Does Capitalized Net Product Equal Discounted Optimal Consumption in Discrete Time? by W.E. Diewert and P. Schreyer.[1]

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Abstract

In 1976, Weitzman derived some provocative results on the equality of discounted consumption to discounted net product produced by the market sector of the economy using continuous time optimization techniques. The question addressed in this note is whether discrete time counterparts to his results exist. Our answer is, in general, no. However, if the one period production function is restricted to certain classes of functions, then the paper shows that a discrete time approximate version of the Weitzman result will hold and under other conditions, his result will hold exactly in discrete time. The problem of obtaining general conditions that will imply a discrete time version of the Weitzman result remains open. An example presented in section 7 shows that if there is (foreseen) technical progress or growth in "fixed" inputs, then a discrete time counterpart to the Weitzman result will not hold in general.

Key Words

Capital, discrete versus continuous time optimization, discounted consumption, national output, intertemporal optimization, income, saddlepoint theorems.

Journal of Economic Literature Classification Codes

C61, D6, D9, E2.

1. Introduction

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Weitzman (1976) developed some very important results that provided a welfare interpretation for net national product (in a closed economy with no government, no technical progress and no tax distortions). He described his results as follows:

"Even granted that consumption is the ultimate end of economic activity, the national income statistician's practice of adding in investment goods to the value of consumption by weighting them with prices measuring their marginal rates of transformation might still be defended as a measure of the economy's power to consume at a constant rate. After all, a standard welfare interpretation of NNP is that it is the largest permanently maintainable value of consumption. If all investment were convertible into consumption at the given price transformation rates, the maximum attainable level of consumption that could be maintained forever without running down capital stocks would appear to be NNP as conventionally measure by $C^* + pdK^*/dt$. Unfortunately, such reasoning is insufficient because *marginal* transformation rates cannot in general be used to change *nonmarginal* amounts of investment into consumption. For this reason, the *consumption* level $C^*(t) + p(t)dK^*(t)/dt$ is undoubtedly not attainable at time t."[2] Martin L. Weitzman (1976; 159).

"All this notwithstanding, it turns out that the maximum welfare actually attainable from time t on along a competitive trajectory, $\int_t^\infty C^*(s)e^{-r(t-s)}ds$, is exactly the same as what *would* be obtained from the *hypothetical* constant consumption level $C^*(t) + p(t)dK^*(t)/dt$. In this sense, the naive interpretation of the current power to consume at a constant rate idea gives the right answer, although for the wrong reason. Net nation product is what might be called the *stationary equivalent* of future consumption and this is its primary welfare interpretation."[3] Martin L. Weitzman (1976; 160).

The gist of the above results are as follows: a very accurate estimate of the discounted value of present and future consumption generated by the market sector of an economy can be obtained by capitalizing current consumption and *net* investment, using a constant real interest rate. This very powerful result was derived using continuous time optimization techniques. The question we ask in this note is whether there is a discrete time counterpart to Weitzman's results.

The short answer to the above question is no: Weitzman's result can hold in discrete time for a certain class of one period production functions but it does not appear to hold for all technologies.

An outline of the paper follows. Section 2 lays out the algebra involved in a simple static production function framework with only one consumption good where we maximize net product less capital service rentals, holding constant labour, land and other fixed factors. In section 3, we consider a simple (primal) intertemporal profit maximization problem where the intertemporal structure of consumption prices is given exogenously by the assumption of an exogenous constant real interest rate. In section 4, we show how the primal intertemporal profit maximization problem can be "decentralized" using duality theory. The results in this section show that the primal intertemporal consumption maximization problem can be supported by decentralized period by period profit

[2] This same point was made very clearly by Samuelson (1961; 45-46).

[3] Weitzman notes that his result will not hold if there is technical progress in the future: "If there is nonattributable or 'atmospheric' technical change, national product will be less than the stationary equivalent of consumption by a term measuring the present discounted value of the pure effect of time alone on increasing output." Martin L. Weitzman (1976; 160).

maximizing behavior on the part of producers. Sections 5-7 present three concrete examples where specific functional forms for the one period production function are assumed and the corresponding intertemporal optimization problems are solved explicitly. Using these explicit solutions, we show in sections 5 and 6, making assumptions similar to those made by Weitzman that the discrete time counterpart to his result can hold (section 6) but it also does not hold (section 5). In section 7, we relax one of Weitzman's assumptions (that there be no technical progress or fixed input growth) and show that under these conditions, there is virtually no chance for counterparts to his result to hold in discrete time. Section 8 concludes.

2. The Single Period Production Function Framework

In this section, we set out our basic assumptions. We follow Weitzman in assuming that there is only a single consumption good C. We simplify Weitzman's model by assuming that the economy utilizes only a single reproducible capital good K (instead of a vector of capital goods) and we let I be the *net change in the capital stock* so that I is *net investment* that occurs in the market sector of the economy in a given period. Production in any given period is characterized by the production function F(K,I) where the maximum amount of consumption C that can be produced in the economy in any period is

(1) $C = F(K,I)$

where K is the initial (net) capital stock available to the economy at the start of the period and I is the amount of (net) investment that takes place during the period. Note that the production function F is assumed to remain the same over all periods.[4] We assume that F(K,I) is a nonnegative, concave, once differentiable function of K and I that is strictly increasing in K and strictly decreasing in I[5] over some closed, convex domain of definition set S.[6]

In each period t, we assume that producers solve the following profit maximization problem, conditional on the beginning of the period fixed factors that they have at their disposal:

(2) $\max_{C,K,I} \{p_C^t C + p_I^t I - p_K^t K : C = F(K,I) ; (K,I) \in S\}$
$\quad = \max_{K,I} \{p_C^t F(K,I) + p_I^t I - p_K^t K : (K,I) \in S\}$

[4] Actually, this assumption is not necessary for the *analysis* which follows: instead, we could assume that producers could perfectly anticipate future technologies and under this assumption, our analysis (which follows) would remain essentially unchanged. In section 7, we adopt this more general framework. However, if we are to obtain a counterpart to Weitzman's result, it is necessary to make the same assumptions as he made.
[5] In fact, we will assume that $\partial F(K,I)/\partial K \geq 0$ and $\partial F(K,I)/\partial I \leq 0$ for all $(K,I) \in S$. We also assume that $\partial F(K,I)/\partial K$ is bounded from above and $\partial F(K,I)/\partial I$ is bounded from below for all $(K,I) \in S$.
[6] Of course, there can be a vector L of fixed (labor and land) factors lurking in the background; i.e., the production function should be written as $C = F(K,I,L)$. However, since we will hold L fixed at the same level from period to period, we can suppress L from the notation.

where p_C^t, p_I^t and p_K^t are the period t prices for consumption C, (net) investment I and capital services K respectively. Let C^t, K^t and I^t solve (2). Set the price of consumption equal to unity; i.e., assume that p_C^t equals 1. Assuming that (K^t,I^t) belongs to the interior of the domain of definition set S, K^t and I^t will satisfy the following first order necessary conditions for the maximization problem defined by (2):[7]

(3) $F_1(K^t,I^t) - p_K^t = \partial F(K^t,I^t)/\partial K - p_K^t = 0$;
(4) $F_2(K^t,I^t) + p_I^t = \partial F(K^t,I^t)/\partial I + p_I^t = 0$.

Thus the rental price of capital in period t, p_K^t, and the price of a new investment good produced in period t, p_I^t, are equal to the following first order partial derivatives of the production function evaluated at the period t equilibrium values for capital and net investment, K^t,I^t:

(5) $p_K^t = F_1(K^t,I^t) > 0$;
(6) $p_I^t = -F_2(K^t,I^t) > 0$.

We will use (5) and (6) in the following sections in order to provide economic interpretations for the first order partial derivatives of the production function in terms of market prices for the economy in period t.

We turn now to the dynamic intertemporal optimization problem that the economy faces starting at period 0.

3. The Production Sector's Period 0 Intertemporal Optimization Problem

We follow Weitzman (1976) in assuming that the market production sector of the economy faces the constant real interest rate r > 0 over all periods. Let the price of the consumption good in period 0 be set equal to 1 and let the price of the consumption good in period t be $1/(1+r)^t$ for t = 1,2, ... The market sector's period 0 intertemporal profit maximization problem is the following constrained optimization problem, where producers attempt to maximize the discounted value of consumption goods sold on the market, subject to their technological production function constraints:[8]

(7) $\max_{I^0,I^1,...;K^1,K^2,...;C^0,C^1,...} \{C^0 + (1+r)^{-1}C^1 + (1+r)^{-2}C^2 + ... : \text{(i) } C^t = F(K^t,I^t) ; t = 0,1,2,...;$
$\text{(ii) } K^1 = K^0 + I^0 ; K^2 = K^1 + I^1 ; K^3 = K^2 + I^2 ; ... \}$

(8) $= \max_{I^0,I^1,...;K^1,K^2,...} \{F(K^0,I^0) + (1+r)^{-1}F(K^1,I^1) + (1+r)^{-2}F(K^1,I^1) + ... :$
$\text{(ii) } K^1 = K^0 + I^0 ; K^2 = K^1 + I^1 ; K^3 = K^2 + I^2 ; ... \}$

[7] Since F is concave and (K^t,I^t) is in the interior of S, these conditions are also sufficient for (K^t,I^t) to solve (2).
[8] For simplicity, we have dropped the domain of definition restrictions $(K^t,I^t)\in S$ for t = 0,1,2, ... from problems (7), (8) and (10).

where we have substituted the constraints (i) into the objective function to obtain (8) from (7). Note that the constraints (ii) provide the dynamics for the model; i.e., the net capital stock at the beginning of period 1, K^1 is equal to the net capital stock at the beginning of period 0, K^0, (note that this is a constant in the maximization problems (7) and (8)) plus net investment in period 0, I^0, and so on. We can rewrite the constraints (ii) in the following more useful form:

(9) $K^1 = K^0 + I^0$; $K^2 = K^0 + I^0 + I^1$; $K^3 = K^0 + I^0 + I^1$; ... .

The constraints (9) can be used to eliminate the capital stock variables K^1, K^2, K^3, ... in the constrained maximization problem (8). When these variables are eliminated, (8) becomes the following unconstrained maximization problem:

(10) $\max_{I^0,I^1,\ldots} \{F(K^0,I^0) + (1+r)^{-1}F(K^0+I^0,I^1) + (1+r)^{-2}F(K^0+I^0+I^1,I^2) + \ldots \}$.

We assume that a net investment solution I^{0*} I^{1*}, I^{2*}, ... to (10) exists and that the resulting optimized objective function is finite.[9] Equations (9) can be used to define the corresponding capital stock solution sequence:[10]

(11) $K^{1*} \equiv K^0 + I^{0*}$; $K^{2*} \equiv K^0 + I^{0*} + I^{1*}$; $K^{3*} \equiv K^0 + I^{0*} + I^{1*}$; ... .

Once the optimal capital stocks have been determined by (11), the optimal consumption sequence can be determined using the production function constraints (i) in (7):

(12) $C^{0*} \equiv F(K^0,I^{0*})$; $C^{t*} \equiv F(K^{t*},I^{t*})$, t = 1,2, ...

Under our regularity conditions, the following sequence of first order necessary conditions for the intertemporal maximization problem (10) must hold:

(13) $F_2(K^0,I^{0*}) + (1+r)^{-1}F_1(K^0+I^{0*},I^{1*}) + (1+r)^{-2}F_1(K^0+I^{0*}+I^{1*},I^{2*}) + \ldots = 0$;
$(1+r)^{-1}F_2(K^0+I^{0*},I^{1*}) + (1+r)^{-2}F_1(K^0+I^{0*}+I^{1*},I^{2*}) + (1+r)^{-3}F_1(K^0+I^{0*}+I^{1*}+I^{2*},I^{3*}) + \ldots = 0$;
... .

Recalling equations (5) and (6) in the previous section, we can recognize the partial derivative $F_2(K^0,I^{0*})$ as minus the period 0 price of an investment good, $-p_I^0$, $F_1(K^0+I^{0*},I^{1*}) = F_1(K^{1*},I^{1*})$ as p_K^1, the price of capital services in period 1, $F_1(K^0+I^{0*}+I^{1*},I^{2*}) = F_1(K^{2*},I^{2*})$ as p_K^2, the price of capital services in period 2, and so on. Using these substitutions, it can be seen that equations (13) are equivalent to the following system of equations:[11]

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[9] Since $F(K^{t*},I^{t*}) \geq 0$, $(1+r)^{-t}F(K^{t*},I^{t*})$ must tend to zero as t tends to infinity.

[10] We assume that (K^{t*},I^{t*}) belongs to the interior of S for each t so that we have an interior solution for (10).

[11] This type of result dates back to Böhm-Bawerk (1891; 342) at least; see Diewert (2005) for more on the history of this result.

$$
\begin{aligned}
(14)\ p_I^0 &= (1+r)^{-1}p_K^1 + (1+r)^{-2}p_K^2 + (1+r)^{-3}p_K^3 + \dots \\
p_I^1 &= (1+r)^{-1}p_K^2 + (1+r)^{-2}p_K^3 + (1+r)^{-3}p_K^4 + \dots \\
p_I^2 &= (1+r)^{-1}p_K^3 + (1+r)^{-2}p_K^4 + (1+r)^{-3}p_K^5 + \dots \\
&\dots .
\end{aligned}
$$

Equations (14) have simple economic interpretations. The first equation in (14) says that the price of an investment good in period 0, p_I^0, is equal to the discounted stream of future expected rentals that the investment good will earn in future periods, $(1+r)^{-1}p_K^1 + (1+r)^{-2}p_K^2 + (1+r)^{-3}p_K^3 + \dots$; the second equation in (14) says that the price of an investment good in period 1, p_I^1, is equal to the discounted stream of future expected rentals that the investment good will earn in future periods, $(1+r)^{-1}p_K^2 + (1+r)^{-2}p_K^3 + (1+r)^{-3}p_K^4 + \dots$ and so on.

In the following section, instead of using the constraints in (8) to eliminate the capital stock variables K^1, K^2, ... from the intertemporal maximization problem, we take a Lagrangian approach to solving the problem. This approach will enable us to relate the solution to (8) to the net product approach used by Weitzman in his continuous time counterpart to our discrete time approach.

4. An Alternative Approach to the Intertemporal Consumption Maximization Problem

The results in this section show that the primal intertemporal consumption maximization problem (10) can be supported by decentralized profit maximizing behavior on the part of producers.

Instead of substituting the constraints (9) into the objective function of (8) to obtain an unconstrained maximization problem, in this section, we use the Lagrangian (or saddle point) approach to solving the problem. Letting λ^i be the Lagrange multiplier that corresponds to the ith constraint in (9), define the *Lagrangian* as follows:

$$
\begin{aligned}
(15)\ L(I^0,I^1,I^2,\dots;K^1,K^2,\dots;u^1,u^2,\dots) \equiv\ & F(K^0,I^0) + (1+r)^{-1}F(K^1,I^1) + (1+r)^{-2}F(K^2,I^2) + \dots \\
&+ (1+r)^{-1}u^1[K^0 + I^0 - K^1] \\
&+ (1+r)^{-2}u^2[K^0 + I^0 + I^1 - K^2] \\
&+ (1+r)^{-2}u^3[K^0 + I^0 + I^1 + I^2 - K^3] \\
&+ \dots
\end{aligned}
$$

where we have replaced the original Lagrange multipliers λ^t by transformed multipliers u^t defined as follows:

$$(16)\ u^t \equiv \lambda^t(1+r)^t ; \qquad t = 1,2, \dots .$$

The first order necessary conditions for $I^* \equiv [I^{0*},I^{1*},I^{2*},...]$ and $K^* \equiv [K^{1*},K^{2*},K^{3*},...]$ to solve (8) are the existence of $u^* \equiv [u^{1*},u^{2*},u^{3*},...]$ such that the following equations are satisfied:

(17) $\partial L(I^*,K^*,u^*)/\partial I^t = 0$; $\quad t = 0,1,2, ...$;
(18) $\partial L(I^*,K^*,u^*)/\partial K^t = 0$; $\quad t = 1,2, ...$;
(19) $\partial L(I^*,K^*,u^*)/\partial u^t = 0$; $\quad t = 1,2, ...$.

Equations (19) are equivalent to equations (9) evaluated at the equilibrium point. Equations (17) simplify into the following system of equations:

(20)
$$- F_2(K^{0*},I^{0*}) = (1+r)^{-1}u^{1*} + (1+r)^{-2}u^{2*} + (1+r)^{-3}u^{3*} + ...$$
$$- F_2(K^{1*},I^{1*}) = (1+r)^{-1}u^{2*} + (1+r)^{-2}u^{3*} + (1+r)^{-3}u^{4*} + ...$$
$$- F_2(K^{2*},I^{2*}) = (1+r)^{-1}u^{3*} + (1+r)^{-2}u^{4*} + (1+r)^{-3}u^{5*} + ...$$
$$... .$$

Equations (18) simplify into the following system of equations:

(21)
$$F_1(K^{1*},I^{1*}) = u^{1*}$$
$$F_1(K^{2*},I^{2*}) = u^{2*}$$
$$F_1(K^{3*},I^{3*}) = u^{3*}$$
$$... .$$

If we substitute equations (21) into (20), it can be seen that we obtain our old system of first order conditions (13). But a more interesting observation that emerges from equations (20) and (21) is the fact that the (transformed) Lagrange multiplier for period t, u^{t*}, can be identified with the price of (net) capital services in period t, p_K^t, and the derivatives, $- F_2(K^{t*},I^{t*})$, can be identified with the period t price of an investment good, p_I^t; i.e., we have:

(22) $- F_2(K^{t*},I^{t*}) = p_I^t$; $\quad t = 0,1,2, ...$;
(23) $F_1(K^{t*},I^{t*}) = p_K^t = u^{t*}$; $\quad t = 1,2,3, ...$.

Now we need to show that the optimized objective function V^* for the reduced form intertemporal consumption maximization problem (10) is equal to a saddle point of the Lagrangian defined by (15); i.e., we want to show that

(24)
$$C^* \equiv C^{0*} + (1+r)^{-1}C^{1*} + (1+r)^{-2}C^{2*} + ...$$
$$= \max_{I^0,I^1,...} \{F(K^0,I^0) + (1+r)^{-1}F(K^0+I^0,I^1) + (1+r)^{-2}F(K^0+I^0+I^1,I^2) + ... \}$$
$$= \max_{I^0,I^1,...;K^1,K^2,...} \min_{u^1,u^2,...} \{L(I^0,I^1,I^2,...; K^1,K^2,...; u^1,u^2,...): (K^t,I^t)\in S \text{ for } t = 0,1, ...; u^t \geq 0 \text{ for } t = 1,2, ...\}$$

where $L(I^0,I^1,I^2,...; K^1,K^2,...; u^1,u^2,...)$ is the Lagrangian defined by (15) above.[12] To see why the last equation in (24) holds, consider the following finite dimensional approximation to the first optimization problem in (24). Let the sequence $\{I^{t*}: t = 0,1,2, ...\}$ solve (10) and let T be large enough so that the following sum of terms is as small as desired:

$$(25)\ (1+r)^{-(T+2)}F(K^0+I^{0*}+I^{1*}+...+I^{T+1*},I^{T+2*}) + (1+r)^{-(T+3)}F(K^0+I^{0*}+I^{1*}+...+I^{T+2*},I^{T+3*}) + ...$$

Thus C^* will be approximately equal to the following maximization problem that involves only the finite number of variables $I^0,I^1,...,I^T$:

$$(26)\ C^* \cong \max_{I^0,I^1,...,I^T} \{F(K^0,I^0) + (1+r)^{-1}F(K^0+I^0,I^1) + (1+r)^{-2}F(K^0+I^0+I^1,I^2) + ... \\ (1+r)^{-(T+1)}F(K^0+I^0+I^1+...I^T,I^{T+1*}): (K^0,I^0)\in S, (K^0+I^0,I^1)\in S, ... , (K^0+I^0+...I^T,I^{T+1*})\in S\}$$

$$(27) = \max_{I^0,I^1,...I^T;K^1,K^2,...,K^T} \min_{u^1,u^2,...u^T} \{L^T(I^0,I^1,I^2,...I^T; K^1,K^2,...K^T; u^1,u^2,...,u^T): \\ (K^t,I^t)\in S \text{ for } t = 0,1,...,T; u^t \geq 0 \text{ for } t = 1,2,...,T\}$$

where the T period approximate Lagrangian L^T is defined as follows:

$$(28)\ L^T(I^0,I^1,I^2,...I^T; K^1,K^2,...K^T; u^1,u^2,...,u^T) \equiv F(K^0,I^0) + (1+r)^{-1}F(K^1,I^1) \\ + (1+r)^{-2}F(K^2,I^2) + ... + (1+r)^{-T}F(K^T,I^T + (1+r)^{-1}u^1[K^0 + I^0 - K^1] \\ + (1+r)^{-2}u^2[K^0 + I^0 + I^1 - K^2] + ... + (1+r)^{-T}u^t[K^0 + I^0 + I^1 + ... + I^T - K^{T+1}].$$

The equality of (27) with the line above follows from the Karlin (1959; 201) Uzawa (1958; 34) Saddle Point Theorem, where we have used the concavity of F and the convexity of the set S.[13] Now let T tend to plus infinity and the last equality in (24) will follow.

Now insert the optimal multipliers u^{t*} into the last equation in (24). From (23), we know u^{t*} is equal to p_K^t, the (anticipated) period t price of capital services. Using these relations (23), we see that the last equation in (24) can be rewritten as follows:

$$(29)\ C^* = \max_{I^0,I^1,...;K^1,K^2,...} \{L(I^0,I^1,I^2,...; K^1,K^2,...; p_K^1,p_K^2,...): (K^t,I^t)\in S \text{ for } t = 0,1, ...\} \\ = \max_{I^0,I^1,...;K^1,K^2,...} \{F(K^0,I^0) + (1+r)^{-1}F(K^1,I^1) + (1+r)^{-2}F(K^2,I^2) + ... \\ + (1+r)^{-1}p_K^1[K^0 + I^0 - K^1] + (1+r)^{-2}p_K^2[K^0 + I^0 + I^1 - K^2] \\ + (1+r)^{-2}p_K^3[K^0 + I^0 + I^1 + I^2 - K^3] + ... : (K^t,I^t)\in S \text{ for } t = 0,1, ...\} \\ \text{using definition (15) for L} \\ = \max_{I^0,I^1,...;K^1,K^2,...} \{F(K^0,I^0) + (1+r)^{-1}F(K^1,I^1) + (1+r)^{-2}F(K^2,I^2) + ...$$

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[12] We have converted the equality constraints $K^1 = K^0 + I^0$; $K^2 = K^0 + I^0 + I^1$; $K^3 = K^0 + I^0 + I^1$; ... in (8) into the inequality constraints $K^1 \leq K^0 + I^0$; $K^2 \leq K^0 + I^0 + I^1$; $K^3 \leq K^0 + I^0 + I^1$; ... .This change will not change the solution to the problem under our assumptions but the change leads to the nonnegativity constraints for the Lagrange multipliers $u^t \geq 0$ for t = 1,2, ... in the saddle point problem.

[13] We also require that the following constraint qualification constraint condition hold: for each $(u^1,u^2,...u^T) > 0_T$, we have some $I^0,I^1,...,I^T$ and $K^1,K^2,...,K^T$ such that $(K^0,I^0)\in S$, $(K^1,I^1)\in S$, ... , $(K^T,I^T)\in S$.

$$
\begin{aligned}
&+ [(1+r)^{-1}p_K^{\,1} + (1+r)^{-2}p_K^{\,2} + (1+r)^{-2}p_K^{\,3} + \ldots\ ][K^0 + I^0] \\
&+ (1+r)^{-1}[(1+r)^{-1}p_K^{\,2} + (1+r)^{-2}p_K^{\,3} + (1+r)^{-2}p_K^{\,4} + \ldots\ ]I^1 \\
&+ (1+r)^{-1}[(1+r)^{-1}p_K^{\,3} + (1+r)^{-2}p_K^{\,4} + (1+r)^{-2}p_K^{\,5} + \ldots\ ]I^2 \\
&+ \ldots \\
&- (1+r)^{-1}p_K^{\,1}K^1 - (1+r)^{-2}p_K^{\,2}K^2 - (1+r)^{-3}p_K^{\,3}K^3 - \ldots : (K^t,I^t)\in S \text{ for } t = 0,1, \ldots\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{rearranging terms} \\
= \max_{I^0,I^1,\ldots;K^1,K^2,\ldots}\ &\{F(K^0,I^0) + (1+r)^{-1}F(K^1,I^1) + (1+r)^{-2}F(K^2,I^2) + \ldots \\
&+ p_I^{\,0}[K^0 + I^0] + (1+r)^{-1}p_I^{\,1}I^1 + (1+r)^{-2}p_I^{\,2}I^2 + (1+r)^{-3}p_I^{\,3}I^3 + \ldots \\
&- (1+r)^{-1}p_K^{\,1}K^1 - (1+r)^{-2}p_K^{\,2}K^2 - (1+r)^{-3}p_K^{\,3}K^3 - \ldots : (K^t,I^t)\in S \text{ for } t = 0,1, \ldots\} \\
&\qquad\qquad \text{using equations (14) which defined the investment prices} \\
&\qquad\qquad \text{in terms of future discounted capital services prices} \\
= \max_{I^0,I^1,\ldots;K^1,K^2,\ldots}\ &\{F(K^0,I^0) + p_I^{\,0}[K^0 + I^0] + (1+r)^{-1}[F(K^1,I^1) + p_I^{\,1}I^1 - p_K^{\,1}K^1] \\
&+ (1+r)^{-2}[F(K^2,I^2) + p_I^{\,2}I^2 - p_K^{\,2}K^2] + (1+r)^{-3}[F(K^3,I^3) + p_I^{\,3}I^3 - p_K^{\,3}K^3] + \ldots. \\
&: (K^t,I^t)\in S \text{ for } t = 0,1, \ldots\} \qquad\qquad\qquad \text{rearranging terms} \\
(30) = &\ C^{0*} + p_I^{\,0}I^{0*} + p_I^{\,0}K^0 + (1+r)^{-1}[C^{1*} + p_I^{\,1}I^{1*} - p_K^{\,1}K^{1*}] \\
&+ (1+r)^{-2}[C^{2*} + p_I^{\,2}I^{2*} - p_K^{\,2}K^{2*}] + (1+r)^{-3}[C^{3*} + p_I^{\,3}I^{3*} - p_K^{\,3}K^{3*}] + \ldots
\end{aligned}
$$

where (30) follows from the line above by substituting the optimal solution to (8) into the line above, since we know the solution to (8) solves the maximization problem in (29) when we substitute in the optimal dual multipliers $u^{t*} = p_K^{\,t}$ for all t. Thus we have two expressions for the optimal discounted value of consumption C^*: namely (24) and (30).

We now indicate how we can obtain a third expression for C^*. From equations (14), we saw that the value of an investment good in a given period was equal to the discounted value of expected future capital services. Now a unit of capital that is available at the beginning of period 0 will be more valuable than a unit of the investment good produced in period 0 because this unit of already available capital will produce $p_K^{\,0}$ units of capital services in addition to the capital services produced by a unit of investment in period 0 where $p_K^{\,0}$ is defined as

$$(31)\ p_K^{\,0} \equiv F_1(K^0,I^{0*}).$$

Let $P_K^{\,0}$ be the (asset) value of a unit of already installed capital services at the beginning of period 0. Then its value, compared to the value of an investment good produced during period 0, $p_I^{\,0}$, is given by:

$$
\begin{aligned}
(32)\ P_K^{\,0} &\equiv p_K^{\,0} + p_I^{\,0} \\
&= p_K^{\,0} + (1+r)^{-1}p_K^{\,1} + (1+r)^{-2}p_K^{\,2} + (1+r)^{-3}p_K^{\,3} + \ldots \qquad \text{using (14).}
\end{aligned}
$$

Now if we add $p_K^{\,0} - p_K^{\,0}$ to the right hand side of (30) and use (32), we obtain the resulting expression for the discounted value of optimal consumption C^*:

$$
\begin{aligned}
(33)\ C^* = &\ P_K^{\,0}K^0 + [C^{0*} + p_I^{\,0}I^{0*} - p_K^{\,0}K^0] + (1+r)^{-1}[C^{1*} + p_I^{\,1}I^{1*} - p_K^{\,1}K^{1*}] \\
&+ (1+r)^{-2}[C^{2*} + p_I^{\,2}I^{2*} - p_K^{\,2}K^{2*}] + (1+r)^{-3}[C^{3*} + p_I^{\,3}I^{3*} - p_K^{\,3}K^{3*}] + \ldots
\end{aligned}
$$

= the value of the initial capital stock plus the value of market sector expected future discounted profits.

The right hand side of (33) represents our third expression for V^*. Note that the period 0 profits, $C^{0*} + p_I^0 I^{0*} - p_K^0 K^0$, represent the period 0 returns to the fixed (labour, land, entrepreneurship and natural resource) factors, the period profits, $C^{1*} + p_I^1 I^{1*} - p_K^1 K^1$, represent the period 1 returns to the fixed factors and so on. Thus equation (33) says that the value of the initial (reproducible) capital stock plus the value of the discounted returns to fixed factors is just sufficient for households to purchase the value of discounted consumption produced by the market sector. This is an intuitively plausible result.

We now define N^* as the (anticipated) *discounted value of the market sector's net output* over the present and all future periods; i.e., we define:

$$(34)\ N^* \equiv [C^{0*} + p_I^0 I^{0*}] + (1+r)^{-1}[C^{1*} + p_I^1 I^{1*}] + (1+r)^{-2}[C^{2*} + p_I^2 I^{2*}] + (1+r)^{-3}[C^{3*} + p_I^3 I^{3*}] + \ldots .$$

It will also prove convenient to define K^* as the *discounted value of the market sector's value of capital services*; i.e., we define:

$$(35)\ K^* \equiv p_K^0 K^0 + (1+r)^{-1} p_K^1 K^{1*} + (1+r)^{-2} p_K^2 K^{2*} + (1+r)^{-3} p_K^3 K^{3*} + \ldots .$$

Using (34) and (35), (33) can be rewritten as follows:

$$(36)\ C^* = N^* + P_K^0 K^0 - K^*.$$

Thus the discounted value of consumption produced by the market sector C^* is equal to the discounted value of the net product produced by the market sector N^* plus the value of the initial capital stock $P_K^0 K^0$ less the discounted value of (net) capital services used by the market sector K^*.

An interesting question that we could ask at this point is whether the discounted value of consumption equals the discounted value of market sector net product; i.e., we ask under what conditions the following equality might hold:

$$(37)\ C^* = N^*.$$

However, in view of (36), in order to obtain (37) we would require that the value of the initial capital stock be equal to the discounted value of (net) capital services used by the market sector over the present and future periods; i.e., we would require that

$$(38)\ P_K^0 K^0 = K^*.$$

In general, it does not appear that condition (38) will hold. In particular, if the initial net capital stock K^0 for the economy is small relative to later optimal values K^{t*} for the economy (think of an economy in the initial stages of development), then

(39) $P_K^0K^0 < K^*$ which implies $C^* < N^*$

so that net product will tend to overstate consumption. The fact that (37) does not hold in general is not surprising. Define the discounted value of optimal net investments, I^*, as follows:

(40) $I^* \equiv p_I^0I^{0*} + (1+r)^{-1}p_I^1I^{1*} + (1+r)^{-2}p_I^2I^{2*} + (1+r)^{-3}p_I^3I^{3*} + \ldots$.

Using definitions (24), (34) and (40), we have:

(41) $N^* = C^* + I^*$.

Thus if the positive optimal net investments I^{t*} are large enough to outweigh any negative I^{t*} so that the discounted present value I^* defined by (40) is positive, then (41) will imply that N^* is greater than C^*.

However, there is one set of conditions which will imply the equalities (38) and (41) and that is if the initial capital stock corresponds to a stationary equilibrium. Thus suppose that

(42) $I^0 \equiv 0$; $C^0 \equiv F(K^0,0)$; $C^t = C^0$, $I^t = 0$, $K^t = K^0$ for $t = 1,2,\ldots$.

The first order necessary and sufficient conditions (13) which rationalize this equilibrium reduce to the following single equation:

$$
\begin{aligned}
(43)\ -F_2(K^0,0) &= (1+r)^{-1}F_1(K^0,0) + (1+r)^{-2}F_1(K^0,0) + \ldots \\
&= (1+r)^{-1}F_1(K^0,0)[1 + (1+r)^{-1} + (1+r)^{-2} + \ldots] \\
&= (1+r)^{-1}F_1(K^0,0)[1- (1+r)^{-1}]^{-1} \\
&= (1+r)^{-1}F_1(K^0,0)(1+r)/r \\
&= F_1(K^0,0)/r.
\end{aligned}
$$

Thus if the positive interest rate r, the initial market sector capital stock K^0 and the initial partial derivatives of the production function evaluated at a 0 investment initial point and $-F_2(K^0,0) > 0$ and $F_1(K^0,0) > 0$ are just of the right magnitude to satisfy equation (43), then we have an optimal stationary equilibrium with $C^* = N^*$ and $P_K^0K^0 = K^*$.[14]

5. A Class of Technologies where there is no Counterpart to Weitzman's Result

[14] If we look at the first equation in (13) and equation (43), it can be seen that the objective function in (10) can be increased when evaluated at the stationary point defined by (42) by increasing I^0 by a marginal amount provided that $F_1(K^0,0)/r > - F_2(K^0,0)$ or provided that $r < F_1(K^0,0)/r\ /[- F_2(K^0,0)]$; i.e., provided that the real interest rate is sufficiently small relative to the technology and initial capital stock. Under these conditions, K^* will tend to be bigger than $P_K^0K^0$ and N^* will tend to overstate C^*.

The results derived in the previous section are not discrete time equivalents to the Weitzman result; his results are much stronger. A discrete time counterpart to Weitzman's (1976; 160) continuous time result is the following equation:

$$(44)\ C_W \equiv [C^{0*} + p_I{}^0 I^{0*}][1 + (1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + ...]$$
$$= [C^{0*} + p_I{}^0 I^{0*}](1+r)/r$$
$$= C^{0*} + (1+r)^{-1}C^{1*} + (1+r)^{-2}C^{2*} + (1+r)^{-3}C^{3*} + ...$$
$$\equiv C^*.$$

Thus on the left hand side of (44), we capitalize the initial period 0 optimal net product, $C^{0*} + p_I{}^0 I^{0*}$; i.e., we assume that this net product persists into the future and we calculate the present value of this constant consumption and investment stream, which we define as the *Weitzman approximation* C_W to discounted consumption. On the right hand side, we calculate the actual discounted present value of optimal consumption which we have previously denoted by C^*.

In this section, we provide an explicit example which shows that (44) does not hold in general.

Let the one period production function F(K,I) have the following form:[15]

$$(45)\ F(K,I) \equiv f(K) - I$$

where f(K) is a once differentiable, strictly concave function defined over a closed convex set S that contains the initial net capital stock, K^0, in its interior. As usual, $r > 0$ is the real interest rate that the market sector faces. We assume that the real interest rate and the initial capital stock satisfy the following inequality:[16]

$$(46)\ df(K^0)/dK = f\,'(K^0) > r.$$

We assume that there exists a $K^{1*} \in S$ such that $K^{1*} = K^0 + I^{0*}$ where $I^{0*} > 0$ such that

$$(47)\ f\,'(K^{1*}) = f\,'(K^0 + I^{0*}) = r.$$

Use the above I^{0*} in order to define the following period 0 consumption, which we assume is positive:

$$(48)\ C^{0*} \equiv f(K^0) - I^{0*} > 0.$$

[15] In view of Weitzman's (1976; 160) diagram, this functional form would seem to be the most favorable case to establish a discrete time counterpart to his results; i.e., with this functional form, the tradeoff between net investment and consumption within a period is represented by a linear frontier. This type of assumption on the intertemporal technology is made frequently in the literature; e.g., see Dorfman, Samuelson and Solow (1958; 309).

[16] This corresponds to an initial capital stock that is too small relative to the optimal capital stock.

Now consider the intertemporal consumption maximization problem (10) with F(K,I) defined by (45). We will show that the following sequence of net investments solves (10):

(49) $I^{0*} > 0$ defined by (47) and $I^{t*} \equiv 0$ for t = 1,2, ...

The corresponding sequence of optimal net capital stocks and consumptions are defined as follows:

(50) $K^{1*} \equiv K^0 + I^{0*}$ and $K^{t*} \equiv K^{1*}$ for t = 2,3, ... ;
(51) C^{0*} defined by (48) and $C^{t*} \equiv C^{1*} \equiv f(K^{1*})$ for t = 2,3, ...

Thus after a making a positive investment in period 0, we end up making no net investments in subsequent periods and in fact, from period 1 on, we are in a stationary equilibrium.

To see that the I^{t*} sequence defined by (49) solves (10), we need only verify that each of the necessary and sufficient conditions in (13) reduces to the following single equation when evaluated at the candidate equilibrium point:

(52) $-1 + (1+r)^{-1}f'(K^0+I^{0*}) + (1+r)^{-2}f'(K^0+I^{0*}) + ... = 0$ or
$f'(K^0+I^{0*})[1 + (1+r)^{-1} + (1+r)^{-2} + ...] = (1+r)$ or
$f'(K^0+I^{0*})(1+r)/r = (1+r)$ or
$f'(K^0+I^{0*}) = r.$

But the last equation in (52) is satisfied using (47). Thus we have exhibited an explicit solution to the intertemporal consumption maximization problem (10) under the functional form assumption on F(K,I) given by assumption (45). We have also shown that under these assumptions, it takes the economy only one period to attain an optimal stationary steady state, which is a somewhat surprising result.

Since we have an explicit solution, we can determine whether the discrete time counterpart to the Weitzman equality, (44), holds. Using this explicit solution to evaluate the right and left hand sides of (44) leads to the following equation:[17]

(53) $[C^{0*} + I^{0*}](1+r)/r = C^{0*} + C^{1*}/r$

Now use equations (48) and (51) to eliminate C^{0*} and C^{1*} from (53). The resulting equation simplifies to the following equation:

(54) $[f(K^0+I^{0*}) - f(K^0)]/I^{0*} = r.$

Assuming that I^{0*} is positive, we see that the left hand side of (54) is a discrete approximation to the derivative of F evaluated at either K^0 (where it is a discrete

[17] Using (6) and (45), we deduce that $p_I^0 = 1$.

approximation to the right of K^0) or at K^{1*} (where it is a discrete approximation to the left of K^{1*}). Using (47), we have $f'(K^{1*}) = r$. Since we have assumed that f is strictly concave, it can be seen that instead of the equality (54), the following inequality holds under our assumptions:[18]

(55) $[f(K^0+I^{0*}) - f(K^0)]/I^{0*} > r$.

Thus our discrete counterpart to the Weitzman result, (44), does not hold for this example. Working backwards from the true inequality (55), we can deduce that

(56) $[C^{0*} + p_I{}^0 I^{0*}](1+r)/r < C^*$;

i.e., under our assumptions on the technology, capitalizing current net product (the left hand side of (56)) *will be strictly less* than the discounted stream of optimal consumption (the right hand side of (56)).[19] From (55), we see that the gap between the left and right hand sides of (56) can be substantial if I^{0*} is large; i.e., if the initial capital stock K^0 is substantially below the period 1 optimal capital stock K^{1*}. Conversely, if I^{0*} is small so that we are close to the optimum capital stock in period 0, then the discretized Weitzman result (53) will hold to a high degree of approximation for this class of models where the one period production function F(K,I) satisfies (45).[20]

6. A Class of Technologies where the Weitzman Result Holds Exactly

In this section, we provide an explicit example which shows that (44) *can* hold in discrete time.

Let the one period production function F(K,I) have the following form:

(57) $F(K,I) \equiv K - g(I)$

where g(I) is a once differentiable, increasing, strictly convex function defined over the closed interval, $0 \le I \le I^{**}$. As usual, $r > 0$ is the real interest rate that the market sector faces. We assume that the real interest rate and g satisfy the following inequality:[21]

(58) $g'(0) < 1/r$.

[18] The derivative $f'(K+\delta)$ of an increasing strictly concave function f will be less than the left hand side discrete approximation $[f(K+\delta) - f(K)]/\delta$ to this derivative for any $\delta > 0$. On the other hand, $f'(K)$ will be greater than $[f(K+\delta) - f(K)]/\delta$.

[19] If the initial capital stock is too large relative to the optimal capital stock $K^0 + I^{0*}$, then the inequality (56) still holds, where we use the second inequality in the above footnote. In this case, I^{0*} is negative and if capital once installed is irreversible (except for the decline implied by depreciation), then in order to attain the optimal capital stock in one period, depreciation has to be equal or greater than $-I^{0*} > 0$.

[20] A close examination of Weitzman's (1976; 160-161) proof shows why his method of proof may fail for our present example. The problem is that the discrete jump in optimal consumption at the starting point means that his proposed solution $Y^*(t)$ defined by his equation (10) does not satisfy his differential equation (14) at the initial point.

[21] This condition means that the initial capital stock is too small relative to the optimal capital stock.

We assume that there exists a I^{0*} such that $0 < I^{0*} \leq I^{**}$ such that

(59) $g'(I^{0*}) = 1/r$.

Now consider the intertemporal consumption maximization problem (10) with F(K,I) defined by (57). We will show that the following sequence of net investments solves (10):

(60) $I^{t*} = I^{0*} > 0$ for t = 1,2, ... where I^{0*} satisfies (59).

The corresponding sequence of optimal net capital stocks and consumptions are defined as follows:

(61) $K^{t*} \equiv K^0 + tI^{0*}$ for t = 1,2,3, ... ;
(62) $C^{0*} \equiv K^0 - g(I^{0*})$ and $C^{t*} \equiv K^{t*} - g(I^{0*}) = C^{0*} + tI^{0*}$ for t = 1,2,3, ...

To see that the constant net investment series I^{t*} defined by (60) solves (10), we need only verify that each of the necessary and sufficient conditions in (13) reduces to the following single equation when evaluated at the candidate equilibrium point:

(63) $-g'(I^{0*}) + (1+r)^{-1}1 + (1+r)^{-2}1 + ... = 0$ or
$g'(I^{0*}) = (1+r)^{-1}(1+r)/r$ or
$g'(I^{0*}) = 1/r$.

But the last equation in (63) is satisfied using (59). Thus (60)-(62) does provide an explicit solution to (10), provided that the production function F(K,I) satisfies (57).

Note that we can use (6) to find the equilibrium price of the investment good in period 0:

(64) $p_I^0 \equiv -F_2(K^0,I^{0*})$
$= g'(I^{0*})$ using (57)
$= 1/r$ using (59).

Since we have an explicit solution to (10) under our present assumptions on the technology, we can determine whether the discrete time counterpart to the Weitzman equality, (44), holds. Using this explicit solution to evaluate the left hand side of (44) leads to the following equation:

(64) $[C^{0*} + p_I^0 I^{0*}][1 + (1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + ...]$
$= [C^{0*} + p_I^0 I^{0*}](1+r)/r$
$= C^{0*}(1+r)/r + I^{0*}(1+r)/r^2$ using (64).

The right hand side of (44) becomes:

(65) $C^* = C^{0*} + (1+r)^{-1}C^{1*} + (1+r)^{-2}C^{2*} + ...$

$$= C^{0*} + (1+r)^{-1} [C^{0*}+1I^{0*}] + (1+r)^{-2}[C^{0*}+2I^{0*}] + (1+r)^{-3}[C^{0*}+3I^{0*}] \quad \text{... using (62)}$$
$$= C^{0*}(1+r)/r + (1+r)^{-1} I^{0*}[1 + (1+r)^{-1}2 + (1+r)^{-2}3 + ...]$$
$$= C^{0*}(1+r)/r + (1+r)^{-1} I^{0*}[1 + c\,2 + c^2\,3 + c^3\,4 + ...] \qquad \text{defining } c \equiv 1/(1+r).$$

Using $0 < c < 1$, it can be seen that

$$(66)\ 1 + c\,2 + c^2\,3 + c^3\,4 + ...$$
$$= 1 + c + c^2 + c^3 + ...$$
$$+ c + c^2 + c^3 + c^4 + ...$$
$$+ c^2 + c^3 + c^4 + c^5 + ...$$
$$+ ...$$
$$= (1-c)^{-1}[1 + c + c^2 + c^3 + ...]$$
$$= (1-c)^{-2}$$
$$= (1+r)^2/r^2 \qquad \text{using } c = 1/(1+r).$$

Substituting (66) into (65) gives us the following equation for the discounted value of future consumption:

$$(67)\ C^* = C^{0*}(1+r)/r + (1+r)\, I^{0*}/r^2 .$$

Comparing (64) and (67), it can be seen that they are exactly the same. Thus for this class of models, the discrete time counterpart to the Weitzman result holds exactly!

7. An Example with Technical Progress

In this section, we draw to the reader's attention that there are some strong assumptions that Weitzman required in order to obtain his result. In particular, he required that all non capital inputs be fixed and that there be no technical progress. In this section, we add technical progress (or labour input growth that is equivalent to the type of technical progress that we have assumed) to the model of the previous section and see if the Weitzman result remains valid.

Thus we now let the one period production function $F(K,I,t)$ depend on time t and have the following form:

$$(68)\ F(K,I,t) \equiv (1+\beta)^t[K - g(I)]$$

where where β is a positive constant satisfying

$$(69)\ 0 < \beta < r$$

and $g(I)$ is a once differentiable, increasing, strictly convex function defined over the closed interval, $0 \le I \le I^{**}$. As usual, r is the real interest rate that the market sector faces. We assume that the real interest rate and g satisfy the following inequality:[22]

[22] This condition means that the initial capital stock is too small relative to the optimal capital stock.

(70) $g'(0) < (1+\beta)/(r-\beta)$.

We assume that there exists a I^{0*} such that $0 < I^{0*} \leq I^{**}$ such that

(71) $g'(I^{0*}) = (1+\beta)/(r-\beta)$.

Now consider the intertemporal consumption maximization problem (10) with F(K,I) defined by (68) for period t. We will show that the following sequence of net investments solves (10):

(72) $I^{t*} = I^{0*} > 0$ for t = 1,2, ... where I^{0*} satisfies (71).

The corresponding sequence of optimal net capital stocks and consumptions are defined as follows:

(73) $K^{t*} \equiv K^0 + tI^{0*}$ for t = 1,2,3, ... ;

To see that the constant net investment series I^{t*} defined by (72) solves (10), we need only verify that first order condition for I^t in (13) reduces to the following equation:

(74) $-g'(I^{t*}) + (1+r)^{-1}(1+\beta) + (1+r)^{-2}(1+\beta)^2 + ... = 0$ for t = 0,1,2, ... or

$g'(I^{t*}) = c[1 + c + c^2 + c^3 + ...]$ defining $c \equiv (1+r)^{-1}(1+\beta)$

$= c(1-c)^{-1}$

$= (1+\beta)/(r-\beta)$.

Using (74) and (71), we see that (72) holds. Hence (73) holds. Turning now to the optimal consumption sequence, we have:

(76) $C^{t*} \equiv F(K^{t*},I^{t*},t)$ for t = 0,1,2, ...

$\equiv (1+\beta)^t[K^{t*} - g(I^{t*})]$ using (68)

$= (1+\beta)^t[K^0 + tI^{0*} - g(I^{0*})]$ using (72) and (73).

As usual, we can use (6) to find the equilibrium price of the investment good in period 0:

(77) $p_I^0 \equiv -F_2(K^0,I^{0*},0)$

$= g'(I^{0*})$ using (57)

$= (1+\beta)/(r-\beta)$. using (74).

Since we have an explicit solution to (10) under our present assumptions on the technology, we can determine whether the discrete time counterpart to the Weitzman equality, (44), holds. Using this explicit solution to evaluate the left hand side of (44) (which we previously called the Weitzman approximation to discounted consumption, C_W) leads to the following equation:

$$\begin{aligned}(78)\ C_W &\equiv [C^{0*} + p_I{}^0 I^{0*}][1 + (1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + ...]\\ &= [C^{0*} + p_I{}^0 I^{0*}](1+r)/r\\ &= [K^0 - g(I^{0*}) + \{(1+\beta)/(r-\beta)\}I^{0*}](1+r)/r \qquad \text{using (77) and (76) for } t = 0\\ &= \{(1+r)/r\}[K^0 - g(I^{0*})] + \{(1+\beta)(1+r)/(r-\beta)r\}I^{0*} \qquad \text{rearranging terms.}\end{aligned}$$

The right hand side of (44) (which is actual discounted consumption) becomes:

$$\begin{aligned}(79)\ C^* &\equiv C^{0*} + (1+r)^{-1}C^{1*} + (1+r)^{-2}C^{2*} + ...\\ &= \textstyle\sum_{t=0}^{\infty} (1+r)^{-1}(1+\beta)^t[K^0 + tI^{0*} - g(I^{0*})] \qquad \text{using (76)}\\ &= \textstyle\sum_{t=0}^{\infty} (1+r)^{-1}(1+\beta)^t[K^0 - g(I^{0*})] + \sum_{t=1}^{\infty} (1+r)^{-1}(1+\beta)^t\, tI^{0*}\\ &= \textstyle\sum_{t=0}^{\infty} c^t\,[K^0 - g(I^{0*})] + \sum_{t=1}^{\infty} c^t\, tI^{0*} \qquad \text{letting } c \equiv (1+\beta)/(1+r)\\ &= (1-c)^{-1}\,[K^0 - g(I^{0*})] + c(1-c)^{-2}\, I^{0*} \qquad \text{using the algebra in (66)}\\ &= \{(1+r)/(r-\beta)\}[K^0 - g(I^{0*})] + [\{(1+\beta)/(1+r)\}\{(1+r)^2/(r-\beta)^2\}I^{0*}\\ &> \{(1+r)/r\}[K^0 - g(I^{0*})] + \{(1+\beta)(1+r)/(r-\beta)r\}I^{0*} \qquad \text{if } 0 < \beta < r\\ &= C_W \qquad \text{using (78).}\end{aligned}$$

Thus if we have a positive rate of technical progress β (which is foreseen) and the one period production function for period t $F(K,I,t)$ has the form defined by (68), then the Weitzman approximation to the present value of discounted consumption *is strictly less* than the actual present value of discounted consumption, provided that expectations are realized. *Note that the gap between actual and approximate discounted consumption can become arbitrarily large as the rate of technical progress β approaches the real interest rate r.*[23]

8. Conclusion

Weitzman (1976; 160) derived some provocative results on the equality of discounted consumption to discounted net product produced by the market sector of the economy using continuous time optimization techniques. The question addressed in this note is whether discrete time counterparts to his results exist. Our answer is, in general, no. However, if the one period production function $F(K,I)$ is restricted to the class of functions defined by (45), then we have shown that there are conditions which would imply a discrete time *approximate* version of the Weitzman result. And if the one period production function $F(K,I)$ is restricted to the class of functions defined by (57), then we have shown that the discrete time counterpart to the Weitzman result holds exactly. Thus the problem of obtaining general conditions on the technology that imply an exact Weitzman type discrete time identity remains open.[24] However, even if these general conditions are found, the example presented in section 7 shows that if there is (foreseen)

[23] This inequality does not contradict Weitzman's result since his result was derived under the assumptions of no growth in non capital factors of production and no technical change. Weitzman (1976; 160) acknowledged that if technical change was anticipated, then the inequality (79) was likely to hold (see footnote 3).

[24] We have followed the example of Weitzman and buried the complications due to depreciation in the production function $F(K,I)$ where I is regarded as a net addition to the initial capital stock K. However, we believe that the above results can be reworked with the role of depreciation made explicit.

technical progress or growth in "fixed" inputs, then we cannot expect to find a discrete time counterpart to the Weitzman result.

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