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The quadratic approximation lemma and decompositions of superlative indexes

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It was shown in 1976 that a difference in a quadratic function of N variables evaluated at two points is exactly equal to the sum of the arithmetic average of the first order partial derivatives of the function evaluated at the two points times the differences in the independent variables. In the present paper, this result is generalized and the resulting generalized quadratic approximation lemma is used to establish all of the superlative index number formulae that were derived in Diewert [4]. In addition, some new exact decompositions of the percentage change in the Fisher and Walsh superlative indexes into N components are derived. Each component in this decomposition represents the contribution of a change in a single independent variable to the overall percentage change in the index. Finally, these components are given economic interpretations.

Keywords: Superlative index, quadratic approximations, percentage change decompositions, price indexes, quantity indexes, Fisher ideal index

1. Introduction

Let $F(z_1, \ldots, z_N)$ be a quadratic function of N variables, $(z_1, \ldots, z_N) \equiv z$; i.e., define F as follows:

$$F(z) \equiv a_0 + \sum_{n=1}^{N} a_n z_n + \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} z_i z_j \tag{1}$$

where $a_{ij} = a_{ji}$ for all i and j. It is well known that a second order Taylor series approximation to a quadratic function will exactly reproduce the quadratic function. It is not so well known that the arithmetic average of two linear approximations will also reproduce a quadratic function exactly. To see this, write the linear approximation to $F(z)$ around the point $z^0 \equiv (z_1^0, \ldots, z_N^0)$ as

$$F(z) \approx F(z^0) + \nabla F(z^0) \cdot [z - z^0] \equiv F(z^0) + \sum_{n=1}^{N} F_n(z^0)[z_n - z_n^0] \tag{2}$$

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where $\nabla F(z^0) \equiv [\partial F(z^0)/\partial z_1, \ldots, \partial F(z^0)/\partial z_N] \equiv [F_1(z^0), \ldots, F_N(z^0)]$ is the vector of first order partial derivatives of F evaluated at the point z^0 and

$$x \cdot y \equiv \sum_{n=1}^{N} x_n y_n$$

denotes the inner product of the vectors x and y. The linear approximation to F around another point z^1 is

$$F(z) \approx F(z^1) + \nabla F(z^1) \cdot [z - z^1] \equiv F(z^1) + \sum_{n=1}^{N} F_n(z^1)[z_n - z_n^1]. \tag{3}$$

Now let $z = z^1$ in Eq. (2) and $z = z^0$ in Eq. (3) and treat the two approximations as equalities. Taking the arithmetic average of the resulting two equations and rearranging terms yields the following equation:

$$\begin{aligned} F(z^1) - F(z^0) &= (1/2)[\nabla F(z^0) + \nabla F(z^1)] \cdot [z^1 - z^0] \\ &= \sum_{n=1}^{N} (1/2)[F_n(z^0) + F_n(z^1)][z_n^1 - z_n^0]. \end{aligned} \tag{4}$$

It can be verified by substituting the F defined by Eq. (1) into Eq. (4) that if F is quadratic, then Eq. (4) holds exactly for any two points, z^0 and z^1.[2] Note that this result shows that taking an average of two first order approximations yields the equivalent of a second order approximation.

Diewert [4, pp. 118–121] used the fact that Eq. (4) holds as an identity for quadratic functions to prove that the Törnqvist [21,22] quantity index is exact for a translog aggregator function and that the Törnqvist price index is exact for a translog unit cost function.[3] Diewert established these results by taking a simple transformation of Eq. (4).

It turns out that other transformations of the quadratic identity Eq. (4) can be used to establish the exactness of all of the major families of superlative index numbers.[4] We show this in Section 4 below.

In Section 2, we provide the economic framework for our index number results.

In Section 3, we provide a rather general transformation of the quadratic identity Eq. (4). We then show how a special case of this general result yields the translog results.

In Section 4, we specialize our general transformation of Eq. (4) to yield the exactness of the quadratic mean of order r indexes. Thus our transformation of

[2] Diewert [4, p. 118] and Lau [16] showed that the converse result also holds; i.e., if Eq. (4) holds, then F must be quadratic. Diewert called this result the quadratic approximation lemma.

[3] The translog functional form was introduced into the economics literature by Christensen et al. [1,2].

[4] A superlative index number formula is exact for a flexible functional form; see Diewert [4].

Eq. (4) provides a unified framework for deriving the commonly used superlative index number formulae.

In Section 5, we specialize the results of Section 4 to the case where r equals 2. This specialization allows us to obtain additive percentage change decompositions for the Fisher [12] ideal price and quantity indexes.

In Section 6, we specialize the results of Section 4 to the case where r equals 1. This specialization allows us to obtain additive percentage change decompositions for some indexes that were originally defined by Walsh [24,25].

In Section 5, we find that our additive percentage change decompositions for the Fisher ideal price and quantity indexes are not unique. Thus it is important to provide some sort of an axiomatic or economic justification for any particular additive percentage change decomposition. In Section 7, we provide economic interpretations for our preferred decompositions.

Section 8 concludes. We conclude that the decompositions that we obtain for the Törnqvist,[5] Fisher and Walsh indexes are particularly attractive.

2. The economic framework

For simplicity, we consider a consumer[6] who minimizes the cost of achieving a given utility level in two periods where the utility or aggregator function $f(q)$ is (positively) linearly homogeneous, positive for positive q, nondecreasing and concave function in the N variables, $q \equiv (q_1, \ldots, q_N)$. We assume that we can observe the price vectors that the consumer faces during these two periods, say $p^t \equiv (p_1^t, \ldots, p_N^t)$ for $t = 0, 1$, and the quantity vectors chosen for the two periods, say $q^t \equiv (q_1^t, \ldots, q_N^t)$ for $t = 0, 1$. The unit cost function c that corresponds to the aggregator function f is defined as the minimum cost of achieving the utility level 1; i.e., for each vector of positive commodity prices p, define c by:[7]

$$c(p) \equiv \min_q \{p \cdot q : f(q) = 1\}. \tag{5}$$

Under our assumptions on consumer behavior, the observed period t expenditure on the N commodities,

$$p^t \cdot q^t = \sum_{n=1}^{N} p_n^t q_n^t,$$

[5] The Törnqvist decomposition was obtained earlier by Diewert and Morrison [7] and Kohli [13].

[6] Alternatively, the same theory applies to a producer who minimizes the cost of achieving a given level of output in two periods, where $f(q_1, \ldots, q_N)$ is the maximum output that can be produced by the vector of inputs $q \equiv (q_1, \ldots, q_N)$. This is the framework used by Shephard [20], Samuelson and Swamy [19] and Diewert [4].

[7] For additional material on unit cost functions and the other theoretical results used in this section, see Diewert [3,5].

will equal the product of the period t utility level, $f(q^t)$, times the period t unit cost, $c(p^t)$:[8]

$$p^t \cdot q^t = c(p^t)f(q^t); \quad t = 0, 1. \tag{6}$$

Thus taking ratios of the period 1 expenditures on the N commodities to the period 0 observed expenditures, we obtain:

$$\begin{aligned} p^1 \cdot q^1/p^0 \cdot q^0 &= [c(p^1)f(q^1)]/[c(p^0)f(q^0)] \\ &= [c(p^1)/c(p^0)][f(q^1)/f(q^0)]. \end{aligned} \tag{7}$$

The term $[c(p^1)/c(p^0)]$ on the right hand side of Eq. (7) can be interpreted as the consumer's "true" price index[9] and the term $[f(q^1)/f(q^0)]$ can be interpreted as the consumer's "true" quantity index.

If the unit cost function $c(p)$ is differentiable with respect to the components of the price vector p, then Shephard's [20, p. 11] Lemma implies the following useful equations:[10]

$$\nabla c(p^t)/c(p^t) = q^t/p^t \cdot q^t; \quad t = 0, 1. \tag{8}$$

If the aggregator function $f(q)$ is differentiable with respect to the components of q, the Wold's [26, pp. 69–71; 27, p. 145] Identity implies the following equally useful equations:[11]

$$\nabla f(q^t)/f(q^t) = p^t/p^t \cdot q^t; \quad t = 0, 1. \tag{9}$$

With the above economic preliminaries out of the way, we can derive a generalization of the Quadratic Identity, Eq. (4).

3. A transformed quadratic identity

We now suppose that our aggregator function $f(q)$ has the following transformed quadratic functional form:

$$g[f(q)] \equiv a_0 + \sum_{n=1}^{N} a_n h(q_n) + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} h(q_i)h(q_j); \quad a_{ij} = a_{ji} \tag{10}$$

[8] See Shephard [20] or Diewert [4, p. 120].
[9] This concept for a price index is due to Konüs [14].
[10] See Diewert [4, p. 120] for more details.
[11] In deriving Eq. (9), we also used $f(q^t) = \nabla f(q^t) \cdot q^t$ which follows from Euler's Theorem on homogeneous functions.

where the a_n and the a_{ij} are constants and the functions g and h are continuous monotonic functions of one variable with nonzero derivatives. Later, we will specialize the general functional form f defined by Eq. (10) by choosing specific functions for g and h and we will place some restrictions on the coefficients a_n and a_{ij} so that the resulting f will be linearly homogeneous.

It is fairly obvious that Eq. (10) can be rewritten as a quadratic function of the type defined by Eq. (1) if we make some transformations of variables. Thus define:

$$z_n \equiv h(q_n); \quad n = 1, 2, \ldots, N. \tag{11}$$

Due to the assumed continuity and monotonicity of the function h, we can invert Eq. (11):

$$q_n = h^{-1}(z_n); \quad n = 1, 2, \ldots, N. \tag{12}$$

We rewrite the N equations in Eq. (12) in vector notation as follows:

$$q = H^{-1}(z) \tag{13}$$

where $q \equiv (q_1, \ldots, q_N)$ and $z \equiv (z_1, \ldots, z_N)$. Now define the function of N variables $F(z)$ by:

$$F(z) \equiv g[f\{H^{-1}(z)\}]. \tag{14}$$

Substituting Eqs (11)–(14) into Eq. (10) shows that the F defined by Eq. (14) is the quadratic function of z defined by Eq. (1).

We now need to express the first order partial derivatives of F, $F_n(z) \equiv \partial F(z)/\partial z_n$, in terms of f, g and h. First note that since $h'(q_n) \neq 0$ by assumption, we have

$$dh^{-1}(z_n)/dz_n = 1/h'(q_n); \quad n = 1, 2, \ldots, N. \tag{15}$$

Now differentiate Eq. (14) with respect to z_n:

$$\begin{aligned} F_n(z) &= g'[f\{H^{-1}(z)\}]f_n\{H^{-1}(z)\}dh^{-1}(z_n)/dz_n \\ &= g'[f(q)]f_n(q)dh^{-1}(z_n)/dz_n && \text{using Eq. (13)} \\ &= g'[f(q)]f_n(q)/h'(q_n) && \text{using Eq. (15).} \end{aligned} \tag{16}$$

Now substitute Eq. (16) into Eq. (4) and we obtain the following identity:

$$\begin{aligned} g[f(q^1)] - g[f(q^0)] &= \sum_{n=1}^{N}(1/2)[F_n(z^0) + F_n(z^1)][h(q_n^1) - h(q_n^0)] \\ &= \sum_{n=1}^{N}(1/2)[\{f_n(q^0)g'[f(q^0)]/h'(q_n^0)\} \\ &\quad +\{f_n(q^1)g'[f(q^1)]/h'(q_n^1)\}][h(q_n^1) - h(q_n^0)]. \end{aligned} \tag{17}$$

Equation (17) is our generalized quadratic identity and it holds as an identity for all functions f defined by Eq. (10).

To illustrate the usefulness of Eq. (17), let g and h be the natural logarithm functions; i.e., define:

$$g(y) \equiv \ln\ y \text{ and } h(y) \equiv \ln\ y. \tag{18}$$

Using $g'(y) = 1/y$ and $h'(y) = 1/y$, Eq. (17) becomes

$$\begin{aligned} \ln\ f(q^1) - \ln\ f(q^0) = \sum_{n=1}^{N} &(1/2)[\{f_n(q^0)q_n^0/f(q^0)\} \\ &+\{f_n(q^1)q_n^1/f(q^1)\}][\ln\ q_n^1 - \ln\ q_n^0] \end{aligned} \tag{19}$$

and Eq. (10) becomes

$$\ln\ f(q) \equiv a_0 + \sum_{n=1}^{N} a_n \ln\ q_n + \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} \ln\ q_i \ln\ q_j; \quad a_{ij} = a_{ji}. \tag{20}$$

Note that the $f(q)$ defined by Eq. (20) becomes the well known translog aggregator function.[12]

In order to make the $f(q)$ defined by Eq. (20) linearly homogeneous, we require the following restrictions:

$$\sum_{n=1}^{N} a_n = 1; \quad \sum_{j=1}^{N} a_{ij} = 0; \quad i = 1, 2, \ldots, N. \tag{21}$$

With the restrictions Eq. (21) imposed, $f(q)$ defined by Eq. (20) is linearly homogeneous and we can apply Wold's identity Eq. (9), $f_n(q^t) = f(q^t)p_n^t/p^t \cdot q^t$, for $t = 0, 1$ and $n = 1, 2, \ldots, N$. Substituting these relations into Eq. (19) yields:

$$\ln\,[f(q^1)/f(q^0)] = (1/2)\sum_{n=1}^{N}[s_n^0 + s_n^1]\ln\,[q_n^1/q_n^0] \tag{22}$$

where $s_n^t \equiv p_n^t q_n^t / p^t \cdot q^t$ is the share of period t expenditure on commodity n for $t = 0, 1$ and $n = 1, 2, \ldots, N$. The right hand side of Eq. (22) is the logarithm of the Törnqvist quantity index, $Q_T(p^0, p^1, q^0, q^1)$, and the left hand side of Eq. (22) is the logarithm of the true quantity index, $f(q^1)/f(q^0)$. Thus we have

$$f(q^1)/f(q^0) = Q_T(p^0, p^1, q^0, q^1). \tag{23}$$

[12] This functional form was introduced into the economics literature by Christensen et al. [1,2].

Note that the right hand sides of Eq. (22) and (23) can be calculated using observable data.

The above algebra can be repeated for the translog unit cost function, which can be defined by Eq. (20), except that $c(p)$ replaces $f(q)$ and $\ln\ p_n$ replaces $\ln\ q_n$. The counterpart to Eq. (19) becomes

$$\begin{aligned}\ln\ c(p^1) - \ln\ c(p^0) = &\sum_{n=1}^{N}(1/2)[\{c_n(p^0)p_n^0/c(p^0)\} \\ &+\{c_n(p^1)p_n^1/c(p^1)\}][\ln\ p_n^1 - \ln\ p_n^0].\end{aligned} \tag{24}$$

Again, we assume that the restrictions Eq. (21) hold so that the translog unit cost function $c(p)$ is linearly homogeneous in the components of p. Now use Shephard's lemma Eq. (8), $c_n(p^t) = c(p^t)q_n^t/p^t \cdot q^t$, for $t = 0, 1$ and $n = 1, 2, \ldots, N$. Substituting these relations into Eq. (24) yields:

$$\ln\,[c(p^1)/c(p^0)] = (1/2)\sum_{n=1}^{N}[s_n^0 + s_n^1]\ln\,[p_n^1/p_n^0]. \tag{25}$$

The right hand side of Eq. (25) is the logarithm of the Törnqvist price index, $P_T(p^0, p^1, q^0, q^1)$, and the left hand side of Eq. (25) is the logarithm of the true price index, $c(p^1)/c(p^0)$. Thus we have

$$c(p^1)/c(p^0) = P_T(p^0, p^1, q^0, q^1). \tag{26}$$

The exact index number results Eqs (23) and (26) illustrate the usefulness of the generalized quadratic identity Eq. (17) even though these results are not new. [13] In the following section, we provide some new applications of Eq. (17).

4. The generalized quadratic identity and mean of order r indexes

Recall the generalized quadratic functional form defined by Eq. (10) above. We now place the following restrictions on the coefficients a_n:

$$a_n = 0; \quad n = 0, 1, \ldots, N. \tag{27}$$

We also assume that the functions g and h which appear in the definition of f are defined as follows:

$$g(y) \equiv y^r; \quad h(y) \equiv y^{r/2}; \quad r \neq 0. \tag{28}$$

[13] See Diewert [4, pp. 119–121].

Using the restrictions Eq. (27) and (28), the function f defined by Eq. (10) becomes the following quadratic mean of order r aggregator function:

$$f(q) \equiv [\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} q_i^{r/2} q_j^{r/2}]^{1/r}; \quad a_{ij} = a_{ji}. \tag{29}$$

It can be shown that $f(q)$ defined by Eq. (29) is linearly homogeneous flexible functional form; that is, it can provide a second order approximation to an arbitrary twice continuously differentiable linearly homogeneous function. [14]

Substituting the restrictions and definitions Eq. (27) and (28) into the generalized quadratic identity Eq. (17) yields the following identity:

$$\begin{aligned} &[f(q^1)]^r - [f(q^0)]r \\ &= (1/2)\sum_{n=1}^{N}[\{f_n(q^0)r[f(q^0)]^{r-1}/[(r/2)(q_n^0)^{(r/2)-1}]\} \\ &\quad +\{f_n(q^1)r[f(q^1)]^{r-1}/[(r/2)(q_n^1)^{(r/2)-1}]\}[(q_n^1)^{r/2} - (q_n^0)^{r/2}] \end{aligned} \tag{30}$$

or

$$\begin{aligned} [f^1]^r - [f^0]^r = \sum_{n=1}^{N}&[f_n^0\{f^0\}^{r-1}(q_n^0)^{1-r/2} \\ &+f_n^1\{f^1\}^{r-1}(q_n^1)^{1-r/2}][(q_n^1)^{r/2} - (q_n^0)^{r/2}] \end{aligned} \tag{31}$$

where we have simplified the notation by defining

$$\begin{aligned} &f^0 \equiv f(q^0); \quad f^1 \equiv f(q^1); \quad f_n^t \equiv f_n(q^t) \equiv \partial f(q^t)/\partial q_n; \\ &t = 0, 1; \quad n = 1, 2, \ldots, N. \end{aligned} \tag{32}$$

Since the $f(q)$ defined by Eq. (29) is linearly homogeneous, we may use Wold's identity Eq. (9) to replace the partial derivatives f_n^t by $p_n^t f^t/p^t \cdot q^t$. The notation will be simplified if we define the period t normalized price for commodity n, w_n^t, as follows:

$$\begin{aligned} w_n^t &\equiv p_n^t/p^t \cdot q^t && t = 0, 1; \quad n = 1, 2, \ldots, N \\ &= [1/f(q^t)]\partial f(q^t)/\partial q_n && \text{using Wold's identity, Eq. (9)} \\ &= \partial \ln f(q^t)/\partial q_n && \\ &= f_n^t/f^t && \text{using the notation in Eq. (32).} \end{aligned} \tag{33}$$

[14] See Diewert [4, p. 130]. Färe and Sung [10] showed that the translog case considered earlier and the present normalized quadratic function are the only special cases of the family of generalized quadratic functions that are also linearly homogeneous.

Thus w_n^t is the period t price for commodity n, p_n^t, divided by period t expenditure on all commodities in the aggregate,

$$p^t \cdot q^t \equiv \sum_{n=1}^{N} p_n^t q_n^t.$$

Using Wold's identity, the normalized price w_n^t is equal to the logarithmic derivative of the aggregator function with respect to commodity n evaluated at the period t data, $\partial \ln f(q^t)/\partial q_n$. Making use of Eq. (33), Eq. (31) can be rewritten as follows:

$$\begin{aligned}[f^1]^r - [f^0]^r = \sum_{n=1}^{N} &[w_n^0 \{f^0\}^r (q_n^0)^{1-r/2} \\ &+ w_n^1 \{f^1\}^r (q_n^1)^{1-r/2}][(q_n^1)^{r/2} - (q_n^0)^{r/2}].\end{aligned} \tag{34}$$

Now divide both sides of Eq. (34) through by $[f^0]^r$ to obtain:

$$\begin{aligned}[f^1/f^0]^r - 1 = \sum_{n=1}^{N} &[w_n^0 (q_n^0)^{1-r/2} \\ &+ w_n^1 \{f^1/f^0\}^r (q_n^1)^{1-r/2}][(q_n^1)^{r/2} - (q_n^0)^{r/2}].\end{aligned} \tag{35}$$

Now f^1/f^0 is the true quantity index, $Q_r \equiv f(q^1)/f(q^0)$. Replace f^1/f^0 in Eq. (35) by Q_r and solve the resulting equation for Q_r. We obtain the following solution:

$$Q_r = [\sum_{n=1}^{N} (q_n^1/q_n^0)^{r/2} s_n^0]^{1/r} [\sum_{n=1}^{N} (q_n^1/q_n^0)^{-r/2} s_n^1]^{-1/r} \tag{36}$$

where s_n^t is the period t expenditure share for commodity n; i.e.,

$$\begin{aligned} s_n^t &\equiv p_n^t q_n^t / p^t \cdot q^t; \quad t = 0, 1; \quad n = 1, 2, \ldots, N \\ &= w_n^t q_n^t \quad \text{using definitions Eq. (33).} \end{aligned} \tag{37}$$

The index number formula on the right hand side of Eq. (36) depends only on the observed prices and quantities pertaining to the two periods under consideration and it is equal to the quadratic mean of order r quantity index defined by Diewert [4, p. 130]. The above results show that it is exactly equal to $f(q^1)/f(q^0)$ where f is the quadratic mean of order r aggregator function defined by Eq. (29). Thus we have used the generalized quadratic identity Eq. (17) in order to establish this exactness result.

The above algebra for quantity indexes has its counterpart for price indexes as we now show. Define the quadratic mean of order r unit cost function $c(p)$ by:

$$c(p) \equiv [\sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} p_i^{r/2} p_j^{r/2}]^{1/r}; \quad a_{ij} = a_{ji}. \tag{38}$$

Define the period t normalized quantity for commodity n, v_n^t, as follows:

$$\begin{aligned} v_n^t &\equiv q_n^t / p^t \cdot q^t && t = 0, 1; \quad n = 1, 2, \ldots, N \\ &= [\partial c(p^t)/\partial p_n]/c(p^t) && \text{using Shephard's lemma, Eq. (8)} \\ &= \partial \ln\ c(p^t)/\partial p_n. \end{aligned} \tag{39}$$

Let the level of unit cost in period 0 be $c^0 \equiv c(p^0)$ and the level of unit cost in period 1 be $c^1 \equiv c(p^1)$. The counterpart to Eq. (35) is now

$$\begin{aligned} [c^1/c^0]^{r-1} = \sum_{n=1}^{N} &[v_n^0 (p_n^0)^{1-r/2} \\ &+ v_n^1 \{c^1/c^0\}^r (p_n^1)^{1-r/2}][(p_n^1)^{r/2} - (p_n^0)^{r/2}]. \end{aligned} \tag{40}$$

Note that $c^1/c^0 \equiv c(p^1)/c(p^0) \equiv P_r$ is the true price index that corresponds to the unit cost function defined by Eq. (38). Now replace c^1/c^0 in Eq. (40) by P_r and solve the resulting equation for P_r. We obtain the following solution:

$$P_r = [\sum_{n=1}^{N} (p_n^1/p_n^0)^{r/2} s_n^0]^{1/r} [\sum_{n=1}^{N} (p_n^1/p_n^0)^{r/2} s_n^1]^{-1/r} \tag{41}$$

where s_n^t is the period t expenditure share for commodity n defined earlier by Eq. (37). The index number formula on the right hand side of Eq. (41) depends only on the observed prices and quantities pertaining to the two periods under consideration and it is equal to the quadratic mean of order r price index defined by Diewert [4, p. 131]. The above results show that it is exactly equal to $c(p^1)/c(p^0)$ where c is the quadratic mean of order r unit cost function defined by Eq. (38). Thus again we have used the generalized quadratic identity Eq. (17) in order to establish this exactness result.

In the following two section, we examine Eqs (35) and (40) more closely for the special cases when $r = 1$ or 2.

5. Additive percentage change decompositions for the Fisher ideal indexes

It can be verified that when $r = 2$, Q_2 defined by Eq. (36) turns out to equal the Fisher [12] ideal quantity index Q_F; i.e., we have

$$Q_2 = Q_F(p^0, p^1, q^0, q^1) \equiv [\{p^0 \cdot q^1 / p^0 \cdot q^0\}\{p^1 \cdot q^1 / p^1 \cdot q^0\}]^{1/2}. \tag{42}$$

Using Eq. (34) when $r = 2$, we have the following decomposition:[15]

$$[f^1]^2 - [f^0]^2 = \sum_{n=1}^{N} [w_n^0 \{f^0\}^2 + w_n^1 \{f^1\}^2][q_n^1 - q_n^0] \tag{43}$$

[15] This decomposition was used already by Reinsdorf et al. [18].

where the normalized prices w_n^t are defined by Eq. (33). From elementary algebra, we have:

$$[f^1]^2 - [f^0]^2 = [f^1 - f^0][f^1 + f^0]. \tag{44}$$

Now divide both sides of Eq. (43) by $f^1 + f^0$. Using Eq. (44), the resulting equation becomes:

$$\begin{aligned} & f^1 - f^0 \\ &= f^0 \sum_{n=1}^{N} \{(f^0/[f^0 + f^1])w_n^0 + (f^1/f^0)\{(f^1/[f^0 + f^1])w_n^1\}\{q_n^1 - q_n^0\}. \end{aligned} \tag{45}$$

Divide both sides of Eq. (45) by f^0 and using $Q_F = f^1/f^0$, we have the following additive percentage change decomposition for the Fisher ideal quantity index: [16]

$$Q_F - 1 = \sum_{n=1}^{N} \{(1/[1 + Q_F])w_n^0 + (Q_F/[1 + Q_F])Q_F w_n^1\}\{q_n^1 - q_n^0\}. \tag{46}$$

In the above decomposition, the term in front of the change in quantity n going from period 0 to 1, Q_{Fn}, the nth percentage change quantity weight, is defined as follows:

$$Q_{Fn} \equiv (1/[1 + Q_F])w_n^0 + (Q_F/[1 + Q_F])Q_F w_n^1. \tag{47}$$

Note that the nth percentage change quantity weight is almost a weighted average (with weights $(1/[1 + Q_F])$ and $(Q_F/[1 + Q_F])$ which sum to unity) of the two normalized prices for commodity n in the two periods under consideration, $w_n^t \equiv p_n^t/p^t \cdot q^t$ for $t = 0, 1$. However, the period 1 normalized price w_n^1 gets an extra weighting factor equal to Q_F, the value of the Fisher quantity index going from period 0 to 1. If $Q_F = 1$, then Q_{Fn} is equal to the arithmetic average of the normalized prices for commodity n, $(1/2)w_n^0 + (1/2)w_n^1$.

In a manner analogous to the derivation of Eq. (46), we can obtain the following additive percentage change decomposition for the Fisher ideal price index:

$$P_F - 1 = \sum_{n=1}^{N} \{(1/[1 + P_F])v_n^0 + (P_F/[1 + P_F])P_F v_n^1\}\{p_n^1 - p_n^0\} \tag{48}$$

where the Fisher ideal price index P_F is defined as follows:

$$\begin{aligned} P_F(p^0, p^1, q^0, q^1) &\equiv [\{p^1 \cdot q^0/p^0 \cdot q^0\}\{p^1 \cdot q^1/p^0 \cdot q^1\}]^{1/2} \\ &= p^1 \cdot q^1/p^0 \cdot q^0 Q_F(p^0, p^1, q^0, q^1) \end{aligned} \tag{49}$$

[16] If we solve Eq. (46) for Q_F, we obtain the Fisher ideal index defined by Eq. (42) as the solution. This shows that Eq. (46) is an identity, which is valid for all p^0, p^1, q^0, q^1.

where Q_F is the Fisher ideal quantity index defined earlier by Eq. (42). In the decomposition Eq. (48), the term in front of the change in price n going from period 0 to 1, P_{Fn}, the nth percentage change price weight, is defined as follows:

$$P_{Fn} \equiv (1/[1+P_F])v_n^0 + (P_F/[1+P_F])P_F v_n^1 \tag{50}$$

where the normalized quantities, v_n^t are equal to $q_n^t/p^t \cdot q^t$ for $t = 0, 1$.

It should be noted that the concept of a price or quantity index number formula having an additive percentage change decomposition is not quite the same as an index number formula having the property of additivity. We now explain the difference.

A price index, $P(p^0, p^1, q^0, q^1)$, is said to be additive if it can be written as follows:

$$P(p^0, p^1, q^0, q^1) = \sum_{n=1}^N q_n^* p_n^1 / \sum_{n=1}^N q_n^* p_n^0 \tag{51}$$

where the "quantity" weights q_n^* are usually taken to be some sort of average of the base and current period quantities for commodity n, q_n^0 and q_n^1. However, in principle, more complicated quantity weighting could be used: the important factor in the definition of additivity given by Eq. (51) is that the quantity weights be the same in the numerator and the denominator of the right hand side of Eq. (51).

In a similar manner, a quantity index, $Q(p^0, p^1, q^0, q^1)$, is said to be additive if it can be written as follows:

$$Q(p^0, p^1, q^0, q^1) = \sum_{n=1}^N p_n^* q_n^1 / \sum_{n=1}^N p_n^* q_n^0 \tag{52}$$

where the "price" weights p_n^* are usually taken to be some sort of average of the base and current period prices for commodity n, p_n^0 and p_n^1. However, in principle, more complicated price weighting could be used: as before, the important factor in the definition of additivity given by Eq. (52) is that the price weights be the same in the numerator and the denominator of the right hand side of Eq. (52).

It is straightforward to show that additive price and quantity indexes have additive percentage change decompositions. For example, suppose we have an additive quantity index of the type defined by Eq. (52) above. Then we have:

$$\begin{aligned} Q(p^0, p^1, q^0, q^1) - 1 &= [\sum_{n=1}^N p_n^* q_n^1 / \sum_{n=1}^N p_n^* q_n^0] - 1 \quad \text{using Eq. (52)} \\ &= [\sum_{n=1}^N p_n^* q_n^1 / \sum_{n=1}^N p_n^* q_n^0] - [\sum_{n=1}^N p_n^* q_n^0 / \sum_{n=1}^N p_n^* q_n^0] \\ &= [\sum_{n=1}^N p_n^* \{q_n^1 - q_n^0\}] / \sum_{n=1}^N p_n^* q_n^0 \end{aligned} \tag{53}$$

$$= \sum_{n=1}^{N} Q_n \{q_n^1 - q_n^0\}$$

where the nth percentage change quantity weight Q_n is defined as

$$Q_n \equiv p_n^* / \sum_{n=1}^{N} p_n^* q_n^0; \quad n = 1, \ldots, N. \tag{54}$$

Thus we can always find an additive percentage change decomposition for an additive price or quantity index. However, it may not always be possible to go from an additive percentage change decomposition to a corresponding additive index number formula.

Unfortunately, the additive percentage change decompositions Eqs (46) and (48) that we obtained for the Fisher quantity and price indexes are not unique. For example, Van Ijzeren [23, p. 6] chose the following values for the quantity weights q_n^* which appear in Eq. (51) above:

$$q_n^* \equiv (1/2)q_n^0 + (1/2)q_n^1/Q_F(p^0, p^1, q^0, q^1); \quad n = 1, 2, \ldots, N \tag{55}$$

where Q_F is the Fisher quantity index defined by Eq. (42). Thus the reference quantity for commodity n in Eq. (51), q_n^*, is chosen to be the arithmetic mean of the period 0 and period 1 observed use of commodity n, except that the period 1 use, q_n^1, is deflated by the Fisher quantity index, Q_F, for the entire group of commodities in the aggregate. To show that the resulting price index defined by Eq. (51) is the Fisher ideal price index, replace Q_F in Eq. (55) by $p^1 \cdot q^1 / p^0 \cdot q^0 P$ if necessary. Now solve Eq. (51) using the weights defined by Eq. (55) for P; i.e., solve the resulting equation for P:

$$\begin{aligned}
P &= \sum_{n=1}^{N} (1/2)[q_n^0 + q_n^1/Q\}]p_n^1 \\
&\quad / \sum_{n=1}^{N} (1/2)[q_n^0 + q_n^1/Q]p_n^0 \\
&= [p^1 \cdot q^0 + p^1 \cdot q^1/Q] \\
&\quad /[p^0 \cdot q^0 + p^0 \cdot q^1/Q] && \text{or} \qquad (56) \\
PQp^0 \cdot q^0 + p^0 \cdot q^1 P &= Qp^1 \cdot q^0 + p^1 \cdot q^1 && \text{or} \\
[p^1 \cdot q^1/p^0 \cdot q^0]p^0 \cdot q^0 &+ p^0 \cdot q^1 P \\
&= Qp^1 \cdot q^0 + p^1 \cdot q^1 && \text{using } PQ = [p^1 \cdot q^1/p^0 \cdot q^0] \text{ or} \\
p^0 \cdot q^1 P &= Qp^1 \cdot q^0 && \text{canceling terms or} \\
p^0 \cdot q^1 P &= [p^1 \cdot q^1/p^0 \cdot q^0 P]p^1 \cdot q^0 && \text{using } Q = p^1 \cdot q^1/p^0 \cdot q^0 P \text{ or} \\
P^2 &= [p^1 \cdot q^1/p^0 \cdot q^1][p^1 \cdot q^0/p^0 \cdot q^0] && \text{or}
\end{aligned}$$

$$P = \{[p^1 \cdot q^1/p^0 \cdot q^1][p^1 \cdot q^0/p^0 \cdot q^0]\}^{1/2} \equiv P_F. \tag{57}$$

Thus Eqs (51) and (55) provide an exact additive representation for the Fisher ideal price index.

In a similar fashion, Van Ijzeren [23, p. 6] showed that if we choose the following values for the price weights p_n^* which appear in Eq. (52) above:

$$p_n^* \equiv (1/2)p_n^0 + (1/2)p_n^1/P_F(p^0, p^1, q^0, q^1); \quad n = 1, 2, \ldots, N \tag{58}$$

where P_F is the Fisher price index defined by Eq. (49), then Eqs (52) and (58) provide an exact additive representation for the Fisher ideal quantity index.

Since an additive representation for index number formula implies an additive percentage change decomposition for the formula, we see that our additive percentage change decompositions for the Fisher ideal price and quantity indexes given by Eqs (46) and (48) above are not unique. [17]

In retrospect, it is not surprising that additive percentage change decompositions of any index number formula are not unique (unless the decomposition has to satisfy further properties). To see this, look at the right hand side of Eq. (52), which is homogeneous of degree 0 in $p_1^*, \ldots, p_N^*$. Thus given an index value Q, we can never determine the scale of the p_n^*. Hence, let us impose a normalization on the p_n^*, such as:

$$\sum_{n=1}^{N} p_n^* q_n^0 = 1. \tag{59}$$

Using Eq. (59), Eq. (52), which defines an additive representation for the quantity index Q, can be rewritten as follows:

$$\sum_{n=1}^{N} p_n^* q_n^1 = Q. \tag{60}$$

Equations (59) and (60) can be regarded as two simultaneous linear equations in the N unknowns, $p_1^*, \ldots, p_N^*$. Obviously, as soon as N exceeds 2, there will be an infinite number of solutions to Eqs (59) and (60) in general. Thus the quest for unique additive representations or unique additive percentage change decompositions of an index number formula is doomed to failure. Hence any particular additive percentage change decomposition needs to be justified on axiomatic grounds or on its economic interpretation. We will return to this topic after the following section.

In the following section, we examine Eqs (35) and (40) for the special case when r equals 1.

[17] In fact, two additional additive percentage change decompositions for the Fisher indexes may be found in Reinsdorf et al. [18]. The first of these two decompositions turns out to be equivalent to the decomposition of Van Ijzeren [23, p. 6], which was also independently derived by Dikhanov [8]. The Van Ijzeren decomposition is currently being used by Bureau of Economic Analysis; see Moulton and Seskin [17, p. 16] and Ehemann et al. [9].

6. Additive percentage change decompositions for the implicit Walsh indexes

Our goal in this section is to provide some additive percentage change decompositions for some indexes defined by Walsh.

Walsh [24, p. 398; 25, p. 97] defined the following very useful price index: [18]

$$P_W(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^{N}[q_n^0 q_n^1]^{1/2}p_n^1 \Big/ \sum_{n=1}^{N}[q_n^0 q_n^1]^{1/2}p_n^0. \tag{61}$$

Thus the Walsh price index is an additive price index of the type defined by Eq. (51) where the quantity weights q_n^* are equal to the geometric means of the period 0 and 1 consumption of commodity n, $[q_n^0 q_n^1]^{1/2}$. As indicated in the previous section, the Walsh price index necessarily has an additive percentage change decomposition.

Using the fact that the price index times the quantity index should equal the value ratio for the two periods under consideration; i.e., using [19]

$$P_W(p^0,p^1,q^0,q^1)Q_W^*(p^0,p^1,q^0,q^1) = p^1\cdot q^1/p^0\cdot q^0, \tag{62}$$

we can define the implicit Walsh quantity index Q_W^* that corresponds to P_W defined by Eq. (61) as follows:

$$\begin{aligned}
&Q_W^*(p^0,p^1,q^0,q^1)\\
&\equiv [p^1\cdot q^1/p^0\cdot q^0]/P_W(p^0,p^1,q^0,q^1)\\
&= \left\{\sum_{n=1}^{N}[q_n^0 q_n^1]^{1/2}p_n^0/p^0\cdot q^0\right\} \Bigg/ \left\{\sum_{n=1}^{N}[q_n^0 q_n^1]^{1/2}p_n^1/p^1\cdot q^1\right\} \quad \text{using Eq. (61)}\\
&= \left\{\sum_{n=1}^{N}[q_n^1/q_n^0]^{1/2}p_n^0 q_n^0/p^0\cdot q^0\right\}\\
&\quad \Bigg/ \left\{\sum_{n=1}^{N}[q_n^0/q_n^1]^{1/2}p_n^1 q_n^1/p^1\cdot q^1\right\}\\
&= \left\{\sum_{n=1}^{N}[q_n^1/q_n^0]^{1/2}s_n^0\right\} \Bigg/ \left\{\sum_{n=1}^{N}[q_n^0/q_n^1]^{1/2}s_n^1\right\} \quad \text{using Eq. (37)}\\
&= Q_1(p^0,p^1,q^0,q^1)
\end{aligned} \tag{63}$$

where Q_1 is a quadratic mean of order r quantity index defined by Eq. (36) when $r = 1$. Thus the Walsh implicit quantity index, Q_W^*, is equal to a special case of the quadratic mean of order quantity indexes defined earlier.

[18] Diewert [6] made a case for this index being the “best” pure price or fixed basket type index. The Australian statistician Knibbs [15, pp. 43–44] was perhaps the first to define the class of fixed basket type indexes, which he called unequivocal indexes.

[19] See Eq. (7) above. Fisher [11] was the first to suggest that that the product of the price and quantity indexes should equal the value ratio between the two periods under consideration.

It is not at all obvious what an additive percentage change decomposition for the implicit Walsh quantity index would look like. However, using the decomposition Eq. (35) for $r = 1$ yields the following equation:

$$Q_1 - 1 = \sum_{n=1}^{N} [w_n^0 (q_n^0)^{1/2} + w_n^1 \{Q_1\} (q_n^1)^{1/2}][(q_n^1)^{1/2} - (q_n^0)^{1/2}] \tag{64}$$

where Q_1 is defined by Eq. (63). Now multiply the numerator and the denominator of the nth term on the right hand side of Eq. (64) by $(q_n^1)^{1/2} + (q_n^1)^{1/2}$ for $n = 1, \ldots, N$. The resulting equation is:

$$\begin{aligned} Q_1 - 1 &= \sum_{n=1}^{N} [w_n^0 \{(q_n^0)^{1/2} / [(q_n^1)^{1/2} + (q_n^1)^{1/2}]\} \\ &\quad + w_n^1 Q_1 \{(q_n^1)^{1/2} / [(q_n^1)^{1/2} + (q_n^1)^{1/2}]\} [q_n^1 - q_n^0] \\ &= \sum_{n=1}^{N} Q_{1n} [q_n^1 - q_n^0] \end{aligned} \tag{65}$$

where the nth Walsh percentage change quantity weight Q_{1n} is defined as

$$\begin{aligned} Q_{1n} &\equiv w_n^0 \{(q_n^0)^{1/2} / [(q_n^1)^{1/2} + (q_n^1)^{1/2}]\} \\ &\quad + w_n^1 Q_1 \{(q_n^1)^{1/2} / [(q_n^1)^{1/2} + (q_n^1)^{1/2}]\}. \end{aligned} \tag{66}$$

Note that the nth percentage change quantity weight is almost a weighted average (with weights $(q_n^0)^{1/2} / [(q_n^1)^{1/2} + (q_n^1)^{1/2}]$ and $(q_n^1)^{1/2} / [(q_n^1)^{1/2} + (q_n^1)^{1/2}]$ which sum to unity) of the two normalized prices for commodity n in the two periods under consideration, $w_n^t \equiv p_n^t / p^t \cdot q^t$ for $t = 0, 1$. However, as was the case with the Fisher decomposition defined earlier by Eqs (46) and (47), the period 1 normalized price w_n^1 gets an extra weighting factor equal to Q_1, the value of the Walsh implicit quantity index going from period 0 to 1.

The counterpart to the Walsh price index defined by Eq. (61) above is the Walsh [25, p. 103] quantity index Q_W defined as follows:[20]

$$Q_W(p^0, p^1, q^0, q^1) \equiv \sum_{n=1}^{N} [p_n^0 p_n^1]^{1/2} q_n^1 / \sum_{n=1}^{N} [p_n^0 p_n^1]^{1/2} q_n^0. \tag{67}$$

It is easy to see that the Walsh quantity index has the additive form defined by Eq. (52) where the nth price weight p_n^* is the geometric mean of the period 0 and

[20] The Walsh quantity index is a special case of Knibb's [15, pp. 43–44] class of unequivocal quantity indexes. See Diewert [6] for further discussion.

1 prices for commodity n, $[p_n^0 p_n^1]^{1/2}$. Thus the Walsh quantity index also has an additive percentage change decomposition; recall Eqs (53) and (54) above.

The Walsh [25, p. 103] implicit price index that corresponds to the Walsh quantity index Q_W defined by Eq. (67) is defined as follows:

$$\begin{aligned}
&P_W^*(p^0,p^1,q^0,q^1)\\
&\equiv [p^1\cdot q^1/p^0\cdot q^0]/Q_W(p^0,p^1,q^0,q^1)\\
&=\left\{\sum_{n=1}^N [p_n^0 p_n^1]^{1/2} q_n^0/p^0\cdot q^0\right\}\\
&\quad\bigg/\left\{\sum_{n=1}^N [p_n^0 p_n^1]^{1/2} q_n^1/p^1\cdot q^1\right\} && \text{using Eq. (67)}\\
&=\left\{\sum_{n=1}^N [p_n^1/p_n^0]^{1/2} p_n^0 q_n^0/p^0\cdot q^0\right\}\\
&\quad\bigg/\left\{\sum_{n=1}^N [p_n^0/p_n^1]^{1/2} p_n^1 q_n^1/p^1\cdot q^1\right\}\\
&=\left\{\sum_{n=1}^N [p_n^1/p_n^0]^{1/2} s_n^0\right\}\bigg/\left\{\sum_{n=1}^N [p_n^0/p_n^1]^{1/2} s_n^1\right\} && \text{using Eq. (37)}\\
&=P_1(p^0,p^1,q^0,q^1)
\end{aligned} \tag{68}$$

where P_1 is a quadratic mean of order r price index defined by Eq. (42) when $r = 1$. Thus the Walsh implicit price index, P_W^*, is equal to a special case of the quadratic mean of order quantity indexes defined earlier.

We can repeat the algebra associated with Eqs (64) and (65) above using the decomposition Eq. (40) in place of Eq. (35) to show that the Walsh implicit price index has the following additive percentage change decomposition:

$$P_1 - 1 = \sum_{n=1}^N P_{1n}[p_n^1 - p_n^0] \tag{69}$$

where the nth Walsh percentage change price weight P_{1n} is defined as

$$\begin{aligned}
P_{1n} &\equiv v_n^0\{(p_n^0)^{1/2}/[(p_n^1)^{1/2}+(p_n^1)^{1/2}]\}\\
&\quad + v_n^1 P_1\{(p_n^1)^{1/2}/[(p_n^1)^{1/2}+(p_n^1)^{1/2}]\}.
\end{aligned} \tag{70}$$

Note that the nth percentage change price weight is almost a weighted average (with weights $(p_n^0)^{1/2}/[(p_n^1)^{1/2} + (p_n^1)^{1/2}]$ and $(p_n^1)^{1/2}/[(p_n^1)^{1/2} + (p_n^1)^{1/2}]$ which sum to unity) of the two normalized quantities for commodity n in the two periods under consideration, $v_n^t \equiv q_n^t/p^t \cdot q^t$ for $t = 0, 1$. However, as was the case with the Fisher decomposition defined earlier by Eq. (48), the period 1 normalized quantity v_n^1 gets an extra weighting factor equal to P_1, the value of the Walsh implicit price index going from period 0 to 1.

The results in this section show that all four of the Walsh price and quantity indexes have additive percentage change decompositions. In the following section, we will attempt to provide economic interpretations for the terms in two of these additive decompositions.

7. Economic interpretations for some additive percentage change decompositions

Given that in general an infinite number of additive percentage change decompositions are possible for any given price or quantity index, it will be useful to find decompositions such that each term in the decomposition has an economic interpretation. In this section, we shall show how this can be done for some of the most commonly used superlative index number formulae.[21]

We first need to provide an exact interpretation for each of the N terms on the right hand side of the quadratic identity Eq. (4) above. Let $F(z)$ be the quadratic function defined by Eq. (1) and consider a change in the vector z from the base period situation $z^0 \equiv (z_1^0, z_2^0, \ldots, z_N^0)$ to $(z_1^1, z_2^0, \ldots, z_N^0)$; i.e., only the first component of z changes from the base period value z_1^0 to the period 1 value z_1^1. Then since $F(z_1, z_2, \ldots, z_N)$ is quadratic in z_1, we can apply the quadratic identity Eq. (4) to this change and get the following equation:

$$\begin{aligned}
&F(z_1^1, z_2^0, \ldots, z_N^0) - F(z_1^0, z_2^0, \ldots, z_N^0) \\
&= (1/2)[F_1(z_1^0, z_2^0, \ldots, z_N^0) + F_1(z_1^1, z_2^0, \ldots, z_N^0)][z_1^1 - z_1^0] \\
&= (1/2)[a_1 + 2a_{11}z_1^0 + \sum_{i=2}^{N} 2a_{1i}z_i^0 + a_1 \\
&\quad +2a_{11}z_1^1 + \sum_{i=2}^{N} 2a_{1i}z_i^0][z_1^1 - z_1^0] \qquad (71) \\
&\quad \text{partially differentiating the } F \text{ defined by (1)} \\
&= (1/2)[F_1(z_1^0, z_2^0, \ldots, z_N^0) + 2a_{11}(z_1^1 - z_1^0) + F_1(z_1^0, z_2^0, \ldots, z_N^0)][z_1^1 - z_1^0] \\
&= [F_1(z_1^0, z_2^0, \ldots, z_N^0) + a_{11}(z_1^1 - z_1^0)][z_1^1 - z_1^0].
\end{aligned}$$

Now consider a change in z from $(z_1^0, z_2^1, \ldots, z_N^1)$ to $z^1 \equiv (z_1^1, z_2^1, \ldots, z_N^1)$. In a manner analogous to our derivation of Eq. (71), we can show that

$$F(z_1^1, z_2^1, \ldots, z_N^1) - F(z_1^0, z_2^1, \ldots, z_N^1)$$

[21] For the Törnqvist price and quantity indexes, we will obtain multiplicative decompositions rather than additive ones.

$$= (1/2)[F_1(z_1^0, z_2^1, \ldots, z_N^1) + F_1(z_1^1, z_2^1, \ldots, z_N^1)][z_1^1 - z_1^0]$$

$$= (1/2)[a_1 + 2a_{11}z_1^0 + \sum_{i=2}^{N} 2a_{1i}z_i^1 + a_1$$

$$+2a_{11}z_1^1 + \sum_{i=2}^{N} 2a_{1i}z_i^1][z_1^1 - z_1^0] \quad (72)$$

partially differentiating the F defined by (1)

$$= (1/2)[F_1(z_1^0, z_2^1, \ldots, z_N^1) + 2a_{11}(z_1^1 - z_1^0) + F_1(z_1^0, z_2^1, \ldots, z_N^1)][z_1^1 - z_1^0]$$

$$= [F_1(z_1^0, z_2^1, \ldots, z_N^1) + a_{11}(z_1^1 - z_1^0)][z_1^1 - z_1^0].$$

Finally, take the arithmetic average of Eqs (71) and (72) and we obtain the following exact identity:

$$(1/2)[F(z_1^1, z_2^0, \ldots, z_N^0) - F(z_1^0, z_2^0, \ldots, z_N^0)]$$
$$+(1/2)[F(z_1^1, z_2^1, \ldots, z_N^1) - F(z_1^0, z_2^1, \ldots, z_N^1)] \quad (73)$$
$$= (1/2)[F1(z_1^0, z_2^0, \ldots, z_N^0) + F_1(z_1^1, z_2^1, \ldots, z_N^1)][z_1^1 - z_1^0].$$

Note that the right hand side of Eq. (73) is the first term on the right hand side of the quadratic identity Eq. (4). Thus this first term is equal to the arithmetic average of two differences in the level of $F(z)$ where only the first component of the z vector changes in each of these two differences.

We define the left hand side of Eq. (73) as the first difference effect, δ_1. In general, define the nth difference effect, δ_n, as follows:

$$\delta_n \equiv (1/2)[F(z_1^0, \ldots, z_{n-1}^0, z_n^1, z_{n+1}^0 \ldots, z_N^0) - F(z_1^0, z_2^0, \ldots, z_N^0)]$$
$$+(1/2)[F(z_1^1, z_2^1, \ldots, z_N^1) - F(z_1^1, \ldots, z_{n-1}^1, z_n^0, z_{n+1}^1, \ldots, z_N^1)]; \quad (74)$$
$$n = 1, 2, \ldots, N.$$

Thus δ_n is the arithmetic average of two hypothetical changes in $F(z)$ where in the first (second) change, only the nth component changes from its period 0 level of z_n^0 to its period 1 level z_n^1 and all other components of z are held constant at their period 0 (1) levels. In a manner analogous to our derivation of Eq. (73), we can show that δ_n is equal to the nth term on the right hand side of the quadratic identity Eq. (4); i.e., we have:

$$\delta_n = (1/2)[F_n(z_1^0, z_2^0, \ldots, z_N^0) + F_n(z_1^1, z_2^1, \ldots, z_N^1)][z_n^1 - z_n^0];$$
$$n = 1, 2, \ldots, N. \quad (75)$$

We now have to translate Eqs (74) and (75) into our generalized quadratic identity framework. If $f(q)$ is defined by Eq. (10), it is straightforward to show that the counterpart to Eq. (75) is

$$\begin{aligned}\delta_n &= (1/2)[\{f_n(q^0)g'[f(q^0)]/h'(q_n^0)\}\\ &\quad +\{f_n(q^1)g'[f(q^1)]/h'(q_n^1)\}][h(q_n^1)-h(q_n^0)];\\ &\quad n=1,2,\ldots,N\end{aligned} \tag{76}$$

where δ_n is now defined as follows:

$$\begin{aligned}\delta_n &\equiv (1/2)\{g[f(q_1^0,\ldots,q_{n-1}^0,q_n^1,q_{n+1}^0,\ldots,q_N^0)]\\ &\quad -g[f(q_1^0,q_2^0,\ldots,q_N^0)]\}+(1/2)\{g[f(q_1^1,q_2^1,\ldots,q_N^1)]\\ &\quad -g[f(q_1^1,\ldots,q_{n-1}^1,q_n^0,q_{n+1}^1\ldots,q_N^1)]\};\\ &\quad n=1,2,\ldots,N.\end{aligned} \tag{77}$$

Note that the right hand side of Eq. (76) is the nth term in our generalized quadratic identity and Eq. (77) gives an economic interpretation for this term in terms of differences in $g[f(q)]$ where only the nth component of q changes. Thus each of the N terms on the right hand side of the generalized quadratic identity Eq. (17) has an economic interpretation as an average of two finite differences in the level of our transformed aggregator function $g[f(q)]$ where only one component of q changes in each of the finite differences.

We now specialize Eqs (76) and (77) by considering specific functions for g and h.

The first special case that we consider is the case where g and h are the natural logarithm functions (recall Eq. (18) above), which gave rise to the translog aggregator function defined by Eqs (20) and (21). In this case, the generalized quadratic identity Eq. (17) became Eq. (22). Thus we have:

$$\begin{aligned}\ln[f(q^1)/f(q^0)] &= \sum_{n=1}^{N}(1/2)[s_n^0+s_n^1]\ln[q_n^1/q_n^0]\\ &= \sum_{n=1}^{N}\delta_n\end{aligned} \tag{78}$$

where δ_n is defined by Eq. (77) where g is the logarithm function in this special case.

It is useful to introduce some additional notation at this point. Define the base period nth quantity effect σ_n^0 as the relative change in the aggregate going from the base period quantities q^0 to new quantities where we only change q_n to the period 1 level, q_n^1; i.e., define σ_n^0 as follows:

$$\begin{aligned}\sigma_n^0 &\equiv f(q_1^0,\ldots,q_{n-1}^0,q_n^1,q_{n+1}^0\ldots,q_N^0)/f(q_1^0,q_2^0,\ldots,q_N^0);\\ &\quad n=1,2,\ldots,N.\end{aligned} \tag{79}$$

Define the current period nth quantity effect σ_n^1 as the relative change in the aggregate going to the current period quantities q^1 from quantities where all quantities are at their period 1 levels except q_n is equal to the period 0 level, q_n^0; i.e., define σ_n^1 as follows:

$$c_n^1 \equiv f(q_1^1, q_2^1, \ldots, q_N^1)/f(q_1^1, \ldots, q_{n-1}^1, q_n^0, q_{n+1}^1 \ldots, q_N^1); \quad n = 1, 2, \ldots, N. \tag{80}$$

Finally, define the nth quantity effect c_n as the geometric mean of the base and current period quantity effects defined by Eqs (79) and (80); i.e., define

$$c_n \equiv [\sigma_n^0 \sigma_n^1]^{1/2}; \quad n = 1, 2, \ldots, N. \tag{81}$$

Using this new notation and exponentiating both sides of Eq. (78), we obtain the following decomposition for the Törnqvist quantity index, $Q_T(p^0, p^1, q^0, q^1)$ (recall Eq. (23) above):[22]

$$\begin{aligned} f(q^1)/f(q^0) &= Q_T(p^0, p^1, q^0, q^1) \\ &= \prod_{n=1}^{N} \exp[\delta_n] \quad \text{where } \exp[y] \equiv e^y \\ &= \prod_{n=1}^{N} [\sigma_n^0 \sigma_n^1]^{1/2} \\ &= \prod_{n=1}^{N} c_n. \end{aligned} \tag{82}$$

Thus we have an exact multiplicative decomposition of the Törnqvist quantity index Q_T into a product of N quantity effects, $\prod_{n=1}^{N} c_n$, where each quantity effect is a quantity index which shows the effect of changing just the nth quantity from q_n^0 to q_n^1; see Eqs (79) to (81) above.

The same algebra works for a multiplicative decomposition for the Törnqvist price index P_T defined earlier by Eqs (25) and (26). Again, we introduce some additional notation in order to define the terms in the decomposition. Define the base period nth price effect ρ_n^0 as the relative change in the aggregate going from the base period prices p^0 to new prices where we only change p_n to the period 1 level, p_n^1; i.e., define ρ_n^0 as follows:

$$\rho_n^0 \equiv c(p_1^0, \ldots, p_{n-1}^0, p_n^1, p_{n+1}^0 \ldots, p_N^0)/c(p_1^0, p_2^0, \ldots, p_N^0); \quad n = 1, 2, \ldots, N \tag{83}$$

[22]For similar decompositions in the profit or revenue function context, see Diewert and Morrison [7, pp. 666–667] and Kohli [13].

where $c(p)$ is the translog unit cost function defined in Section 3 above. Define the current period nth price effect ρ_n^1 as the relative change in the aggregate going to the current period prices p^1 from prices where all prices are at their period 1 levels except p_n is equal to the period 0 level, p_n^0; i.e., define ρ_n^1 as follows:

$$\rho_n^1 \equiv c(p_1^1, p_2^1, \ldots, p_N^1)/c(p_1^1, \ldots, p_{n-1}^1, p_n^0, p_{n+1}^1, \ldots, p_N^1); \quad n = 1, 2, \ldots, N. \tag{84}$$

Finally, define the nth price effect b_n as the geometric mean of the base and current period quantity effects defined by Eqs (83) and (84); i.e., define

$$b_n \equiv [\rho_n^0 \rho_n^1]^{1/2}; \quad n = 1, 2, \ldots, N. \tag{85}$$

Using this new notation, we obtain the following decomposition for the Törnqvist price index, $P_T(p^0, p^1, q^0, q^1)$ (recall Eq. (26) above):[23]

$$\begin{aligned} c(p^1)/c(p^0) &= P_T(p^0, p^1, q^0, q^1) \\ &= \prod_{n=1}^{N} \exp\{(1/2)[s_n^0 + s_n^1] \ln [p_n^1/p_n^0]\} \\ &= \prod_{n=1}^{N} [\rho_n^0 \rho_n^1]^{1/2} \\ &= \prod_{n=1}^{N} b_n. \end{aligned} \tag{86}$$

Thus we have an exact multiplicative decomposition of the Törnqvist price index P_T into a product of N price effects, $\prod_{n=1}^{N} b_n$, where each price effect is a price index which shows the effect of changing just the nth price from p_n^0 to p_n^1.

We turn now to our second special case of Eqs (76) and (77) where g and h are defined by Eq. (28) for $r \neq 0$ and the restrictions Eq. (27) are satisfied. Thus $f(q)$ is the quadratic mean of order r aggregator function defined by Eq. (29) for $r \neq 0$. Using Eqs (76) and (77) above, the generalized quadratic identity Eq. (34) in this case becomes:

$$\begin{aligned} &[f^1]^r - [f^0]^r \\ &= \sum_{n=1}^{N} [w_n^0 \{f^0\}^r (q_n^0)^{1-r/2} + w_n^1 \{f^1\}^r (q_n^1)^{1-r/2}][(q_n^1)^{r/2} - (q_n^0)^{r/2}] \\ &= \sum_{n=1}^{N} \delta_n \end{aligned} \tag{87}$$

[23] See Diewert and Morrison [7, pp. 666–667] and Kohli [13] for similar decompositions.

where δ_n defined in general by Eq. (77) becomes the following expression when the restrictions Eqs (27) and (28) are satisfied:

$$\begin{aligned}
&\delta_n \\
&= (1/2)\{[f(q_1^0, \ldots, q_{n-1}^0, q_n^1, q_{n+1}^0, \ldots, q_N^0)]^r - [f(q_1^0, q_2^0, \ldots, q_N^0)]^r\} \\
&\quad + (1/2)\{[f(q_1^1, q_2^1, \ldots, q_N^1)]^r - [f(q_1^1, \ldots, q_{n-1}^1, q_n^0, q_{n+1}^1 \ldots, q_N^1)]r\}; \quad (88) \\
&\quad n = 1, 2, \ldots, N \\
&= (1/2)[f^0]^r\{[\sigma_n^0]^r - 1\} + (1/2)[f^1]^r\{1 - [1/\sigma_n^1]^r\}
\end{aligned}$$

where the quantity effects σ_n^0 and σ_n^1 are defined by Eqs (79) and (80).

Now specialize Eq. (87) to the case where $r = 1$. Upon dividing both sides of Eq. (87) by f^0, we obtain the following additive percentage change decomposition for the implicit Walsh quantity index Q_1 defined earlier by Eq. (63):

$$Q_1 - 1 = \sum_{n=1}^{N} \delta_n / f^0 \qquad (89)$$

where δ_n/f^0 is defined as

$$\begin{aligned}
\delta_n/f^0 &\equiv (1/2)\{\sigma_n^0 - 1\} + (1/2)Q_1\{1 - [1/\sigma_n^1]\}; \quad n = 1, 2, \ldots, N \\
&= Q_{1n}[q_n^1 - q_n^0]
\end{aligned} \qquad (90)$$

and where the nth Walsh percentage change quantity weight Q_{1n} was defined by Eq. (66). Thus the nth term in the additive percentage change decomposition for Q_1 given by Eq. (65), $Q_{1n}[q_n^1 - q_n^0]$, can be interpreted as a weighted sum of the percentage changes in the two single variable changes, $\sigma_n^0 - 1$ and $1 - [1/\sigma_n^1]$, where σ_n^0 and σ_n^1 are defined by Eqs (79) and (80). The weighted sum is an arithmetic average of the changes $\sigma_n^0 - 1$ and $1 - [1/\sigma_n^1]$ if the index Q_1 is equal to one.[24]

The above algebra can be adapted to provide an economic interpretation for the terms in the additive percentage change decomposition Eq. (69) that we obtained earlier for the Walsh implicit price index P_1. Thus we have

$$\begin{aligned}
P_{1n}[p_n^1 - p_n^0] &= (1/2)\{\rho_n^0 - 1\} + (1/2)P_1\{1 - [1/\rho_n^1]\}; \\
&\qquad n = 1, 2, \ldots, N
\end{aligned} \qquad (91)$$

where the nth Walsh percentage change price weight P_{1n} was defined earlier by Eq. (70) and the base period nth price effects ρ_n^0 and the current period nth price effects ρ_n^1 were defined by Eqs (83) and (84). Thus the nth term in the additive

[24] If σ_n^1 is close to one, then $1 - [1/\sigma_n^1]$ will be close to $\sigma_n^1 - 1$. These two expressions have the same first order Taylor series approximations around the point of approximation $\sigma_n^1 = 1$.

percentage change decomposition for P_1 given by Eq. (69), $P_{1n}[p_n^1 - p_n^0]$, can be interpreted as a weighted sum of the percentage changes in the two single variable changes, $\rho_n^0 - 1$ and $1 - [1/\rho_n^1]$, where ρ_n^0 and ρ_n^1 are defined by Eqs (83) and (84). The weighted sum is an arithmetic average of the changes $\rho_n^0 - 1$ and $1 - [1/\rho_n^1]$ if the overall price index P_1 is equal to one.

Now specialize Eq. (87) to the case where $r = 2$. Upon dividing both sides of Eq. (87) by $f^0[f^0 + f^1]$, we obtain the following additive percentage change decomposition for the Fisher ideal quantity index $Q_2 = Q_F$ defined earlier by Eq. (42):

$$Q_F - 1 = \sum_{n=1}^{N} \delta_n / f^0[f^0 + f^1] \tag{92}$$

where $\delta_n / f^0[f^0 + f^1]$ is defined for $n = 1, 2, \ldots, N$ as

$$\begin{aligned}
&\delta_n / f^0[f^0 + f^1] \\
&\equiv [(1/2)[f^0]^2\{[\sigma_n^0]^2 - 1\} + (1/2)[f^1]^2\{1 - [1/\sigma_n^1]^2\}/f^0[f^0 + f^1] \\
&= (1/2)\{f^0/[f^0 + f^1]\}\{[\sigma_n^0]^2 - 1\} \\
&\quad +(1/2)Q_F\{f^1/[f^0 + f^1]\}\{1 - [1/\sigma_n^1]^2\} \\
&= (1/2)\{1/[1 + Q_F]\}\{[\sigma_n^0]^2 - 1\} \\
&\quad +(1/2)[Q_F]^2\{1/[1 + Q_F]\}\{1 - [1/\sigma_n^1]^2\} \\
&= Q_{Fn}[q_n^1 - q_n^0]
\end{aligned} \tag{93}$$

and where the nth Fisher percentage change quantity weight Q_{Fn} was defined by Eq. (47). Thus the nth term in the additive percentage change decomposition for Q_F given by Eq. (46), $Q_{Fn}[q_n^1 - q_n^0]$, can be interpreted as a weighted sum of the changes in the two single variable changes, $[\sigma_n^0]^2 - 1$ and $1 - [1/\sigma_n^1]^2$, where σ_n^0 and σ_n^1 are defined by Eqs (79) and (80). If the Fisher quantity index Q_F equals 1, then Eq. (93) becomes:

$$\begin{aligned}
&Q_{Fn}[q_n^1 - q_n^0] \\
&= (1/4)\{[\sigma_n^0]^2 - 1\} + (1/4)\{1 - [1/\sigma_n^1]^2\} \\
&= (1/4)\{[\sigma_n^0 - 1][\sigma_n^0 + 1]\} + (1/4)\{[1 - (1/\sigma_n^1)][1 + (1/\sigma_n^1)]\} \\
&\approx (1/2)[\sigma_n^0 - 1] + (1/2)[1 - (1/\sigma_n^1)]
\end{aligned} \tag{94}$$

where the last approximation follows if the two quantity effects σ_n^0 and σ_n^1 are close to one. Thus under normal conditions when all of the quantity indexes are close to one, the nth term in the additive percentage change decomposition for Q_F given by

Eq. (46), $Q_{Fn}[q_n^1 - q_n^0]$, will be approximately equal to the arithmetic average of the two single variable index changes, $\sigma_n^0 - 1$ and $1 - (1/\sigma_n^1)$.[25]

Of course, the above algebra can be adapted to provide an economic interpretation for the terms in the additive percentage change decomposition Eq. (48) that we obtained earlier for the Fisher price index $P_2 = P_F$. Thus we have for $n = 1, 2, \ldots, N$:

$$P_{Fn}[p_n^1 - p_n^0] = (1/2)\{1/[1 + P_F]\}\{[\rho_n^0]^2 - 1\} \\ +(1/2)[P_F]^2\{1/[1 + P_F]\}\{1 - [1/\rho_n^1]^2\} \tag{95}$$

where the nth Fisher percentage change price weight P_{Fn} was defined earlier by Eq. (50) and the base period nth price effects ρ_n^0 and the current period nth price effects ρ_n^1 were defined by Eqs (83) and (84). Thus if P_F, ρ_n^0 and ρ_n^1 are all close to one, then the nth term in the additive percentage change decomposition for P_F given by Eq. (48), $P_{Fn}[p_n^1 - p_n^0]$, is approximately equal to the arithmetic average of the percentage changes in the two single variable changes, $\rho_n^0 - 1$ and $1 - [1/\rho_n^1]$, where ρ_n^0 and ρ_n^1 are defined by Eqs (83) and (84).

8. Conclusion

The results in the previous sections demonstrate that the quadratic identity Eq. (4) and its generalizations provide a unifying framework for deriving all of the most commonly used superlative index number formula. In addition, the single variable quadratic identity Eq. (73) and its generalizations have proven to be very useful in providing economic interpretations for some additive percentage change decompositions for these commonly used superlative indexes.[26]

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