)$.

Under the assumption of cost minimizing behavior (34), Konüs (1924) established the following bounds:

$$(35)\quad P_K(p^0,p^1,q^0) \le p^1\cdot q^0/p^0\cdot q^0 \equiv P_L(p^0,p^1,q^0,q^1);$$

$$(36)\quad P_K(p^0,p^1,q^1) \ge p^1\cdot q^1/p^0\cdot q^1 \equiv P_P(p^0,p^1,q^0,q^1),$$

where P_L and P_P are the Laspeyres and Paasche price indexes defined earlier by (8) and (9). If in addition, the aggregator function is homothetic, then Frisch (1936) showed that for any reference vector $q > 0_N$,

$$(37)\quad P_P \equiv p^1\cdot q^1/p^0\cdot q^1 \le P_K(p^0,p^1,q) \le p^1\cdot q^0/p^0\cdot q^0 \equiv P_L.$$

In the consumer context, it is unlikely that preferences will be homothetic; hence the bounds (37) cannot be justified in general. However, Konüs (1924) showed that bounds similar to (37) would hold even in the general nonhomothetic case, provided that we choose a reference vector $q \equiv \lambda q^0 + (1-\lambda)q^1$ which is a λ, $(1-\lambda)$ weighted average of the two observed quantity points. Specifically, Konüs showed that there exists a λ between 0 and 1 such that if $P_P \le P_L$, then

$$(38)\quad P_P \le P_K[p^0,p^1,\lambda q^0 + (1-\lambda)q^1] \le P_L$$

or if $P_P > P_L$, then

$$(39)\quad P_L \le P_K[p^0,p^1,\lambda q^0 + (1-\lambda)q^1] \le P_P.$$

The bounds on the microeconomic price index P_K given by (37) in the homothetic case and (38)-(39) in the nonhomothetic case are the best bounds that we can obtain without making further assumptions on F. In the time series context, the bounds given by (38) or (39) are usually quite satisfactory: the Paasche and Laspeyres price indexes for consecutive time periods will usually differ by less than 1 percent (and hence taking the Fisher geometric average will generally suffice for most practical purposes). However, in the cross section context where the observations represent, for example, production data for two producers in the same industry but in different regions, the bounds are often not very useful since P_L and P_P can differ by 50 percent or more in the cross sectional context; see Ruggles (1967) and Hill (2006).

For generalizations of the above single household theory to many households, see Pollak (1980; 276) (1981; 328), Diewert (1983a) (2001) and chapter 18 in ILO (2004).

In Section 7 below, we will make additional assumptions on the aggregator function F or its cost function dual C that will enable us to determine P_K exactly. Before we do this, in the next section, we will define various quantity indexes that have their origins in microeconomic theory.

6. Economic Approaches to Quantity Indexes

In the one commodity case, a natural definition for a quantity index is q_1^1/q_1^0, the ratio of the single quantity in period 1 to the corresponding quantity in period 0. This ratio is also equal to the expenditure ratio, $p_1^1q_1^1/p_1^0q_1^0$, divided by the price ratio, p_1^1/p_1^0. This suggests that in the N commodity case, a reasonable definition for a quantity index would be the expenditure ratio divided by the Konüs price index, P_K. This type of index was suggested by Pollak (1983). Thus the *Konüs-Pollak quantity index*, Q_K, is defined by:

$$(40)\ Q_K(p^0,p^1,q^0,q^1,q) \equiv p^1\cdot q^1/p^0\cdot q^0\ P_K(p^0, p^1,q)$$
$$= \{C[F(q^1),p^1]/C[F(q),p^1]\}/\{C[F(q^0),p^0]/C[F(q),p^0]\}$$

where the second line follows from the definition of P_K, (32), and the assumption of cost minimizing behavior in the two periods, (34).

The definition of Q_K depends on the reference vector q which appears in the definition of P_K. The general definition of Q_K simplifies considerably if we choose the reference q to be q^0 or q^1. Thus define the Laspeyres-Konüs quantity index as

$$(41)\ \ Q_K(p^0,p^1,q^0,q^1,q^0) \equiv C[F(q^1),p^1]/C[F(q^0),p^1]$$

and the *Paasche-Konüs* quantity index as

$$(42)\ \ Q_K(p^0,p^1,q^0,q^1,q^1) \equiv C[F(q^1),p^0]/C[F(q^0),p^0].$$

It turns out that the indexes defined by (41) and (42) are special cases of another class of quantity indexes. For any reference price vector $p >> 0_N$, define the *Allen* (1949) *quantity index* by

$$(43)\ \ Q_A(q^0,q^1,p) \equiv C[F(q^1),p]/C[F(q^0),p].$$

If p is chosen to be p^0, (43) becomes (42) and if $p = p^1$, then (43) becomes (41).

Using the properties of cost functions, it can be shown that if $F(q^1) \geq F(q^0)$, then $Q_A(q^0,q^1,p) \geq 1$ while if $F(q^1) \leq F(q^0)$, then $Q_A(q^0,q^1,p) \leq 1$. Thus the Allen quantity index correctly indicates whether the commodity vector q^1 is larger or smaller than q^0. It can also be seen that Q_A satisfies a counterpart to the time reversal test; i.e., $Q_A(q^1,q^0,p) = 1/Q_A(q^0,q^1,p)$.

Just as the price index P_K depended on the unobservable aggregator function, so also do the quantity indexes Q_K and Q_A. Thus it is useful to develop bounds for the quantity indexes that do not depend on the particular functional form for F.

Samuelson (1947) and Allen (1949) established the following bounds for (41) and (42):

$$(44)\ Q_A(q^0,q^1,p^0) = Q_K(p^0,p^1,q^0,q^1,q^1) \leq p^0\cdot q^1/p^0\cdot q^0 \equiv Q_L;$$

(45) $Q_A(q^0,q^1,p^0) = Q_K(p^0,p^1,q^0,q^1,q^0) \geq p^1\cdot q^1/p^1\cdot q^0 \equiv Q_P$.

Note that the observable *Laspeyres* and *Paasche quantity indexes*, Q_L and Q_P, appear on the right hand sides of (44) and (45).

Diewert (1981), utilizing some results of Pollak (1983) and Samuelson and Swamy (1974), established the following results: if the underlying aggregator function F is neoclassical and (32) holds, then for all $p >> 0_N$ and $q >> 0_N$,

(46) $Q_P \leq Q_A(q^0,q^1,p) = Q_K(p^0,p^1,q^0,q^1,q) = F(q^1)/F(q^0) \leq Q_L$.

Thus if the aggregator function F is neoclassical, then the Allen quantity index for all reference vectors p equals the Konüs quantity index for all reference quantity vectors q which in turn equals the ratio of aggregates, $F(q^1)/F(q^0)$. Moreover, Q_A and Q_K are bounded from below by the Paasche quantity index Q_P, and bounded from above by the Laspeyres quantity index Q_L in the neoclassical case.

In the general nonhomothetic case, Diewert (1981) showed that there exists a λ between 0 and 1 such that $Q_K(p^0,p^1,q^0,q^1,\lambda q^0 +(1-\lambda)q^1]$ lies between Q_P and Q_L and there exists a λ^* between 0 and 1 such that $Q_A(q^0,q^1,\lambda^* p^0+(1-\lambda^*)p^1)$ also lies between Q_P and Q_L. Thus the observable Paasche and Laspeyres quantity indexes bound both the Konüs quantity index and the Allen quantity index, provided that we choose appropriate reference vectors between q^0 and q^1 and p^0 and p^1 respectively.

Using the linear homogeneity property of the cost function in its price arguments, we can show that the Konüs price index has the desirable homogeneity property, $P_K(p^0,\lambda p^0,q) = \lambda$ for all $\lambda > 0$; i.e., if period 1 prices are proportional to period 0 prices, then P_K equals this common proportionality factor. It would be desirable for an analogous homogeneity property to hold for quantity indexes. Unfortunately, it is not in general true that $Q_K(q^0,\lambda q^0,p^0,p^1,q) = \lambda$ or that $Q_A(q^0,\lambda q^0,p) = \lambda$. Thus we turn to a third economic approach to defining a quantity index which does have the desirable quantity proportionality property.

Let q^1 and q^2 be the observable quantity vectors in the two situations as usual, let F(q) be an increasing, continuous aggregator function, and let $q >> 0$ be a reference quantity vector. Then the *Malmquist* (1953) *quantity index* Q_M is defined as:

(47) $Q_M(q^0,q^1,q) \equiv D[F(q),q^1]/D[F(q),q^0]$

where $D(u,q^t) \equiv \max_k \{k : F(q^t/k) \geq u, k > 0\}$ is the *deflation* or *distance function* which corresponds to F. Thus $D[F(q),q^1]$ is the biggest number which will just deflate the quantity vector q^1 onto the boundary of the utility (or production) possibilities set $\{z : F(z) \geq F(q)\}$ indexed by the reference quantity vector q while $D[F(q),q^0]$ is the biggest number which will just deflate the quantity vector q^0 onto the set $\{z : F(z) \geq F(q)\}$ and Q_M is the ratio of these two deflation factors. Note that there is no optimization problem involving prices in the definition of the Malmquist quantity index but the definition of the distance function involves certain deflation problems that can be interpreted as technical efficiency optimization problems.

Q_M depends on the unobservable aggregator function F and as usual, we are interested in bounds for Q_M.

Diewert (1981) showed that Q_M satisfied bounds analogous to (33); i.e.,

(48) $\min_n\{q_n^1/q_n^0\} \leq Q_M(q^0,q^1,q) \leq \max_n\{q_n^1/q_n^0\}$.

As noted above, the assumption of cost minimizing behavior is not required in order to define the Malmquist quantity index or to establish the bounds (46). However, in order to establish the following bounds due to Malmquist (1953) for Q_M, we do need the assumption of cost minimizing behavior (32) for the two periods under consideration and we require the reference vector q to be q^0 or q^1:

(49) $Q_M(q^0,q^1,q^0) \leq p^0 \cdot q^1 / p^0 \cdot q^0 \equiv Q_L$;

(50) $Q_M(q^0,q^1,q^1) \geq p^1 \cdot q^1 / p^1 \cdot q^0 \equiv Q_P$.

Diewert (1981) showed that under the hypothesis of cost minimizing behavior, there exists a λ between 0 and 1 such that $Q_M(q^0,q^1,\lambda q^0+(1-\lambda)q^1)$ lies between Q_P and Q_L. Thus the Paasche and Laspeyres quantity indexes provide bounds for a Malmquist quantity index for some reference indifference or product surface indexed by a quantity vector which is a λ, $(1-\lambda)$ weighted average of the two observable quantity vectors, q^0 and q^1.

Pollak (1983) showed that if F is neoclassical, then we can extend the string of equalities in (46) to include the Malmquist quantity index $Q_M(q^0,q^1,q)$, for any reference quantity vector q. Thus in the case of a linearly homogeneous aggregator function, all three theoretical quantity indexes coincide and this common theoretical index is bounded from below by the Paasche quantity index Q_P and bounded from above by the Laspeyres quantity index Q_L.

In the general case of a nonhomothetic aggregator function, our best theoretical quantity index, the Malmquist index, is also bounded by the Paasche and Laspeyres indexes, provided that we choose a suitable reference quantity vector. In order to improve upon the bounding approach, Caves, Christensen and Diewert (1982b) show that if one is willing to assume optimizing behavior and make certain functional form assumptions about the underlying technology, then it is possible to obtain exact expressions for the Malmquist quantity index.

We noted in the price index context that the Paasche and Laspeyres price indexes were usually quite close in the time series context. A similar remark also applies to the Paasche and Laspeyres quantity indexes. Thus taking an average of the Paasche and Laspeyres indexes, such as the Fisher price and quantity indexes, will generally approximate underlying microeconomic price and quantity indexes sufficiently accurately for most practical purposes. However, this observation does not apply to the cross sectional context, where the Paasche and Laspeyres indexes can differ widely. In the following section, we offer another microeconomic justification for using the Fisher indexes that also applies in the context of making interregional and cross country comparisons.

7. Exact and Superlative Indexes

Assume that the producer or consumer is maximizing a neoclassical aggregator function f subject to a budget constraint during the two periods. Under these conditions, it can be shown that the economic agent is also minimizing cost subject to a utility or output constraint. Moreover, the cost function C that corresponds to f can be written as $C[f(q),p] = f(q)c(p)$ where c is the unit cost function (recall (28) above).

Suppose a bilateral price index $P(p^0,p^1,q^0,q^1)$ and the corresponding quantity index $Q(p^0,p^1,q^0,q^1)$ that satisfy (5) are given. The quantity index Q is defined to be *exact* for a neoclassical aggregator function f with unit cost dual c if for every $p^0 \gg 0_N$, $p^1 \gg 0_N$ and $q^i \gg 0_N$ which is a solution to the aggregator maximization problem $\max_q\{f(q) : p^i \cdot q \leq p^i \cdot q^i\} = f(q^i) > 0$ for $i = 0,1$, we have

$$(51)\quad Q(p^0,p^1,q^0,q^1) = f(q^1)/f(q^0).$$

Under the same hypothesis, the price index P is *exact* for f and c if we have

$$(52)\quad P(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0).$$

In (51) and (52), the price and quantity vectors are not regarded as being independent. The p^i can be independent, but the q^i are solutions to the corresponding aggregator maximization problem involving p^i, for $i = 0,1$. Note that if Q is exact for a neoclassical f, then Q can be interpreted as a Konüs, Allen or Malmquist quantity index and the corresponding P defined implicitly by (5) can be interpreted as a Konüs price index.

The concept of exactness is due to Konüs and Byushgens (1926). Below, we shall give some examples of exact index number formulae. Additional examples may be found in Afriat (1972), Pollak (19783), Samuelson and Swamy (1974) and Diewert (1976) (1992b).

Konüs and Byushgens (1926) showed that Irving Fisher's ideal price index P_F defined by (14) and the corresponding quantity index Q_F defined implicitly by (5) are exact for the homogeneous quadratic aggregator function f defined by

$$(53)\quad f(q_1,\ldots,q_N) \equiv (\textstyle\sum_{n-1}^N \sum_{m-1}^N a_{nm}q_nq_m)^{1/2} \equiv (q \cdot Aq)^{1/2}$$

where $A \equiv [a_{nm}]$ is a symmetric $N \times N$ matrix of constants. Thus under the assumption of maximizing behavior, we can show that $f(q^1)/f(q^0) = Q_F$ and $c(p^1)/c(p^0) = P_F$ where f is defined by (51) and c is the unit cost function that corresponds to f. The important point to note is that f depends on N(N+1)/2 unknown a_{nm} parameters but we do not need to know these parameters in order to be able to calculate $f(q^1)/f(q^0)$ and $c(p^1)/c(p^0)$.

Diewert (1976) showed that the Törnqvist Theil price index P_T defined by (22) is exact for the unit cost function c(p) defined by:

(54) $\ln c(p) \equiv \alpha_0 + \sum_{n=1}^N \alpha_n \ln p_n + (1/2) \sum_{m=1}^N \sum_{n=1}^N \alpha_{mn} \ln p_m \ln p_n$

where the parameters α_n and α_{mn} satisfy the following restrictions:

(55) $\sum_{n=1}^N \alpha_n = 1$, $\sum_{n=1}^N \alpha_{mn} = 0$ for m = 1,..., N and $\alpha_{mn} = \alpha_{nm}$ for all m, n.

Thus we may calculate $c(p^1)/c(p^0) = P_T$ and $f(q^1)/f(q^0) = p^1\cdot q^1/p^0\cdot q^0\, P_T \equiv Q_T$ where c is the unit cost function defined by (54), f is the aggregator function which corresponds to this c, and Q_T is the implicit Törnqvist Theil quantity index. Note that we do not have to know the parameters α_n and α_{mn} in order to evaluate $c(p^1)/c(p^0)$ and $f(q^1)/f(q^0)$.

The unit cost function defined by (54) is the *translog* unit cost function defined by Christensen, Jorgenson and Lau (1971). Since P_T is exact for this translog functional form, P_T is sometimes called the *translog price index*.

Define the following family of quantity indexes Q_r that depend on a number, $r \neq 0$:

(56) $Q_r(p^0,p^1,q^0,q^1) \equiv [\sum_{n=1}^N s_n^0(q_n^1/q_n^0)^{r/2}]^{1/r} [\sum_{m=1}^N s_m^1(q_m^1/q_m^0)^{-r/2}]^{-1/r}$

where $s_n^i \equiv p_n^i q_n^i/p^i\cdot q^i$ is the period i expenditure share for good n. For each $r \neq 0$, define the corresponding implicit price index by:

(57) $P_r^*(p^0,p^1,q^0,q^1) \equiv p^1\cdot q^1/p^0\cdot q^0\, Q_r(p^0,p^1,q^0,q^1)$.

A bit of algebra will show that when r = 2, $P_2^* = P_F$, the Fisher price index defined by (14) and when r equals 1, P_1^* equals:

(58) $P_1^* = \sum_{n=1}^N p_n^1(q_n^0 q_n^1)^{1/2} / \sum_{m=1}^N p_m^0(q_m^0 q_m^1)^{1/2} = P_W$

where P_W is the Walsh price index defined earlier by (17).

Diewert (1976) showed that Q_r and P_r^* are exact for the *quadratic mean of order* r *aggregator function* f_r defined as follows:

(59) $f_r(q_1,...,q_N) \equiv (\sum_{m=1}^N \sum_{n=1}^N a_{mn}\, q^{r/2}\, q_n^{r/2})^{1/r}$

where $A \equiv [a_{mn}]$ is a symmetric matrix of constants. Thus the Walsh and Fisher price indexes, P_W and P_F, are exact for $f_1(q)$ and $f_2(q)$ respectively, defined by (59) when r = 1 and 2.

Diewert (1974) defined a linearly homogeneous function f of N variables to be *flexible* if it could provide a second order approximation to an arbitrary twice continuously differentiable linearly homogeneous function. It can be shown that f defined by (53), c defined by (54) and (55) and f_r defined by (59) for each $r \neq 0$ are all examples of flexible functional forms.

Let the price and quantity indexes P and Q satisfy the product test equality, (5). Then Diewert (1976) defined P and Q to be *superlative indexes* if either P is exact for a flexible unit cost function c or Q is exact for a flexible aggregator function f. Thus P_F, P_W, P_T and P_r^* are all superlative price indexes. Thus from the viewpoint of the economic approach to index number theory, all of these indexes can be judged to be equally good.

At this point, it is useful to review the various approaches to bilateral index number theory discussed in the previous sections. In section 2, it was found that the "best" average basket approaches led to the Fisher or Walsh price indexes. In section 3, the "best" index from the viewpoint of the stochastic approach was the Törnqvist Theil index. In section 4, the test approach led to the Fisher or the Törnqvist Theil indexes as being "best". Finally, in this section, the economic approach led to the Fisher, Walsh and Fisher or the Törnqvist Theil indexes as being equally good. *Thus all four major approaches to index number theory led to the same three indexes as being best*. But which one of these three formulae, P_F, P_W and P_T, should we choose? Fortunately, it does not matter very much which of these formulae we choose to use in applications; they will all give the same answer to a reasonably high degree of approximation. Diewert (1978; 889) showed that all known superlative index number formulae approximate each other to the second order when each index is evaluated at an equal price and quantity point. This means the P_F, P_W, P_T and each P_r^* have the same first and second order partial derivatives with respect to all 4N arguments when the derivatives are evaluated at a point where $p^0 = p^1$ and $q^0 = q^1$. A similar string of equalities also holds for the corresponding implicit quantity indexes defined using the product test (5). In fact, these derivative equalities are still true provided that $p^1 = \lambda p^0$ and $q^1 = \mu q^0$ for any numbers $\lambda > 0$ and $\mu > 0$. However, although Diewert's approximation result is mathematically true, Hill (2006) has shown that superlative indexes of the form P_r^* for r very large in magnitude do not necessarily empirically approximate the standard superlative indexes P_F, P_W and P_T very closely. But these standard superlative indexes typically approximate each other to something less than 0.2 percent in the time series context and to about 2 percent in the cross section context; see Fisher (1922), Ruggles (1967), Diewert (1978; 894-895) and Hill (2006) for empirical evidence on this point.

Diewert (1978) also showed that the Paasche and Laspeyres indexes approximate the superlative indexes to the first order at an equal price and quantity point. In the time series context, for adjacent periods, the Paasche and Laspeyres price indexes typically differ by less than 0.5 percent; hence these indexes may provide acceptable approximations to a superlative index.

Having considered the case of two observations at length, the many observation case is considered in the following two sections.

8. The Fixed Base Versus the Chain Principle

In this section, the merits of using the chain system for constructing price indexes in the time series context versus using the fixed base system are discussed.

The chain system, introduced independently into the economics literature by Lehr (1885; 45-46) and Marshall (1887; 373), measures the change in prices going from one period to a subsequent period using a bilateral index number formula involving the prices and quantities pertaining to the two adjacent periods. These one period rates of change (the links in the chain) are then cumulated to yield the relative levels of prices over the entire period under consideration. Thus if the bilateral price index is P, the chain system generates the following pattern of price levels for the first three periods:

(60) $1, P(p^0,p^1,q^0,q^1), P(p^0,p^1,q^0,q^1)\, P(p^1,p^2,q^1,q^2)$.

On the other hand, the fixed base system of price levels using the same bilateral index number formula P simply computes the level of prices in period t relative to the base period 0 as $P(p^0,p^t,q^0,q^t)$. Thus the fixed base pattern of price levels for periods 0,1 and 2 is:

(61) $1, P(p^0,p^1,q^0,q^1), P(p^0,p^2,q^0,q^2)$.

Due to the difficulties involved in obtaining current period information on quantities (or equivalently, on expenditures), as was indicated in section 2, many statistical agencies loosely base their Consumer Price Index on the use of the Laspeyres formula and the fixed base system. Therefore, it is of some interest to look at some of the possible problems associated with the use of fixed base Laspeyres indexes.

The main problem with the use of the fixed base Laspeyres index is that the period 0 fixed basket of commodities that is being priced out in period t can often be quite different from the period t basket. Thus if there are systematic *trends* in at least some of the prices and quantities in the index basket, the fixed base Laspeyres price index $P_L(p^0,p^t,q^0,q^t)$ can be quite different from the corresponding fixed base Paasche price index, $P_P(p^0,p^t,q^0,q^t)$. This means that both indexes are likely to be an inadequate representation of the movement in average prices over the time period under consideration.

As Hill (1988) noted, the fixed base Laspeyres quantity index cannot be used forever: eventually, the base period quantities q^0 are so far removed from the current period quantities q^t that the base must be changed. Chaining is merely the limiting case where the base is changed each period.

The main advantage of the chain system is that under normal conditions, chaining will reduce the spread between the Paasche and Laspeyres indexes; see Diewert (1978; 895) and Hill (1988) (1993; 387-388). These two indexes each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration and it could be expected that a single point estimate of the aggregate price change should lie between these two estimates. Thus the use of either a chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth".

Hill (1993; 388), drawing on the earlier research of Szulc (1983) and Hill (1988; 136-137), noted that it is not appropriate to use the chain system when prices oscillate or "bounce" to use Szulc's (1983; 548) term. This phenomenon can occur in the context of regular seasonal fluctuations or in the context of price wars. However, in the context of roughly monotonically changing prices and quantities, Hill (1993; 389) recommended the use of chained symmetrically weighted indexes. The Fisher, Walsh and Törnqvist Theil indexes are examples of symmetrically weighted indices.

It is possible to be more precise under what conditions one should chain or not chain. Following arguments due to Walsh (1901; 206) (1921a; 84-85) and Fisher (1911; 204 and 423-424), one should chain if the prices and quantities pertaining to adjacent periods are *more similar* than the prices and quantities of more distant periods, since this strategy will lead to a narrowing of the spread between the Paasche and Laspeyres indices at each link. Of course, one needs a measure of how similar are the prices and quantities pertaining to two periods. The similarity measures could be *relative* ones or *absolute* ones. In the case of absolute comparisons, two vectors of the same dimension are similar if they are identical and dissimilar otherwise. In the case of relative comparisons, two vectors are similar if they are proportional and dissimilar if they are nonproportional. Once a similarity measure has been defined, the prices and quantities of each period can be compared to each other using this measure and a "tree" or path that links all of the observations can be constructed where the most similar observations are compared with each other using a bilateral index number formula. Fisher (1922; 271-276) informally suggested this strategy. However, the recent literature on this approach is due to Robert Hill. Initially, Hill (1999a) (1999b) (2001) defined the price structures between the two countries to be more dissimilar the bigger is the spread between P_L and P_P; i.e., the bigger is max $\{P_L/P_P, P_P/P_L\}$. The problem with this measure of dissimilarity in the price structures of the two countries is that it could be the case that $P_L = P_P$ (so that the Hill measure would register a maximal degree of similarity) but p^0 could be very different than p^t. Thus there is a need for a more systematic study of similarity (or dissimilarity) measures in order to pick the "best" one that could be used as an input into Hill's (1999a) (1999b) (2001) (2004) (2006b) (2007) spanning tree algorithm for linking observations; see Diewert (2007a).

The method of linking observations explained in the previous paragraph based on the similarity of the price and quantity structures of any two observations may not be practical in a statistical agency context since the addition of a new period may lead to a reordering of the previous links. However, the above "scientific" method for linking observations may be useful in deciding whether chaining is preferable or whether fixed base indexes should be used while making month to month comparisons within a year.

Some index number theorists have objected to the chain principle on the grounds that it has no counterpart in the spatial context:

"They [chain indexes] only apply to intertemporal comparisons, and in contrast to direct indices they are not applicable to cases in which no natural order or sequence exists. Thus the idea of a chain index for example has no counterpart in interregional or international price comparisons, because countries cannot be sequenced in a 'logical' or 'natural' way (there is no k+1 nor k−1country to be compared with country k)." Peter von der Lippe (2001; 12).

This is of course correct but the approach of Robert Hill does lead to a "natural" set of spatial links. Applying the same approach to the time series context will lead to a set of links between periods which may not be month to month but it will in many cases justify year over year linking of the data pertaining to the same month.

It is of some interest to determine if there are index number formulae that give the same answer when either the fixed base or chain system is used. Comparing the sequence of chain indexes defined by (60) above to the corresponding fixed base indexes defined by (61), it can be seen that we will obtain the same answer in all three periods if the index number formula P satisfies the following functional equation for all price and quantity vectors:

(62) $P(p^0,p^2,q^0,q^2) = P(p^0,p^1,q^0,q^1)\, P(p^1,p^2,q^1,q^2)$.

If a bilateral index number formula P satisfies (62), then P satisfies the *circularity test*; see Westergaard (1890; 218-219) and Fisher (1922; 413).

If it is assumed that the index number formula P satisfies certain properties or tests in addition to the circularity test above, then Funke, Hacker and Voeller (1979) showed that P must have the following functional form due originally to Konüs and Byushgens (1926; 163-166):

(63) $\ln P_{KB}(p^0,p^1,q^0,q^1) \equiv \sum_{i=1}^{N} \alpha_i \ln(p_i^1/p_i^0)$

where the N constants α_i satisfy the following restrictions:

(64) $\sum_{i=1}^{N} \alpha_i = 1$ and $\alpha_i > 0$ for $i = 1,\ldots,N$.

Thus under very weak regularity conditions, the only price index satisfying the circularity test is a weighted geometric average of all the individual price ratios, the weights being constant through time. This result vindicates Irving Fisher's (1922; 274) intuition who asserted that "the only formulae which conform perfectly to the circular test are index numbers which have *constant weights*…".

The problem with the indexes defined by Konüs and Byushgens is that the individual price ratios, p_n^1/p_n^0, have weights that are *independent* of the economic importance of commodity n in the two periods under consideration. Put another way, these price weights are independent of the quantities of commodity n consumed or the expenditures on commodity n during the two periods. Hence, these indexes are not really suitable for use by statistical agencies at higher levels of aggregation when expenditure share information is available.

The above results indicate that it is not useful to ask that the price index P satisfy the circularity test *exactly*. However, it is of some interest to find index number formulae that satisfy the circularity test to some degree of *approximation* since the use of such an index number formula will lead to measures of aggregate price change that are more or less the same no matter whether we use the chain or fixed base systems. Irving Fisher (1922; 284) found that deviations from circularity using his data set and the Fisher ideal price index P_F were quite small. This relatively high degree of correspondence between fixed base and chain indexes has been found to hold for other symmetrically weighted formulae like the Walsh index P_W defined earlier. It is possible to give a theoretical explanation for the approximate satisfaction of the circularity test in the time series context for symmetrically weighted index number formulae, such as P_F and P_W. Another symmetrically weighted formula is the Törnqvist Theil index P_T. Alterman, Diewert and Feenstra (1999; 61) showed that if the logarithmic price ratios ln (p_n^t/p_n^{t-1}) trend linearly with time t and the expenditure shares s_n^t also trend linearly with time, then the Törnqvist index P_T will satisfy the circularity test exactly. Since many economic time series on prices and quantities satisfy these assumptions approximately, then the Törnqvist index P_T will satisfy the circularity test approximately. As was noted earlier, the Törnqvist index generally closely approximates the symmetrically weighted Fisher and Walsh indexes, so that for many economic time series (with smooth trends), all three of these symmetrically weighted indexes will satisfy the circularity test to a high enough degree of approximation so that it will not matter whether we use the fixed base or chain principle.

Walsh (1901; 401) (1921a; 98) (1921b; 540) introduced the following useful variant of the circularity test:

$$(65)\ 1 = P(p^0,p^1,q^0,q^1)\ P(p^1,p^2,q^1,q^2)\ldots P(p^{T-1},p^T,q^{T-1},q^T)\ P(p^T,p^0,q^T,q^0).$$

The motivation for this test is the following one. Use the bilateral index formula $P(p^0,p^1,q^0,q^1)$ to calculate the change in prices going from period 0 to 1, use the same formula evaluated at the data corresponding to periods 1 and 2, $P(p^1,p^2,q^1,q^2)$, to calculate the change in prices going from period 1 to 2, … , use $P(p^{T-1},p^T,q^{T-1},q^T)$ to calculate the change in prices going from period T−1 to T, introduce an artificial period T+1 that has exactly the price and quantity of the initial period 0 and use $P(p^T,p^0,q^T,q^0)$ to calculate the change in prices going from period T to 0. Finally, multiply all of these indexes together and since we end up where we started, then the product of all of these indexes should ideally be one. Diewert (1993a; 40) called this test a *multiperiod identity test*. Note that if T = 2 (so that the number of periods is 3 in total), then Walsh's test reduces to Fisher's (1921; 534) (1922; 64) *time reversal test*.

Walsh (1901; 423-433) showed how his circularity test could be used in order to evaluate how "good" any bilateral index number formula was. What he did was invent artificial price and quantity data for 5 periods and he added a sixth period that had the data of the first period. He then evaluated the right hand side of (65) for various bilateral formula, $P(p^0,p^1,q^0,q^1)$, and determined how far from unity the results were. His "best" formulae had products that were close to one. Fisher (1922; 284) later used this methodology as well.

This same framework is often used to evaluate the efficacy of chained indexes versus their direct counterparts. Thus if the right hand side of (65) turns out to be different than unity, the chained indexes are said to suffer from "chain drift". If a formula does suffer from chain drift, it is sometimes recommended that fixed base indexes be used in place of chained ones. However, this advice, if accepted would *always* lead to the adoption of fixed base indexes, provided that the bilateral index formula satisfies the identity test, $P(p^0,p^0,q^0,q^0) = 1$. Thus it is not recommended that Walsh's circularity test be used to decide whether fixed base or chained indexes should be calculated. However, it is fair to use Walsh's circularity test as he originally used it i.e., as an approximate method for deciding how "good" a particular index number formula is. In order to decide whether to chain or use fixed base indexes, one should decide on the basis of how similar are the observations being compared and choose the method which will best link up the most similar observations.

Robert Hill's method for linking observations can be regarded as a multilateral index number method; one which is based on a suitable bilateral formula, a measure of the similarity of any two price and quantity vectors and an algorithm for linking the observations via a path that links the most similar observations. In the following section, we review some other multilateral methods.

9. Multilateral Indexes

Assume that there are I positive price vectors $p^i \equiv (p_1^i,\ldots,p_N^i)$ and I quantity vectors $q^i \equiv (q_1^i,\ldots,q_N^i)$ with $p^i \cdot q^i > 0$ for i = 1,…,I. We wish to find 2I positive numbers P^i (price indexes) and Q^i (quantity indexes) such that $P^iQ^i = p^i \cdot q^i$ for i = 1,…,I. The I data points (p^i,q^i) will typically be observations on production or consumption units that are separated spatially but yet are still comparable. For the sake of definiteness, we shall refer to the I data points as countries. Each commodity n is supposed to be the same across all countries. This can always be done by a suitable extension of the list of commodities.

Our first approach to the construction of a system of multilateral price and quantity indexes is based on the use of a bilateral quantity index Q. In this method, the first step is to pick the 'best' bilateral index number formula: e.g., the Fisher quantity index Q_F defined by (14) and (5) or the implicit Törnqvist Theil quantity index Q_T defined by (22) and (5). Secondly, pick a numeraire country, say country 1, and then calculate the aggregate quantity for each country i relative to country 1 by evaluating the quantity index $Q(p^1,p^i,q^1,q^i)$. In order to put these relative quantity measures on a symmetric footing, we convert each relative to country 1 quantity measure into a share of world quantity by dividing through by $\sum_{k=1}^{I} Q(p^1,p^k,q^1,q^k)$. For a general numeraire country j, define the *share of world quantity for country i, using country j as the numeraire country*, by:

$$(66)\quad \sigma_i^j(p,q) \equiv Q(p^j,p^i,q^j,q^i) / \sum_{k=1}^{I} Q(p^j,p^k,q^j,q^k) ; \qquad i = 1,\ldots,I,$$

where $p \equiv (p^1,\ldots,p^I)$ is the N by I matrix of price data and $q \equiv (q^1,\ldots,q^I)$ is the N by I matrix of quantity data. Once the numeraire country j has been chosen and the country i shares σ_i^j calculated, we may set $Q^i \equiv \sigma_i^j$ and $P^i \equiv p^i \cdot q^i/Q^i$ for $i = 1,\ldots,I$. Thus we have provided a solution to the multilateral index number problem (1). Of course, one is free to renormalize the resulting P^i and Q^i if desired; i.e., all Q^i can be multiplied by a number provided all P^i are divided by this same number. Kravis (1984) called this method the *star system*, since the numeraire country plays a starring role: all countries are compared with it and it alone.

Of course, the problem with the star system for making multilateral comparisons is its lack of invariance to the choice of the numeraire or star country. Different choices for the base country will in general give rise to different indexes P^i and Q^i. This problem can be traced to the lack of circularity of the bilateral formula Q: if Q satisfies the time reversal test and the circular test for quantity indexes, then $\sigma_i^j = \sigma_i^k$ for all i, j and k; i.e., the shares σ_i^j defined by (66) do not depend on the choice of the numeraire country j. However, given that the chosen "best" bilateral formula does not satisfy the circularity test (as is the case with Q_F and Q_T), how can we generate multilateral indexes that treat each country symmetrically?

Fisher (1922; 305) recognized that the simplest way of achieving symmetry was to average base specific index numbers over all possible bases. Thus define country i's share of world output $S_i(p,q)$ by

$$(67)\quad S_i(p,q) \equiv \sum_{j=1}^{I} \sigma_i^j(p,q)/I, \qquad i = 1,\ldots,I$$

where the σ_i^j are defined by (66). We can now define country i quantities and prices by

$$(68)\quad Q^i \equiv S_i(p,q);\ P^i \equiv p^i \cdot q^i/Q^i, \qquad i =1,\ldots,I.$$

Fisher (1922; 305) called this method of constructing multilateral indexes the *blend method* while Diewert (1986) called it the *democratic weights method*, since each share of world output using each country as the base is given an equal weight in the formation of the average.

Of course, there is no need to use an arithmetic average of the σ_i^j as in (67); one can use a geometric average:

$$(69)\quad \sigma_i(p,q) \equiv [\prod_{j=1}^{I} \sigma_i^j(p,q)]^{1/I}, \qquad i = 1,\ldots,I.$$

Using (69), the resulting shares no longer sum to one in general, so country i's share of world output is now defined as:

$$(70)\quad S_i(p,q) \equiv \sigma_i(p,q) / \sum_{k=1}^{I} \sigma_k(p,q), \qquad i = 1,\ldots,I.$$

If the Fisher index Q_F is used in the definition of the σ_i^j, then

$$(71)\quad S_i(p,q)/S_j(p,q) = [\prod_{k=1}^{I} Q_F(p^k,p^i,q^k,q^i) / \prod_{m=1}^{I} Q_F(p^m,p^j,q^m,q^j)]^{1/I}$$

and in this case, the multilateral method defined by (71) reduces to a method recommended by Gini (1924) (1931), Eltetö and Köves (1964) and Szulc (1964), the *GEKS method*. Instead of using the Fisher formula in (71), Caves, Christensen and Diewert (1982a) advocated the use of the (direct) Törnqvist Theil quantity index while Diewert (1986) suggested the use of the implicit translog quantity index Q_T defined by (5) when P is P_T defined by (22), since Q_T is well defined even in the case where some quantities q_n^i are negative. We call the indexes generated by (69) and (70) for a general bilateral index Q *generalized GEKS* indexes.

When forming averages of the σ_i^j as in (67) or (69), there is no necessity to use equal weights: one can define country j's value share of world output as $\beta_j \equiv p^j \cdot q^j / \sum_{k=1}^{I} p^k \cdot q^k$ (this requires all prices to be measured in units of a common currency) and then we may define a plutocratic share weighted average of the σ_i^j:

$$(72)\ S_i(p,q) \equiv \sum_{j=1}^{I} \beta_j(p,q)\ \sigma_i^j(p,q).$$

Diewert (1986) called this method of constructing multilateral indexes the *plutocratic weights method*.

Another multilateral method that is based on a bilateral index Q may be described as follows. Define

$$(73)\ \sigma_i(p,q) \equiv \sum_{j=1}^{I} [Q(p^j,p^i,q^j,q^i)^{-1}]^{-1}\ ; \qquad i = 1,\ldots,I.$$

If there is only one commodity so that N = 1 and the bilateral index Q satisfies quantity counterparts to tests T3 and T5, then $\alpha_i = [\sum_{j=1}^{I}(q^i/q^j)^{-1}]^{-1} = [\sum_{j=1}^{I} q^j/q^i]^{-1} = q^i/\sum_{j=1}^{I} q^j$ which is country i's share of world product. In the general case where N > 1, the "shares" α_i do not necessarily sum up to unity, so it is necessary to normalize them:

$$(74)\quad S_i(p,q) \equiv \alpha_i(p,q) / \sum_{k=1}^{I} \alpha_k(p,q)\ ; \qquad i = 1,\ldots,I.$$

Diewert (1986) (1988) (1999b) called this the *own share method* for making multilateral comparisons.

The above methods for achieving consistency and symmetry rely on averaging over various bilateral index number comparisons. Fisher (1922; 307) realized that symmetry could be achieved by making comparisons with an average; he called this broadening the base. Thus the *average basket method* (see Walsh (1901; 431), Gini (1931; 8)) Fisher (1922; 307), Ruggles (1967) and Diewert (1999b; 24-25)) may be described as follows. The price level of country I relative to country j is set equal to $p^i\cdot(\sum_{k=1}^J q^k/I)/p^j\cdot(\sum_{k=1}^J q^k/I)$. Now define $Q^{ji} \equiv [p^i\cdot q^i/p^j\cdot q^j]/[p^i\cdot(\sum_k q^k)/p^j\cdot(\sum_k q^k)]$ to be the implicit output of country i relative to j. Choose a j as a numeraire country and calculate country i's share of world output as:

$$(75)\ S_i(p,q) \equiv Q^{ji} / \sum_{k=1}^I Q^{jk} = (p^i\cdot q^i/p^i\cdot\sum_k q^k) / \sum_{m=1}^I (p^m\cdot q^m/p^m\cdot\sum_k q^k)\ ; \qquad i = 1,\ldots,I.$$

Note that the final expression for S_i does not depend on the choice of the numeraire country j. As usual, once the share functions, S_i, have been defined, the aggregate Q^i and P^i may be defined by (68).

A variation on the basket method due to Geary (1958) and Khamis (1972) is defined by (76)-(78) below:

$$(76)\ \pi_n \equiv \sum_{i=1}^I p_n^i q_n^i / P^i \sum_{k=1}^I q_n^k, \qquad n = 1,\ldots,N;$$
$$(77)\ P^i \equiv \sum_{n=1}^N p_n^i q_n^i / \sum_{m=1}^N \pi_m q_m^i, \qquad i = 1,\ldots,I;$$
$$(78)\ Q^i \equiv p^i\cdot q^i/P^i, \qquad i = 1,\ldots,I.$$

π_n is interpreted as an average international price for good n. From (77), it can be seen that P^i, the price level or purchasing power parity for country i, is a Paasche-like price index for country i except that the base prices are chosen to be the international prices π_n. The π_n and $(P^i)^{-1}$ can be solved for as a system of simultaneous linear equations (up to a scalar normalization) or the $(P^i)^{-1}$ may be determined as the components of the eigenvector that corresponds to the maximal positive eigenvalue of a certain matrix. The P^i can be normalized so that the quantities Q^i defined by (78) sum up to unity. This GK *method* for making multilateral comparisons has been widely used in empirical applications; e.g., see Kravis, Kenessey, Heston and Summers (1975).

We have defined seven methods for making multilateral comparisons: the star method (66), the democratic (67) and plutocratic (72) weights methods, the GEKS method (71), the own share method (74), the average basket method (75) and the GK method (78). Many additional methods have been suggested; e.g., see Hill (1997), Diewert (1986) (1988) (1999b), Rao (1990) and Balk (1996). How can we discriminate among them? One helpful approach would be to define a system of *multilateral tests* and then evaluate how the above methods satisfy these tests. Space does not permit the development of this approach in this short survey; for applications of this approach, see Diewert (1988) (1999b) and Balk (1996). A clear consensus on the "best" multilateral method has not yet emerged.

We conclude this section by looking at a stochastic or descriptive statistics approach to making multilateral comparisons: namely Summer's (1973) Country Product Dummy (CPD) method for making multilateral comparisons. If there are I countries in the comparison and N products, the relationship of the prices between the various countries using the CPD model is given (approximately) by the following model:

(79) $p_n^c \approx \alpha_c\beta_n$; $c = 1,....,I$; $n = 1,...,N$;
(80) $\alpha_1 = 1$

where p_n^c is the price (in domestic currency) of commodity n in country c. Quantities for each commodity in each country are assumed to be measured in the same units. Equation (80) above is an identifying normalization; i.e., we measure the price level of each country relative to the price level in country 1. Note that there are IN prices in the model and there are $I - 1 + N$ parameters to "explain" these prices. Note also that the basic hypothesis that is implied by (79) is that commodity prices are approximately proportional between the two countries. Taking logarithms of both sides of (79) and adding error terms leads to the following CPD regression model:

(81) $\ln p_n^c = \ln \alpha_c + \ln \beta_n + \varepsilon_n^c$; $c = 1,....,I$; $n = 1,...,N$.

The main advantage of the CPD method for comparing prices across countries over traditional index number methods is that we can obtain *standard errors* for the country price levels $\alpha_2, \alpha_3,..., \alpha_I$. This advantage of the stochastic approach to index number theory was stressed by Summers (1973) and more recently by Selvanathan and Rao (1994).

The recent literature on the CPD method notes that it is a special case of a hedonic regression model and this recent literature makes connections between weighted hedonic regressions and traditional index number formulae; see Triplett and McDonald (1977), Diewert (2003) (2005b) (2005c) (2007b), de Haan (2004a) (2004b), Silver (2003) and Silver and Heravi (2005).

10. Other Aspects of Index Number Theory

There are many important recent developments in index number theory that we cannot cover in any depth in this brief survey. Some of these developments are:

- *Sampling problems and the construction of indexes at the first stage of aggregation*; see Dalén (1992), Diewert (1995a), ILO (2004) and IMF (2004).
- *The treatment of seasonality*; see Turvey (1979), Balk (1980) (2005), Diewert (1983c) (1998b) (1999a), Hill (1996), Alterman, Diewert and Feenstra (1999), ILO (2004) and Armknecht and Diewert (2004).
- *The analysis of sources of bias in consumer price indexes*. This topic was greatly stimulated by the Boskin Commission Report; see Boskin, Dullberger, Gordon, Griliches and Jorgenson (1996). For additional contributions to this subject, see

Diewert (1987) (1998a), Reinsdorf (1993), Schultze and Mackie (2002), Lebow and Rudd (2003), Balk and Diewert (2004) and ILO (2004).

- *Productivity indexes*. As more and more countries start programs to measure sectoral and economy wide productivity, this topic has become more important . The original methodology for measuring productivity using index number techniques is due to Jorgenson and Griliches (1967) (1972) and it was first adopted by the U.S, Bureau of Labor Statistics (1983) and subsequently by Canada, Australia and more recently by New Zealand and Switzerland. Diewert (1976) (1983b) Caves, Christensen and Diewert (1982b), Diewert and Morrison (1986), Kohli (1990), Morrison and Diewert (1990), Balk (1998) (2003), Schreyer (2001), Diewert and Fox (2004), Diewert and Nakamura (2003) and Diewert and Lawrence (2006) all made contributions connecting productivity measurement with index number theory.
- *Contribution analysis*. Suppose an aggregate price or quantity index shows a certain change over a certain period. Many analysts want to be able to compute the contribution of price or quantity change of specific components of the overall index and the problem of precisely defining such contributions has given rise to a fairly substantial recent literature. Contributors to this literature include Diewert (1983b) (2002a), Diewert and Morrison (1986), van IJzeren (1987), Kohli (1990) (2003) (2004) (2007), Morrison and Diewert (1990), Fox and Kohli (1998) and Reinsdorf, Diewert and Ehemann (2002).
- *Quality change*. The analysis thus far has assumed that the list of commodities in the aggregate is fixed and is unchanging and thus it is not able to deal with the problem of quality change. For extensive discussions of this problem, see Triplett (2004) and the chapters on quality change in ILO (2004) and IMF (2004).
- *Index number theory in terms of differences rather than ratios*. Hicks (1941-42) noticed the similarities between measuring welfare change (difference measures) and index numbers of quantity change (ratio measures). The early literature on the difference approach dates back to Bennet (1920) and Montgomery (1929) (1937). More recent contributions to this subject may be found in Diewert (1992b) (2005a).

The last 20 years has seen an increase in interest in index number theory and economic measurement problems in general. Perhaps influenced by Hill (1993), national statistical agencies are moving towards using chained superlative indexes as their target indexes; see Moulton and Seskin (1999) and Cage, Greenlees and Jackman (2003) for U.S. developments. International agencies have also endorsed the use of superlative indexes as target indexes; see the Manuals produced by the ILO (2004) and the IMF (2004). These Manuals are a useful development since they help disseminate best practices and they help to harmonize statistics across countries, leading to a higher degree of accuracy and comparability. Hopefully, these positive developments will continue.

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